ON RATIONAL DYCK PATHS AND THE ENUMERATION
OF FACTOR-FREE DYCK WORDS

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ABSTRACT. Motivated by independent results of Bizley and Duchon, we study rational Dyck paths and their subset of factor-free elements. On the one hand, we give a bijection between rational Dyck paths and regular Dyck paths with ascents colored by factor-free words. This bijection leads to a new statistic based on the reducibility level of the paths for which we provide a corresponding formula. On the other hand, we prove an inverse relation for certain sequences defined via partial Bell polynomials, and we use it to derive a formula for the enumeration of factor-free words. In addition, we give alternative formulas for various enumerative sequences that appear in the context of rational Dyck paths.

1. INTRODUCTION

In his paper [3] of 1954, Bizley proved a formula (accredited to Howard Grossman) for the number ϕ_n of lattice paths from $(0,0)$ to $(\alpha n, \beta n)$ which may touch but never rise above the line $\alpha y = \beta x$, where n, α, and β are positive integers with $\gcd(\alpha, \beta) = 1$. Such a path is called $\frac{\beta}{\alpha}$-Dyck path of length $(\alpha + \beta)n$. Using the notation $f_j = \frac{1}{(\alpha+\beta)j}\binom{(\alpha+\beta)j}{\alpha j}$ for $j \in \mathbb{N}$, Bizley's formula reads

$$\phi_n = \sum \frac{f_1^{k_1} f_2^{k_2} \cdots}{k_1! k_2! \cdots}, \tag{1.1}$$

where the sum runs over all $k_j \in \mathbb{N}_0$ such that $k_1 + 2k_2 + \cdots + nk_n = n$.

On a more recent paper, Duchon [5] studied generalized Dyck languages and discussed the particular case of a two-letter language with alphabet $\{a, b\}$, having valuations $h(a) = \beta$ and $h(b) = -\alpha$ with $\gcd(\alpha, \beta) = 1$. In this case, associating the letter a with the step $(1, 0)$ and the letter b with the step $(0, 1)$, the set of all Dyck words (words with total valuation equal to 0) is in one-to-one correspondence with the set of $\frac{\beta}{\alpha}$-Dyck paths.

In [5, Theorem 9], Duchon established a key connection between the set $\mathcal{D}_{\beta/\alpha}$ of generalized Dyck words with slope β/α and its subset of corresponding factor-free words,[1] denoted by $\tilde{\mathcal{D}}_{\beta/\alpha}$. More precisely, he proved the identity

$$\phi_n = [t^n]D_{\beta/\alpha}(t) = \frac{1}{1 + (\alpha + \beta)n}[t^n]\tilde{D}_{\beta/\alpha}^{1+(\alpha+\beta)n}(t), \tag{1.2}$$

where $D_{\beta/\alpha}(t)$ and $\tilde{D}_{\beta/\alpha}(t)$ are the generating functions enumerating (by word length) the elements of $\mathcal{D}_{\beta/\alpha}$ and $\tilde{\mathcal{D}}_{\beta/\alpha}$, respectively. The identities (1.1) and (1.2) are the foundation for the results obtained in this paper.

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[1]A word in a language L is said to be factor-free if it has no proper factor in L.

On the one hand, as a consequence of a result given in [1, Theorem 3.5], the right-hand side of (1.2) also counts the elements of $\mathfrak{D}_n^\Theta(\alpha + \beta, 0)$, the set of Dyck words of semilength $(\alpha + \beta)n$ created from strings of the form "d" and "$u^{(\alpha+\beta)j}d$" for $j = 1, \dots, n$, such that each maximal ascent $u^{(\alpha+\beta)j}$ may be colored in as many different ways as the number θ_j of factor-free words of length $(\alpha + \beta)j$. In other words,

> there is a bijection between the set of $\frac{\beta}{\alpha}$-Dyck paths of length $(\alpha + \beta)n$ and
> the set of colored Dyck paths in $\mathfrak{D}_n^\Theta(\alpha + \beta, 0)$.

