Three notes on Ser's and Hasse's representations for the zeta-functions

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Abstract

This paper contains three notes concerning Ser's and Hasse's series representations for the zeta-functions. All notes are presented as theorems. The first theorem shows that the famous Hasse's series for the Euler–Riemann zeta-function, derived in 1930 and named after the German mathematician Helmut Hasse, is equivalent to an earlier expression given by a little-known French mathematician Joseph Ser in 1926. The second theorem provides a comparatively simple series representation for the zeta-function in terms of the Cauchy numbers of the second kind (Nørlund numbers). This series is complimentary to another Ser's result and also gives rise to a new series expansion for the Stieltjes constants and for the MacLaurin coefficients of the regularized zeta-function. In the third theorem, previous results are generalized to the Hurwitz zeta-function and involve special polynomials (close to the Bernoulli polynomials), of which Cauchy numbers of the second kind and Gregory's coefficients are simple particular cases. In particular, in this theorem, three series representations for the Hurwitz zeta–function and two series representations for the zeta–functions are obtained. These representations are complimentary to Hasse's series (they contain the same finite differences) and also generalize Ser's results. These expansions lead to various series expansions for the generalized Stieltjes constants (including series with rational terms for Euler's constant), for the MacLaurin coefficients of the regularized Hurwitz zeta-function, for the logarithm of the gamma-function and for the digamma function. Finally, several "unpublished" contributions of Charles Hermite related to these results are also mentioned.

Keywords: Zeta-function, Hasse, Ser, Finite difference, Generalized Bernoulli numbers, Bernoulli polynomials, Stirling numbers, Cauchy numbers, Norlund numbers, Gregory coefficients, Logarithmic numbers, Stieltjes constants, Interpolation.

I. Introduction

The Euler–Riemann zeta-function

$$\zeta(s) \equiv \sum_{n=1}^{\infty} n^{-s} = \prod_{n=1}^{\infty} \left(1 - p_n^{-s}\right)^{-1}, \qquad \begin{array}{l} \operatorname{Re} s > 1 \\ p_n \in \mathbb{P} \equiv \{2,3,5,7,11,\ldots\} \end{array}$$

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and its most common generalization the Hurwitz zeta-function

$$\zeta(s,v) \equiv \sum_{n=0}^{\infty}(n+v)^{-s}\,, \qquad \begin{array}{l} \operatorname{Re} s > 1 \\ v \neq 0, -1, -2, \ldots \end{array}$$

$\zeta(s) = \zeta(s,1)$, are some of the most important special functions in analysis and number theory. They were studied by many famous mathematicians, including Stirling, Euler, Malmsten, Clausen, Kinkelin, Riemann, Hurwitz, Lerch, Landau, and continue to receive considerable attention from modern researchers. In 1930, the German mathematician Helmut Hasse published a paper [28], in which he obtained and studied these globally convergent series for the ζ–functions

$$\zeta(s) \;=\; \frac{1}{s-1}\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+1)^{1-s} \tag{1}$$

$$\zeta(s,v) \;=\; \frac{1}{s-1}\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+v)^{1-s} \tag{2}$$

containing finite differences $\Delta^n 1^{1-s}$ and $\Delta^n v^{1-s}$ respectively.[1] Hasse also remarked[2] that the first series is quite similar to the Euler transformation of the η-function series $\sum\limits_{n=1}^{\infty}(-1)^{n+1}n^{-s} = \big(1-2^{1-s}\big)\,\zeta(s)\,,\ \operatorname{Re} s > 0\,$, i.e.

$$\zeta(s) \;=\; \frac{1}{1-2^{1-s}}\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+1)^{-s} \tag{3}$$

which also contain the finite difference $\Delta^n 1^{1-s}$.[1] Formulæ (1)–(2) have become widely known, and in literature they are often referred to as *Hasse's formulæ* for the ζ-functions. Moreover, some of them were subsequently rediscovered several times throughout the following decades (e.g., by Jonathan Sondow [58], [52], [20]). At the same time, it is much less known that 4 years earlier, a little-known French mathematician Joseph Ser published a paper [50] containing very similar results. In particular, he showed that

$$\zeta(s) \;=\; \frac{1}{s-1}\sum_{n=0}^{\infty}\frac{1}{n+2}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+1)^{-s} \tag{4}$$

as well as gave this curious series

$$\begin{aligned}\zeta(s) \;&=\; \frac{1}{s-1}+\sum_{n=0}^{\infty}\big|G_{n+1}\big|\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+1)^{-s} \;= \\ &=\; \frac{1}{s-1}+\frac{1}{2}+\frac{1}{12}\big(1-2^{-s}\big)+\frac{1}{24}\big(1-2\cdot 2^{-s}+3^{-s}\big)+\ldots\end{aligned} \tag{5}$$

[1] For the definition of the finite difference operator, see (28).
[2] Stricktly speaking, the remark was communicated to him by Konrad Knopp.

[50, Eq. (4), p. 1076][3], [5, p. 382], to which Charles Hermite was also very close already in 1900.[4] Numbers G_n appearing in the latter expansion are known as *Gregory's coefficients* and may also be called by some authors *(reciprocal) logarithmic numbers*, *Bernoulli numbers of the second kind*, normalized *generalized Bernoulli numbers* $B_n^{(n-1)}$ and normalized *Cauchy numbers of the first kind* $C_{1,n}$. They are rational and may be defined either via their generating function

$$\frac{z}{\ln(1+z)} = 1 + \sum_{n=1}^{\infty} G_n z^n, \qquad |z| < 1, \tag{6}$$

or explicitly

$$G_n = \frac{C_{1,n}}{n!} = \lim_{s\to n} \frac{-B_s^{(s-1)}}{(s-1)\,s!} = \frac{1}{n!}\sum_{l=1}^{n} \frac{S_1(n,l)}{l+1} = \frac{1}{n!}\int_0^1 (x-n+1)_n\,dx\,, \qquad n = 1,2,3,\ldots \tag{7}$$

where $(x)_n \equiv x\,(x+1)\,(x+2)\cdots(x+n-1)$ stands for the Pochhammer symbol (also known as the rising factorial), so that we have

$$(x-n+1)_n = x\,(x-1)\,(x-2)\cdots(x-n+1) = \sum_{l=1}^{n} S_1(n,l)\,x^l\,, \tag{8}$$

and where $S_1(n,l)$ are the signed Stirling numbers of the first kind.[5] Gregory's coefficients are alternating $G_n = (-1)^{n-1}|G_n|$ and decreasing in absolute value; they behave as $\frac{1}{n\ln^2 n}$ at $n \to \infty$ and may be bounded from below and from above accordingly to formulæ (55)–(56) from [6].[6] The first few coefficients are: $G_1 = +1/2$, $G_2 = -1/12$, $G_3 = +1/24$, $G_4 = -19/720$, $G_5 = +3/160$, $G_6 = -863/60\,480$, ...[7] For more information about these important numbers, see [6, pp. 410–415], [5, p. 379], and the literature given therein (nearly 50 references).

II. On the equivalence between Ser's and Hasse's representations for the Euler–Riemann zeta-function

One may readily remark that Hasse's representation (1) and Ser's representation (4) are very similar, so one may question whether these expressions are equivalent or not. The paper written by Ser [50] is much less cited than that by Hasse [28], and in the few works in which both of them are cited,

[3] Our formula (5) is a corrected version of the original Ser's formula (4) [50, p. 1076] (we also gave the correct variant of this formula in our previous work [5, p. 382]). For Ser's formula (4), the corrections which need to be done are the following. First, in the second line of (2) the last term should be $(-1)^n(n+1)^{-s}$ and not $(-1)^n n^{-s}$. Second, in equation (3), "(1-x((2-x)" should read "(1-x)(2-x)". Third, the region of convergence of formula (4), p. 1076, should be $s \in \mathbb{C}\setminus\{1\}$ and not $s < 1$. Note also that Ser's p_{n+1} are equal to our $|G_n|$. We, actually, carefully examined 5 different hard copies of [50] (from the *Institut Henri Poincaré*, from the *École Normale Supérieure Paris*, from the *Université Pierre-et-Marie-Curie*, from the *Université de Strasbourg* and from the *Bibliothèque nationale de France*), and all of them contained the same misprints.

