

GRAPHS WITH OBSTACLE NUMBER GREATER THAN ONE

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ABSTRACT. An *obstacle representation* of a graph G is a straight-line drawing of G in the plane together with a collection of connected subsets of the plane, called *obstacles*, that block all non-edges of G while not blocking any of the edges of G . The *obstacle number* $\text{obs}(G)$ is the minimum number of obstacles required to represent G .

We study the structure of graphs with obstacle number greater than one. We show that the icosahedron has obstacle number 2, thus answering a question of Alpert, Koch, & Laison asking whether all planar graphs have obstacle number at most 1. We also show that the 1-skeleton of a related polyhedron, the *gyroelongated 4-bipyramid*, has obstacle number 2. The order of this graph is 10, which is also the order of the smallest known graph with obstacle number 2.

Some of our methods involve instances of the Satisfiability problem; we make use of various “SAT solvers” in order to produce computer-assisted proofs.

1. INTRODUCTION

All graphs will be finite, simple, and undirected. Following Alpert, Koch, & Laison [2], we define an *obstacle representation* of a graph G to be a straight-line drawing of G in the plane, together with a collection of connected subsets of the plane, called *obstacles*, such that no obstacle meets the drawing of G , while every non-edge of G intersects at least one obstacle. By *non-edge*, we mean a pair of distinct vertices a, b of G where ab is not an edge of G . The least number of obstacles required to represent G is the *obstacle number* of G , denoted $\text{obs}(G)$. For clarity, we will sometimes refer to this as the *ordinary* obstacle number. In an obstacle representation of a graph G , an *outside obstacle* is an obstacle that is contained in the unbounded component of the complement of the drawing of G . Any other obstacle is an *interior obstacle*. Note that by [10], determining whether $\text{obs}(G) \leq k$ for a given k is *not* in NP.

Figure 1 shows several obstacle representations for the Petersen graph P . The usual drawing of P requires five obstacles. This is illustrated in Figure 1a (edges of P are shown as thick line segments, while *non-edges* are shown as thin line segments).

An obstacle may be any connected plane region; however, it is convenient to expand each obstacle until it is as large as possible. Each of the resulting obstacles is an open plane region forming a connected component of the complement of the drawing of the graph. The boundary of the obstacle—which is not part of the obstacle itself—is formed by appropriate parts of the edges of the graph. Figure 1b illustrates such obstacles for P .

However, P can be represented using fewer obstacles; $\text{obs}(P)$ is in fact 1. Figure 1c shows an obstacle representation of P using only a single obstacle (adapted from Laison [13]).

A related parameter, in which each obstacle is a single point, was studied by Matoušek [14] and by Dumitrescu, Pach, and Tóth [7]. The study of the obstacle number *per se* was

John Gimbel’s research was partially supported by the Czech Ministry of Education Grant #ERC-CZ 1201.

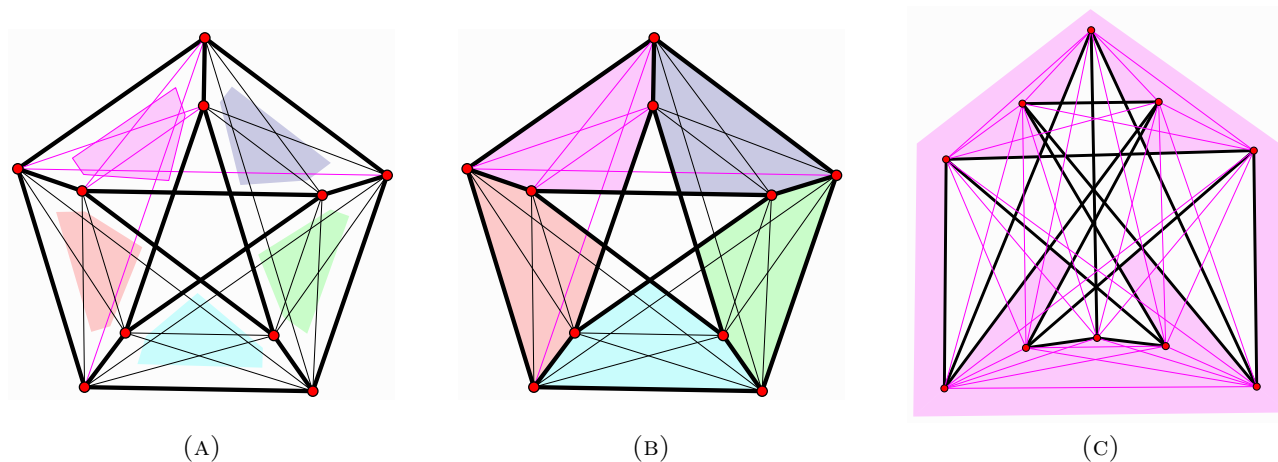


FIGURE 1. (A) The familiar embedding of the Petersen graph, which requires five obstacles; the non-edges blocked by the pink obstacle are colored pink. (B) Here, we expand the obstacles as much as possible, until each is a maximal connected region in the complement of the drawing. (C) However, $\text{obs}(P) = 1$, using a single outside obstacle.

initiated by Alpert, Koch, & Laison [2]. These parameters have since been investigated by others [8, 10, 16, 17, 18, 19].

Alpert, Koch, & Laison [2, Thm. 2] showed that there exist graphs with arbitrarily high obstacle number and asked [2, p. 229] for the smallest order of a graph with obstacle number greater than 1. They proved [2, Thm. 4] that the graph $K_{5,7}^*$ has obstacle number 2, where $K_{a,b}^*$ (with $a \leq b$) denotes the graph obtained by removing a matching of size a from the complete bipartite graph $K_{a,b}$. Pach & Sariöz [19, Thm. 2.1] found a smaller example of a graph with obstacle number 2.

Theorem 1.1 (Pach & Sariöz). $\text{obs}(K_{5,5}^*) = 2$.

The Pach-Sariöz example, of order 10, is apparently the smallest known graph with obstacle number greater than 1. In section 2 we consider the question of whether any graph of smaller order has obstacle number greater than 1. We do not answer this question; however, in Proposition 5.3 we will provide an example of a *planar* graph of order 10 with obstacle number 2.

We define the *outside obstacle number* of G to be the least number of obstacles required to represent G , such that one of the obstacles is an outside obstacle—or zero if G has obstacle number zero. We denote the outside obstacle number of G by $\text{obs}_{\text{out}}(G)$.

Clearly we have $\text{obs}(G) \leq \text{obs}_{\text{out}}(G) \leq \text{obs}(G) + 1$ for every graph G . Alpert, Koch, & Laison [2, p. 229] asked (using different terminology) whether every graph G with $\text{obs}(G) = 1$ also has $\text{obs}_{\text{out}}(G) = 1$. We ask a more general question.

Question 1.2. *Is it true that $\text{obs}(G) = \text{obs}_{\text{out}}(G)$ for every graph G ?*

Alpert, Koch, & Laison [2, p. 231] asked whether every planar graph has obstacle number at most 1 (also see a series of questions in the Open Problem Garden [9]). They further

asked for the obstacle numbers of the icosahedron and the dodecahedron. In Sections 3, 4, and 5 we develop tools for determining the obstacle numbers of particular graphs, and we use them to address these questions.

In Section 3 we consider the outside obstacle number. We make use of instances of the Satisfiability problem (SAT) to produce computer-assisted proofs that the outside obstacle number of certain graphs is at least 2.

In Section 4 we consider the ordinary obstacle number. We show that a lower bound on the outside obstacle number of a graph implies a lower bound on the obstacle number of a different graph. We use this to answer the first of the above-mentioned questions by constructing a planar graph with obstacle number 2.

In Section 5 we develop methods similar to those of Section 3 for producing computer-assisted proofs that the ordinary obstacle number of certain graphs is at least 2. We answer another of the above-mentioned questions by showing that the obstacle number of the icosahedron is 2. We also show that the obstacle number of the dodecahedron is 1. We further describe a graph of order 10 that has obstacle number 2. This graph, which we call X_4 , has the same order as the Pach-Sariöz example, but, unlike that graph, it is planar.

We conclude this section with some easy observations, which we will make use of throughout the remainder of this paper.

Observation 1.3. *Given an obstacle representation of a graph G , we can perturb all vertices an arbitrarily small distance to obtain an essentially equivalent obstacle representation in which no three vertices are collinear.*

Because of the above observation, we will generally assume that our obstacle representations have the property that no three vertices are collinear. Lest there be any confusion, only obstacles block edges, vertices do not, so this has no impact on the definition of obstacle number.

Observation 1.4. *Obstacles are not required to be simply connected, but adding this requirement does not change the obstacle number.*

To see why this is true we note two facts: (1) whether an edge is blocked or not by an obstacle is determined only by whether the edge somewhere crosses a portion of the boundary of the obstacle; (2) if an obstacle is not simply connected, then it may be replaced with a simply connected obstacle that blocks those same edges by removing sufficiently small open sets about curves connecting portions of the complement that do not intersect the points on the boundary where edges of the graph cross the obstacle (the simplest case is cutting a small wedge out of an annulus blocking an edge connecting a vertex inside the annulus to a vertex outside the annulus).