In Section 2 we give an explicit bijection that reveals, in geometric terms, the factor-free factorization of a rational Dyck path. Specifically, we use the number of peaks of a path in $\mathfrak{D}_n^\Theta(\alpha+\beta, 0)$ to define the *reducibility level* of the associated $\frac{\beta}{\alpha}$-Dyck path. As a application, we give a formula (Theorem 2.6) to compute the number of $\frac{\beta}{\alpha}$-Dyck paths with a given reducibility level.

On the other hand, by rewriting (1.1) and (1.2) in terms of partial Bell polynomials, and using an interesting inverse relation that we prove in Proposition 3.4,

> we obtain a formula for the number θ_n of factor-free $\frac{\beta}{\alpha}$-Dyck words of length
> $(\alpha + \beta)n$ in terms of $f_1, f_2, \dots$ in the spirit of (1.1).

Our formula, given in Theorem 4.4, appears to be new for cases other than $\alpha = 2$ and $\beta = 3$. For the case of slope $3/2$, Duchon already observed (cf. [5, Prop. 10]) that there are $\theta_n = C_n + C_{n-1}$ factor-free Dyck words of length $5n$, where C_n is the nth Catalan number.

We finish the paper with various formulas connecting the sequences (f_n), (ϕ_n), and (θ_n) with the related sequence (ψ_n) that counts the number of $\frac{\beta}{\alpha}$-Dyck paths of length $(\alpha + \beta)n$ that never touch the line $\alpha y = \beta x$ (except at the initial and terminal points).

2. Rational Dyck paths as colored regular Dyck paths

In this paper, we will follow the terminology used in [5] for the study of generalized Dyck words. We consider the alphabet $U = \{a, b\}$ and assume the valuations $h(a) = \beta$ and $h(b) = -\alpha$ for positive integers α and β with $\gcd(\alpha, \beta) = 1$. A Dyck word w with slope β/α is a string of letters from U such that $h(w) = 0$, and for each left factor u of w, $h(u) \geq 0$. Let $\mathcal{D}_{\beta/\alpha}$ denote the set of all such words. Note that the length $|w|$ of a word in $\mathcal{D}_{\beta/\alpha}$ is always a multiple of $\alpha + \beta$.

A word $w' \in \mathcal{D}_{\beta/\alpha}$ is called a factor of w if there are words u and v (possibly empty words) such that $w = uw'v$ and $uv \in \mathcal{D}_{\beta/\alpha}$. If u and v are both not empty words, then w' is called a *proper factor* of w. A word $w \in \mathcal{D}_{\beta/\alpha}$ is said to be *factor-free* if it has no proper factors in $\mathcal{D}_{\beta/\alpha}$. Let ε denote the empy word.

Let ϕ_n be the number of elements in $\mathcal{D}_{\beta/\alpha}$ of length $(\alpha + \beta)n$, and let θ_n be the number of factor-free words in $\mathcal{D}_{\beta/\alpha}$ of length $(\alpha + \beta)n$. Applying Faà di Bruno's formula to the right-hand side of (1.2), we get the equivalent representation

$$\phi_n = \sum_{k=1}^n \binom{(\alpha + \beta)n}{k - 1} \frac{(k - 1)!}{n!} B_{n,k}(1!\theta_1, 2!\theta_2, \dots), \tag{2.1}$$

where $B_{n,k}(x_1, \dots, x_{n-k+1})$ denotes the (n, k)-th partial Bell polynomial.

As shown in [1, Theorem 3.5], the right-hand side of (2.1) gives the number of regular Dyck paths of semilength $(\alpha + \beta)n$ whose ascents have length a multiple of $\alpha + \beta$ and such that an ascent of length $(\alpha + \beta)j$ may be colored in θ_j different ways. We denote this set of Dyck paths/words by $\mathfrak{D}_n^\Theta(\alpha + \beta, 0)$. The purpose of this section is to provide an explicit bijection between elements in $\mathcal{D}_{\beta/\alpha}$ of length $(\alpha + \beta)n$ and elements of $\mathfrak{D}_n^\Theta(\alpha + \beta, 0)$. To this end, we first introduce some notation and discuss one example.

