Discrete Calculus of Finite Sequences

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Abstract

The calculus of finite differences is a solid foundation to the development of operations like the derivative and the integral for infinite sequences here we showed that is also possible to define it for finite sequences. Thus we could define convexity of finite sequences.

1 A little introduction on calculus of finite differences

We will introduce some definitions and notation from the calculus of finite differences. Mostly all notation are from Jordan [2], and, like the notation, mostly definitions on the subject are the same, as we can find in Jordan's [2] book or any other book on finite differences, like Boyle [1].

The definition of symbol from Boyle [1, pp. 16–18] will be generalized, and it will be used later on the study of discrete calculus of sequences.

1.1 Difference

The main definition of the calculus of finite differences is the difference. The *difference* of a function $f(x)$, which is given for $x_1, x_2, \ldots, x_n$; such that $x_{i+1} - x_i = h$ for all i between 1 and $n-1$, is

$$\underset{x,h'}{\Delta} f = f(x+h') - f(x).$$

1.2 Operation of displacement

The displacement operation is an important operation in calculus of finite differences, and consist in increasing the argument of the function by some amount. Then, if we denote this operation by $\mathbf{E}$, we have

$$\underset{h}{\mathbf{E}} f(x) = f(x+h). \tag{1}$$

Note that, in $\mathbf{E}$ we are omitting h and x as we will do most time with such operations, and as is vastly done in literature, like in Jordan's book [2, p. 6], let us define $\mathbf{E}^n$ by

$$\mathbf{E}^n f(x) = \mathbf{E}^{n-1}[\mathbf{E} f(x)] = f(x + nh),$$

where n are a positive integer, and for a negative integer $-n$, we have

$$\mathbf{E}^{-n} f(x) = f(x - nh).$$

1.3 Operation of the mean

The operation of the mean will be more important later in this paper than displacement operation, because it is invariant on the interchange of $\mathbf{E}$ and $\mathbf{1}$. This operation will be denoted by $\mathbf{M}$, and is defined as follows:

$$\underset{h}{\mathbf{M}} f(x) = \frac{f(x) + f(x+h)}{2}. \tag{2}$$

1.4 Symbolic Calculus

Let $\mathcal{O}$ be the set of all operation already defined and also contains more one operations denoted by $\mathbf{1}$. Where $\mathbf{1}$ is the *identity operation* which take a function to itself. We note that any operation from $\mathcal{O}$ are linear and commute with each others.

We can define addiction of operation as $[\mathbf{A} + \mathbf{B}] f(x) = \mathbf{A} f(x) + \mathbf{B} f(x)$, for any $\mathbf{A}$ and $\mathbf{B}$, and multiplication of a operation by a real number, in this case λ, $[\lambda \mathbf{A}] f(x) = \lambda \mathbf{A} f(x)$, as well as multiplication of operation that for consistency must be defined as $\mathbf{AB} f(x) = \mathbf{A}[\mathbf{B} f(x)]$.

Then we see that for operation in $\mathcal{O}$ some proprieties such as associativity, distribution of multiplication over addiction, linearity and commutativity are satisfied, moreover is easy to see that for addiction the order of the operations does not matter.

Definition 1. Let O be a set of unary operations $\mathbf{O} : F \to F$. Where F is any set of function. We say that O is a *symbolic set over* F if all proprieties below are satisfied

(i) Linearity: If $\mathbf{S} \in O$. Then

$$\mathbf{S}(\lambda f + g) = \lambda \mathbf{S} f + \mathbf{S} g,$$

where f and g belong to F, and λ is a real number.

(ii) Commutativity (multiplication): For any $\mathbf{S}$ and $\mathbf{O}$ in O. We have

$$\mathbf{SO} = \mathbf{OS}.$$

(iii) Commutativity (addiction): Given $\mathbf{S}$ and $\mathbf{O}$ in O. Then

$$\mathbf{S} + \mathbf{O} = \mathbf{O} + \mathbf{S}.$$

(iv) Associativity: If $\mathbf{S}$, $\mathbf{O}$ and $\mathbf{G}$ are elements of $\mathcal{O}$. Hence

$$\mathbf{S} + (\mathbf{O} + \mathbf{G}) = (\mathbf{S} + \mathbf{O}) + \mathbf{G}.$$

(v) Distribution over addiction: Let $\mathbf{S}$, $\mathbf{G}$ and $\mathbf{O}$ be in $\mathcal{O}$. We must have that

$$\mathbf{S}(\mathbf{G} + \mathbf{O}) = \mathbf{SG} + \mathbf{SO}.$$

It is easy to prove that $\mathcal{O}$ is a symbolic set over the set of all continuous functions. If $\mathbf{S}$ belongs to a symbolic set $\mathbf{O}$, then $\mathbf{S}$ is *symbol under O*. But, $\mathbf{S}$ still can be a symbol under O even if $\mathbf{S}$ does not belongs to O.

Definition 2. Let O be any symbolic set over F, and let $\mathbf{S}$ by any unitary operation $S : F \to F$. Hence, $\mathbf{S}$ is a *symbol under O over F* if the set $O' = O \cup \{\mathbf{S}\}$ is also a symbolic set over F.