Observation 1.5. *Let G be a graph, and let H be an induced subgraph of G . Given an obstacle representation of G , removing all vertices of G that are not in H , and keeping the same obstacles, results in an obstacle representation of H .*

The next result follows easily from the above observation.

Proposition 1.6. *Let k be a nonnegative integer. The class of all graphs G such that $\text{obs}(G) \leq k$ is closed under taking induced subgraphs, and similarly for obs_{out} .*

Proposition 1.6 does not hold if the word “induced” is removed, as the classes in question are generally not closed under taking arbitrary subgraphs. For example, Alpert, Koch, &

Laison [2, Thm. 2] showed that there are graphs with arbitrarily large obstacle number. But every graph is a subgraph of a complete graph, and $\text{obs}(K_n) = 0$ for every n .

2. SMALL GRAPHS

As noted in the previous section, Pach & Sariöz showed that $\text{obs}(K_{5,5}^*) = 2$ (see Theorem 1.1). What can we say about graphs of small order with obstacle number greater than 1?

Proposition 2.1. *Let G be a graph. Let H be a graph obtained by starting with G and adding a new vertex of degree at most 1. If G is complete and H is not complete, then $\text{obs}(H) = 1 = \text{obs}(G) + 1$. Otherwise, $\text{obs}(H) = \text{obs}(G)$. Furthermore, these continue to hold if obs is replaced by obs_{out} .*

Proof. We give the proof for obs ; the proof for obs_{out} is essentially the same.

We have $\text{obs}(G) = 0$ if and only if G is complete; the cases when G is complete follow easily. Suppose that G is not complete, and let v be the new vertex in H .

Begin with an obstacle representation of G . First suppose that the degree of v in H is 0. In this case, Let T be any obstacle of G . Let p be a point in the interior of T , replace T by $T - p$, and place vertex v at the point p . Then $\text{obs}(H) = \text{obs}(G)$ since the edges between v and the vertices of G are blocked by T .

If instead v has degree 1, then let $q \in G$ be the neighbor of v in H . Choose a point p in the interior of an obstacle T so that the line segment $S = pq$ intersects no other obstacle of G ; note that S may cross edges of G . (To find such a T choose any obstacle R and a point r on its interior, and then let T be the first obstacle crossed by the ray \overrightarrow{qr} .) Now replace T by $T - S$, and place vertex v at the point p . (See Figure 2 for an illustration.) This results in an obstacle representation of H using the same number of obstacles as that for G , and also using an outside obstacle if the representation of G used one. \square

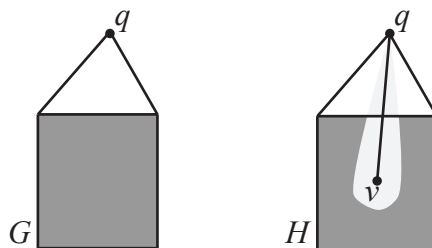


FIGURE 2. The graph H obtained from G by adding a single pendant vertex. The dark gray shaded area corresponds to an obstacle for each graph.

We restate [2, Thm. 5] as follows.

Theorem 2.2. *The obstacle number of an outerplanar graph is at most 1.*

Corollary 2.3. *Let G be a graph with the property that every cycle in G has length 3. Then $\text{obs}_{\text{out}}(G) \leq 1$.*

Proof. Graph G can have no minor isomorphic to K_4 or $K_{2,3}$, since these graphs both have 4-cycles. Hence G is outerplanar, so by Theorem 2.2, the obstacle number is at most 1. In their proof, Alpert, Koch, & Laison construct an obstacle representation using a single outside obstacle, thus showing that obs_{out} is at most 1. The result follows. \square

Proposition 2.4. *If a graph G has order at most 5, then $\text{obs}_{\text{out}}(G) \leq 1$.*

Proof. By Proposition 2.1, we may assume that G has minimum degree at least 2, since vertices of degree at most 1 do not affect the obstacle number. There are only four graphs of order at most 4 whose minimum degree is at least 2: K_3 , C_4 , K_4 , and $K_4 - e$. All of these are easily checked. So assume that G has order 5.

If G contains no 4- or 5-cycle, then we may apply Corollary 2.3. If G has a 5-cycle, then arrange the 5 vertices on a circle in the plane so that each edge in the 5-cycle joins vertices that are not consecutive on the circle (so the 5-cycle is a pentagram). No matter what other edges lie in G , an outside obstacle can block all non-edges (see Figure 3 for an illustration).

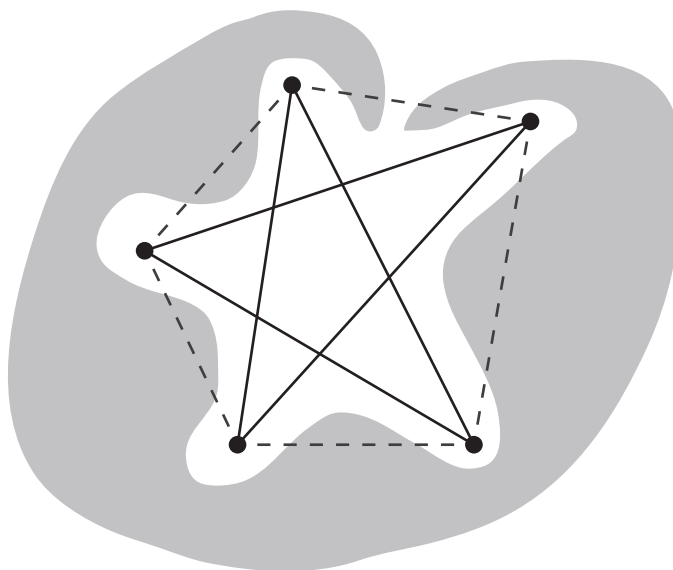


FIGURE 3. An embedding of a 5-cycle permitting the blocking of any remaining possible edges (drawn as dashed lines) by a single outside obstacle.

So we may assume that G contains no 5-cycle, but does contain a 4-cycle. Choose a 4-cycle in G , and let x be the vertex of G that does not lie on this 4-cycle. Since G has minimum degree at least 2, x must be adjacent to at least 2 vertices of the 4-cycle. The neighbors of x cannot include two consecutive vertices on the 4-cycle, since then G would have a 5-cycle. Consequently, x has degree exactly 2, and its neighbors are nonconsecutive vertices on the 4-cycle. It remains to consider adjacencies among vertices in the 4-cycle. The non-neighbors of x cannot be adjacent, as this would form a 5-cycle in G .

The only question left is whether the neighbors of x are adjacent. Thus there are exactly 2 graphs that satisfy our assumptions: $K_{2,3}$ and $K_{1,1,3}$. Both are easily shown to have outside obstacle number 1. \square

Corollary 2.5. *Let k be the minimum order of a graph G with $\text{obs}(G) = 2$. Then $6 \leq k \leq 10$. Similarly, the minimum order k of a graph G with $\text{obs}_{\text{out}}(G) = 2$ satisfies $6 \leq k \leq 10$.*

Proof. The lower bound follows from Proposition 2.4. For the ordinary obstacle number, the upper bound follows from the result of Pach & Sariöz (Theorem 1.1). The obstacle

representation constructed by Pach & Sariöz [19, Fig. 1] uses an outside obstacle, and so $\text{obs}_{\text{out}}(K_{5,5}^*) = 2$ as well. The upper bound for the outside obstacle number follows. \square

We now consider what properties a small graph with $\text{obs}(G) > 1$ must have. We show that such a graph cannot be a subgraph of $K_{4,4}$.

Proposition 2.6. *Let G be a subgraph of $K_{4,4}$. Then $\text{obs}_{\text{out}}(G) \leq 1$.*

Proof. Let a, b be the smallest values such that G is a subgraph of $K_{a,b}$. Let H be the subgraph of $K_{a,b}$ formed by those edges that do *not* lie in G .

If G has a vertex of degree 0 or 1, then we may apply induction, by Proposition 2.1. So assume G has minimum degree at least 2. Then H has maximum degree at most 2.

We observe that, if there is a set S of vertices containing at most 2 vertices from each partite set, such that S contains an endpoint of each edge in H , then we can draw G so that every non-edge is blocked by an outside obstacle, by placing the two partite sets in (near) lines, with the vertices in S at the ends of their respective lines. (See Figure 4 for an illustration.) We will use this observation repeatedly in the remainder of this proof.

