SHEDDING VERTICES OF VERTEX DECOMPOSABLE GRAPHS

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Abstract. Let G be a vertex decomposable graph. Inspired by a conjecture of Villarreal, we investigate when $\mathrm{Shed}(G)$, the set of shedding vertices of G, forms a dominating set in the graph G. We show that this property holds for many known families of vertex decomposable graphs. A computer search shows that this property holds for all vertex decomposable graphs on eight or less vertices. However, there are vertex decomposable graphs on nine or more vertices for which $\mathrm{Shed}(G)$ is not a dominating set. We describe three new infinite families of vertex decomposable graphs, each with the property that $\mathrm{Shed}(G)$ is not a dominating set.

1. Introduction

This paper was initially motivated by a conjecture of R. Villarreal [22] about Cohen-Macaulay graphs. Let $G = (V, E)$ be a finite simple graph on the vertex set $V = \{x_1, \ldots, x_n\}$ and edge set E. Villarreal [22] introduced the notion of an *edge ideal* of G, that is, in the polynomial ring $R = k[x_1, \ldots, x_n]$ over a field k, let $I(G)$ denote the square-free quadratic monomial ideal $I(G) = \langle x_ix_j \mid \{x_i, x_j\} \in E\rangle$. A graph G is *Cohen-Macaulay* if the the quotient ring $R/I(G)$ is a Cohen-Macaulay ring, that is, the depth of $R/I(G)$ equals the Krull dimension of $R/I(G)$. The goal of [22] was to determine necessary and sufficient conditions for a graph to be Cohen-Macaulay.

Based upon computer experiments on all graphs on six or less vertices, Villarreal proposed a two-part conjecture:

Conjecture 1.1 ([22, Conjectures 1 and 2]). *Let G be a Cohen-Macaulay graph and let*

$$S = \{x \in V \mid G \setminus x \text{ is a Cohen-Macaulay graph}\}.$$

Then (i) $S \neq \emptyset$, and (ii) S is a dominating set of G.

In Conjecture 1.1, $G \setminus x$ denotes the graph formed from G by removing the vertex x and all of the edges adjacent to x. A subset $D \subseteq V$ is a *dominating set* if every vertex $x \in V \setminus D$ is adjacent to a vertex of D. Notice that (ii) will not hold if (i) does not hold.

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It is known that Conjecture 1.1 (i) is false. One example is due to Terai [23, Exercise 6.2.24]. However Terai's example depends upon the characteristic of the field k. Earl and the last two authors [8] found an example of a circulant graph G on 16 vertices with the property that G is Cohen-Macaulay in all characteristics, but there is no vertex x such that $G \setminus x$ is Cohen-Macaulay.

Although Conjecture 1.1 is false in general, Villarreal's work suggests that there may exist some nice subset of Cohen-Macaulay graphs for which the Conjecture 1.1 still holds. Dochtermann-Engström [7] and Woodroofe [24] independently showed that many of the algebraic questions studied by Villarreal can be answered by studying the independence complex of G and applying the tools of combinatorial algebraic topology. The independence complex of a graph G is the simplicial complex whose faces are the independent sets of G. Equivalently, it is the simplicial complex associated to $I(G)$ via the Stanley-Reisner correspondence.

Via combinatorial algebraic topology, there are a number of families of pure simplicial complexes that are known to be Cohen-Macaulay (e.g., shellable, constructible). Of interest to this paper are the pure vertex decomposable simplicial complexes. We say a graph G is *vertex decomposable* if its independence complex is a pure vertex decomposable simplicial complex (see the next section for more details; note that we will use an equivalent definition of vertex decomposable that does not require the language of simplicial complexes). Since vertex decomposable graphs are Cohen-Macaulay, it is reasonable to consider the following variation of Conjecture 1.1:

Question 1.2. *Let G be a vertex decomposable graph, and let*

$$\mathrm{Shed}(G) = \{x \in V \mid G \setminus x \textit{ is a vertex decomposable graph}\}.$$

Is $\mathrm{Shed}(G)$ *a dominating set of G?*

The set $\mathrm{Shed}(G)$ denotes the set of all *shedding vertices* of G. It will follow from the definition of vertex decomposable graphs that $\mathrm{Shed}(G) \neq \emptyset$, so we do not need an analog of Conjecture 1.1 (i). Technically, a vertex x is a shedding vertex of a vertex decomposable graph G if and only if $G \setminus x$ and $G \setminus N[x]$ (the graph with the closed neighbourhood of x removed) are both vertex decomposable, but we explain why it suffices to only consider $G \setminus x$ (see Theorem 2.5).