We already state that $\mathcal{O}$ is a symbolic set over continuous functions, and as any differentiable function is continuous we have that $\mathcal{O}$ is a symbolic set over differentiable functions too. If we consider the differentiation operation denoted by $\mathbf{D}$ we can show that it is a symbol under $\mathcal{O}$, obviously over differentiable functions.

Definition 3. Let $\mathbb{D}$ denote the *discrete differentiation* operation that act in a function as follows:

$$\underset{h}{\mathbb{D}} f(x) = \frac{f(x+h) - f(x)}{h}.$$

First, we see that $\lim_{h \to 0} \underset{h}{\mathbb{D}} = \mathbf{D}$, which given as a hint about proprieties and interpretation of $\mathbb{D}$. Second, by the definition is easy to see that $\mathbb{D}$ is a symbol under $\mathcal{O}$, because $\mathbb{D}$ is the product of the symbol[1] $\underset{h}{\Delta} f$, and h^{-1} which act as a real. A equivalent operation is mentioned by Boyle [1, p. 3] given by the ratio $\frac{\underset{h}{\Delta}}{\Delta x}$, including that $\underset{1}{\mathbb{D}}$ is equal to $\underset{1}{\Delta}$.

Remark 4. It very important to remember that $\mathbf{D}$ act on any differentiable function, on the other hand $\underset{h}{\mathbb{D}}$ act in a wider set of function including differentiable function and discrete function such as sequences.

1.5 Relation between symbols under $\mathcal{O}$

Now is clear that with respect to addiction, subtraction and multiplication, as we defined for any operation, the symbols works as algebraic quantities. Then we can find relation between symbols, and obtain more symbols by addiction and multiplication of symbols.

[1] Usually we will omit the symbolic set when it does not cause any ambiguity.

We can give some well know equalities as follows:

$$\Delta = \mathbf{E} - \mathbf{1}, \tag{3}$$

$$\mathbf{M} = \frac{\mathbf{1} + \mathbf{E}}{2} \quad \text{and} \tag{4}$$

$$\underset{h}{\mathbb{D}} = h^{-1} \underset{h}{\Delta}. \tag{5}$$

2 Finite Sequences

Now we will work mostly with finite integer sequences. For clarity let us define addiction and multiplication of sequences:

$$(S + G)(i) = S(i) + G(i),$$

$$(SG)(i) = S(i)G(i),$$

we also can define multiplication by a real number; $(\lambda S)(i) = \lambda S(i)$, where $\lambda \in \mathbb{R}$.

2.1 Top, middle and bottom operations

We let $\mathcal{I}$ denote *top operation* that is defined by cutting off the last term of a n-tuple as result we obtain a subsequence of length $n - 1$.

Definition 5. Let $S = (s_i)$ be a finite sequence of length $n > 0$. The *top operation* is defined as follows:

$$[\mathcal{I}S](i) = s_i \quad \text{for all } i \in \{1, \ldots, n-1\}.$$

The resulting subsequence $\mathcal{I}S$ is called the *top* of S.

Meanwhile, the *bottom operation* $\mathcal{E}$ is defined by cutting off the first term of a n-tuple resulting in a subsequence of length $n - 1$.

Definition 6. Let $S = (s_i)$ be a finite sequence of length $n > 0$. The *bottom operation* is defined as follows

$$[\mathcal{E}S](i) = s_{i+1} \quad \text{for all } i \in \{1, \ldots, n-1\}.$$

The resulting subsequence $\mathcal{E}S$ is called the *bottom* of S.

Remark 7. The bottom operation shift each term to the following term, and act like $\underset{1}{\mathbf{E}}$ for finite sequences.

The *middle operation* denoted by $\mathcal{M}$ act in a sequence S resulting in the mean of the top and the bottom of S.

Definition 8. The *middle operation* $\mathcal{M}$ is defined by his action in a sequence $S = (s_i)$ of length $n > 0$ as follows:

$$[\mathcal{M}S](i) = \frac{s_i + s_{i+1}}{2} \quad \text{for all } i \in \{1, \ldots, n-1\}.$$

The subsequence $\mathcal{M}S$ is called the *middle* of S.

Remark 9. If a sequence R have a unit length. Hence $\mathcal{I}R$, $\mathcal{E}R$ and $\mathcal{M}R$ are equal to the empty sequence that we will denote by $(\varnothing)$. Any operation above are defined for empty sequences, but we can extend all definitions, such that, we have

$$\mathcal{I}(\varnothing) = \mathcal{M}(\varnothing) = \mathcal{E}(\varnothing) = (\varnothing).$$

We already point out that $\mathcal{M}S = \frac{1}{2}(\mathcal{E}S + \mathcal{I}S)$. If we compare with (4) is clear that $\mathcal{M}$ act in a finite sequence as $\mathbf{M}$. At the same time is clear that $\mathbf{1}$ does not act as $\mathcal{I}$, clearly $\mathbf{1}S = S$ meanwhile the top of S have a different length. Then we need the top operation $\mathcal{I}$ to work with finite sequences.