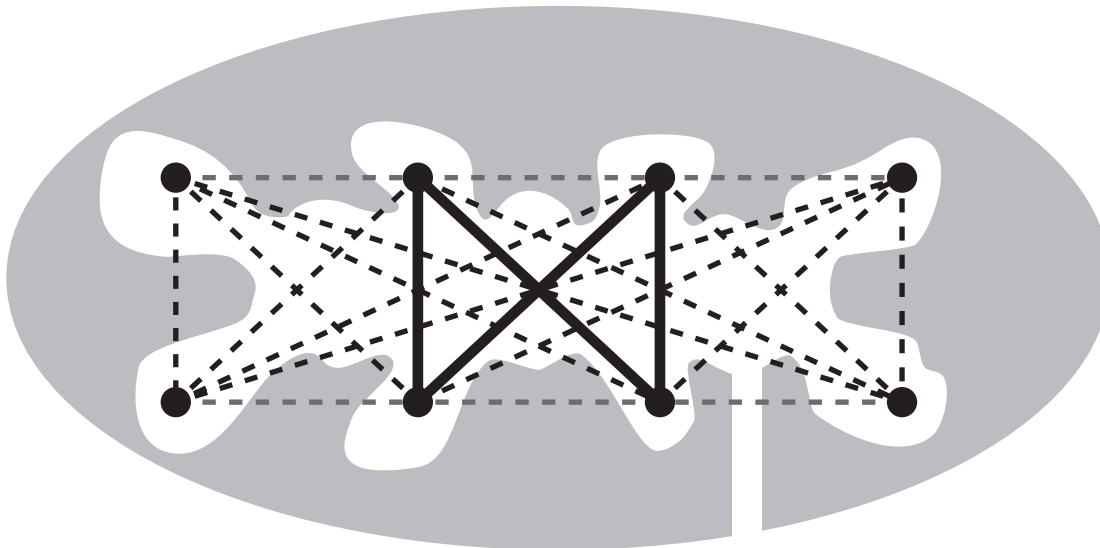


FIGURE 4. Illustration of part of the proof of Proposition 2.6. The two partite sets of G are formed by the upper and lower four vertices, respectively. The set S consists of the two rightmost and the two leftmost vertices. Light dashed lines are non-edges within each partite set. Dark dashed lines are the non-edges between the partite sets, that is, the edges of H ; each has at least one endpoint in S . The drawing illustrates how the gray outside obstacle can block every non-edge of G .

If H does not contain a matching of size 4, then H has a vertex cover C containing at most 3 vertices, by the König-Egerváry Theorem (see, e.g., Bondy & Murty [5, p. 74]). If C includes members of both partite sets, then let $S = C$. Otherwise, replace one of the vertices in C with the other endpoints of the (at most 2) edges of H it is incident with; again, we have our set S .

So we may assume that H has a matching of size 4. Thus $a = b = 4$. Since G has minimum degree at least 2, we have two cases: in the first case G contains two vertex-disjoint 4-cycles; in the second G contains an 8-cycle. In the former case, the vertices of one of the 4-cycles form our set S . In the latter case, if G contains at least one edge $e = xy$ that is not in the 8-cycle, then e and 3 edges of the 8-cycle form a 4-cycle in G . Let S consist of the 4 vertices of G that do not lie on this 4-cycle.

It remains only to handle the case when G is an 8-cycle with no additional edges. In this case there is no set S with the properties we are looking for. However, we can draw G with every edge blocked by an outside obstacle as in Figure 5. \square

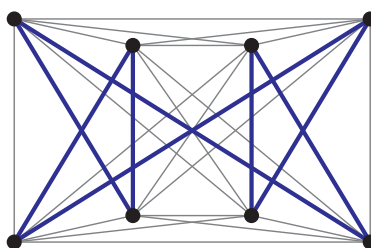


FIGURE 5. An obstacle representation of an 8-cycle using a single outside obstacle. Thin lines represent non-edges.

3. OUTSIDE OBSTACLE NUMBER

We now begin our development of tools for explicitly determining the (outside) obstacle number of a particular graph. We will apply these tools to various planar graphs. In this section, we cover results related to outside obstacle number; we will consider the ordinary obstacle number in the following sections.

Our ideas are based on the Satisfiability Problem (SAT). An instance of this problem is a particular kind of Boolean formula. In this context, a *variable* is a Boolean variable; i.e., its value is either *true* or *false*. A *literal* is either a variable or the negation of a variable. A *clause* is the inclusive-OR of one or more literals. For example, given variables x_1 , x_2 , x_3 and x_4 , the following is an example of a clause:

$$x_1 \vee \neg x_2 \vee x_4$$

An instance of SAT consists of a number of clauses. The instance is said to be *satisfiable* if there exists a truth assignment for the variables such that every clause in the instance is true; i.e., a truth assignment in which each clause contains at least one true literal.

We will show how to construct, for each graph G , a SAT instance encoding necessary conditions for the existence of an obstacle representation of G using no interior obstacles. Thus, if we can show that the instance is not satisfiable, then we know that $\text{obs}_{\text{out}}(G) \geq 2$. There are a number of freely available, high-quality implementations of algorithms to determine satisfiability of a SAT instance. Using these, we will construct computer-aided proofs that $\text{obs}_{\text{out}} \geq 2$ for various planar graphs.

When interpreting the effect of the satisfiability or non-satisfiability of the SAT instances on the obstacle number problem, it is important to note that *if* the SAT instance is *not* satisfiable, then we are guaranteed that it is impossible to draw the graph using a single outside obstacle; i.e., $\text{obs}_{\text{out}} \geq 2$. However, if the SAT instance *is* satisfiable, then it does not

follow that $\text{obs}_{\text{out}} \leq 1$; satisfiability is a necessary but not sufficient condition for concluding that $\text{obs}_{\text{out}} \leq 1$.

For a, b and $c \in \mathbb{R}^2$, we say that abc is a *clockwise triple* if a, b and c appear in clockwise order. More formally, identifying each point in \mathbb{R}^2 with the corresponding point in the plane $z = 1$ in \mathbb{R}^3 via the map $a = (a_1, a_2) \mapsto (a_1, a_2, 1)$, we denote the determinant of the 3×3 matrix with columns a, b and c by $[abc]$. Then abc is a clockwise triple if $[abc] < 0$. We similarly define *counter-clockwise triple*. Note that for a triple abc , exactly one of the following is true: abc is clockwise, abc is counter-clockwise, or a, b and c are collinear (in which case $[abc] = 0$).

The following two lemmas give properties that hold for all point arrangements in the plane. We will use these properties to construct clauses for our SAT instances.

Lemma 3.1 (4-Point Rule). *Let a, b, c, d be distinct points in \mathbb{R}^2 . If abc, acd , and adb are clockwise triples, then bcd must also be a clockwise triple.*

Proof. The 4-Point Rule is equivalent to what D. Knuth called the *interiority* property of triples of points; see Knuth [12, p. 4, Axiom 4]. In addition, the 4-Point Rule is a consequence of the fact that a point configuration is a rank 3 *acyclic* oriented matroid. Intuitively, the 4-Point Rule follows from the observation that if abc is a clockwise-oriented triple, then d cannot be “outside” all the lines ab, bc, cd (where “outside the lines” means on the negative side of the closed halfplanes determined by the orientation of abc). See Figure 6. \square

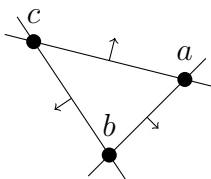


FIGURE 6. If abc is a clockwise-oriented triangle, then d cannot be on the negative side with respect to the orientation of abc (negative side indicated by the arrows) of each of the halfplanes determined by the sides of abc . Therefore, if acd and adb are also clockwise-oriented, then bcd must be clockwise too.

We encode the 4-Point Rule as follows. For each triple abc we introduce a variable x_{abc} representing the statement that abc is a clockwise triple. By Observation 1.3 we may assume that no three vertices are collinear, so $\neg x_{abc}$ represents the statement that abc is a counterclockwise triple. The 4-point rule is represented by the clause

$$(1) \quad \neg x_{abc} \vee \neg x_{acd} \vee \neg x_{adb} \vee x_{bcd}.$$

Lemma 3.2 (5-Point Rule). *Let a, b, c, d and e be distinct points in \mathbb{R}^2 . If abc, acd, ade , and abe are clockwise triples, then either*

- (i) *both abd and ace are clockwise triples, or*
- (ii) *both abd and ace are counter-clockwise triples.*

Proof. This is equivalent to what D. Knuth called the *transitivity* property of triples of points; see Knuth [12, p. 4, Axiom 5]. It also follows from the Grassman-Plücker relation

$$[abc][ade] - [abd][ace] + [abe][acd] = 0;$$

see Björner *et al.* [4, pp. 6–7]. \square

The 5-Point Rule is represented by the following two clauses:

$$(2) \quad \neg x_{abc} \vee \neg x_{acd} \vee \neg x_{ade} \vee \neg x_{abe} \vee x_{abd} \vee \neg x_{ace};$$

$$(3) \quad \neg x_{abc} \vee \neg x_{acd} \vee \neg x_{ade} \vee \neg x_{abe} \vee \neg x_{abd} \vee x_{ace}.$$

In constructing our SAT instance, we will make use of the following lemma, which describes a restriction that must hold for an obstacle representation without interior obstacles.