The goal of this paper is to explore Question 1.2. The next result summarizes some of our findings.

Theorem 1.3. *Suppose that G is a vertex decomposable graph. If G is*

- (i) *a bipartite graph, or*
- (ii) *a chordal graph, or*
- (iii) *a very well-covered graph, or*
- (iv) *a circulant graph, or*
- (v) *a Cameron-Walker graph, or*
- (vi) *a clique-whiskered graph, or*
- (vii) *a graph with girth at least five,*

then $\mathrm{Shed}(G)$ *is a dominating set.*

In particular, (i) is Corollary 4.3, (ii) is Theorem 2.12, (iii) is Theorem 4.2, (iv) is Theorem 2.10, (v) is Corollary 3.2, (vi) is Theorem 3.3, and (vii) Theorem 5.3.

The number of positive answers to Question 1.2 initially suggested a positive answer for all vertex decomposable graphs. However, a computer search has revealed a counterexample on nine vertices. We use this counterexample (and others) to build three new infinite families of vertex decomposable graphs which do not currently appear in the literature.

We outline the structure of this paper. Section 2 contains the requisite background material plus a proof that chordal and circulant vertex decomposable graphs satisfy Question 1.2. In Section 3, we consider two constructions of vertex decomposable graphs, and show that any vertex decomposable graph G constructed via either construction satisfies the property that $\mathrm{Shed}(G)$ is a dominating set. In Section 4, we consider all the very well-covered graphs that are vertex decomposable. In Section 5, we focus on all vertex decomposable graphs with girth at least five. In Section 6, we describe three infinite families of graphs where each graph G is vertex decomposable, but $\mathrm{Shed}(G)$ is not a dominating set. In Section 7, we show how to take a graph G which is vertex decomposable but $\mathrm{Shed}(G)$ is not a dominating set and duplicate a vertex to construct a larger graph with the same properties. Section 8 complements Section 6 by describing the results of our computer search on all graphs on 10 or less vertices. We find the smallest graph that gives a negative answer to Question 1.2. In fact, this example also provides a minimal counterexample to Conjecture 1.1. As part of our computer search, we also show that the set of vertex decomposable graphs is the same as the set of Cohen-Macaulay graphs for all the graphs on 10 vertices or less.

2. Background definitions and first results

2.1. **Vertex decomposable graphs.** Let G be a finite simple graph with vertex set $V = \{x_1, \ldots, x_n\}$ and edge set E. We may sometimes write $V(G)$, respectively $E(G)$, for V, respectively E, if we wish to highlight that we are discussing the vertices, respectively edges, of G. A subset $W \subseteq V$ is an *independent* set if no two vertices of W are adjacent. An independent set W is a *maximal independent set* if there is no independent set U such that W is a proper subset of U. If $W \subseteq V$ is an independent set, then $V \setminus W$ is a *a vertex cover*. A vertex cover C is a *minimal vertex cover* if $V \setminus C$ is a maximal independent set. A graph is *well-covered* if all the maximal independent sets have the same cardinality, or equivalently, if every minimal vertex cover has the same cardinality.

For any $x \in V$, let $G \setminus x$ denote the graph G with the vertex x and incident edges removed. The *neighbours* of a vertex $x \in V$ in G, is the set $N(x) = \{y \mid \{x, y\} \in E\}$. The *closed neighbourhood* of a vertex x is $N[x] = N(x) \cup \{x\}$. For $S \subseteq V$, we let $G \setminus S$ denote the graph obtained by removing all the vertices of S and their incident edges.

Definition 2.1. A graph G is *vertex decomposable* if G is well-covered and

- (i) G consists of isolated vertices, or G is empty, or
- (ii) there exists a vertex $x \in V$, called a *shedding vertex*, such that $G \setminus x$ and $G \setminus N[x]$ are vertex decomposable.