2.1.1 Proprieties

For all similarity with symbols under $\mathcal{O}$, such as $\mathbf{E}$ and $\mathbf{M}$, if we let $O = \{\mathbf{1}, \mathcal{I}, \mathcal{E}, \mathcal{M}\}$. Then, for any finite sequence S and G of same length, we have some proprieties as follows;

(i) Linearity: For any $\mathbf{O}$ in O and λ a real number, we have

$$\mathbf{O}(\lambda S + G) = \lambda \mathbf{O}S + \mathbf{O}G.$$

Proof. We just need to show that $\mathcal{I}$ and $\mathcal{E}$ are linear. By definition, we have

$$\mathcal{I}\left[\lambda S + G\right](i) = \lambda S(i) + G(i) = \lambda \mathcal{I}S + \mathcal{I}G \quad \text{for } i = 1, \ldots, n-1;$$

and for $\mathcal{E}$ is almost the same:

$$\mathcal{E}[\lambda S + G](i) = \lambda S(i+1) + G(i+1) = \lambda \mathcal{E}S + \mathcal{E}G \quad \text{for } i = 1, \ldots, n-1. \qquad \square$$

(ii) Commutativity (multiplication): Let be $\mathbf{O}$ and $\mathbf{O}'$ two elements of $\mathcal{O}$. The follow equality hold

$$\mathbf{O}\mathbf{O}' = \mathbf{O}'\mathbf{O}.$$

Proof. Proving that $\mathcal{I}$ and $\mathcal{E}$ commute are enough as $\mathcal{M}$ is a linear combination between them, and $\mathbf{1}$ is know to commute with any operation. So, by multiplication definition

$$\mathcal{I}\mathcal{E}S(i) = \mathcal{I}[S(i+1)] = S(i+1) \quad \text{where } i = 1, \ldots, n-2, \text{ and}$$

$$\mathcal{E}\mathcal{I}S(i) = \mathcal{E}[S(i)] = S(i+1) \quad \text{where } i = 1, \ldots, n-2.$$

Therefore, $[\mathcal{I}, \mathcal{E}]S = 0$. $\square$

(iii) Commutativity (addiction): For any $\mathbf{O}$ and $\mathbf{O}'$ in $\mathcal{O}$. We have

$$\mathbf{O} + \mathbf{O}' = \mathbf{O}' + \mathbf{O}.$$

Proof. For real sequences and by definition of addiction of sequences is clear that the commutativity for addiction holds. $\square$

(iv) Associativity: Given $\mathbf{O}$, $\mathbf{O}'$ and $\mathbf{O}''$ elements of $\mathcal{O}$. Then

$$\mathbf{O} + (\mathbf{O}' + \mathbf{O}'') = (\mathbf{O} + \mathbf{O}') + \mathbf{O}''.$$

Proof. As above is easy to prove that associativity is valid for real sequences. □

(v) Distribution over addiction: If $\mathbf{O}$, $\mathbf{O}'$ and $\mathbf{O}''$ are in $\mathcal{O}$. Then the equality must hold

$$\mathbf{O}(\mathbf{O}' + \mathbf{O}'') = \mathbf{O}\mathbf{O}' + \mathbf{O}\mathbf{O}''.$$

Proof. We have that, for example,

$$\mathcal{I}(\mathcal{E}+\mathcal{M})S(i) = \mathcal{I}\left[S(i+1) + \frac{S(i)+S(i+1)}{2}\right] = \mathcal{I}\mathcal{E}+\mathcal{I}\mathcal{M} \quad \text{for } i = 1, \ldots, n-2.$$

□

We see by Definition 1 that the set O is a symbolic set over the set of all finite sequences denoted by $\mathcal{F}$. In the same way we saw in Section 1 that symbols works as algebraic quantities, and we could have relation between them. We have, for example, by definition 8 that

$$\mathcal{M} = \frac{\mathcal{I} + \mathcal{E}}{2}. \tag{6}$$

3 Derivative of finite sequences

A good way to define the derivative would use the know discrete derivative. Let S be any finite sequence of length equal to n. If $\mathbb{D}$ is applied to S this would result in a sequence of length equal to $n-1$, and $\mathbb{D}S(i)$ would be equal to $S(i+1) - S(i)$ for $0 < i < n$.

Definition 10. The *derivative of a sequence* S denoted by $\mathcal{D}S$ is defined as follows:

$$(\mathcal{D}S)(i) = \underset{1}{\mathbb{D}}S(i) = S(i+1) - S(i) \quad i = 1, 2, \ldots, (n-1); \tag{7}$$

where n is the length of S. The derivative is the result of the differentiation operation denoted by $\mathcal{D}$.

We will show later that $\mathcal{D}$ is very similar to the usual differentiation. We will find product and quotient rules like in calculus, and use first and second derivatives just like in calculus to classify sequences.

With the use of bottom and top operation we have that differentiation can be defined as follows

$$\mathcal{D} = \mathcal{E} - \mathcal{I}. \tag{8}$$

Proof.

$$(\mathcal{E} - \mathcal{I})S(i) = (\mathcal{E}S)(i) - (\mathcal{I}S)(i) = S(i+1) - S(i) = (\mathcal{D}S)(i)$$

□

The equality (8) is the equivalent of the equality (3) for finite sequences. Still by (8) we have that $\mathcal{D}$ is a symbol under O, because $\mathcal{D}$ is a linear combination of the symbols $\mathcal{I}$ and $\mathcal{E}$.