[4] Sticktly speaking, Charles Hermite aimed to obtain a more general expression (see, for more details, Theorem 3 and footnote 14).

[5] These numbers are defined as coefficients in expansion (8). For more information about these numbers, see [6, Sect. 2] and numerous references given therein. Here we use exactly the same definition for $S_1(n,l)$ as in the cited reference.

[6] For the asymptotics of G_n and its history, see [7, Sect. 3].

[7] Numerators and denominators of G_n may also be found in OEIS A002206 and A002207 respectively.

these series are treated as different and with no connection between.[8] However, as we come to show later, this not true.

In one of our previous works [5, p. 382], we already noticed that these two series are, in fact, equivalent, but this was stated in a footnote and without a proof.[9] Below, we provide a rigorous proof of this statement.

Theorem 1. *Ser's representation for the ζ–function [50, p. 1076, Eq. (7)]*

$$\zeta(s) \;=\; \frac{1}{s-1}\sum_{n=0}^{\infty}\frac{1}{n+2}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+1)^{-s} \tag{9}$$

and Hasse's representation for the ζ–function [28, pp. 460–461]

$$\zeta(s) \;=\; \frac{1}{s-1}\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+1)^{1-s} \tag{10}$$

are equivalent in the sense that one series is a rearranged version of the other.

Proof 1. In view of the fact that

$$\frac{1}{k+1}\cdot\binom{n}{k} = \frac{1}{n+1}\cdot\binom{n+1}{k+1} \quad \text{and that} \quad \frac{1}{(n+2)(n+1)} \;=\; \frac{1}{n+1}-\frac{1}{n+2}\,,$$

Ser's formula (9) multiplied by $(s-1)$ may be written as

$$\begin{aligned}
(s-1)\zeta(s) \;&=\; \sum_{n=0}^{\infty}\frac{1}{n+2}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+1)^{-s} = \sum_{n=0}^{\infty}\frac{1}{(n+2)(n+1)}\sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}(k+1)^{1-s} \\
&=\; \sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}(k+1)^{1-s} - \sum_{n=0}^{\infty}\frac{1}{n+2}\sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}(k+1)^{1-s} \\
&=\; 1+\sum_{n=1}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n+1}{k+1}(k+1)^{1-s} - \sum_{n=1}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n-1}(-1)^k\binom{n}{k+1}(k+1)^{1-s} \\
&=\; 1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)^s} + \sum_{n=1}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n-1}\frac{(-1)^k}{(k+1)^{s-1}}\left\{\binom{n+1}{k+1}-\binom{n}{k+1}\right\} \\
&=\; 1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)^s} + \sum_{n=1}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n-1}\frac{(-1)^k}{(k+1)^{s-1}}\binom{n}{k} \\
&=\; 1+\sum_{n=1}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}\frac{(-1)^k}{(k+1)^{s-1}}\binom{n}{k} \;=\; \sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}\frac{(-1)^k}{(k+1)^{s-1}}\binom{n}{k}
\end{aligned}$$

where, between the fourth and fifth lines, we resorted to the well–known recurrent property of the binomial coefficients. The last line is identical with Hasse's formula (10). □

[8] A simple internet search with "Google scholar" indicates that Hasse's paper [28] is cited more than 50 times, while Ser's paper [50] is cited only 12 times (including several on-line resources such as [57], as well as some incorrect items), and all the citations are very recent. These citations mainly regard an infinite product for e^{γ}, e.g. [53], [8], [15], [14], and we found only two works [20], [23], where both articles [50] and [28] were cited simultaneously and in the context of series representations for $\zeta(s)$.

[9] We, however, indicated that a recurrence relation for the binomial coefficients should be used for the proof.

Corollary 1a. Series (9) and (10) are equivalent in the sense that (10) may be obtained by an appropriate rearrangement of terms in (9) and *vice-versa*. However, it is not difficult to see that their rates of convergence differ: the rate of convergence of Ser's series for the argument s corresponds to that of Hasse's series for the argument $s-1$.

Corollary 1b. Series (9), (10) and (5) may be readily used to get corresponding expressions for the Stieltjes constants γ_m. We recall that the latter γ_m , $m=0,1,2,\ldots$, are the coefficients appearing in the regular part of the Laurent series expansion of $\zeta(s)$ about its unique pole $s=1$

$$\zeta(s) \,=\, \frac{1}{s-1} + \gamma + \sum_{m=1}^{\infty} \frac{(-1)^m \gamma_m}{m!} \, (s-1)^m \,, \qquad\qquad s \neq 1\,, \tag{11}$$

and $\gamma_0 = \gamma$.[10] Since the function $\zeta(s) - (s-1)^{-1}$ is holomorphic on the entire complex s–plane, it may be expanded into the Taylor series. The latter expansion, applied to (9), (10) and (5) in a neighbourhood of $s=1$, yields

$$\gamma_m \,=\, -\frac{1}{m+1} \sum_{n=0}^{\infty} \frac{1}{n+2} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\ln^{m+1}(k+1)}{k+1} \,, \qquad m=0,1,2\ldots$$

$$\gamma_m \,=\, -\frac{1}{m+1} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \ln^{m+1}(k+1) \,, \qquad m=0,1,2\ldots \tag{12}$$

$$\gamma_m \,=\, \sum_{n=0}^{\infty} \big|G_{n+1}\big| \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\ln^{m}(k+1)}{k+1} \,, \qquad\qquad m=1,2,3,\ldots$$

respectively.

Corollary 1c. Analogously to the Stieltjes constants γ_m, may be introduced the normalized MacLaurin coefficients δ_m of $\zeta(s) - (s-1)^{-1}$

$$\zeta(s) \,=\, \frac{1}{s-1} + \frac{1}{2} + \sum_{m=1}^{\infty} \frac{(-1)^m \delta_m}{m!} \, s^m \,, \qquad\qquad s \neq 1. \tag{13}$$

[11]

Using Ser's formula (5), it is easy to see that δ_m admit the following series representation

$$\delta_m \,=\, \sum_{n=0}^{\infty} \big|G_{n+1}\big| \sum_{k=0}^{n} (-1)^k \binom{n}{k} \ln^{m}(k+1) \,, \qquad m=1,2,3,\ldots \tag{14}$$

III. A series for the zeta-function with the Cauchy numbers of the second kind (Nørlund numbers)

An appropriate rearrangement of terms in another series of Ser, formula (5), also leads to an interesting result. In particular, one may obtain a series very similar to (5), but containing the normalized Cauchy numbers of the second kind C_n instead of Gregory's coefficients G_n.

The normalized Cauchy numbers of the second kind C_n, related to the ordinary Cauchy numbers of the second kind $C_{2,n}$ as $C_n \equiv C_{2,n}/n!$ (numbers $C_{2,n}$ are also known as signless *generalized Bernoulli*

[10]For more information on γ_m , see [24, p. 166 *et seq.*], [6], [4], and the literature given therein.

[11]Numbers δ_m were studied by numerous authors, including some modern writers, e.g. [37], [51], [21]. They are also related to γ_m; in particular: $\delta_1 = \frac{1}{2}\ln 2\pi - 1$, $\delta_2 = \gamma_1 + \frac{1}{2}\gamma^2 - \frac{1}{2}\ln^2 2\pi - \frac{1}{24}\pi^2 + 2$, and so on, see e.g. [24, p. 168 *et seq.*].

numbers $|B_n^{(n)}|$ and signless *Nørlund numbers*), appear in the series expansion of $\frac{z}{(1\pm z)\ln(1\pm z)}$ and of $\ln\ln(1\pm z)$ in a neighbourhood of zero

$$\begin{cases} \dfrac{z}{(1+z)\ln(1+z)} \;=\; 1+\displaystyle\sum_{n=1}^{\infty} C_n\,(-z)^n\,, & |z|<1\,, \\[3ex] \ln\ln(1+z) \;=\; \ln z+\displaystyle\sum_{n=1}^{\infty}\frac{C_n}{n}\,(-z)^n\,, & |z|<1\,, \end{cases} \tag{15}$$

and may be also defined explicitly

$$C_n \;\equiv\; \frac{C_{2,n}}{n!} \;=\; \frac{|B_n^{(n)}|}{n!} \;=\; \frac{1}{n!}\sum_{l=1}^{n}\frac{|S_1(n,l)|}{l+1} \;=\; \frac{1}{n!}\int\limits_0^1 (x)_n\,dx\,, \qquad n=1,2,3,\ldots \tag{16}$$

Numbers C_n are positive rational and always decrease with n; they behave as $\frac{1}{\ln n}$ at $n\to\infty$ and may be bounded from below and from above accordingly to formulæ (53)–(54) from [6]. The first few values are: $C_1={}^1/_2$, $C_2={}^5/_{12}$, $C_3={}^3/_8$, $C_4={}^{251}/_{720}$, $C_5={}^{95}/_{288}$, $C_6={}^{19\,087}/_{60\,480}$, $\ldots$[12] For more information on the Cauchy numbers of the second kind, see [44, pp. 150–151], [22, p. 12], [45], [2, vol. III, pp. 257–259], [18, pp. 293–294, nº 13], [30, 1, 62, 46], [6, pp. 406, 410, 414–415, 428–430], [5].