Remark 2.2. Vertex decomposability was first introduced by Provan and Billera [19] for simplicial complexes. Our definition of vertex decomposability is equivalent to the statement that the *independence complex* of a graph G is a vertex decomposable simplicial complex. The independence complex, denoted $\mathrm{Ind}(G)$, is the simplicial complex

$$\mathrm{Ind}(G) = \{W \subseteq V \mid W \text{ is an independent set}\}.$$

One can use [7, Lemma 2.4] to show the equivalence of definitions. Provan and Billera's definition required that the simplicial complex be pure (which translates in the graph case to the condition that G is well-covered). A non-pure version of vertex decomposability was introduced by Björner and Wachs [3]. In the literature, a graph is sometimes called vertex decomposable if $\mathrm{Ind}(G)$ satisfies Björner-Wachs's definition, that is, G need not be well-covered. However, when we say that G is vertex decomposable, it must also be well-covered.

To determine vertex decomposability, it is enough to consider connected components.

Lemma 2.3 ([24, Lemma 20])**.** *Suppose G and H are disjoint graphs. Then $G \cup H$ is vertex decomposable if and only if G and H are each vertex decomposable.*

The following construction allows us to make vertex decomposable graphs from a given graph. For any graph G, let $S \subseteq V$, and after relabeling, let $S = \{x_1, \ldots, x_s\}$. We let $G \cup W(S)$ denote the graph with the vertex set $V \cup \{z_1, \ldots, z_s\}$ and edge set $E \cup \{\{x_i, z_i\} \mid i = 1, \ldots, s\}$. The graph $G \cup W(S)$ is called the *whiskered graph at S* since we are adding leaves or "whiskers" to all the vertices of S. Biermann, Francisco, Hà, and Van Tuyl [1] showed that if we carefully choose S, the new graph $G \cup W(S)$ will be vertex decomposable in the non-pure sense. We can adapt their result as follows:

Theorem 2.4. *Let G be a graph and $S \subseteq V$. If the induced graph on $V \setminus S$ is a well-covered chordal graph and if $G \cup W(S)$ is well-covered, then $G \cup W(S)$ is vertex decomposable.*

Proof. The above statement is [1, Corollary 4.6], but without the adjectives "well-covered". However, the proofs of [1] will also work if we require all of our graphs to be well-covered. □

2.2. **Shedding vertices.** If G is a vertex decomposable, then the set of shedding vertices is denoted by:

$$\mathrm{Shed}(G) = \{x \in V \mid G \setminus x \text{ and } G \setminus N[x] \text{ are vertex decomposable}\}.$$

However, if it is known that G is vertex decomposable, to determine if $x \in \mathrm{Shed}(G)$, then it is enough to check if $G \setminus x$ is vertex decomposable.

Theorem 2.5. *If G is vertex decomposable, then $G \setminus N[x]$ is vertex decomposable for all $x \in V$. Consequently,* $\mathrm{Shed}(G) = \{x \in V \mid G \setminus x$ *is vertex decomposable*$\}$.

Proof. We sketch out the main idea. The graph $G \setminus N[x]$ is vertex decomposable if and only if the independence complex $\mathrm{Ind}(G \setminus N[x])$ is a vertex decomposable simplicial complex. It can be shown that $\mathrm{Ind}(G \setminus N[x])$ equals the simplicial complex $\mathrm{link}_{\mathrm{Ind}(G)}(x)$, the link of the element x in $\mathrm{Ind}(G)$. Then one uses [19, Proposition 2.3] which shows that every link of a vertex decomposable simplicial complex is also vertex decomposable. □

We now provide some tools that enable us to identify some elements of $\mathrm{Shed}(G)$. For any $W \subseteq V$, the *induced graph* of G on W, denoted $G[W]$, is the graph with vertex set W and edge set $\{e \in E \mid e \subseteq W\}$. The *complete graph* on n vertices, denoted K_n, is the graph on the vertices $\{x_1, \ldots, x_n\}$ with edge set $\{\{x_i, x_j\} \mid i \neq j\}$. A *clique* in G is an induced subgraph of G that is isomorphic to K_m for some $m \geq 1$.

Definition 2.6. A vertex $x \in V$ is a *simplicial vertex* if the induced graph on $N(x)$ is a clique; equivalently the vertex x appears in exactly one maximal clique of the graph. A *simplex* is a clique containing at least one simplicial vertex of G. A graph G is *simplicial* if every vertex of G is a simplicial vertex or adjacent to one.

Example 2.7. (i) A vertex x is a *leaf* if it has degree one. Since a leaf has exactly one neighbour, which is a K_1, it is a simplicial vertex.

(ii) The graph in Figure 1 is simplicial. The simplicial vertices are x_1, x_2, x_3 and x_4, and each vertex is either a simplicial vertex or adjacent to one.