3.1 Differentiation rules

Beside all general proprieties that $\mathcal{D}$ already satisfy as a symbol under O. We can obtain some important equalities that make clear the relation between differential calculus, calculus of finite differences and discrete calculus of sequences.

Lemma 11. *Let S be a sequence. The derivative of S is equal to a constant sequence $\mathcal{D}S = (0_i)$, if and only if, S is a constant sequence.*

Proof. If S is a constant sequence, then $S(i+1) = S(i)$. Therefore, $(\mathcal{D}S)(i) = (0_i)$. Meanwhile if $(\mathcal{D}S)(i) = 0$, then $S(i+1) - S(i) = 0$. Therefore, S is a constant sequence as we want to prove. □

Below we listed some remarks regarding the derivative of sequences:

(i) Let $\lambda \in \mathbb{R}$, hence $\mathcal{D}(\lambda_i) = (0_i)$. We already proved it in the first part of the proof of the Lemma 11.

(ii) If S is a arithmetics sequence with common difference d. Hence $\mathcal{D}S = (d_i)$. In the other way, if $\mathcal{D}S = (d_i)$ and d is a real number different from 0. Then, S is a arithmetics progression with common difference equal to d. We will omit the proof, because it is obvious from Definition 10 and the definition of arithmetics progression.

(iii) For a geometric sequence S with common ratio $q \neq 0$, we have that $\mathcal{D}S = (q-1)\mathcal{I}S$.

Proof. We have, by definition of geometric progression, that

$$\frac{\mathcal{E}S}{\mathcal{I}S} = q \Rightarrow \frac{\mathcal{E}S - \mathcal{I}S}{\mathcal{I}S} = ([q-1]_i),$$

therefore $\mathcal{D}S = (q-1)\mathcal{I}S$, remembering that the product of a constant sequence (λ_i) and any sequence of same length S is equal to the product of S and the real number λ. □

Let us take a real function f, such as, $f(j) = a_1 q^j$, where j, a_1 and q are real numbers. We have that the derivative is

$$f'(j) = f(j)\log q,$$

and the discrete derivative is

$$\underset{1}{\mathbb{D}} f(j) = f(j+1) - f(j) = a_j(q-1).$$

Hence, we note that studying the discrete derivative of a real function given us a way to find the derivative for the sequence of images $f(i)$ with $i = i_0, i_0 + 1 \cdots$. This relation between the operation from O and $\mathcal{O}$, which we already already mentioned for some isolated cases, will be study later in this paper.

(iv) Let S be a sequence, such as, $S(i) \neq 0$ for all i. We let S^{-1} denote the inverse of S which is defined as follows:

$$S^{-1}(i) = \frac{1}{S(i)}.$$

So, the derivative of S^{-1} is

$$\mathcal{D}S^{-1}(i) = \frac{1}{S(i+1)} - \frac{1}{S(i)} = \frac{S(i) - S(i+1)}{S(i)S(i+1)}.$$

Finally, we have that

$$\mathcal{D}S^{-1} = -\frac{\mathcal{D}S}{\mathcal{I}S\,\mathcal{E}S},$$

which looks like the derivative of the inverse of a function f given by $-\frac{f'}{f^2}$.

3.1.1 Higher order derivatives

From (8) we must have that

$$\mathcal{D}^m = (\mathcal{E} - \mathcal{I})^m = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \mathcal{I}^k \mathcal{E}^{m-k}. \tag{9}$$

Hence, for example,

$$\mathcal{D}^2 S(i) = 1S(i+2) - 2S(i+1) + 1S(i), \tag{10}$$

with (6) is possible obtain others equations for higher order of $\mathcal{D}$ that can be useful in some situations.

3.1.2 Product rule

Let S and G be two sequences of same length. The derivative of SG is, by (8)

$$\mathcal{D}(SG) = \mathcal{E}(SG) - \mathcal{I}(SG) = \mathcal{E}S\,\mathcal{E}G - \mathcal{I}S\,\mathcal{I}G, \tag{11}$$

$$\mathcal{D}(SG) = \mathcal{D}S\,\mathcal{E}G + \mathcal{I}S\,\mathcal{D}G \quad \text{and} \tag{12}$$

$$\mathcal{D}(SG) = \mathcal{D}S\,\mathcal{I}G + \mathcal{E}S\,\mathcal{D}G. \tag{13}$$

The equality (11) can be easy proved as the generalization

$$\mathbf{O}\prod_{i=1}^{n} S_i = \prod_{i=1}^{n} \mathbf{O}S_i,$$

where $\mathbf{O}$ can be both $\mathcal{I}$ or $\mathcal{E}$. Then, we obtain that $\mathcal{D}(SG)$ is equal to (12) and (13) which are similar to product rule from usual calculus. However, (12) and (13) are not symmetric.