Theorem 2. *The ζ–function may be represented by the following globally convergent series*

$$\zeta(s) \;=\; \frac{1}{s-1}+1-\sum_{n=0}^{\infty} C_{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+2)^{-s} \;= \tag{17}$$

$$=\;\frac{1}{s-1}+1-2^{-s-1}-\frac{5}{12}\big(2^{-s}-3^{-s}\big)-\frac{3}{8}\big(2^{-s}-2\cdot 3^{-s}+4^{-s}\big)-\ldots$$

where C_n are the normalized Cauchy numbers of the second kind.

Proof 2. Using Fontana's identity $\sum|G_n|=1$, where the summation extends over $n=[1,\infty)$, see e.g. [6, p. 410, Eq. (20)], Ser's formula (5) takes the form

$$\zeta(s) \;=\; \frac{1}{s-1}+1+\sum_{n=1}^{\infty}\big|G_{n+1}\big|\sum_{k=1}^{n}(-1)^k\binom{n}{k}(k+1)^{-s} \;= \tag{18}$$

$$=\;\frac{1}{s-1}+1+\frac{1}{12}\big(-2^{-s}\big)+\frac{1}{24}\big(-2\cdot 2^{-s}+3^{-s}\big)+\ldots$$

Now, by taking into account the joint recurrent property relating the Cauchy numbers of the second kind to Gregory's coefficients $C_{n-1}-C_n=|G_n|$, see [22, p. 12, Eq. (5)], [6, pp. 429–430], and by employing that of the binomial coefficients

$$\binom{n+1}{k}-\binom{n}{k}=\binom{n}{k-1},$$

[12] See also OEIS A002657 and A002790, which are the numerators and denominators respectively of $C_{2,n}=n!\,C_n$.

we find that

$$\begin{aligned}
\zeta(s)-\frac{1}{s-1}-1 &= \sum_{n=1}^{\infty} C_n \sum_{k=1}^{n}(-1)^k\binom{n}{k}(k+1)^{-s}-\sum_{n=1}^{\infty} C_{n+1}\sum_{k=1}^{n}(-1)^k\binom{n}{k}(k+1)^{-s} \\
&= \sum_{n=0}^{\infty} C_{n+1}\sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}(k+1)^{-s}-\sum_{n=1}^{\infty} C_{n+1}\sum_{k=1}^{n}(-1)^k\binom{n}{k}(k+1)^{-s} \\
&= -C_1\,2^{-s}+\sum_{n=1}^{\infty} C_{n+1}\left\{\sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}(k+1)^{-s}-\sum_{k=1}^{n}(-1)^k\binom{n}{k}(k+1)^{-s}\right\} \\
&= -C_1\,2^{-s}+\sum_{n=1}^{\infty} C_{n+1}\left\{\frac{(-1)^{n+1}}{(n+2)^s}+\sum_{k=1}^{n}(-1)^k\binom{n}{k-1}(k+1)^{-s}\right\} \\
&= -C_1\,2^{-s}-\sum_{n=1}^{\infty} C_{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+2)^{-s} = -\sum_{n=0}^{\infty} C_{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+2)^{-s}
\end{aligned}$$

which is identical with (17). The global convergence of (17) follows from that of (5). □

Corollary 2a. Proceeding analogously to Corollary 1b, we obtain for the Stieltjes constants the following series

$$\gamma_m = -\sum_{n=0}^{\infty} C_{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\ln^m(k+2)}{k+2}, \qquad m=1,2,3,\ldots \tag{19}$$

For Euler's constant γ, we have an expression which may be simplified thanks to the fact that $(-1)^n\Delta^n x^{-1}\big|_{x=2}=\frac{1}{(n+1)\,(n+2)}$

$$\begin{aligned}
\gamma_0 = \gamma &= 1-\sum_{n=0}^{\infty} C_{n+1}\sum_{k=0}^{n}\frac{(-1)^k}{k+2}\binom{n}{k} = 1-\sum_{n=0}^{\infty}\frac{C_{n+1}}{(n+1)\,(n+2)} = \\
&= 1-\frac{1}{4}-\frac{5}{72}-\frac{1}{32}-\frac{251}{14\,400}-\frac{19}{1728}-\frac{19\,087}{2\,540\,160}-\ldots
\end{aligned}$$

the series which we already encountered in earlier works [6, p. 380, Eq. (34)], [5, p. 429, Eq. (95)].

Corollary 2b. Similarly, from Theorem 2 it follows that

$$\delta_m = -\sum_{n=0}^{\infty} C_{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\ln^m(k+2), \qquad m=1,2,3,\ldots \tag{20}$$

IV. Generalizations of series with Gregory's coefficients and Cauchy numbers of the second kind to the Hurwitz zeta–function

Theorem 3. *The Hurwitz zeta–function $\zeta(s,v)$ may be represented by the following series with the finite difference $\Delta^n v^{-s}$*

$$\zeta(s,v) = \frac{v^{1-s}}{s-1}+\sum_{n=0}^{\infty}|G_{n+1}|\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+v)^{-s} = \frac{v^{1-s}}{s-1}+\frac{1}{2}v^{-s}+ \tag{21}$$

$$+\frac{1}{12}\big[v^{-s}-(1+v)^{-s}\big]+\frac{1}{24}\big[v^{-s}-2(1+v)^{-s}+(2+v)^{-s}\big]+\ldots, \qquad \operatorname{Re} v>0,$$

$$\zeta(s,v) \;=\; \frac{(v-1)^{1-s}}{s-1} - \sum_{n=0}^{\infty} C_{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (k+v)^{-s} \;=\; \frac{(v-1)^{1-s}}{s-1} - \frac{1}{2}v^{-s} - \tag{22}$$

$$-\frac{5}{12}\big[v^{-s} - (1+v)^{-s}\big] - \frac{3}{8}\big[v^{-s} - 2\,(1+v)^{-s} + (2+v)^{-s})\big] - \ldots\,, \qquad \operatorname{Re} v > 1\,,$$

and

$$\zeta(s,v) = \frac{1}{m\,(s-1)} \sum_{n=0}^{m-1} (v+a+n)^{1-s} + \frac{1}{m}\sum_{n=0}^{\infty} (-1)^n N_{n+1,m}(a) \sum_{k=0}^{n} (-1)^k \binom{n}{k} (k+v)^{-s} \tag{23}$$

$\operatorname{Re} v > -\operatorname{Re} a\,,\ \operatorname{Re} a \geqslant -1$ *containing Gregory's coefficients G_n, normalized Cauchy numbers of the second kind C_n and polynomials*

$$N_{n,m}(a) \;\equiv\; \frac{1}{n!}\int_{a}^{a+m} (x-n+1)_n\, dx \;=\; \psi_{n+1}(a+m) - \psi_{n+1}(a)\,, \qquad \begin{array}{l} n, m \in \mathbb{N} \\ \operatorname{Re} a \geqslant -1 \end{array} \tag{24}$$

which may also be given via their generating function

$$\frac{(1+z)^{a+m} - (1+z)^a}{\ln(1+z)} \;=\; m \,+ \sum_{n=1}^{\infty} N_{n,m}(a)\, z^n\,, \qquad |z| < 1\,, \tag{25}$$

respectively, function $\psi_n(x)$ being the normalized antiderivative of the falling factorial, also known as the Bernoulli polynomials of the second kind. *All these series are complimentary to Hasse's series* (2)*, which contains the same finite difference $\Delta^n v^{-s}$.*