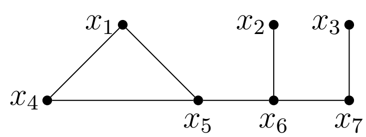

FIGURE 1. A simplicial graph

Lemma 2.8. *Suppose G is well-covered. If x is a simplicial vertex, then for every $y \in N(x)$, the graph $G \setminus y$ is also well-covered.*

Proof. Let H be a maximal independent set of $G \setminus y$. Then H is also an independent set of G. If H was not maximal in G, then $H \cup \{y\}$ must still be independent in G. This implies $(N[x] \setminus \{y\}) \cap H = \emptyset$. But then $H \cup \{x\}$ would be an independent set of $G \setminus y$, contradicting the maximality of H. So H is also a maximal independent set of G, and since G is well-covered, all the maximal independent sets of $G \setminus y$ have the same cardinality. □

Lemma 2.9. *Let G be a vertex decomposable graph. If x is a simplicial vertex, then $N(x) \subseteq \mathrm{Shed}(G)$.*

Proof. Let x be a simplicial vertex of G and suppose $y \in N(x)$. By [24, Corollary 7], y is a shedding vertex of G, although [24] uses the non-pure definition of vertex decomposable. However, if G is a well-covered graph, then $G \setminus y$ is also well-covered by Lemma 2.8, so $G \setminus y$ is vertex decomposable. □

2.3. **Circulant and chordal graphs.** We end this section by giving a positive answer to Question 1.2 for two classes of graphs, circulant graphs and chordal graphs.

Let $n \geq 1$ and $S \subseteq \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$. The *circulant graph* $C_n(S)$ is the graph on the vertex set $\{0, \ldots, n-1\}$ with all edges $\{a, b\}$ that satisfy $|a-b| \in S$ or $n - |a-b| \in S$.

Theorem 2.10. *Suppose G is a circulant graph. If G is vertex decomposable, then $\mathrm{Shed}(G)$ is a dominating set.*

Proof. If G is vertex decomposable, then there exists some vertex i such that $G \setminus i$ is vertex decomposable. By the symmetry of the graph $G \setminus j$ is isomorphic to $G \setminus i$ for all $i \neq j$. But then $\mathrm{Shed}(G) = V$, and hence $\mathrm{Shed}(G)$ is a dominating set. □

A *chordal graph* is a graph G such that every induced cycle in G has length three. We have the following classification of vertex decomposable chordal graphs.

Theorem 2.11. *Let G be a chordal graph. Then the following are equivalent:*

(*i*) *G is vertex decomposable.*
(*ii*) *G is well-covered.*
(*iii*) *Every vertex of G belongs to exactly one simplex of G.*

Proof. ($(i) \Rightarrow (ii)$) If G is vertex decomposable, then by definition, G is well-covered.

($(ii) \Rightarrow (i)$) Woodroofe ([24, Corollary 7]) (and independently by Dochtermann and Engström [7]) showed that every chordal graph G is also vertex decomposable, in the non-pure sense of vertex decomposable due to Björner-Wachs [3]. But if G is well-covered, one can adapt this proof to show that G is vertex decomposable as we have defined it.

($(ii) \Leftrightarrow (iii)$) This is [18, Theorem 2]. □

We can now prove the following result.

Theorem 2.12. *Suppose G is a chordal graph. If G is vertex decomposable, then* $\mathrm{Shed}(G)$ *is a dominating set.*

Proof. By Lemma 2.3, we can assume that G is connected and has at least two vertices. Since G is vertex decomposable, by Theorem 2.11, the simplexes of G partition V, i.e., $V = V_1 \cup \cdots \cup V_t$ where the induced graph on each V_i is a simplex. So, every V_i contains at least one simplicial vertex.

For each $i = 1, \ldots, t$, let $x_i \in V_i$ be a simplicial vertex. Note that this means that $N(x_i) = V_i \setminus \{x_i\}$ for each i. By Lemma 2.9, $N(x_i) \subseteq \mathrm{Shed}(G)$. So $N(x_1) \cup \cdots \cup N(x_t) \subseteq \mathrm{Shed}(G)$. But then $\mathrm{Shed}(G)$ is a dominating set. Indeed, if $x \neq x_i$ for any i, then x is a neighbour of some x_j, and so is in $\mathrm{Shed}(G)$. If $x = x_i$ for some i, then all of its neighbours belong to $\mathrm{Shed}(G)$. □

3. Vertex Decomposable Constructions

Given a graph G, there are two known constructions (see [6, 14]) that enable one to build a new vertex decomposable graph that contains G as an induced subgraph. We show that the resulting graph in either construction has the property that its set of shedding vertices is a dominating set.