Lemma 3.3. *Suppose we are given an obstacle representation of a graph G using no interior obstacles. Let ab be a non-edge of G . Then there exists a half-plane H determined by the line \overleftrightarrow{ab} such that, for each a, b -path P in G , some internal vertex of P lies in H .*

Proof. Denote by H^+ and H^- the two closed half-planes determined by the line \overleftrightarrow{ab} .

Suppose that there exist two distinct a, b -paths P_1 and P_2 in G such that (without loss of generality) $P_1 \subseteq H^+$ and $P_2 \subseteq H^-$. Perturbing slightly if necessary, we assume that no internal vertex of P_1 or P_2 lies on the line \overleftrightarrow{ab} (see Observation 1.3). We observe then that $P_1 \cup P_2$ forms a closed curve in the plane; the open line segment from a to b lies in a bounded component of the complement of this closed curve. By assumption there is no obstacle lying in such a bounded component, and thus there is nothing to block the line segment ab , a contradiction. \square

Given a graph G , we use Lemmas 3.1, 3.2, and 3.3 to create a SAT instance whose satisfiability is a necessary condition for the existence of an obstacle representation of G using no interior obstacles. If this SAT instance is not satisfiable, then we may conclude that graph G requires an interior obstacle, that is, that $\text{obs}_{\text{out}}(G) > 1$.

Since we are placing the vertices of G in the plane, the 4-Point Rule (Lemma 3.1) must hold for any 4 vertices in the graph, the 5-Point Rule (Lemma 3.2) must hold for any 5 vertices in the graph, and by Observation 1.3 we may assume that no three vertices are collinear. Our SAT instance includes clauses from the 4-Point Rule corresponding to clause (1) for every set of 4 vertices of our graph, and every permutation of these 4 vertices. It also includes clauses from the 5-Point Rule corresponding to clauses (2) and (3) for every set of 5 vertices of our graph, and every permutation of these 5 vertices.

Note that there are six ways to say vertices a, b and c lie in clockwise order. Variables corresponding to even permutations of abc ($x_{abc}, x_{bca}, x_{cab}$) represent equivalent statements. Variables corresponding to odd permutations of abc ($x_{bac}, x_{acb}, x_{cba}$) represent their negations.

When we construct our SAT instance, we may choose one of the variables from among $\{x_{abc}, x_{bca}, x_{cab}, x_{bac}, x_{acb}, x_{cba}\}$ as the canonical variable; we represent the other five using either the canonical variable or its negation, as appropriate.

Additionally, the actions of even permutations of $\{b, c, d\}$ on clauses (1 and (2) result in statements equivalent to the original; likewise an even permutation of $\{c, d, e\}$ does not alter clause (3). This reduces the number of clauses required by a factor of 3. Thus, for an n -vertex graph, our SAT instance includes $\binom{n}{4} \cdot 4!/3 = 8\binom{n}{4}$ clauses based on the 4-Point Rule and $2\binom{n}{5} \cdot 5!/3 = 80\binom{n}{5}$ clauses based on the 5-Point Rule.

We similarly construct SAT clauses based on Lemma 3.3. Since this lemma concerns a line defined by two points, and which side of this line certain other points lie on, the lemma can be stated in terms of clockwise or counter-clockwise triples. Specifically, this lemma implies that, for each pair of nonadjacent vertices a, b of G , one of the two half-planes determined by segment ab is “special”; that is, this half-plane contains at least one internal vertex from

each a, b -path in G . We create a new variable s_{ab} representing the statement that the special half-plane is the half-plane containing points p such that abp is a clockwise triple.

Let a, s, t, \dots, u, b be the sequence of vertices in some a, b -path P . Then the following clauses represent the statement of Lemma 3.3 for P :

$$(4) \quad \neg s_{ab} \vee x_{abs} \vee x_{abt} \vee \dots \vee x_{abu}$$

$$(5) \quad s_{ab} \vee \neg x_{abs} \vee \neg x_{abt} \vee \dots \vee \neg x_{abu}$$

As with the x variables, we choose one of s_{ab} and s_{ba} to be the canonical variable, and we represent the other by its negation.

Observation 3.4. *Let G be a graph. If $\text{obs}_{\text{out}}(G) \leq 1$, then the SAT instance consisting of all clauses of the forms (2)–(5)—using canonical variables, as discussed above—is satisfiable.*

Note that if all that is known is that the SAT instance is satisfiable, then we have no definitive information about $\text{obs}_{\text{out}}(G)$, although the satisfiability of the instance may provide us with a starting point for finding—by hand—a useful embedding.

Using these ideas, we can determine the exact value of the outside obstacle number for some special planar graphs.

Following [11], for $n \geq 3$ we define the *gyroelongated n -bipyramid* to be a convex polyhedron formed by adding pyramids to the top and bottom base of the n -antiprism; see Figure 7. The gyroelongated square bipyramid, when constructed using equilateral triangles, is also known as the Johnson solid J_{17} . The gyroelongated pentagonal bipyramid, again when constructed of equilateral triangles, is the regular icosahedron. We denote the skeleton of the gyroelongated n -bipyramid by X_n . We also refer to X_5 as I , since it is the icosahedron.

The graph X_n can be constructed as two disjoint n -wheels, connected by a $2n$ -cycle that alternates between vertices of the wheel boundaries taken cyclically. Figures 7, 8, 9, and 11 show gyroelongated n -bipyramids for various values of n , while Figure 10 illustrates the general case. In each of these figures, the wheel boundaries are labeled with consecutive numbers and are shown in green, the spokes of the wheel in blue, and the connecting cycle in black; the cycle connecting the two wheel boundaries correspond to the sequence of labeled vertices $1, 1, 2, 2, \dots, n, n$. Non-edges are shown with thin gray or pink lines.

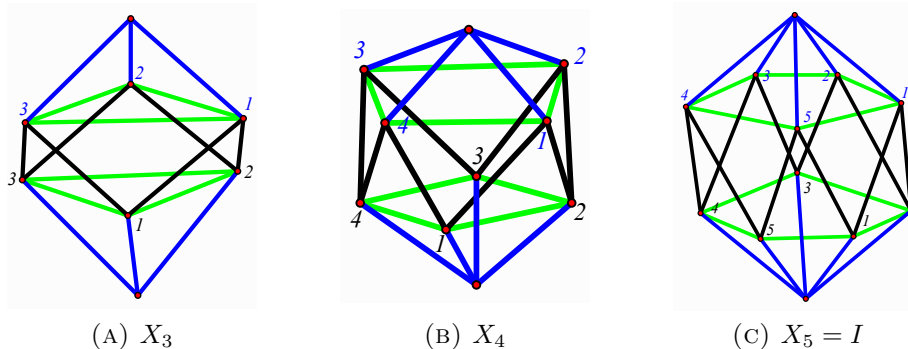


FIGURE 7. Gyroelongated n -bipyramid skeleta; the antiprisms are highlighted in black and green, while the pyramids erected on the bases are shown in blue.

Proposition 3.5. *All of the following hold.*

- (1) $\text{obs}_{\text{out}}(X_4) = 2$.
- (2) $\text{obs}_{\text{out}}(I) = \text{obs}_{\text{out}}(X_5) = 2$.
- (3) $\text{obs}_{\text{out}}(X_6) = 2$.

Proof. The lower bounds were found using a computer. Two of the authors independently created SAT instances as described above, using clauses derived from Lemmas 3.1 (the 4-Point Rule), 3.2 (the 5-Point Rule), and 3.3, as discussed above. See [6] for software to generate the SAT instances. For each graph, standard SAT solvers (we used MiniSat [15], PicoSAT [3], and zChaff [20]) indicate that the SAT instance is not satisfiable. Thus, by Observation 3.4, we have $\text{obs}_{\text{out}} \geq 2$ for each graph.

For the upper bounds, we exhibit an obstacle representation of each graph using two obstacles, one of which is an outside obstacle. Figure 8 shows embeddings of the icosahedron, while Figure 9 shows embeddings of X_4 and X_6 . \square

Note that Figure 8a exhibits 3-fold dihedral symmetry, while the other representations have only one vertical mirror of symmetry.

Figure 10 shows how the embeddings in Figures 8b and 9 can be generalized to a 2-obstacle embedding of X_n for arbitrarily large n . Thus we observe the following.

Theorem 3.6. *Let $n \geq 3$. Then $\text{obs}_{\text{out}}(X_n) \leq 2$.*

For the gyroelongated 3-bipyramid we have $\text{obs}_{\text{out}}(X_3) = 1$ (see Figure 11), while for $n = 4, 5, 6$ we have $\text{obs}_{\text{out}}(X_n) = 2$ by Proposition 3.5. It seems likely that, for all larger values of n , the graph X_n cannot be represented without an interior obstacle; however the corresponding SAT instances are too large for our computational methods to be feasible.

Conjecture 3.7. *If $n \geq 4$, then $\text{obs}_{\text{out}}(X_n) = 2$.*

4. OBSTACLE NUMBER, PART I

We have shown that there exist planar graphs with outside obstacle number greater than 1. We now turn our attention to the ordinary obstacle number.