Every word $w \neq \varepsilon$ in $\mathcal{D}_{\beta/\alpha}$ can be written, uniquely, as $w = uw'v$, where w' is the left-most, nonempty factor-free subword of w. In this case, we say that the word uv is a *reduction* of w by w' and write $w \underset{w'}{\rightarrow} uv$.

Conversely, if w and w' are Dyck words and $1 \leq j \leq |w|$, then we can write $w = uv$ with $|u| = j$ and insert w' between u and v to form the new word $\hat{w} = uw'v$. We call $\hat{w}$ the *extension* of w by w' at the position j and write $\hat{w} = (w \underset{j}{\leftarrow} w')$.

Example 2.2. Consider the word $w = aabbabbaababaabbbbbb$ in $\mathcal{D}_{3/2}$, which represents the following $\frac{3}{2}$-Dyck path of length 20:
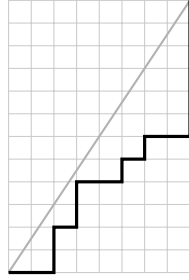

Factor w as $w = u_1 w_1' v_1 = aabbabbaabab(aabbb)bbb$ noting that $w_1' = aabbb$ is the left-most, nonempty factor-free subword of w. Thus the reduction $w \rightarrow u_1 v_1$ gives

$$aabbabbaabab(aabbb)bbb \underset{aabbb}{\longrightarrow} aabbabbaababbbb.$$

Similarly, $w_2' = ababb$ is factor-free, so we can reduce $aabbabba(ababb)bb \underset{ababb}{\longrightarrow} aabbabbabb$. Observe that $w_3' = aabbabbabb$ is factor-free. If we let ℓ_j be the length of the factor u_j in the jth reduction of w, then $\ell_1 = |u_1| = |aabbabbaabab| = 12$ and $\ell_2 = |u_2| = |aabbabba| = 8$.

Finally, we construct the colored Dyck path D_w associated with w as follows:
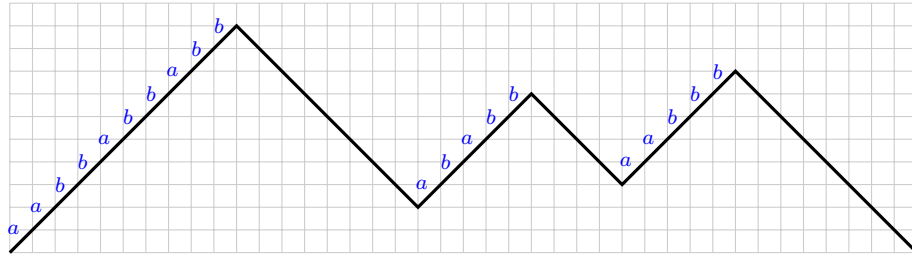
$$D_w = u^{|w_3'|} d^{\ell_2} u^{|w_2'|} d^{\ell_1 - \ell_2} u^{|w_1'|} d^{20 - \ell_1} = u^{10} d^8 u^5 d^4 u^5 d^8,$$

where the ascents are colored (from left to right) by $aabbabbabb$, $ababb$, and $aabbb$.

The above construction is reversible. Note that for a given Dyck path D of semilength 20, colored by $aabbabbabb$, $ababb$, and $aabbb$, one can create a unique word w_D by successively inserting the factor-free words at positions determined by the downs of the Dyck path.

Theorem 2.3. *Let $\alpha, \beta \in \mathbb{N}$ with $\gcd(\alpha, \beta) = 1$. The algorithm outlined in Example 2.2 provides a bijection between the set of $\frac{\beta}{\alpha}$-Dyck paths of length $(\alpha + \beta)n$ and $\mathfrak{D}_n^\Theta(\alpha + \beta, 0)$.*

Proof. Let w be a word in $\mathcal{D}_{\beta/\alpha}$ of length $(\alpha + \beta)n$. Without loss of generality, we assume that the corresponding lattice path stays strongly below the line $y = \frac{\beta}{\alpha}x$ (except at the endpoints). For a general path, we just look at its connected components separately.