It is not hard to see that if we change $\mathcal{I}$ by $\mathcal{E}$ in (12) as result we would have (13), and vice versa. Now, let us sum the equations (12) and (13), and rewrite $\mathcal{D}(SG)$ as follows:

$$\mathcal{D}(SG) = \mathcal{D}S\,\mathcal{M}G + \mathcal{M}S\,\mathcal{D}G. \tag{14}$$

Equation (14), differently of (12) and (13), is symmetric, and analogous to the usual product rule from differential calculus. Moreover, the *product rule* (14) is equal to the product rule of calculus of finite differences, where we have

$$\Delta(fg) = \Delta f \mathbf{M} g + \mathbf{M} f \Delta g.$$

3.1.3 Quotient rule

It is easy to show that if S and G are sequences of same length, then

$$\mathcal{D}\left(\frac{S}{G}\right) = \frac{\mathcal{D}S\,\mathcal{M}G - \mathcal{D}G\,\mathcal{M}S}{\mathcal{I}G\,\mathcal{E}G}.$$

We have two way to show it: go for it straight by Definition 10 or use the product rule that we obtained above.

Using the equality proved in the item (iv) at the beginning of the section,

$$\mathcal{D}G^{-1} = -\frac{\mathcal{D}G}{\mathcal{I}G\,\mathcal{E}G},$$

and with the product rule, we have

$$\mathcal{D}(SG^{-1}) = \mathcal{D}S\,\mathcal{M}G^{-1} + \mathcal{M}S\,\mathcal{D}G^{-1} = \frac{\mathcal{D}S\,\mathcal{M}G^{-1}\,\mathcal{I}G\,\mathcal{E}G - \mathcal{M}S\,\mathcal{D}G}{\mathcal{I}G\,\mathcal{E}G}.$$

Finally, by definition of the middle operation:

$$\left(\mathcal{M}G^{-1}\right)(i) = \frac{1}{2}\left(\frac{1}{G(i+1)} + \frac{1}{G(i)}\right) = \frac{(\mathcal{M}G)(i)}{(\mathcal{I}G)(i)(\mathcal{E}G)(i)},$$

thus we have the quotient rule as expected

$$\mathcal{D}\left(\frac{S}{G}\right) = \frac{\mathcal{D}S\,\mathcal{M}G - \mathcal{D}G\,\mathcal{M}S}{\mathcal{I}G\,\mathcal{E}G}. \tag{15}$$

4 Integral of a finite sequence

Now with a definition for derivative of sequences we can try now look for a inverse of the differentiation operation. We expect, if it exists, be some operation like integration in calculus.

4.1 Indefinite integral and integration operation

Let $\mathcal{J}$ denote the operation of *integration* of sequences defined as the inverse of the derivative $\mathcal{D}$. Hence, we have

$$\mathcal{J}\mathcal{D} = \mathcal{D}\mathcal{J} = \mathbf{1}. \tag{16}$$

If we take the derivative of a n-tuple S, and then integrate we obtain the following equality

$$(\mathcal{J}\mathcal{D}S)(i) = \mathcal{J}(S(i+1) - S(i)) = S(i) = \mathbf{1}S.$$

Therefore we have that

$$(\mathcal{J}S)(i+1) = (\mathcal{J}S)(i) + S(i),$$

finally we can conclude that

$$(\mathcal{J}S)(i) = (\mathcal{J}S)(1) + \sum_{j=1}^{i-1} S(j) \quad \text{for } 1 < i \leq n.$$

The value $(\mathcal{J}S)(1)$ is equivalent to the addiction of a constant sequence to $\mathcal{J}S$ like in the indefinite integral in differential calculus.

Definition 12. We let $\sum S\, d\mathbb{N}$ denote the *indefinite derivative* of the n-tuple S that is defined as the following sequence:

$$\left(\sum S\, d\mathbb{N}\right)(i) = \left(\sum S\, d\mathbb{N}\right)(1) + \sum_{j=1}^{i-1} S(j) \quad \text{for } 1 < i \leq n+1. \tag{17}$$

Where $\left(\sum S\, d\mathbb{N}\right)(1)$ is some real number.

Theorem 13. *The indefinite integral of the sequence S is equal to the resulting sequence of the integration of S.*

Proof. By definition $\mathcal{J}(\mathcal{D}S) = \mathbf{1}$. Hence, we only need to prove that $\mathcal{D}$ and $\mathcal{J}$ commute. The derivative of $\sum S\, d\mathbb{N}$, where S is a sequence of length n, from (17), is

$$\left(\mathcal{D}\sum S\, d\mathbb{N}\right)(i) = \sum_{j=i}^{i} S(j) = S(i) = (\mathbf{1}S)(i).$$ □

Remark 14. The bottom of the indefinite integral of a sequence S is equal to the sequence of partial sum of the elements of S up to a constant sequence.

4.2 Integration Rules

It is not hard to prove that $\mathcal{J}$ is a symbol under O, and, like $\mathcal{D}$, the integration of sequences looks like the integration of Calculus.