Knowing that certain graphs require an interior obstacle allows us to reason about the obstacle number of a larger graph; in particular this will allow us to argue that certain planar graphs have obstacle number greater than or equal to 2.

We denote the disjoint union of graphs G_1, G_2 by $G_1 \dot{\cup} G_2$.

Lemma 4.1. *Let G_1 and G_2 be graphs such that $\text{obs}_{\text{out}}(G_1) > 1$ and $\text{obs}_{\text{out}}(G_2) > 1$. Then $\text{obs}(G_1 \dot{\cup} G_2) > 1$.*

Proof. Let G_1, G_2 be as in the statement of the result, and let $G = G_1 \dot{\cup} G_2$. Suppose for a contradiction that we have an obstacle representation of G using only one obstacle O .

Based on this representation of G , construct an obstacle representation of G_1 by eliminating those vertices and edges that do not lie in G_1 (see Observation 1.5). Let O_1 be the largest connected open set that contains O but does not meet the drawing of G_1 . We may replace O by O_1 in our obstacle representation of G_1 . Since $\text{obs}_{\text{out}}(G_1) > 1$, this obstacle O_1 must be an interior obstacle. We similarly construct an obstacle representation of G_2 using a single interior obstacle O_2 . Note that each of O_1, O_2 is the interior of a bounded polygonal region, and that $O \subseteq O_1 \cap O_2$.

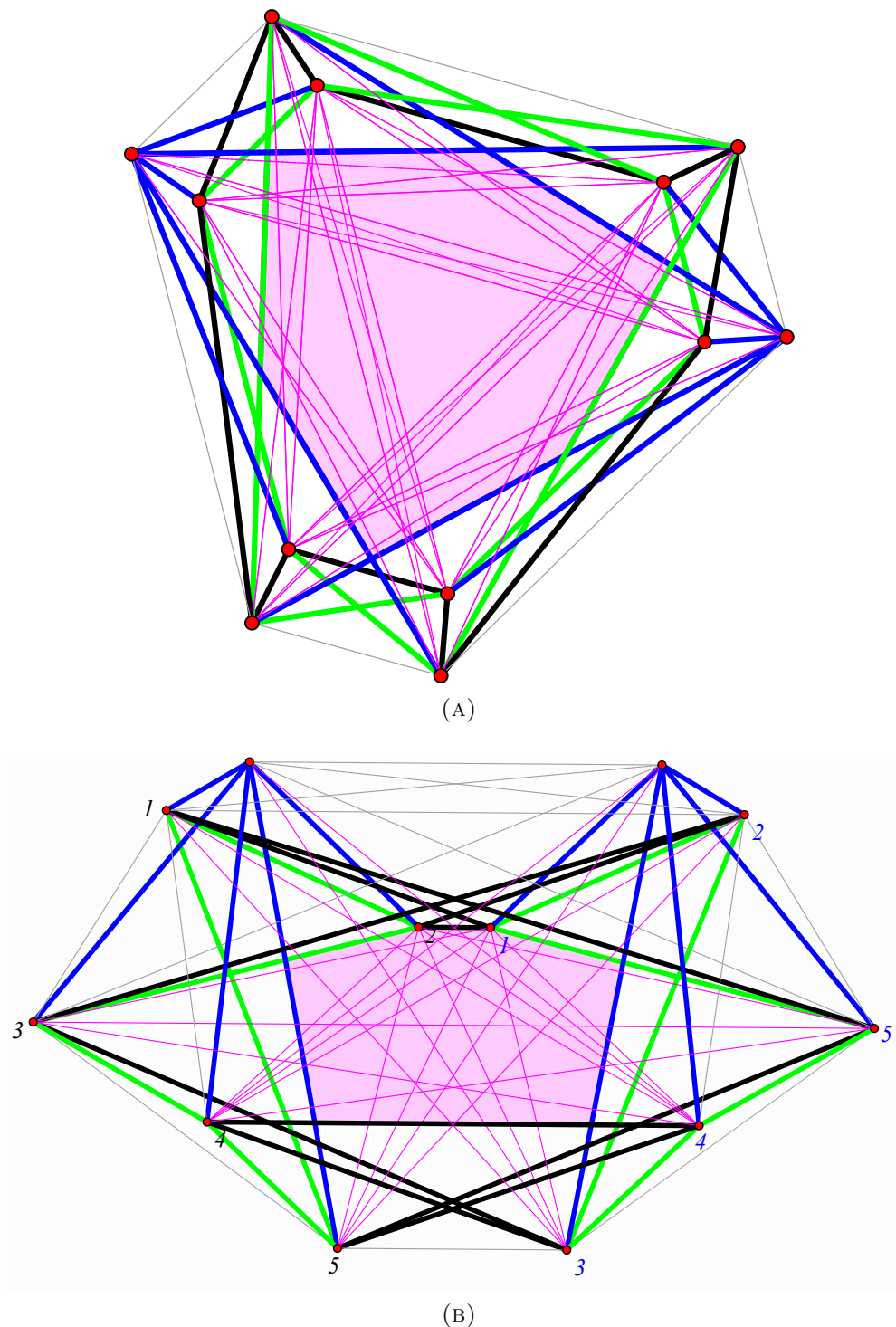
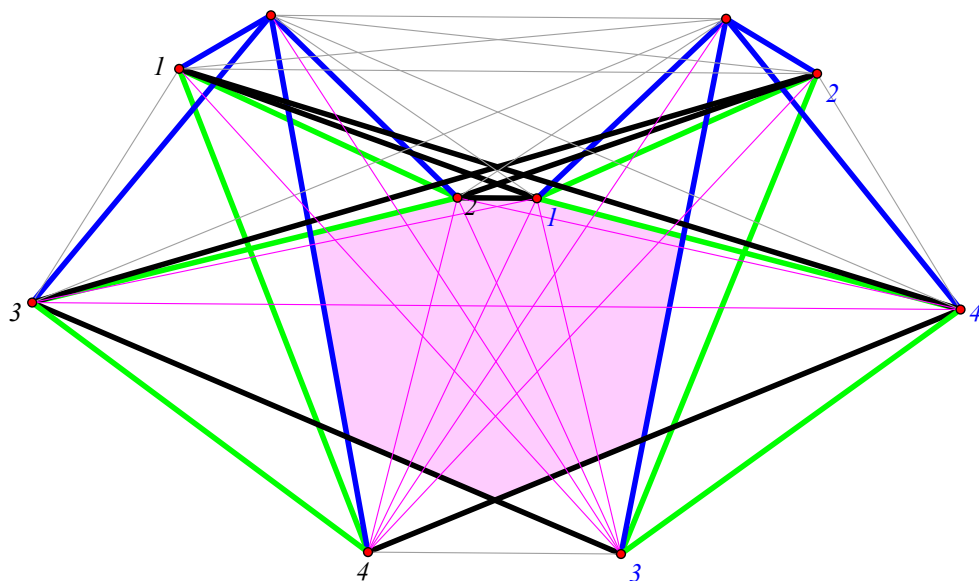
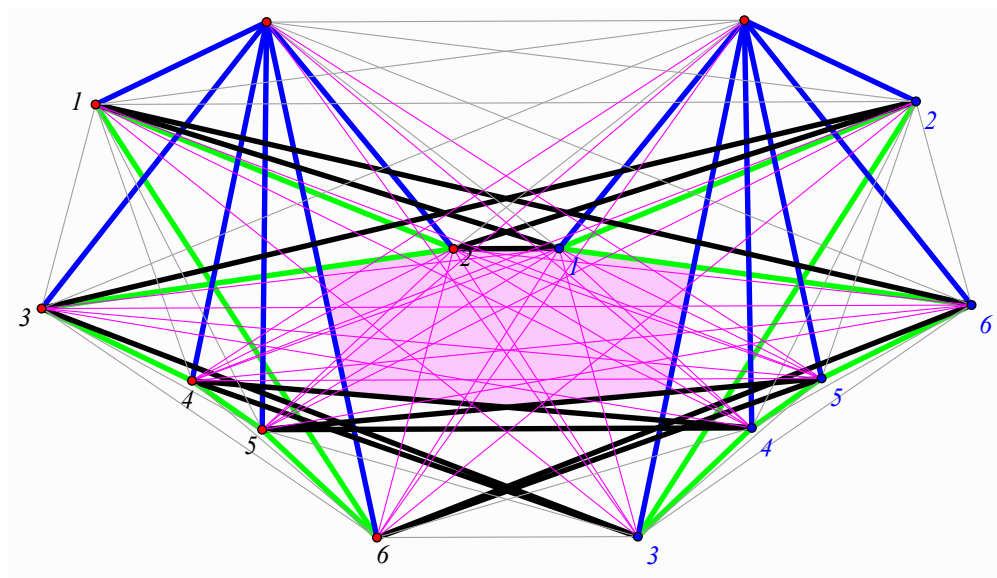


FIGURE 8. Two 2-obstacle embeddings of the icosahedron. The interior obstacle is highlighted in pale magenta, and non-edges blocked by that obstacle are shown with thin magenta lines. The other obstacle is the outside obstacle, and non-edges blocked by that obstacle are shown with thin gray lines.