We first prove the above expansions, and then, in Remark 1, perform a detailed study of the polynomials $N_{n,m}(a)$, which, as we come to show, are closely related to the Bernoulli polynomials of several varieties. In Remark 2 we show that all these formulas may be further generalized. The latter generalization is quite theoretical, but some results, such as, for example,

$$\zeta(s,v) \;= \sum_{l=1}^{k-1} \frac{(-1)^{l+1}\cdot(k-l+1)_l}{(1-s)_l} \cdot \zeta(s-l,v) \,+ \sum_{l=1}^{k} \frac{(-1)^{l}\cdot(k-l+1)_l}{(1-s)_l} \cdot v^{l-s} \,+ \tag{26}$$

$$+\,k\sum_{n=0}^{\infty} (-1)^n G^{(k)}_{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (k+v)^{-s}\,, \qquad G^{(k)}_n \;\equiv\; \frac{1}{n!}\sum_{l=1}^{n} \frac{S_1(n,l)}{l+k}\,,$$

with Gregory's coefficients of higher order $G^{(k)}_n$, may be interesting.

Proof 3. *First variant of proof of* (21)*:* Expanding the function $\zeta(s,x)$ into the *Gregory–Newton interpolation series* (also known as the *forward difference formula*) in a neighborhood of $x = v$, yields

$$\zeta(s,x+v) \;=\; \zeta(s,v) + \sum_{n=1}^{\infty} \frac{(x-n+1)_n}{n!}\, \Delta^n \zeta(s,v)\,. \tag{27}$$

where $\Delta^n f(v)$ is the nth *finite forward difference* of $f(x)$ at point v

$$\Delta^n f(v) \;\equiv\; \Delta^n f(x)\Big|_{x=v} \;=\; \Delta^{n-1} f(v+1) - \Delta^{n-1} f(v) \;=\; \ldots \tag{28}$$

$$\ldots \;=\; \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(n-k+v) \;=\; (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(v+k)$$

with $\Delta^0 f(v) \equiv f(v)$ by convention[13]. Since the operator of finite difference Δ^n is linear and because $\zeta(s,v+1) = \zeta(s,v) - v^{-s}$, it follows from (28) that $\Delta^n \zeta(s,v) = -\Delta^{n-1} v^{-s}$. Formula (27) therefore becomes

$$\zeta(s,v) = \sum_{n=0}^{\infty}(x+v+n)^{-s} + \sum_{n=1}^{\infty}\frac{(x-n+1)_n}{n!}\,\Delta^{n-1}v^{-s}\,. \tag{29}$$

Integrating the latter equality over $x \in [0,1]$ and accounting for the fact that

$$\int_0^1 (x+v)^{-s}dx = \frac{1}{s-1}\left\{v^{1-s} - (v+1)^{1-s}\right\}, \tag{30}$$

as well as using (7), we have

$$\begin{aligned}\zeta(s,v) &= \frac{1}{s-1}\left\{\sum_{n=0}^{\infty}(v+n)^{1-s} - \sum_{n=0}^{\infty}(v+1+n)^{1-s}\right\} + \sum_{n=1}^{\infty} G_n\,\Delta^{n-1}v^{-s} \\ &= \frac{v^{1-s}}{s-1} + \sum_{n=0}^{\infty} G_{n+1}\,\Delta^n v^{-s}\,,\end{aligned} \tag{31}$$

which is identical with (21), because $G_{n+1} = (-1)^n |G_{n+1}|$ and

$$\Delta^n v^{-s} \equiv \Delta^n x^{-s}\big|_{x=v} = (-1)^n \sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+v)^{-s}\,. \tag{32}$$

The reader may also note that if we put $v = 1$ in (29) and (32), then we obtain the correct variant of Ser's formulas (3) and (2) respectively. ☐

Second variant of proof of (21)*:* Consider the generating equation for the numbers G_n, formula (6). Dividing it by z and then putting $z = e^{-x} - 1$ yields

$$\frac{1}{1-e^{-x}} = \frac{1}{x} + |G_1| + \sum_{n=1}^{\infty}|G_{n+1}|\,(1-e^{-x})^n = \tag{33}$$

$$= \frac{1}{x} + |G_1| + \sum_{n=1}^{\infty}|G_{n+1}|\sum_{k=0}^{n}(-1)^k\binom{n}{k}e^{-kx}\,, \qquad x > 0\,, \tag{34}$$

since $|G_{n+1}| = (-1)^n G_{n+1}$. Now, using the well–known integral representation of the Hurwitz ζ-function

$$\zeta(s,v) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{e^{-vx}\,x^{s-1}}{1-e^{-x}}\,dx$$

and Euler's formulæ

$$\frac{v^{1-s}}{s-1} = \frac{1}{\Gamma(s)}\int_0^{\infty} e^{-vx}\,x^{s-2}\,dx\,, \qquad v^{-s} = \frac{1}{\Gamma(s)}\int_0^{\infty} e^{-vx}\,x^{s-1}\,dx$$

[13]Note, however, that due to the fact that finite differences may be defined in slightly different ways and that there also exist *forward*, *central*, *backward* and other finite differences, our definition for $\Delta^n f(v)$ may not be shared by others. Thus, some authors call the quantity $(-1)^n \Delta^n f(v)$ the nth finite difference, see e.g. [56, p. 270, Eq. (14.17)] (we also employed the latter definition in [6, p. 413, Eq. (39)]). For more details on the Gregory–Newton interpolation formula, see e.g. [31, § 9·02], [41, pp. 57–59], [9, Ch. III], [27, Ch. 1 & 9], [44], [36, Ch. 3], [39, Ch. V], [34, Ch. III, pp. 184–185].

we obtain

$$\zeta(s,v)-\frac{v^{1-s}}{s-1}-\big|G_1\big|\,v^{-s}\;=\;\frac{1}{\Gamma(s)}\int\limits_0^\infty e^{-vx}\,x^{s-1}\left\{\frac{1}{1-e^{-x}}-\frac{1}{x}-\big|G_1\big|\right\}dx\;=$$

$$=\;\frac{1}{\Gamma(s)}\sum_{n=1}^{\infty}\big|G_{n+1}\big|\sum_{k=0}^{n}(-1)^k\binom{n}{k}\int\limits_0^\infty e^{-(k+v)x}\,x^{s-1}\,dx\;=\;\sum_{n=1}^{\infty}\big|G_{n+1}\big|\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+v)^{-s}$$

Remarking that $\big|G_1\big|\,v^{-s}$ is actually the term corresponding to $n=0$ in the sum on the right yields (21).[14] □

First variant of proof of (22)*:* Integrating (29) over $x\in[-1,0]$ and remarking that

$$\frac{1}{n!}\int\limits_{-1}^{0}(x-n+1)_n\,dx\;=\;(-1)^n C_n\,,\qquad n=1,2,3,\ldots\,, \tag{35}$$

we have

$$\begin{aligned}\zeta(s,v)\;=&\;\frac{1}{s-1}\left\{\sum_{n=0}^{\infty}(v-1+n)^{1-s}-\sum_{n=0}^{\infty}(v+n)^{1-s}\right\}+\sum_{n=1}^{\infty}(-1)^n C_n\,\Delta^{n-1}v^{-s}\\ =&\;\frac{(v-1)^{1-s}}{s-1}-\sum_{n=0}^{\infty}C_{n+1}\,(-1)^n\Delta^n v^{-s}\,,\end{aligned} \tag{36}$$

which coincide with (22) because of (32).

Second variant of proof of (22)*:* In order to obtain (22), we also may proceed analogously to the demonstration of Theorem 2, in which we replace (5) by (21). The unity appearing from Fontana's series in (17) becomes v^{-s} and the term $(k+2)^{-s}$ becomes $(k+1+v)^{-s}$, that is to say

$$\zeta(s,v)\;=\;\frac{v^{1-s}}{s-1}+v^{-s}-\sum_{n=0}^{\infty}C_{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+v+1)^{-s}\,,\qquad \operatorname{Re}v>0\,. \tag{37}$$

Using the recurrence relation $\zeta(s,v)=\zeta(s,v+1)+v^{-s}$ and rewriting the final result for v instead of $v+1$, we immediately obtain (22).