If w is factor-free, then we define

$$D_w = u^{(\alpha+\beta)n}d^{(\alpha+\beta)n}$$

and color the ascent with w. Clearly $D_w \in \mathfrak{D}_n^\Theta(\alpha + \beta, 0)$.

If w is not a factor-free word, then it can be factored uniquely as $w = u_1 w_1' v_1$, where w_1' is a factor-free word in $\mathcal{D}_{\beta/\alpha}$, $h(u_1) > 0$, and $h(v_1) < 0$. Let $\ell_1 = |u_1|$ and let $u_1 v_1$ be the reduction of w by w_1'. If $u_1 v_1$ is factor-free, then we denote it by w_2' and define

$$D_w = u^{|w_2'|}d^{\ell_1}u^{|w_1'|}d^{(\alpha+\beta)n-\ell_1}.$$

Otherwise, we write $u_1 v_1$ as $u_2 w_2' v_2$, where w_2' is the left-most factor-free word contained in $u_1 v_1$. We let $\ell_2 = |u_2|$ and look at the reduction $u_2 v_2$. We continue this process inductively until we get a factor-free reduction.

Suppose $w \in \mathcal{D}_{\beta/\alpha}$ is a word of length $(\alpha + \beta)n$ such that after $k - 1$ reductions

$$w \xrightarrow[w_1']{} u_1 v_1 \xrightarrow[w_2']{} u_2 v_2 \xrightarrow[w_3']{} \cdots \xrightarrow[w_{k-1}']{} u_{k-1} v_{k-1}$$

we arrive at a factor-free word $w_k' = u_{k-1} v_{k-1}$. Let $\ell_j = |u_j|$ for $j \leq k - 1$, and define

$$D_w = u^{|w_k'|}d^{\ell_{k-1}}u^{|w_{k-1}'|}d^{(\ell_{k-2}-\ell_{k-1})} \cdots u^{|w_2'|}d^{(\ell_1-\ell_2)}u^{|w_1'|}d^{(\alpha+\beta)n-\ell_1}.$$

By construction, there is a total of $(\alpha + \beta)n$ downs, $|w_1'| + \cdots + |w_k'| = (\alpha + \beta)n$, and for every $j = 1, \ldots, k - 1$:

- $|w_j'| \equiv 0 \mod (\alpha + \beta)$,
- $\ell_j - \ell_{j+1} \geq 1$ (letting $\ell_k = 0$),
- $\ell_j \leq |w_{j+1}'| + \cdots + |w_k'|$.

In other words, D_w represents a Dyck path of semilength $(\alpha + \beta)n$. Coloring each ascent $u^{|w_j'|}$ with the factor-free word w_j', we get an element of $\mathfrak{D}_n^\Theta(\alpha + \beta, 0)$.

The above algorithm can be easily reversed. Let D be an element of $\mathfrak{D}_n^\Theta(\alpha + \beta, 0)$ with k peaks, whose ascents are colored by the factor-free words $w_1, \ldots, w_k$. Thus D must be of the form

$$D = u^{|w_1|}d^{j_1}u^{|w_2|}d^{j_2} \cdots u^{|w_k|}d^{j_k},$$

where each $|w_j|$ is a multiple of $\alpha + \beta$, $|w_1| + \cdots + |w_k| = (\alpha + \beta)n$, and for every i we have $|w_1| + \cdots + |w_i| \geq j_1 + \cdots + j_i$. Once again, without loss of generality, we assume that D stays strongly above the x-axis (it has no interior touch point).

We define w_D by repeated insertion of the coloring factor-free words:

$$w_D = \left((\cdots ((w_1 \underset{j_1}{\leftarrow} w_2) \underset{j_1+j_2}{\leftarrow} w_3) \cdots) \underset{j_1+\cdots+j_{k-1}}{\longleftarrow} w_k \right)$$

going through the ascents of D from left to right. By construction, w_D is clearly a Dyck word of length $(\alpha + \beta)n$ with slope β/α. This finishes our bijection. $\square$

Definition 2.4. Every word $w \in \mathcal{D}_{\beta/\alpha}$ has a maximal number of factor-free subwords by which the word can be reduced. We call the number of factor-free "factors" contained in w the *reducibility level* of w, and denote it by $\mathrm{rl}(w)$. Thus $\mathrm{rl}(w) = 1$ if and only if w is a factor-free word. For example, for $w = aabbabbaababaabbbbbb$ as in Example 2.2, we have $\mathrm{rl}(w) = 3$. We define $\mathrm{rl}(\varepsilon) = 0$.