(A) X_4



(B) X_6

FIGURE 9. Two-obstacle embeddings of gyroelongated n -bipyramids; $n = 4, 6$.

Denote the topological closure of a subset X of the plane by \overline{X} . Let \vec{v} be a nonzero 2-vector, and consider the family \mathcal{H} of parallel closed half-planes determined by \vec{v} . As we sweep this family across G , beginning outside of G , there will be some $H \in \mathcal{H}$ that is the first member of \mathcal{H} with the property that H meets both $\overline{O_1}$ and $\overline{O_2}$.

For at least one of O_1, O_2 , half-plane H meets only the boundary of the obstacle. Say this holds for O_1 ; then $H \cap O_1 = \emptyset$, and so $H \cap O = \emptyset$.

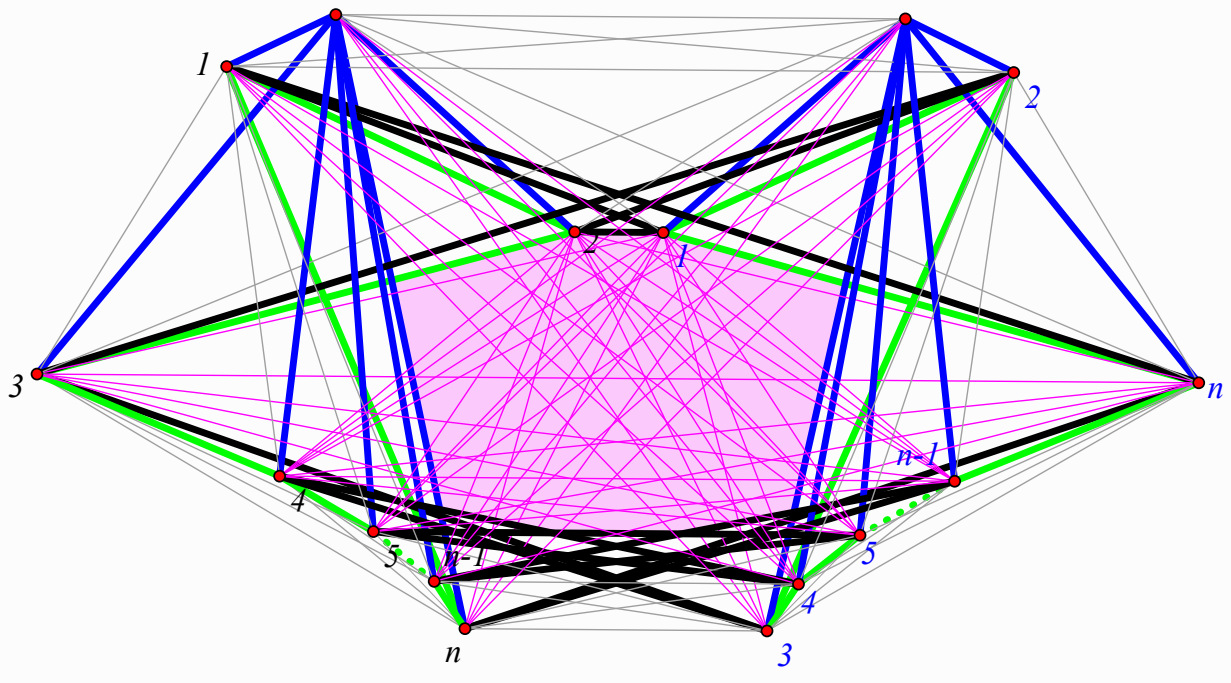


FIGURE 10. A two-obstacle embedding of a gyroelongated n -bipyramid.

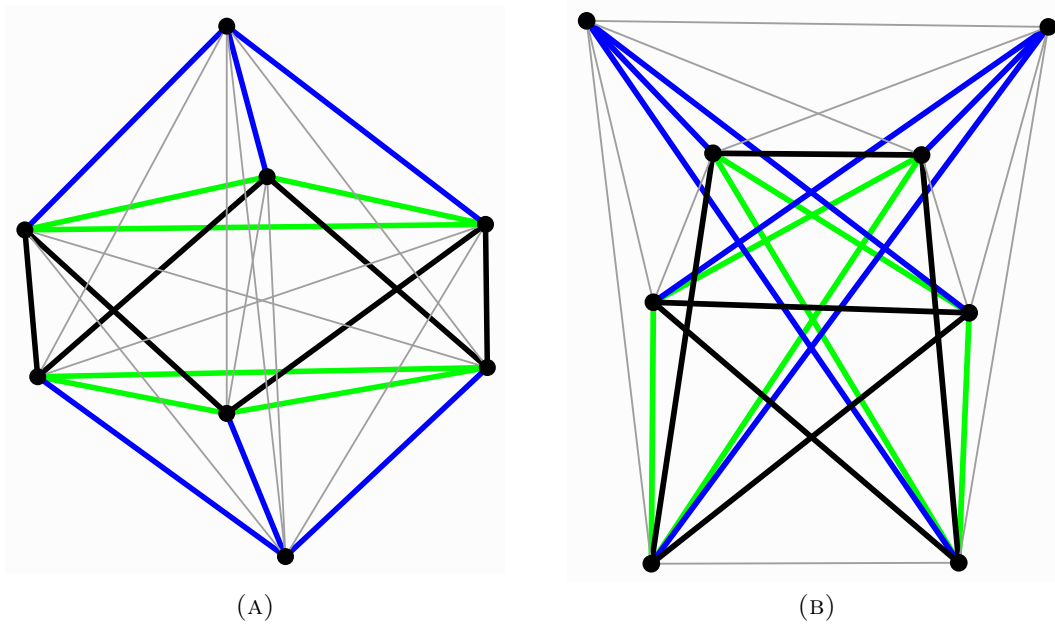


FIGURE 11. (A) The gyroelongated 3-bipyramid X_3 and (B) a one-obstacle embedding.

The set $\overline{O_1}$ is bounded by line segments contained in the edges of G_1 . Thus some vertex p of G_1 lies in H . Similarly, some vertex q of G_2 lies in H . Since pq is not an edge of G , the line segment pq must be blocked by obstacle O . However, this line segment lies entirely in H , and so it does not meet O , a contradiction. \square

As a consequence, we see that each of the graphs $X_4 \dot{\cup} X_4$, $I \dot{\cup} I$, and $X_6 \dot{\cup} X_6$ has obstacle number at least 2 and at most 3. The lower bounds are a consequence of Lemma 4.1, using the previously determined outside obstacle numbers for X_j , $j = 4, 5, 6$ (see Proposition 3.5). The upper bounds are established by using the previously determined outside obstacle embeddings for X_j and embedding two copies of X_j using disjoint interior obstacles (e.g., by placing the disjoint copies far apart).

However, we can show that each of these graphs actually has obstacle number 2. Figure 12 shows a two-obstacle embedding of $X_4 \dot{\cup} X_4$, found by grouping certain related vertices in the 2-obstacle embedding of X_4 . The other two graphs can be drawn similarly, and we have the following.

Corollary 4.2. *All of the following hold.*

- (1) $\text{obs}(X_4 \dot{\cup} X_4) = 2$.
- (2) $\text{obs}(I \dot{\cup} I) = 2$.
- (3) $\text{obs}(X_6 \dot{\cup} X_6) = 2$.

We note that each of X_4 , I , and X_6 is the skeleton of a polyhedron, and thus is planar. Further, the disjoint union of planar graphs is planar; each of the graphs in Corollary 4.2 is a planar graph. Thus we have answered (negatively!) a question of Alpert, Koch, & Laison [2, p. 231], asking whether every planar graph has obstacle number at most 1. Note that $n(X_4 \dot{\cup} X_4) = 20$, $n(I \dot{\cup} I) = 24$, and $n(X_6 \dot{\cup} X_6) = 28$.

Corollary 4.3. *There exists a planar graph with obstacle number greater than 1.*

5. OBSTACLE NUMBER, PART II

In this section we develop a variation on the SAT instance constructed in Section 3. This new SAT instance will encode statements about arbitrary obstacles, instead of only outside obstacles. It will enable us to show directly that certain graphs have obstacle number greater than 1.

In addition to the 4-Point Rule (Lemma 3.1) and the 5-Point Rule (Lemma 3.2), we use a new lemma, Lemma 5.1, which plays role similar to that of Lemma 3.3 in Section 3. For clarity in the next lemma, given distinct points a and b , we denote the two closed halfplanes determined by line \overleftrightarrow{ab} as H_{ab}^+ and H_{ab}^- , where H_{ab}^+ contains all points y such that either y is on line \overleftrightarrow{ab} or aby is oriented clockwise. An ab key path with respect to cd , denoted $P_{ab}(cd)$, is a path from a to b that does not cross the line \overleftrightarrow{cd} ; that is, a path in G from a to b that is entirely contained in one of the closed halfplanes H_{cd}^+ or H_{cd}^- ; see Figure 13.