Proof of (23)*:* Our method of proof, which uses the Gregory–Newton interpolation formula, may be further generalized. By introducing polynomials $N_{n,m}(a)$ accordingly to (24), and then by integrating (29) over $x\in[a,a+m]$, we have

$$\zeta(s,v)=\;\frac{1}{m\,(s-1)}\Big\{\zeta(s-1,v+a)-\zeta(s-1,v+a+m)\Big\}+\frac{1}{m}\sum_{n=1}^{\infty}N_{n,m}(a)\,\Delta^{n-1}v^{-s} \tag{38}$$

[14]It seems appropriate to note here that Charles Hermite in 1900 tried to use a similar method to derive a series with Gregory's coefficients for $\zeta(s,v)$, but his attempt was not succesfull. A carefull analysis of his derivations [29, p. 69], [54, vol. IV, p. 54], reveals that Hermite's errors is due to the incorrect expansion of $\left(1-e^{-x}\right)^{-1}$ into the series with ω_n, which, in turn, leaded him to an incorrect formula for $R(a,s)\equiv\zeta(s,a)$.[15] These results have never been published during Hermite's lifetime and appeared only in epistolary exchanges with the Italian mathematician Salvatore Pincherle, who published them in [29] several months after Hermite's death. Later, these letters were reprinted in [54].

[15]On p. 69 in [29] and p. 540 in [54, vol. IV] in the expansion for $\left(1-e^{-x}\right)^{-1}$ the term ω_1 should be replaced by ω_2 and ω_n by ω_{n+1}. Note that Hermite's $\omega_n=|G_n|$.

Simplifying the expression in curly brackets and reindexing the latter sum immediately yields

$$\zeta(s,v) = \frac{1}{m\,(s-1)}\sum_{n=0}^{m-1}(v+a+n)^{1-s} + \frac{1}{m}\sum_{n=0}^{\infty} N_{n+1,m}(a)\,\Delta^n v^{-s} \tag{39}$$

which is identical with (23). Note that expansions (21)–(22) are both particular cases of (23) at $m=1$. Formula (21) is obtained by setting $a=0$, while (22) corresponds to $a=-1$. □

Remark 1, related to the polynomials $N_{n,m}(a)$ Polynomials $N_{n,m}(a)$ generalize many special numbers and have a variety of interesting properties. First of all, we remark that $N_{n,m}(a)$ are polynomials of degree n in a with rational coefficients. This may be seen from the fact that

$$\begin{aligned} N_{n,m}(a) \;\equiv\; \frac{1}{n!}\int\limits_a^{a+m}(x-n+1)_n\,dx \;&=\; \frac{1}{n!}\sum_{l=1}^{n}\frac{S_1(n,l)}{l+1}\Big\{(a+m)^{l+1}-a^{l+1}\Big\} \\ &=\; \frac{1}{n!}\sum_{l=1}^{n}\frac{S_1(n,l)}{l+1}\sum_{k=0}^{l}a^k m^{l+1-k}\binom{l+1}{k}\,, \end{aligned} \tag{40}$$

Formula (40) is also very handy for the calculation of $N_{n,m}(a)$ with the help of CAS. It is therefore clear that for any $a\in\mathbb{Q}$, polynomials $N_{n,m}(a)$ are simply rational numbers. Some of such examples may be of special interest

$$N_{n,1}(-1) \;=\; (-1)^n C_n\,, \qquad N_{n,1}(0) \;=\; G_n\,, \qquad N_{2n,1}(n-1) \;=\; M_{2n}\,,$$

where M_n are central difference coefficients $M_2 = {}^{-1}\!/_{12}$, $M_4 = {}^{+11}\!/_{720}$, $M_6 = {}^{-191}\!/_{60\,480}$, $M_8 = {}^{+2497}\!/_{3\,628\,800}$, ..., see e.g. [49], [31, § 9·084], [41, p. 186], OEIS A002195 and A002196. Moreover,

$$N_{n+1,1}(a) \;=\; G_{n+1} + aP_n(a)$$

where $P_n(a)$ is a polynomial of degree n in a. The derivative of $N_{n,m}(a)$ is simply

$$\frac{\partial N_{n,m}(a)}{\partial a} \;=\; \frac{(a+m-n+1)_n-(a-n+1)_n}{n!}$$

Polynomials $N_{n,m}(a)$ are related to many other similar polynomials. For instance, at $a=0$ they equal Coffey's polynomials $P_{n+1}(y)$ from Proposition 11 [17, p. 450] $N_{n,m}(0)=P_{n+1}(m)$ and also are close to Nemes' polynomials $c_v(a)$ from [42, p. 601]. Much more similitudes can be found from the generating equation for $N_{n,m}(a)$, which we gave in (25) without proof. Let us prove it. Using (40) and accounting for the absolute convergence, we have

$$\begin{aligned} \sum_{n=1}^{\infty} N_{n,m}(a)\,z^n \;&=\; \sum_{l=1}^{\infty}\frac{(a+m)^{l+1}-a^{l+1}}{l+1}\underbrace{\sum_{n=1}^{\infty}\frac{S_1(n,l)}{n!}\,z^n}_{\ln^l(1+z)/l!} \;=\; \frac{1}{\ln(1+z)}\,\cdot \\ \cdot\left\{\sum_{l=2}^{\infty}\frac{\big[(a+m)\ln(1+z)\big]^l}{l!}-\sum_{l=2}^{\infty}\frac{\big[a\ln(1+z)\big]^l}{l!}\right\} \;&=\; \frac{(1+z)^{a+m}-(1+z)^a}{\ln(1+z)} \;-\; m \end{aligned}$$

in virtue of the generation equation for the Stirling numbers of the first kind, see e.g. [6, p. 408], [5, p. 369]. Polynomials $N_{n,m}(a)$ are, therefore, close to the Stirling polynomials, to Van Veen's polynomials $K_n^{(z)}$ [55], to various generalizations of the Bernoulli numbers/polynomials, including the so–called *Nörlund polynomials* [42, p. 602], which are also known as the generalized Bernoulli polynomials of the second kind [13, p. 324, Eq. (2.1)], and to many other special polynomials, see e.g. [2, Vol. III, § 19], [41, Ch. VI], [38, Vol. I, § 2.8], [12], [44], [45], [25], [26], [60] [11], [10], [47], [35], [48]. The most close connection seems to exist with the *Bernoulli polynomials of the second kind*, denoted by $\psi_n(x)$ by Jordan [32], [33, Ch. 5], [13, p. 324],[16] which have the following generation function[17]

$$\frac{z\,(z+1)^x}{\ln(z+1)} \;=\; \sum_{n=1}^{\infty} \psi_n(x)\, z^n\,, \qquad |z|<1\,, \tag{41}$$

and with the Bernoulli polynomials of higher order, denoted usually by $B_n^{(s)}(x)$, and defined via

$$\frac{z^s\, e^{xz}}{(e^z-1)^s} \;=\; \sum_{n=0}^{\infty} \frac{B_n^{(s)}(x)}{n!}\, z^n\,, \qquad |z|<2\pi\,, \tag{42}$$

see e.g. [41, p. 127], [13, p. 323, Eq. (1.4)], [43], [44], [45]. Indeed, using formula (41), we have for the left part of (25)

$$\frac{(1+z)^{a+m}-(1+z)^a}{\ln(1+z)} \;=\; \sum_{n=1}^{\infty}\big\{\psi_n(a+m)-\psi_n(a)\big\}\,z^{n-1} \;=\; m+\sum_{n=1}^{\infty}\big\{\psi_{n+1}(a+m)-\psi_{n+1}(a)\big\}\,z^n$$

since $\psi_1(x)=x+\frac{1}{2}$. Comparing the latter expression to the right part of (25) immediately yields

$$N_{n,m}(a) \;=\; \psi_{n+1}(a+m)-\psi_{n+1}(a)\,, \qquad n=1,2,3,\ldots \tag{43}$$