Lemma 15. *Let S be a sequence. The indefinite integral of S is a constant sequence, if and only if, S is equal to (0_i).*

Proof. It is easy to see, see Remark 14, that $\sum (0_i)\, d\mathbb{N} = (\lambda_i)$ where λ is some real number. Meanwhile if $\sum S\, d\mathbb{N} = (\lambda_i)$ then by Theorem 13, we know that

$$\mathcal{D} \sum S\, d\mathbb{N} = S = \mathcal{D}\left(\lambda_i\right),$$

therefore by Lemma 11 follows that $S = (0_i)$. □

4.2.1 Integration by parts

Now that we have a product rule let us integrate (14), like we would do in calculus to find the rule of integration by parts, we have by the definition of $\mathcal{J}$:

$$SG = \mathcal{J}\left(\mathcal{D}S\, \mathcal{M}G\right) + \mathcal{J}\left(\mathcal{M}S\, \mathcal{D}G\right),$$

using the Theorem 13 we can rewrite the above equation as follows:

$$\sum \mathcal{D}S\, \mathcal{M}G\, d\mathbb{N} = SG - \sum \mathcal{M}S\, \mathcal{D}G\, d\mathbb{N}. \tag{18}$$

Above equation works as the integration by parts of calculus. Moreover, if S is a sequence and we let $dS \equiv \mathcal{D}S d\mathbb{N}$, then (18) is even more similar:

$$\sum \mathcal{M}S\, dG = SG - \sum \mathcal{M}G\, dS.$$

4.3 Definite integral

As we have in calculus, we will define a real number called definite integral. A theorem like the second fundamental theorem of calculus should be satisfied.

Definition 16. Let S be a sequence of length n. If a and b are two integers between 0 and $n+1$, then the *definite integral* from a to b of S is denoted by $\sum_a^b S\, d\mathbb{N}$. Where

$$\sum_a^b S\, d\mathbb{N} = \sum_{j=a}^{b} S(j).$$

The definition given above is intuitive taking in consideration that a sum is the discrete version of the integral. We see that the definite integral of the derivative of a n-tuple S from 1 to $n-1$ is

$$\sum_1^{n-1} \mathcal{D}S\, d\mathbb{N} = \sum_{j=1}^{n-1} (S(j+1) - S(j)) = S(n) - S(1),$$

then we can state a theorem like the second fundamental theorem of the calculus.

Theorem 17 (Second Fundamental Theorem of Discrete Calculus of Sequences)**.** *Let S be a sequence with n terms, and let I be a sequence, such as, $\mathcal{D}I$ is equal to S. If a and b are integers greater than 0 and less than or equal to n, then*

$$\sum_{a}^{b} S\, d\mathbb{N} = I(b+1) - I(a).$$

Proof. If $\mathcal{D}I = S$, then $I = \mathcal{D}^{-1}S$. Hence, from (16) we have that I is the result of the integration of S. Thus, we only have to show that $\mathcal{J}S|_a^{b+1} \equiv \mathcal{J}S(b+1) - \mathcal{J}S(a)$ is equal to the definite integral of S from a to b.

By Theorem 13 and Definition 12, we have

$$\mathcal{J}S|_a^{b+1} = \sum_{j=1}^{b} S(j) - \sum_{j=1}^{a} S(j) = \sum_{j=a}^{b} S(j) = \sum_{a}^{b} S\, d\mathbb{N}.$$

Then as we have $I = \mathcal{J}S$, thus

$$\sum_{a}^{b} S\, d\mathbb{N} = I|_a^{b+1} = I(b+1) - I(a). \qquad \square$$

Some results are listed below.

(i) Let S be a geometric progression of length n. The sequence S satisfy the discrete equation $\mathcal{D}S = (q-1)\mathcal{I}S$, where q is the common ratio of S. Taking the integral of the discrete equation from 1 to $n-1$, we have

$$\sum_{1}^{n-1} \mathcal{I}S\, d\mathbb{N} = \sum_{j=1}^{n-1} S(j) = \frac{S(n) - S(1)}{q-1} = S(1)\frac{1-q^{n-1}}{1-q}.$$

(ii) If S is a arithmetics progression of length n and common difference d. Then $\mathcal{D}S = (d_i)$, let integrate it from 1 to $i \leq n$ as follows:

$$S(i+1) - S(1) = (i-1)d \therefore S(i+1) = S(i) + (i-1)d.$$

5 Application of the derivative of sequences

5.1 Increasing and decreasing sequences

The Lemma 11 tell us that in case the first derivative of S is (0_i) then S does not increase nor decrease. Hence to cover all possibilities we have that:

(i) If $\mathcal{D}S(i) > 0$ for all i, then S is *strictly monotonically increasing*.

(ii) If $\mathcal{D}S(i) < 0$ for all i, then S is *strictly monotonically decreasing*.

(iii) If $\mathcal{D}S(i) \geq 0$ for all i, then S is *monotonically increasing*.

(iv) If $\mathcal{D}S(i) \leq 0$ for all i, then S is *monotonically decreasing*.

Exactly as we do in calculus for functions using the first derivative.

5.2 Convexity

The calculus of finite differences is a well know way to generalize the conception of convexity of sequences. It is done in many papers like [3] and [6]. We will use this concept with respect to the second derivative that can be take as the second difference Δ^2.

Definition 18. A sequence S is *convex* when the second derivative of S is monotonically increasing. However, if $-S$ is convex, then S is *concave*.