Lemma 5.1. *Suppose we are given an obstacle representation of a graph G that uses at most one obstacle. Then, for each non-edge ab , and for each non-edge $cd \neq ab$ (ab and cd may share one vertex), there exists a halfplane $H_{ab}(cd) \in \{H_{ab}^+, H_{ab}^-\}$ such that, for every ab key path with respect to cd , $P_{ab}(cd)$, some internal vertex of $P_{ab}(cd)$ lies in the interior of $H_{ab}(cd)$.*

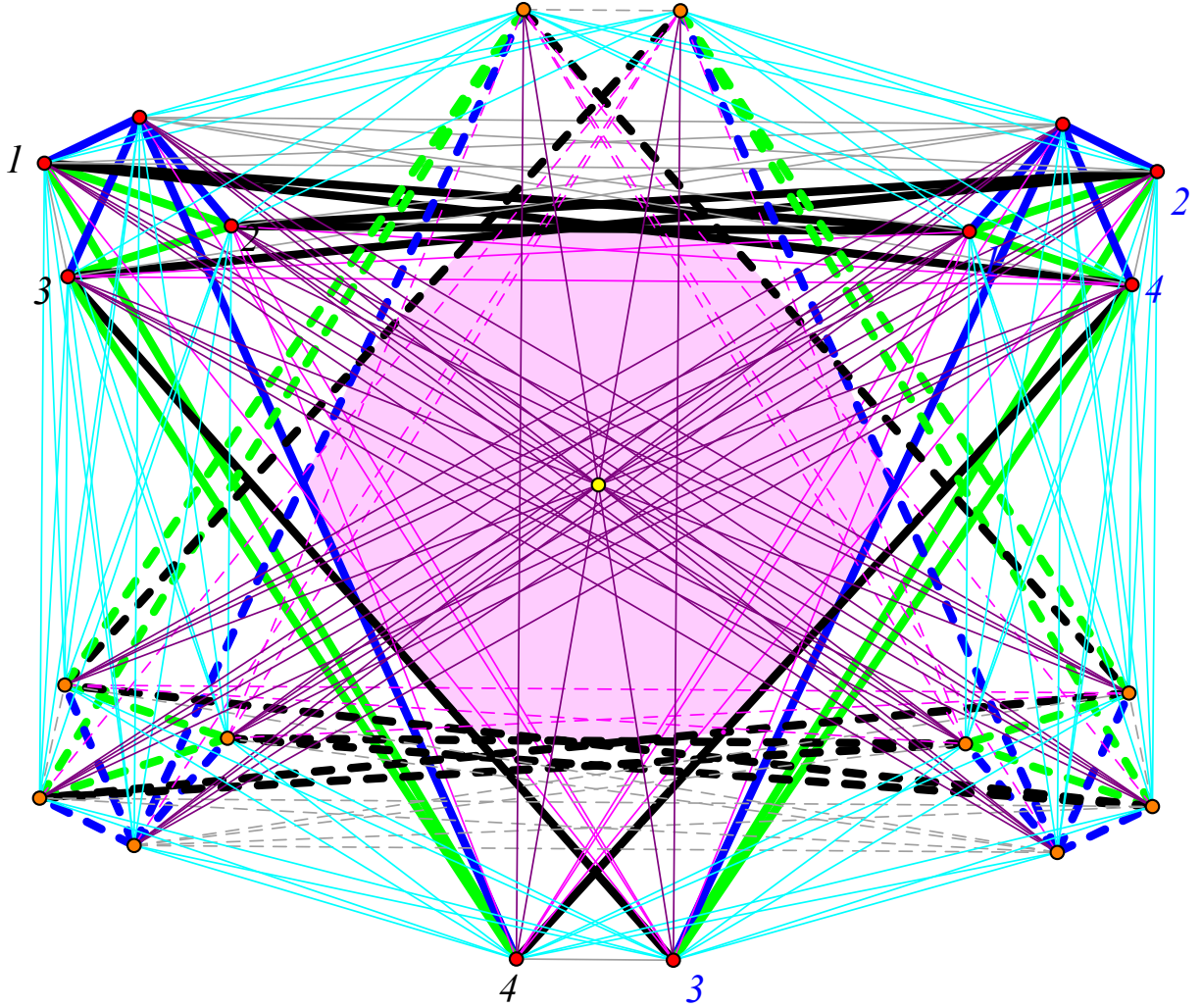


FIGURE 12. A two-obstacle embedding of $X_4 \dot{\cup} X_4$. One copy of X_4 is shown with thick solid lines and red vertices; the other copy is shown with thick dashed lines and orange vertices. The interior obstacle is highlighted in pink. Non-edges that are blocked by the interior obstacle that are non-edges from the individual copies of X_4 are shown with magenta lines (dashed as appropriate), while non-edges between the two copies that are blocked by the interior obstacle are drawn in purple. Non-edges of the individual copies that are blocked by the outside obstacle are drawn in gray (dashed and solid), and non-edges between the two copies that are blocked by the outside obstacle are drawn in cyan.

Proof. Choose any non-edge ab , and then choose a second non-edge cd distinct from ab . Perturbing slightly if necessary, we assume that $\overleftrightarrow{ab} \neq \overleftrightarrow{cd}$ as well (see Observation 1.3). Suppose for a contradiction that there exist two distinct ab paths P_1 and P_2 in G , both ab key paths with respect to cd , that lie in different closed halfplanes determined by \overleftrightarrow{ab} . Without loss of generality, we may assume that $P_1 \subseteq H_{ab}^+$ and $P_2 \subseteq H_{ab}^-$.

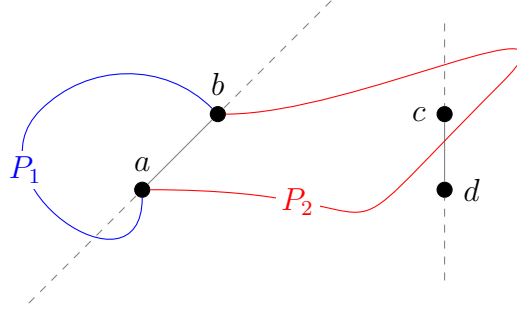


FIGURE 13. An ab key path with respect to cd , denoted $P_{ab}(cd)$, is a path from a to b that does not cross the line \overleftrightarrow{cd} . The path P_1 (blue) is an ab key path with respect to cd , but the path P_2 (red) is not.

Now, each of P_1, P_2 is contained in one of the two halfplanes H_{cd}^+, H_{cd}^- , because they are key paths with respect to cd . Since P_1 and P_2 have common endpoints, namely a and b , we see that P_1 and P_2 , must be contained in the *same* halfplane determined by \overleftrightarrow{cd} .

Therefore $P_1 \cup P_2$ forms a closed path in the plane; the open line segment \overline{ab} lies in one component of the complement of this closed path; while the open line segment \overline{cd} lies in a different component. Since G has only one obstacle, it is impossible for both segments to be blocked, a contradiction. \square

Given a graph G , we can use Lemmas 5.1, 3.1 (the 4-Point rule), and 3.2 (the 5-Point Rule) to create a SAT instance encoding necessary conditions for the existence of an obstacle representation of G using at most 1 obstacle. Note that we are saying nothing here about outside obstacles. Thus, if this SAT instance is not satisfiable, then we may conclude that graph G requires at least two obstacles, that is, that $\text{obs}(G) > 1$.

We represent triples of vertices as we did before, in Section 3, and we encode the 4-Point Rule and the 5-Point Rule using the same clauses as in our previous SAT instance. We construct new SAT clauses based on Lemma 5.1, much as, in Section 3, we constructed clauses based on Lemma 3.3. We now look at how this is done.

Choose any non-edge ab and choose any non-edge cd distinct from ab . We illustrate the encoding of Lemma 5.1 in terms of SAT clauses by showing how to encode statements about a particular path. Let a, s, t, \dots, u, b be the sequence of vertices in some a, b -path P (not necessarily a key path), where vertices a, b are not adjacent (so that ab is a non-edge). Let c, d be nonadjacent vertices not lying on P .