Another way to show it is to recall that $\binom{x}{n}\,dx \,=\, d\psi_{n+1}(x)$, see e.g. [32, p. 130], [33, p. 265]. Hence, the antiderivative of the falling factorial is, up to several constants, precisely the function $\psi_{n+1}(x)$, namely

$$\psi_{n+1}(x) \;=\; \frac{1}{n!}\int (x-n+1)_n\, dx \,+\, f_n\,, \tag{44}$$

where f_n is the constant of integration. In virtue of this important property, which is often not mentioned in relation to $\psi_{n+1}(x)$, formula (43) follows immediately from the definition of $N_{n,m}(a)$. Furthermore, from (42) it follows that $B_n^{(n)}(x+1) \,=\, n!\,\psi_n(x)$, see e.g. [41, pp. 129–135], [13, Eq. (2.1) & (2.11)], whence

$$N_{n,m}(a) \;=\; \frac{1}{(n+1)!}\Big\{B_{n+1}^{(n+1)}(a+m+1)-B_{n+1}^{(n+1)}(a+1)\Big\}\,, \qquad n=1,2,3,\ldots \tag{45}$$

This clearly displays a close connection between $N_{n,m}(a)$ and the Bernoulli polynomials of both varieties. The latter have been the object of much research by Nörlund [43], [44], [45], [41, Ch. VI], Jordan [32], [33, Ch. 5], Carlitz [13] and some other authors. At this stage, it may also be useful to provide explicit expression for the first few polynomials $N_{n,m}(a)$

$$\begin{aligned}
N_{1,m}(a) \;&=\; ma+\frac{m^2}{2}\\
N_{2,m}(a) \;&=\; \frac{6ma^2+6am^2-6am+2m^3-3m^2}{12}\\
N_{3,m}(a) \;&=\; \frac{4ma^3+4m^3a^2-12ma^2+6m^2a+8ma-12m^2a+m^4-4m^3+4m^2}{24}
\end{aligned}$$

[16] These polynomials and/or those equivalent to them, were rediscovered in numerous works and by numerous authors (see e.g. [40, p. 1916], [61, p. 3998], [47, § 5.3.2]), so we give here only the most frequent notations and definitions for them.

[17] In [33, Ch. 5, p. 279, Eq. (8)] and in [13, p. 324, Eq. (1.11)], the summation may also start at $n=1$ since $\psi_0(a)$ vanishes identically.

and so on.

Finally, we remark that the complete asymptotics of the polynomials $N_{n,m}(a)$ at large n are given by

$$N_{n,m}(a) \;\sim\; \frac{(-1)^n}{\pi\, n^{a+1}} \sum_{l=0}^{\infty} \frac{1}{\ln^{l+1} n} \cdot \Big[\sin \pi x \cdot \Gamma(x)\Big]^{(l)}_{x=a+1}\,, \qquad \begin{array}{l} n\to\infty \\ \operatorname{Re} a \geqslant -1 \end{array}\,, \tag{46}$$

where ${}^{(l)}$ stands for the lth derivative, and m is natural and finite. In particular, retaining first two terms, we have

$$N_{n,m}(a) \;\sim\; \frac{(-1)^{n+1}}{\pi\, n^{a+1}\ln n} \cdot \left\{ \sin a\pi \cdot \Gamma(a+1) + \frac{\pi \cos \pi a \cdot \Gamma(a+1) + \sin \pi a \cdot \Gamma(a+1) \cdot \Psi(a+1)}{\ln n} \right\} \tag{47}$$

at $n\to\infty$. Both results can be obtained without difficulty from the complete asymptotics of $B_n^{(n)}(x)$ given by Nörlund [45, p. 38]. Note that if $a\in\mathbb{N}_0$, the first term of asymptotics (46)–(47) vanishes, and thus $N_{n,m}(a)$ decreases faster. Remark also that making $a\to 0$ and $a\to -1$ in (46)–(47), we find asymptotics of numbers G_n and C_n respectively.[18]

Remark 2, related to the generalization of our previous results. Formula (23) may be further generalized. Let $\rho(x)$ be the normalized weight such that

$$\int\limits_a^{a+m} \rho(x)\, dx \;=\; 1\,, \quad \text{and let denote} \quad N_{n,m}^{(\rho)}(a) \;\equiv\; \frac{1}{n!}\int\limits_a^{a+m} (x-n+1)_n\, \rho(x)\, dx$$

Performing the same procedure as in the case of (23) and assuming the uniform convergence, we obtain

$$\zeta(s,v) \;=\; \underbrace{\sum_{n=0}^{\infty} \int\limits_a^{a+m} \frac{\rho(x)}{(v+x+n)^s}\, dx}_{F_{n,m,a}[\rho(x)]} + \sum_{n=1}^{\infty} N_{n,m}^{(\rho)}(a)\, \Delta^{n-1} v^{-s} \tag{48}$$

Albeit this generalization appears rather theoretical, it, however, may be useful if the functional $F_{n,m,a}[\rho(x)]$ admits a suitable closed–form and if the series $\sum F_{n,m,a}[\rho(x)]$ converges. Thus, if we simply put $\rho(x)=1/m$, then we retrieve our formula (23). If we put $\rho(x)=k\,x^{k-1}$, where $k\in\mathbb{N}$, and set $a=0$, $m=1$, then it is not difficult to see that

$$N_{n,1}^{(kx^{k-1})}(0) \;=\; \frac{k}{n!}\sum_{l=1}^{n} \frac{S_1(n,l)}{l+k} \;\equiv\; k\, G_n^{(k)} \tag{49}$$

where numbers $G_n^{(k)}$, generalizing Gregory's coefficients $G_n = G_n^{(1)}$ (we call them Gregory's coefficients of higher order), were already studied in our previous work [6, p. 414].[19] Now, remarking that

[18] In [6, p. 414, Eq. (51)], we obtained the complete asymptotics for $C_n = C_{2,n}/n! = |B_n^{(n)}|/n!$ at large n. However, it seems appropriate to notice that the equivalent result may be straightforwardly derived from Nörlund's asymptotics of $B_n^{(n)}(x)$, since $B_n^{(n)}(0) = B_n^{(n)}$, see [45, pp. 27, 38, 40].

[19] In the latter, we, *inter alia*, showed that $|G_n^{(k)}| \sim n^{-1}\ln^{-k-1} n$ at $n\to\infty$.

the repeated integration by parts yields

$$\int x^{k-1}(v+x+n)^{-s}\,dx \;=\; \frac{1}{k}\sum_{l=1}^{k-1}\frac{(-1)^{l+1}(v+x+n)^{l-s}\cdot x^{k-l}\cdot(k-l+1)_l}{(1-s)_l}+ \\ +\frac{(-1)^{k+1}(v+x+n)^{k-s}\cdot(k-1)!}{(1-s)_k}\,, \tag{50}$$

and evaluating the infinite series $\sum F_{n,1,0}[k\,x^{k-1}]$, formula (48) reduces to (26). Note that we have (21) as a particular case of (26) at $k=1$. Moreover, if we put $v=1$ and simplify the second sum in the first line, then we arrive at this curious formula for the ζ–function

$$\zeta(s) \;=\sum_{l=1}^{k-1}\frac{(-1)^{l+1}\cdot(k-l+1)_l}{(1-s)_l}\cdot\zeta(s-l)\;+\;\frac{k}{s-k}\;+k\sum_{n=0}^{\infty}(-1)^n G_{n+1}^{(k)}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+1)^{-s} \tag{51}$$

It is interesting that formula (48) from [6, p. 414] also contains similar shifted values of the ζ–functions.[20]

Corollary 3a. The Euler–Riemann ζ–function admits the following general expansions

$$\zeta(s) = \frac{1}{m\,(s-1)}\sum_{n=1}^{m}(a+n)^{1-s}+\frac{1}{m}\sum_{n=0}^{\infty}(-1)^n N_{n+1,m}(a)\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+1)^{-s} \tag{52}$$

where $a>-1$, $m\in\mathbb{N}$, and

$$\zeta(s) = 1+\frac{1}{m\,(s-1)}\sum_{n=1}^{m}(a+1+n)^{1-s}+\frac{1}{m}\sum_{n=0}^{\infty}(-1)^n N_{n+1,m}(a)\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+2)^{-s} \tag{53}$$

where $a>-2$, $m\in\mathbb{N}$, containing finite differences $\Delta^n 1^{-s}$ and $\Delta^n 2^{-s}$ respectively. Ser's series (5) and our series from Theorem 2 are simple particular cases of the above expansions.[21]