Proposition 2.5 (Integer slope). *If $\alpha = 1$, then ab^β is the only factor-free word in $\mathcal{D}_\beta$, and every word $w \in \mathcal{D}_\beta$ of length $(1+\beta)n$ has reducibility level n. Conversely, if $\beta > \alpha$ and $\mathrm{rl}(w) = n$ for every $w \in \mathcal{D}_{\beta/\alpha}$ with $|w| = (\alpha + \beta)n$, then $\alpha = 1$.*

Proof. Let $w \in \mathcal{D}_\beta$ be of length $(1+\beta)n$ with $n > 1$. By definition, $h(a) = \beta$ and $h(b) = -1$. Thus the right-most a in w must be followed by a string containing b^β. Hence w can be reduced by ab^β, so it is not factor-free and $\mathrm{rl}(w) = n$.

In order to prove the last statement, assume $\alpha > 1$ and let $w = a^\alpha b^{\beta-1} ab a^{\alpha-1} b^\beta$. Then we have $|w| = 2(\alpha + \beta)$ and $\mathrm{rl}(w) = 1$, which contradicts the assumption that $\mathrm{rl}(w) = 2$ for every $w \in \mathcal{D}_{\beta/\alpha}$ with $|w| = 2(\alpha + \beta)$. $\square$

Remark. From the previous proposition, it is obvious that any reducibility level statistic is only meaningful for rational Dyck paths with non-integer slopes.

Theorem 2.6. *The number $r_{n,k}$ of $\frac{\beta}{\alpha}$-Dyck paths of length $(\alpha + \beta)n$ that have reducibility level equal to k is given by*

$$r_{n,k} = \binom{(\alpha + \beta)n}{k - 1} \frac{(k - 1)!}{n!} B_{n,k}(1!\theta_1, 2!\theta_2, \dots), \tag{2.7}$$

where θ_j is the number of factor-free words in $\mathcal{D}_{\beta/\alpha}$ of length $(\alpha + \beta)j$.

Proof. As a direct consequence of the bijection given in Theorem 2.3, we get that $r_{n,k}$ is equal to the number of Dyck paths in $\mathfrak{D}_n^\Theta(\alpha + \beta, 0)$ having exactly k peaks. Thus (2.7) follows from [1, Theorem 3.5]. $\square$

Example 2.8. If $\alpha = 1$, then ab^β is the only factor-free word in $\mathcal{D}_\beta$, so $\theta_1 = 1$ and $\theta_j = 0$ for $j \neq 1$. Therefore, the number of words w of length $(1+\beta)n$ with $\mathrm{rl}(w) = k$ is zero unless $k = n$. In that case (2.7) gives

$$r_{n,n} = \binom{(1 + \beta)n}{n - 1} \frac{(n - 1)!}{n!} = \frac{1}{\beta n + 1} \binom{(\beta + 1)n}{n},$$

which is, as expected, the total number of words of length $(1 + \beta)n$ in $\mathcal{D}_\beta$.

Example 2.9. As mentioned in the introduction, for $\alpha = 2$ and $\beta = 3$, the number of factor-free words of length $5n$ is given by the sum of adjacent Catalan numbers $C_n + C_{n-1}$. Thus the number of $\frac{3}{2}$-Dyck paths of length $5n$ having reducibility level k is giving by

$$r_{n,k} = \binom{5n}{k - 1} \frac{(k - 1)!}{n!} B_{n,k}(1!(C_0 + C_1), 2!(C_1 + C_2), \dots),$$

which by means of [7, Example 3.2] can be written as

$$r_{n,k} = \binom{5n}{k-1} \sum_{j=0}^{k} \frac{(-1)^{k-j}(2j-k)}{nk} \binom{k}{j} \binom{2(n+j)-k-1}{n-1}.$$

For example, among the 23 words of length 10 ($n = 2$) there are $r_{2,1} = 3$ factor-free words and $r_{2,2} = 20$ words w with $\mathrm{rl}(w) = 2$. And among the 377 words of length 15 there are 7 factor-free words, 90 words with $\mathrm{rl}(w) = 2$, and 280 words with $\mathrm{rl}(w) = 3$.