We introduce a new variable $k_{P(cd)}$ to represent the statement that P is an ab key path with respect to cd . If all vertices v of P produce triangles cdv having the same orientation, then P is a key path. This is encoded by the following two clauses:

$$(6) \quad x_{cda} \vee x_{cbs} \vee x_{cst} \vee \dots \vee x_{cdu} \vee x_{cdb} \vee k_{P(cd)}$$

$$(7) \quad \neg x_{cda} \vee \neg x_{cbs} \vee \neg x_{cst} \vee \dots \vee \neg x_{cdu} \vee \neg x_{cdb} \vee k_{P(cd)}$$

Next we encode the statement that given this ab and cd we can find a special side of ab so that every ab key path with respect to cd lies on the special side of ab . The special side is encoded in another variable, $s_{ab,cd}$. We canonically choose that $s_{ab,cd}$ represents the statement that the special halfplane is H_{ab}^+ ; thus $\neg s_{ab,cd}$ represents the statement that the

special halfplane is H_{ab}^- . Note that the special side of ab depends on the choice of non-edge cd . The following clauses encode the desired statement:

$$(8) \quad \neg k_{P(cd)} \vee \neg s_{ab,cd} \vee x_{abs} \vee x_{abt} \vee \cdots \vee x_{abu}$$

$$(9) \quad \neg k_{P(cd)} \vee s_{ab,cd} \vee \neg x_{abs} \vee \neg x_{abt} \vee \cdots \vee \neg x_{abu}$$

Observation 5.2. *Let G be a graph. If $\text{obs}(G) \leq 1$, then the SAT instance consisting of all clauses of the forms (1), (2), and (3) from Section 3, and (6)–(9) shown above—using canonical variables—is satisfiable.*

Using these ideas, we can determine the exact value of the obstacle number for the icosahedron and some similar graphs.

Proposition 5.3. *All of the following hold.*

$$(1) \text{ obs}(X_4) = 2.$$

$$(2) \text{ obs}(I) = \text{obs}(X_5) = 2.$$

$$(3) \text{ obs}(X_6) = 2.$$

Proof. The lower bounds were found using a computer. We create a SAT instance as described above, using clauses representing the 4-Point Rule, the 5-Point Rule, and the statement of Lemma 5.1. See [6] for software to generate the SAT instances. For each graph, a standard SAT solver (as before, we used MiniSat [15], PicoSAT [3], and zChaff [20]) indicates that the SAT instance is not satisfiable. Thus, by Observation 5.2, we have $\text{obs}_{\text{out}} \geq 2$ for each graph.

For the upper bound, we apply Proposition 3.5, which states that each graph considered here has $\text{obs}_{\text{out}} = 2$, and note that $\text{obs} \leq \text{obs}_{\text{out}}$ for every graph. \square

Note that Part (2) of Proposition 5.3 answers a question of Alpert, Koch, & Laison [2, p. 231] asking for the obstacle number of the icosahedron.

Note also that the graph X_4 , mentioned in part (1) of Proposition 5.3, has order 10. This is thus the second known example of a graph of order 10 with obstacle number 2 (the first being $K_{5,5}^*$, shown to have obstacle number 2 by Pach & Sariöz [19, Thm. 2.1]). But unlike the Pach-Sariöz example, graph X_4 is planar. We do not know whether there is any planar graph—or, indeed, any graph at all—of smaller order that has obstacle number 2.

Alpert, Koch, & Laison [2, p. 231] asked for the obstacle number of the dodecahedron. We answer this question as follows.

Proposition 5.4. *Let D be the dodecahedron. Then $\text{obs}(D) = \text{obs}_{\text{out}}(D) = 1$.*

Proof. Figure 14 shows an obstacle representation of the dodecahedron, using a single outside obstacle. \square

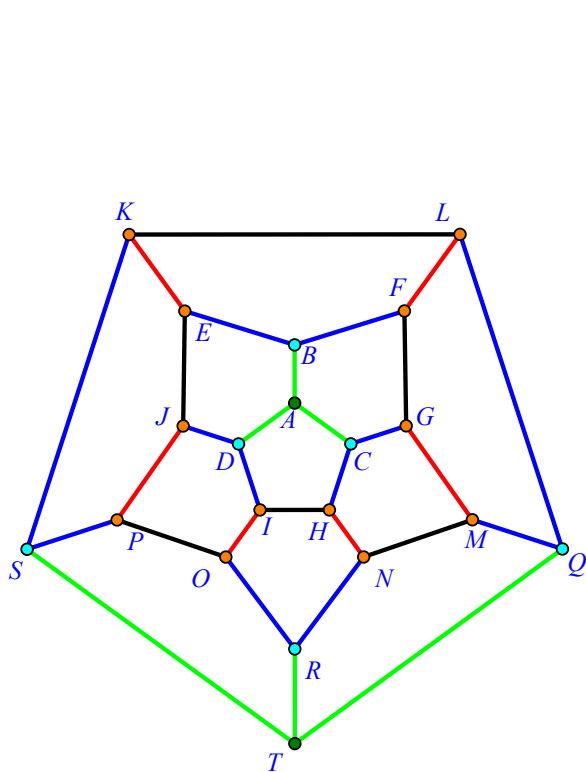
6. OPEN QUESTIONS

There are several interesting open questions related to obstacle numbers of graphs with small numbers of vertices.

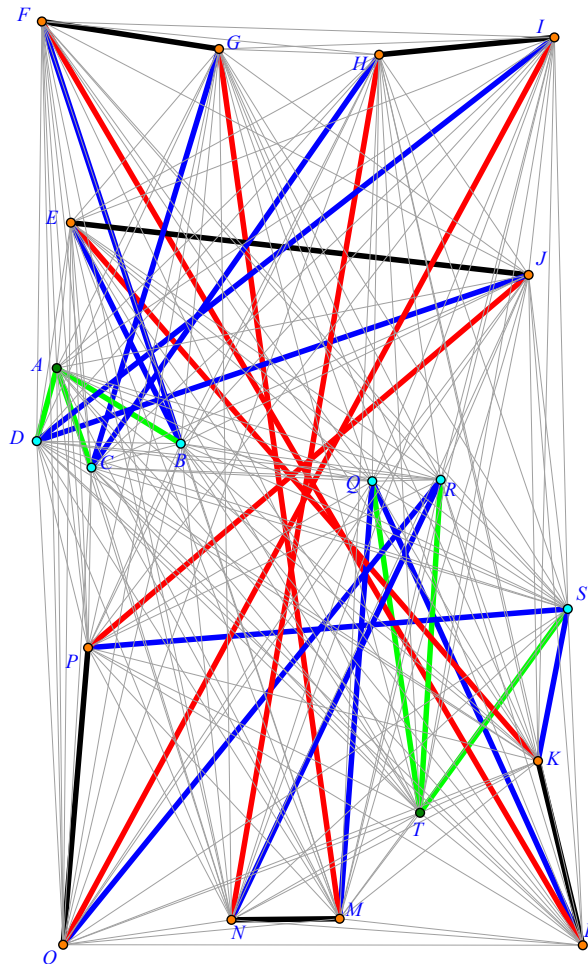
Question 6.1. *What is the minimum order of a planar graph with obstacle number 2?*

By Proposition 2.4, the above minimum must be at least 6.

We have not been able to show that there exists a planar graph with obstacle number greater than 2. It is natural to ask whether there exists an upper bound on the obstacle



(A) A drawing of the dodecahedron



(B) An obstacle representation using a single outside obstacle.

FIGURE 14. A drawing of the dodecahedron, with useful edge-colorings, and an obstacle representation of the dodecahedron using a single outside obstacle (with corresponding edges and vertices).

numbers of planar graphs, and, if so, what it is. It seems likely that either there is no such upper bound, or else the maximum obstacle number of a planar graph is 2. We conjecture that the latter option holds.

Conjecture 6.2. *If G is a planar graph, then $\text{obs}(G) \leq 2$.*

In general, little is known about the least order of a graph with any particular (outside) obstacle number.

Question 6.3. *What is the smallest order of a graph G with $\text{obs}_{\text{out}}(G) = 2$? With $\text{obs}(G) = 2$? With $\text{obs}_{\text{out}}(G)$ or $\text{obs}(G) > 2$?*

Again, by Proposition 2.4, in each case the answer must be at least 6.

We have found the above questions quite resistant to solution. Perhaps this is unsurprising since Johnson and Sariöz showed in [10] that computing the obstacle number of a plane graph is NP-hard. Two examples of graphs of order 10 with obstacle number 2 have been found, namely $K_{5,5}^*$ and X_4 ; none of smaller order are known. If we knew of a single graph of order 9 or less for which one of the SAT instances we construct is not satisfiable, then we could reduce our current bound of 10; however we have found no such graph.

It seems plausible that an approach to answering the above questions would be a brute-force application of SAT instances to all graphs with order strictly less than 10. However, this naive approach has two flaws. First, there are a large number of graphs of order at most 9 (for example, there are 11117 connected graphs of order 8 and 261080 connected graphs of order 9 [1, Sequence A001349]), and there are significant time and computational issues involved in processing the SAT instances for all these graphs.

Second, while non-satisfiability of the SAT instance for one of these graphs would imply that the corresponding (outside) obstacle number was strictly greater than 1, satisfiability of an instance does not imply any bound on the corresponding obstacle number. The solution of one of our SAT instances gives only a specification of clockwise/counter-clockwise orientation for each triple of points. This might not correspond to any actual point placement in the plane. Or it may correspond to many point placements. And even if one of these gives the desired obstacle representation, others may not; or none of them may. Furthermore, a single SAT instance can have exponentially many solutions, each of which may need to be checked, in order to find an obstacle representation. In any case, satisfiability provides us only with a starting point in the search for an obstacle representation; we know of no efficient, reliable technique for actually finding such a representation without human intervention and invention.

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