On the one hand, setting $v=1$ in (39) *we immediately obtain* (52)*. On the other hand, putting $v+1$ instead of v and using the relation $\zeta(s,v+1)=\zeta(s,v)-v^{-s}$, equality* (39) *takes the form*

$$\zeta(s,v) = v^{-s}+\frac{1}{m\,(s-1)}\sum_{n=1}^{m}(v+a+n)^{1-s}+\frac{1}{m}\sum_{n=0}^{\infty}N_{n+1,m}(a)\,\Delta^n(v+1)^{-s} \tag{54}$$

At point $v=1$, this equality becomes (53)*. Continuing the process, we may also obtain similar formulas for $\zeta(s)$ containing finite differences $\Delta^n 3^{-s}$, $\Delta^n 4^{-s}$,...*

Corollary 4b. The generalized Stieltjes constants $\gamma_m(v)$, $m=0,1,2,\ldots$, $v\neq 0,-1,-2,\ldots$, are introduced analogously to the ordinary Stieltjes constants

$$\zeta(s,v) \;=\; \frac{1}{s-1}-\Psi(v)+\sum_{m=1}^{\infty}\frac{(-1)^m\gamma_m(v)}{m!}(s-1)^m\,, \qquad s\neq 1. \tag{55}$$

with $\gamma_0(v)=-\Psi(v)$, see e.g. [4, p. 541, Eq. (14)].[22] Following the same line of reasoning as in Corollary 2a, we conclude that the generalized Stieltjes constants admit the following series representations

$$\gamma_m(v) \;=\; -\frac{\ln^{m+1}v}{m+1}+\sum_{n=0}^{\infty}\left|G_{n+1}\right|\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\ln^m(k+v)}{k+v}\,, \tag{56}$$

[20] Formula (48) from [6] and its proof were first released on 5 January 2015 in the 6th arXiv version of the paper. 28 September 2015, a particular case of the same formula for nonnegative integer s was also presented by Xu, Yan and Shi [59, p. 94, Theorem 2.9], who, apparently, were not aware of the arXiv preprint of our work [6] (we have not found the preprint of [59]).

[21] We have Ser's formula when putting $a=0$, $m=1$ in (52), and our Theorem 2 if setting $a=-1$, $m=1$ in (53).

[22] For more information on $\gamma_m(v)$, see [3], [4], and the literature given in the last reference. Note that since $\zeta(s,1)=\zeta(s)$, the generalized Stieltjes constants $\gamma_m(1)=\gamma_m$.

where $\operatorname{Re} v > 0$,

$$\gamma_m(v) = -\frac{\ln^{m+1}(v-1)}{m+1} - \sum_{n=0}^{\infty} C_{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\ln^m(k+v)}{k+v}, \tag{57}$$

where $\operatorname{Re} v > 1$, and

$$\gamma_m(v) = -\frac{1}{r\,(m+1)} \sum_{l=0}^{r-1} \ln^{m+1}(v+a+l) + \frac{1}{r} \sum_{n=0}^{\infty} (-1)^n N_{n+1,r}(a) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\ln^m(k+v)}{k+v}, \tag{58}$$

where $r \in \mathbb{N}$, $\operatorname{Re} a > -1$ and $\operatorname{Re} v > -\operatorname{Re} a$. Since $(-1)^n \Delta^n v^{-1} = \frac{n!}{(v)_{n+1}}$, we have for the zeroth Stieltjes constant, and hence for the digamma function $\Psi(v)$, the following simple expansions

$$\Psi(v) = \ln v - \sum_{n=1}^{\infty} \frac{|G_n|\,(n-1)!}{(v)_n}, \qquad \operatorname{Re} v > 0 \tag{59}$$

$$\Psi(v) = \ln(v-1) + \sum_{n=1}^{\infty} \frac{C_n\,(n-1)!}{(v)_n}, \qquad \operatorname{Re} v > 1 \tag{60}$$

$$\Psi(v) = \frac{1}{r}\sum_{l=0}^{r-1} \ln(v+a+l) + \frac{1}{r}\sum_{n=1}^{\infty} \frac{(-1)^n\, N_{n,r}(a)\,(n-1)!}{(v)_n}, \qquad \begin{array}{l} r \in \mathbb{N} \\ \operatorname{Re} a > -1 \\ \operatorname{Re} v > -a \end{array} \tag{61}$$

respectively. First two representations coincide with the not well-known Binet–Nørlund expansions for the digamma function [6, pp. 428–429, Eqs. (91)–(94)],[23] while the third one seems to be new. From (59) and (60), it also immediately follows that $\ln(v-1) < \Psi(v) < \ln v$ for $v > 1$, since the sums with G_n and C_n keep their sign.[24] We may also obtain series expansions for the digamma function from (26), but the resulting expressions strongly depend on k. For instance, putting $k = 2$ and expanding both sides into the Laurent series (55), we obtain the following formula

$$\Psi(v) = 2\ln\Gamma(v) - 2v\ln v + 2v + 2\ln v - \ln 2\pi + 2\sum_{n=1}^{\infty} \frac{(-1)^n\, G_n^{(2)}\,(n-1)!}{(v)_n}, \qquad \operatorname{Re} v > 0 \tag{62}$$

which relates the Γ–function to its logarithmic derivative.[25] For higher k these expressions become quite cumbersome and also imply derivatives of the ζ–function at negative integers. In particular, for $k = 3$, we deduce

$$\Psi(v) = 3\ln\Gamma(v) - 6\zeta'(-1,v) + 3v^2\ln v - \frac{3}{2}v^2 - 6v\ln(v) + \tag{63}$$

$$+3v + 3\ln v - \frac{3}{2}\ln 2\pi + \frac{1}{2} + 3\sum_{n=1}^{\infty} \frac{(-1)^n\, G_n^{(3)}\,(n-1)!}{(v)_n}, \qquad \operatorname{Re} v > 0$$

[23] Formula (60) reduces to [6, p. 429, Eq. (94), first formula] by putting v instead of $v-1$ and by making use of the recurrence relationship for the digamma function $\Psi(v+1) = \Psi(v) + v^{-1}$.

[24] This simple and important result is not new, but its derivation from (59) and (60) seems to be novel, and in addition, is elementary.

[25] There were many attempts aiming to find possible relationships between these two functions. For instance, in 1842 Carl Malmsten, by trying to find such a relationship, obtained a variant of Gauss' theorem for $\Psi(v)$ at $v \in \mathbb{Q}$, see [3, p. 37, Eq. (23)], [4, p. 584, Eq. (B.4)].

Formula (62) is also interesting in that it gives series with rational terms for $\ln\Gamma(v)$ if $v \in \mathbb{Q}$ (we only need to use Gauss' digamma theorem for this [4, p. 584, Eq. (B.4)]). Note also that all series (59)–(63) converge very rapidly for large v.

Returning to our formulas for γ_m, we may also mention that the particular case $m = 1$ of (56) was earlier given by Coffey [16, p. 2052, Eq. (1.18)], but the general case of (56), as well as (57), seem to be novel. Note also that formulæ (56)–(57) may be rewritten in a slightly different form by means of the recurrent relationship for the generalized Stieltjes constants $\gamma_m(v+1) = \gamma_m(v) - v^{-1}\ln^m v$.