3. An inverse relation involving partial Bell polynomials

The purpose of this section is to prove an inverse relation for a family of sequences defined through partial Bell polynomials. The main ingredients are Faà di Bruno's formula, expressed in terms of partial Bell polynomials as in [4, Sec. 3.4, Theorem A], together with a substitution formula introduced by the authors in [2]. For convenience, we recall here these results as lemmas.

Lemma 3.1 (Faà di Bruno). *Let f and g be two formal power series:*

$$f = f_0 + \sum_{k=1}^{\infty} f_k \frac{u^k}{k!} \quad and \quad g = \sum_{\ell=1}^{\infty} g_\ell \frac{t^\ell}{\ell!}.$$

If $h = \sum_{n=0}^{\infty} h_n \frac{t^n}{n!}$ is the formal power series of the composition $f \circ g$, then the coefficients h_n are given by

$$h_0 = f_0, \quad h_n = \sum_{k=1}^{n} f_k B_{n,k}(g_1, g_2, \dots).$$

Lemma 3.2 ([2, Theorem 15]). *Let $a, b \in \mathbb{Z}$. Given any sequence (x_n), define (y_n) by*

$$y_n = \sum_{k=1}^{n} \binom{an+bk}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!x_1, 2!x_2, \dots)$$

for every $n \in \mathbb{N}$. Then, for any $\lambda \in \mathbb{C}$ we have

$$\sum_{k=1}^{n} \binom{\lambda}{k-1}(k-1)! B_{n,k}(1!y_1, 2!y_2, \dots) = \sum_{k=1}^{n} \binom{\lambda+an+bk}{k-1}(k-1)! B_{n,k}(1!x_1, 2!x_2, \dots).$$

Corollary 3.3. *If y_n is given by*

$$y_n = \sum_{k=1}^{n} \binom{an}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!x_1, 2!x_2, \dots),$$

then

$$x_n = \sum_{k=1}^{n} \binom{-an}{k-1}(k-1)! B_{n,k}(1!y_1, 2!y_2, \dots).$$

For the results in Section 4, we also need the following inverse relation.

Proposition 3.4. *Let $a \in \mathbb{Z}$. Given any sequence (x_n), define (y_n) by*

$$y_n = \sum_{k=1}^{n} \binom{an-1}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!x_1, 2!x_2, \dots).$$

Then, for every $n \in \mathbb{N}$ we have

$$x_n = \sum_{k=1}^{n} \frac{(1-an)^{k-1}}{n!} B_{n,k}(1!y_1, 2!y_2, \dots).$$

Proof. Let (z_n) be the sequence defined by

$$z_n = \sum_{k=1}^{n} \binom{an}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!x_1, 2!x_2, \dots).$$

By Lemma 3.2 with $b = 0$, we have

$$\sum_{k=1}^{n} \binom{\lambda-1}{k-1}(k-1)! B_{n,k}(1!z_1, 2!z_2, \dots) = \sum_{k=1}^{n} \binom{\lambda+an-1}{k-1}(k-1)! B_{n,k}(1!x_1, 2!x_2, \dots) \qquad (3.5)$$

for any $\lambda \in \mathbb{C}$, and

$$\sum_{k=1}^{n} \binom{-1}{k-1}(k-1)! B_{n,k}(1!z_1, 2!z_2, \dots) = \sum_{k=1}^{n} \binom{an-1}{k-1}(k-1)! B_{n,k}(1!x_1, 2!x_2, \dots). \qquad (3.6)$$