We see that $\mathcal{D}^2$ given us information about a sequence like the second derivative $\mathbf{D}^2$ in calculus. The geometric interpretation of convexity is clear let take the term of the second derivative of a sequence S which is given by (10):

$$\mathcal{D}^2 S(i) = S(i+2) - 2S(i+1) + S(i) \quad \text{for all i},$$

if S is, for example, convex. We have that

$$S(i+1) \leq \frac{S(i+2) + S(i)}{2} \quad \text{for all i} \leq n-3,$$

where n is the length of S, then the central term $S(i+1)$ is less than the mean of the extreme terms.

In the other hand, let take the graph of the sequences which is given by the pairs $(j, S(j))$, a point in a Cartesian coordinate system. We note that the point of the central term $(i+1, S(i+1))$ is always below to the line passing through extremities points $(i, S(i))$ and $(i+2, S(i+2))$, as we have for continuous functions.

Moreover, let $L_2(x) = ax^2 + bx + c$ be a quadratic function such that $L_2(i) = S(i)$, $L_2(i+1) = S(i+1)$ and $L_2(i+2) = (i+2)$.The polynomial L_2 exists, for all i greater than 0 and less than or equal to $n-2$, only if $\mathcal{D}S(i) \neq 0$, and $\mathcal{D}^2 S(i) \neq 0$. It is easy to check that, if exists, $L_2(x)$ is given by

$$L_2(x) = \frac{(x-i-1)(x-i-2)}{2} S(i) - \frac{(x-i)(x-i-2)}{1} S(i+1) + \frac{(x-i)(x-i-1)}{2} S(i+2).$$

The second derivative is

$$L_2''(x) = S(i+1) + S(i) - 2S(i+1) = 2a = \mathcal{D}^2 S(i),$$

so L_2 is convex (concave), if and only if, S is convex (concave).

The function L_2 is a good way to understand the convexity of sequences, and we will study it more later. Now going back to a more geometric view of convexity. Let us calculate the area of the triangle whose vertices are $(i+j, S(i+j))$, where i is a integer greater than 0 and less than to the length of S minus one, and $j = 0, 1, 2$. Let A be the following determinant

$$A = \begin{vmatrix} i & S(i) & 1 \\ i+1 & S(i+1) & 1 \\ i+2 & S(i+2) & 1 \end{vmatrix} = S(i) + S(i+2) - 2S(i) = \mathcal{D}^2 S(i),$$

we know that $\frac{1}{2}|A|$ is equal to the area of the triangle, so the area of the triangle is proportional to the absolute value of the second derivative. The determinant A defined in a natural way can be viewed as the value of a oriented area, when positive S is convex, and when negative S is concave. Beside, if A' is the determinant A for $-S$, we have that $A' = -A$, in agreement with the definition.

Definition 19. A sequence S is *strictly convex* if the second derivative of S is strictly monotonically increasing. But, if $-S$ is strictly convex, hence S is *strictly concave.*

All statements below are equivalent:

(i) The sequence S, with n terms, is strictly convex (concave).

(ii) The three points $(i, S(i))$, $(i+1, S(i+1))$ and $(i+2, S(i+2))$ are not collinear, for all i greater than 0 and less than $n-1$.

(iii) The determinant A is greater (less) than zero, for all i such that $0 < i \leq n-2$.

Proof. It is well know that (iii) $\Rightarrow$ (ii). If (ii) is true, we have that $S(i+1) \neq \frac{1}{2}(S(i) + S(i+2))$ for all i, hence $\mathcal{D}^2 S(i) \neq 0$ for all i. Thus (ii) $\Rightarrow$ (i). Finally, we note that if (iii) is false then $\det A = 0$ for some i. So there is one i such that $\mathcal{D}^2 S(i) = 0$. Therefore (i) $\Rightarrow$ (iii). □

Definition 20. Let S be a sequence of length greater than or equal to 3. Hence S is *continuously convex*, if and only if, S is strictly convex and $\mathcal{D}S(i) \neq 0$ for all i greater than 0 and less than the length of S. If $-S$ is continuously convex, then S is *continuously concave.*

6 Lagrange polynomials

Let $S_n : I_n \to \mathbb{R}$, where $I_n = \{\, x \in \mathbb{N} : x \leq n \,\}$. Consider $m+1 \leq n$ consecutive points $(i, S_n(i))$ with $i = n_0, \ldots, n_0 + m$ where $1 \leq n_0 \leq n - m$. The Lagrange polynomial is given by

$$L(x; n_0, m) = \sum_{j=n_0}^{n_0+m} S_n(j) \prod_{\substack{k=n_0 \\ k \neq j}}^{n_0+m} \frac{x-k}{j-k}. \tag{19}$$

We used $L(x; i, 2)$ to study convexity, the function L_2. The reason for it came from the general fact that

$$\frac{d^m}{dx^m} L(x; n_0, m) = \mathcal{D}^m S_n(n_0).$$

Proof. From Eq. (19), we have

$$\frac{d^m}{dx^m} L(x; n_0, m) = \sum_{j=n_0}^{n_0+m} S_n(j) \frac{d^m}{dx^m} \prod_{\substack{k=n_0 \\ k \neq j}}^{n_0+m} \frac{x-k}{j-k},$$

now using the general Leibniz rule for the derivative of the product:

$$f_m = \frac{d^m}{dx^m} \prod_{\substack{k=n_0 \\ k\neq j}}^{n_0+m} \frac{x-k}{j-k} = \sum_{w_1+w_2+\cdots+w_m=m} \frac{m!}{w_1!w_2!\cdots w_m!} \prod_{\substack{k=n_0 \\ k\neq j}}^{n_0+m} \frac{d^{w_k}}{dx^{w_k}} \frac{x-k}{j-k}$$