Corollary 4c. Putting in the previous formulas for the digamma function argument $v \in \mathbb{Q}$, we may also obtain series with rational terms for Euler's constant. The most simple is to put $v = 1$. In this case, formula (59) reduces to the famous Fontana–Mascheroni series

$$\Psi(1) = -\gamma = -\sum_{n=1}^{\infty}\frac{|G_n|}{n}, \tag{64}$$

see e.g. [4, p. 539], [6, pp. 406, 413,429–430], [5, p. 379], while (61) gives us

$$\gamma = -\frac{1}{m}\sum_{l=1}^{m}\ln(a+l) - \frac{1}{m}\sum_{n=1}^{\infty}\frac{(-1)^n N_{n,m}(a)}{n}, \qquad \begin{array}{l} m \in \mathbb{N} \\ a > -1 \end{array}. \tag{65}$$

This series generalizes (64) to a large family of series (we have the aforementioned series at $a = 0$ since $G_n = N_{n,1}(0)$). For example, setting $a = -\frac{1}{2}$ and $m = 1$ (the mean value between $a = -1$ corresponding to the coefficients C_n and $a = 0$ corresponding to G_n), we have the following series

$$\gamma = \ln 2 - 0 - \frac{1}{48} - \frac{1}{72} - \frac{223}{23\,040} - \frac{103}{14\,400} - \frac{32\,119}{5\,806\,080} - \frac{1111}{250\,880} - \ldots \tag{66}$$

relating two fundamental constants γ and $\ln 2$. This series, however, converges quite slowly (the first term in (47) is maximal). A more rapidly convergent series may be obtained by setting large integer a. At the same time, it should be noted that precisely for large a, first terms of the series may unexpectedly grow, but after some term they decrease and the series converges. For instance, taking $a = 7$, we have

$$\gamma = -3\ln 2 + \frac{15}{2} - \frac{293}{24} + \frac{1079}{72} - \ldots - \frac{8183}{9\,331\,200} - \frac{530\,113}{4\,790\,016\,000} - \tag{67}$$

Also, the pattern of the sign is not obvious. By the way, adding the series with $a = -\frac{1}{2}$ to that with $a = 1$, we eliminate $\ln 2$ and thus get a series with rational terms only for Euler's constant

$$\gamma = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n}\Big\{N_{n,1}\big(-\tfrac{1}{2}\big) + N_{n,1}(1)\Big\} = \frac{3}{4} - \frac{11}{96} - \frac{1}{72} - \frac{311}{46\,080} - \frac{5}{1152} - \frac{7291}{2\,322\,432} - \ldots \tag{68}$$

Other choices of a are also possible in order to get series with rational terms only for γ. In fact, it is not difficult to show that we can eliminate the logarithm by properly choosing a, namely

$$\gamma = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n}\Big\{N_{n,1}(a) + N_{n,1}\Big(-\frac{a}{1+a}\Big)\Big\}, \qquad a > -1, \tag{69}$$

which, at $a \in \mathbb{Q}$, represents a huge family of series with rational terms only for Euler's constant. More generally, from (65) it follows that if $a_1, \ldots, a_k$ are chosen so that $(1+a_1)_m \cdots (1+a_k)_m = 1$, then

$$\gamma = \frac{1}{m\,k}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sum_{l=1}^{k}N_{n,m}(a_l), \qquad a_1, \ldots, a_k > -1, \tag{70}$$

Furthermore, if $a_1,\ldots,a_k$ may be chosen so that $(1+a_1)_m^{q_1}\cdots(1+a_k)_m^{q_k}=1$ for some $q_1,\ldots,q_k$, then we have a more general formula

$$\gamma \;=\; \frac{1}{m\,(q_1+\ldots+q_k)}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sum_{l=1}^{k} q_l\,N_{n,m}(a_l)\,,\qquad a_1,\ldots,a_k>-1\,, \tag{71}$$

which is the most complete generalization of the Fontana–Mascheroni series (64).

It is also possible to deduce series expansions with rational terms for Euler's constant from series (62). Putting $v=1$, we obain the following series

$$\gamma \;=\; \ln 2\pi - 2 - 2\sum_{n=1}^{\infty}\frac{(-1)^n\,G_n^{(2)}}{n} \;=\; \ln 2\pi - 2 + \frac{2}{3}+\frac{1}{24}+\frac{7}{540}+\frac{17}{2880}+\frac{41}{12\,600}+\ldots \tag{72}$$

converging at the same rate as $\sum n^{-2}\ln^{-3} n$ (see footnote 19).

Lastly, the reader may easily verify that all these series are new and at the moment of writing of this paper were not known to the OEIS.

Corollary 4d. Generalizing expansion (13) to the Hurwitz ζ–function, we may introduce $\delta_m(v)$ as the coefficients in

$$\zeta(s,v) \;=\; \frac{1}{s-1}+\frac{3}{2}-v+\sum_{m=1}^{\infty}\frac{(-1)^m\delta_m(v)}{m!}\,s^m\,,\qquad s\neq 1. \tag{73}$$

It is, therefore, not difficult to see that $\delta_m(v)=(-1)^m\big\{\zeta^{(m)}(s,v)+m!\big\}$. From (21)–(23), it follows that for $m=1,2,3,\ldots$ and for other parameters defined exactly as in (56)–(61), we have

$$\delta_m(v) \;=\; f_m(v)+\sum_{n=0}^{\infty}\big|G_{n+1}\big|\sum_{k=0}^{n}(-1)^k\binom{n}{k}\ln^m(k+v)\,, \tag{74}$$

$$\delta_m(v) \;=\; f_m(v-1)-\sum_{n=0}^{\infty}C_{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\ln^m(k+v)\,, \tag{75}$$

$$\delta_m(v) \;=\; \frac{1}{r}\sum_{l=0}^{r-1}f_m(v+a+l)+\frac{1}{r}\sum_{n=0}^{\infty}(-1)^n\,N_{n+1,r}(a)\sum_{k=0}^{n}(-1)^k\binom{n}{k}\ln^m(k+v)\,, \tag{76}$$

where we denoted

$$f_m(v)\equiv(-1)^m m!\left\{1-v-v\sum_{k=1}^{m}(-1)^k\frac{\ln^k v}{k!}\right\}$$

for brevity. Similarly to the generalized Stieltjes constants, functions $\delta_m(v)$ enjoy a recurrent property $\delta_m(v+1)=\delta_m(v)-\ln^m v$, which may be used to rewrite (74)–(75) in a slightly different form if necessary. Finally, recalling that $\delta_1(v)=-\ln\Gamma(v)+\frac{1}{2}\ln 2\pi-1$ and noticing that $f_1(v)=v-1-v\ln v$, gives us these three series expansions

$$\ln\Gamma(v) \;=\; v\ln v - v+\frac{1}{2}\ln 2\pi-\sum_{n=0}^{\infty}\big|G_{n+1}\big|\sum_{k=0}^{n}(-1)^k\binom{n}{k}\ln(k+v)\,,\quad \operatorname{Re} v>0\,, \tag{77}$$

$$\ln\Gamma(v) \;=\; (v-1)\ln(v-1)-v+1+\frac{1}{2}\ln 2\pi+\sum_{n=0}^{\infty}C_{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\ln(k+v)\,,\quad \operatorname{Re} v>1\,, \tag{78}$$

$$\ln\Gamma(v) = \frac{1}{r}\sum_{l=0}^{r-1}(v+a+l)\ln(v+a+l) - v - a - \frac{r}{2} + \frac{1}{2}\ln 2\pi + \\ + \frac{1}{2} - \frac{1}{r}\sum_{n=0}^{\infty}(-1)^n N_{n+1,r}(a)\sum_{k=0}^{n}(-1)^k\binom{n}{k}\ln(k+v)\,, \quad \operatorname{Re} v > -\operatorname{Re} a\,, \tag{79}$$

$r = 1, 2, 3, \ldots$ and Re > -1, for the logarithm of the Γ–function. First of these representations is equivalent to a little-known formula for the logarithm of the Γ–function, which appears in epistolary exchanges between Charles Hermite and Salvatore Pincherle dating back to 1900 [29, p. 63, two last formulæ], [54, vol. IV, p. 535, third and fourth formulæ], while the second and the third representations seem to be novel. Note that the parameter a, in all the expansions in which it appears, plays the role of the "rate of convergence": the greater this parameter, the faster the convergence, especially if a is integer.

Nota Bene

Finally, it may be of interest to note here that Donal Connon [20], [19], by using Hasse's series (2) and some other identities, obtained several expressions of a similar nature for the generalized Stieltjes constants, for the logarithm of the Γ-function and for the digamma function:

$$\gamma_m(v) = -\frac{1}{m+1}\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\ln^{m+1}(k+v)\,, \qquad m = 1, 2, 3\ldots \tag{80}$$

$$\ln\Gamma(v) = -v + \frac{1}{2} + \frac{1}{2}\ln 2\pi + \sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+v)\ln(k+v) \tag{81}$$

$$\Psi(v) = \sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\ln(k+v) \tag{82}$$

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