If we denote $y(t) = \sum_{n=1}^{\infty} y_n t^n$ and $z(t) = \sum_{n=1}^{\infty} z_n t^n$, then (3.6) means (via Faà di Bruno's formula) that the generating functions $y(t)$ and $z(t)$ are related by the identity

$$y(t) = \log(1 + z(t)).$$

Thus, if we let $f_1(t) = e^{\lambda t}$ and $f_2(t) = (1+t)^{\lambda}$, then

$$f_1(y(t)) = f_1(\log(1 + z(t))) = f_2(z(t)),$$

and Faà di Bruno's formula gives

$$[t^n](f_1 \circ y) = \sum_{k=1}^{n} \lambda^k B_{n,k}(1!y_1, 2!y_2, \dots), \text{ and}$$

$$[t^n](f_2 \circ z) = \sum_{k=1}^{n} (\lambda)_k \, B_{n,k}(1!z_1, 2!z_2, \dots) = \sum_{k=1}^{n} \lambda \binom{\lambda-1}{k-1}(k-1)! B_{n,k}(1!z_1, 2!z_2, \dots).$$

This implies

$$\sum_{k=1}^{n} \lambda^k B_{n,k}(1!y_1, 2!y_2, \dots) = \sum_{k=1}^{n} \lambda \binom{\lambda-1}{k-1}(k-1)! B_{n,k}(1!z_1, 2!z_2, \dots).$$

Combining this identity with (3.5), we obtain

$$\sum_{k=1}^{n} \lambda^{k-1} B_{n,k}(1!y_1, 2!y_2, \dots) = \sum_{k=1}^{n} \binom{\lambda+an-1}{k-1}(k-1)! B_{n,k}(1!x_1, 2!x_2, \dots).$$

Finally, if $\lambda = 1 - an$, then most terms on the right-hand side of the equation vanish and the sum reduces to $B_{n,1}(1!x_1, 2!x_2, \dots) = n!x_n$. This gives the claimed identity. $\qquad \square$

4. Enumeration of factor-free Dyck words

Let α and β be positive integers with $\gcd(\alpha, \beta) = 1$. As noted in the introduction, Bizley's formula (1.1) for the number of $\frac{\beta}{\alpha}$-Dyck paths of length $(\alpha+\beta)n$ can be conveniently written using partial Bell polynomials as

$$\phi_n = \sum_{k=1}^{n} \frac{1}{n!} B_{n,k}(1!f_1, 2!f_2, \dots),$$

where $f_j = \frac{1}{(\alpha+\beta)j}\binom{(\alpha+\beta)j}{\alpha j}$ for $j \in \mathbb{N}$. Inverting the above identity, we get

$$f_n = \sum_{k=1}^{n} \binom{-1}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!\phi_1, 2!\phi_2, \dots). \tag{4.1}$$

On the other hand, as claimed in (2.1), we have the alternative representation

$$\phi_n = \sum_{k=1}^{n} \binom{(\alpha+\beta)n}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!\theta_1, 2!\theta_2, \dots), \tag{4.2}$$

where θ_j is the number of factor-free Dyck words with slope β/α and length $(\alpha+\beta)j$. Using Lemma 3.2 with $a = \alpha + \beta$, $b = 0$, and $\lambda = -1$, we get

$$\sum_{k=1}^{n} \binom{-1}{k-1}(k-1)! B_{n,k}(1!\phi_1, 2!\phi_2, \dots) = \sum_{k=1}^{n} \binom{(\alpha+\beta)n-1}{k-1}(k-1)! B_{n,k}(1!\theta_1, 2!\theta_2, \dots),$$

which together with (4.1) gives the identity

$$f_n = \sum_{k=1}^{n} \binom{(\alpha+\beta)n-1}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!\theta_1, 2!\theta_2, \dots) \tag{4.3}$$

for every $n \in \mathbb{N}$.

As a consequence of the inverse relation given in Proposition 3.4, we obtain:

Theorem 4.4. *For every $n \in \mathbb{N}$, the number of factor-free Dyck words with slope β/α and length $(\alpha + \beta)n$ is given by*

$$\theta_n = \sum_{k=1}^{n} \frac{(1 - (\alpha + \beta)n)^{k-1}}{n!} B_{n,k}(1!f_1, 2!f_2, \dots),$$

where $f_j = \frac{1}{(\alpha+\beta)j}\binom{(\alpha+\beta)j}{\alpha j}$ for every j.