From the product of derivatives of linear functions we see that w_i is equal to 0 or 1, for all $i = 1, \ldots, m$. Then with the constraint that the sum of all w's must be m follows that $w_i = 1$ for all i. So now f_m can be simply evaluated as

$$f_m = m! \prod_{\substack{k=n_0 \\ k\neq j}}^{n_0+m} \frac{1}{j-k} = (-1)^{m+j-n_0} \binom{m}{j-n_0}.$$

Thus

$$\frac{d^m}{dx^m} L(x; n_0, m) = \sum_{j=n_0}^{n_0+m} (-1)^{m+j-n_0} \binom{m}{j-n_0} S_n(j),$$

it is clear from (9) that

$$\mathcal{D}^m S(n_0) = (-1)^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} \mathcal{I}^{m-k} \mathcal{E}^k S_n(n_0) = \sum_{j=n_0}^{m+n_0} (-1)^{m+j-n_0} \binom{m}{j-n_0} S_n(j),$$

therefore, we have

$$\frac{d^m}{dx^m} L(x; n_0, m) = \mathcal{D}^m S_n(n_0).$$

□

The order of $L_m \equiv L(n_0, m)$ is less than or equal to m. Then, we have that

$$L(x; n_0, m) = \sum_{i=0}^{m} l_i x^i,$$

where l_i depends on i and the terms of S. The order of L_m can be find as the maximum integer j, such that, $l_j \neq 0$.

From definition, we must have the following equality

$$\sum_{i=0}^{m} l_1 x^i = \sum_{j=n_0}^{n_0+m} S_n(j) \prod_{\substack{k=n_0 \\ k\neq j}}^{n_0+m} \frac{x-k}{j-k},$$

after derivate multiple times, we obtain

$$l_m = \frac{1}{m!} \mathcal{D}^m S_n(n_0).$$

Thus the order of the polynomial $L(n_0, m)$ is m, where m is the index that identify L, if and only if, $\mathcal{D}^m S(n_0) \neq 0$. This explain, why L_2 exists as a quadratic function, if and only if, the sequence associated with L is strictly convex (concave).

We also can generalize the result of the determinant A for higher order of derivative. This fact cames from the polynomial interpolation of a set of data. For a set as the graph of a sequence S_n, where $n > m$, the following system of equation must be satisfied

$$\begin{bmatrix} i^m & i^{m-1} & i^{m-2} & \dots & i & 1 \\ (i+1)^m & (i+1)^{m-1} & (i+1)^{m-2} & \dots & i+1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ (i+m)^m & (i+m)^{m-1} & (i+m)^{m-2} & \dots & i+m & 1 \end{bmatrix} \begin{bmatrix} l_m \\ l_{m-1} \\ \vdots \\ l_0 \end{bmatrix} = \begin{bmatrix} S(i) \\ S(i+1) \\ \vdots \\ S(i+m) \end{bmatrix},$$

where i is any integer less than or equal to $n - m$.

The solution for l_m is given by Cramer's rule as follows

$$l_m = \frac{1}{m!} \begin{vmatrix} S(i)^m & i^{m-1} & i^{m-2} & \dots & i & 1 \\ S(i+1)^m & (i+1)^{m-1} & (i+1)^{m-2} & \dots & i+1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ S(i+m)^m & (i+m)^{m-1} & (i+m)^{m-2} & \dots & i+m & 1 \end{vmatrix},$$

therefore we have

$$\mathcal{D}^m S_n(i) = \begin{vmatrix} S(i)^m & i^{m-1} & i^{m-2} & \dots & i & 1 \\ S(i+1)^m & (i+1)^{m-1} & (i+1)^{m-2} & \dots & i+1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ S(i+m)^m & (i+m)^{m-1} & (i+m)^{m-2} & \dots & i+m & 1 \end{vmatrix}.$$

7 Discrete calculus for infinite sequences

In the previous sections we studied how to find a discrete derivative for finite sequences, and we saw that the calculus of finite differences gives a good basis for it. For infinite sequences we can simply apply any concept of the calculus of finite differences. We will try make it clear as we study the top operation on infinite sequences.

7.1 Top operation action on finite sequences

It's clear that if S is a infinite sequence, then the top of S is simply itself, that is,

$$\mathcal{I}S = \mathbf{1}S.$$

Let S be infinite sequence, and let S_n be the sequence of the first n terms of S. Note that $\mathcal{I}S_n = S_{n-1}$. We can state that

$$\lim_{m\to\infty} S_m = S,$$

in this sense, we can say that:

$$\lim_{m\to\infty} \mathcal{I}S_m = \lim_{m\to\infty} S_{m-1} = S = \mathbf{1}S.$$

So now we can simply use the familiar operation of the calculus of finite differences. The equalities (3), (4) now hold for infinite sequences, where $\underset{1}{\Delta} S$ can be identify as $\mathcal{D}S$.

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