This formula is easy to implement in any of the mainstream computer algebra systems. For example, in Sage, Maple, and Mathematica, partial Bell polynomials are implemented as `bell_polynomial`, `IncompleteBellB`, and `BellY`, respectively. The following table shows a few terms of the sequence (θ_n) for various slopes β/α.

β/α	OEIS	Sequence (θ_n) of factor-free words with slope β/α and length $(\alpha + \beta)n$
3/2	A005807	2, 3, 7, 19, 56, 174, 561, 1859, 6292, 21658, 75582, 266798, ...
5/2	A274052	3, 13, 94, 810, 7667, 76998, 805560, 8684533, 95800850, 1076159466, ...
7/2	A274244	4, 34, 494, 8615, 165550, 3380923, 71999763, 1580990725, 35537491360, ...
9/2	A274256	5, 70, 1696, 49493, 1593861, 54591225, 1950653202, 71889214644, ...
4/3	A274257	5, 52, 880, 17856, 399296, 9491008, 235274240, 6014201600, ...
5/3	A274258	7, 133, 4140, 154938, 6398717, 281086555, 12882897819, 609038885805, ...
7/3	A274259	12, 570, 44689, 4223479, 441010458, 49014411306, 5685822210429, ...

The other class of lattice paths considered by Bizley [3] is the set of $\frac{\beta}{\alpha}$-Dyck paths that stay *strongly* below the line $y = \frac{\beta}{\alpha}x$. He proved that the number ψ_n of such paths of length $(\alpha + \beta)n$ is given by

$$\psi_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{n!} B_{n,k}(1!f_1, 2!f_2, \dots).$$

Written in terms of (ψ_n), the sequences (f_n), (ϕ_n), and (θ_n) show an interesting pattern.

Proposition 4.5. *The following identities hold for every $n \in \mathbb{N}$:*

$$f_n = \sum_{k=1}^{n} \frac{(k-1)!}{n!} B_{n,k}(1!\psi_1, 2!\psi_2, \dots), \tag{4.6}$$

$$\phi_n = \sum_{k=1}^{n} \frac{k!}{n!} B_{n,k}(1!\psi_1, 2!\psi_2, \dots), \tag{4.7}$$

$$\theta_n = \sum_{k=1}^{n} \binom{-(\alpha+\beta)n + k}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!\psi_1, 2!\psi_2, \dots). \tag{4.8}$$

Note that the above expressions are instances of

$$\sum_{k=1}^{n} \binom{r+k-1}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!\psi_1, 2!\psi_2, \dots)$$

for $r = 0, 1, 1 - (\alpha + \beta)n$, respectively. It is worth mentioning that the sequence (ϕ_n) is the INVERT transform of the sequence (ψ_n).

Proof of Proposition 4.5. Let $f(t)$, $\phi(t)$, and $\psi(t)$ be the generating functions of (f_n), (ϕ_n), and (ψ_n), respectively. Since $\psi(t) = 1 - e^{-f(t)}$ and $1 + \phi(t) = e^{f(t)}$, we get

$$f(t) = -\log(1 - \psi(t)) \text{ and } 1 + \phi(t) = \frac{1}{1 - \psi(t)},$$

which give identities (4.6) and (4.8) via Faà di Bruno's formula.

Using (4.7) we now write ϕ_n as

$$\phi_n = \sum_{k=1}^{n} \binom{k}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!\psi_1, 2!\psi_2, \dots)$$

and use Lemma 3.2 with $a = 0$, $b = 1$, and $\lambda = -(\alpha + \beta)n$ to conclude that

$$\sum_{k=1}^{n} \binom{-(\alpha+\beta)n}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!\phi_1, 2!\phi_2, \dots) = \sum_{k=1}^{n} \binom{-(\alpha+\beta)n+k}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!\psi_1, 2!\psi_2, \dots).$$

Finally, because of the representation (4.2), Corollary 3.3 implies that the left-hand side of the above identity is precisely θ_n. Thus (4.8) holds. $\qquad \square$

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