3.1. Appending cliques.

We first consider a construction of Hibi, Higashitani, Kimura, and O'Keefe [14] that builds a vertex decomposable graph by appending a clique at each vertex. More precisely, let G be a graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$. Let $k_1, \ldots, k_n$ be n positive integers with $k_i \geq 2$ for $i = 1, \ldots, n$. We now construct a graph $\widetilde{G} = (V(\widetilde{G}), E(\widetilde{G}))$ with

$$V(\widetilde{G}) = \{x_{1,1}, x_{1,2}, \ldots, x_{1,k_1}\} \cup \{x_{2,1}, \ldots, x_{2,k_2}\} \cup \{\ldots, x_{n,1}, \ldots, x_{n,k_n}\}$$

and edge set

$$E(\widetilde{G}) = \{\{x_{i,1}, x_{j,1}\} \mid \{x_i, x_j\} \in E(G)\} \cup \bigcup_{i=1}^{n} \{\{x_{i,j}, x_{i,l}\} \mid 1 \leq j < l \leq k_i\}.$$

That is, $\widetilde{G}$ is the graph obtained from G by attaching a clique of size k_i at the vertex x_i.

Starting from any graph G, the graph $\widetilde{G}$ will always be a vertex decomposable graph by [14, Theorem 1]. Moreover, any graph arising arising from this construction gives a positive answer to Question 1.2.

Theorem 3.1. *Given any graph G, the vertex decomposable graph $\widetilde{G}$ has the property that* $\mathrm{Shed}(\widetilde{G})$ *is a dominating set.*

Proof. For any $i \in \{1, \ldots, n\}$, $x_{i,k_i} \neq x_{i,1}$ because $k_i \geq 2$. The vertex x_{i,k_i} is a simplicial vertex, so by Lemma 2.9 $x_{i,1} \in N(x_{i,k_i}) \subseteq \mathrm{Shed}(\widetilde{G})$. Thus $T = \{x_{1,1}, \ldots, x_{n,1}\} \subseteq \mathrm{Shed}(\widetilde{G})$, and T is a dominating set of $\widetilde{G}$. $\square$

Hibi et al. [14] developed the above construction to study Cameron-Walker graphs. A graph G is a *Cameron-Walker* graph if the induced matching number G equals the matching number of G (see [14] for precise definitions). One of the main results of [14] is the fact that a Cameron-Walker graph G is a vertex decomposable graph if and only if $G = \widetilde{H}$ for some graph H (with some hypotheses on the k_i's that appear in the construction of $\widetilde{H}$). Consequently, we can immediately deduce the following corollary.

Corollary 3.2. *Suppose G is a Cameron-Walker graph. If G is vertex decomposable, then* $\mathrm{Shed}(G)$ *is a dominating set.*

3.2. **Clique-whiskering.** A second construction of vertex decomposable graphs is due to Cook and Nagel [6]. Let G be a graph on the vertex set $V = \{x_1, \ldots, x_n\}$. A *clique vertex partition* of V is a set $\pi = \{W_1, \ldots, W_t\}$ of disjoint subsets that partition V such that each induced graph $G[W_i]$ is a clique. A *clique-whiskered* graph G^π constructed from the graph G with clique partition $\pi = \{W_1, \ldots, W_t\}$ is the graph with $V(G^\pi) = \{x_1, \ldots, x_n, w_1, \ldots, w_t\}$ and $E(G^\pi) = E \cup \{\{x, w_i\} \mid x \in W_i\}$. In other words, for each clique in the partition π, we add a new vertex w_i, and join w_i to all the vertices in the clique.

Note that if $\widetilde{G}$ is the graph obtained from G by appending cliques with $k_1 = \cdots = k_n = 2$, then $\widetilde{G}$ is isomorphic to the clique-whiskered graph G^π using the clique partition $\pi = \{\{x_1\}, \{x_2\}, \ldots, \{x_n\}\}$.

Cook and Nagel ([6, Theorem 3.3]) showed that for any graph G and any clique partition π of G, the graph G^π is always vertex decomposable. Like the previous construction, any graph constructed via this method gives a positive answer to Question 1.2.

Theorem 3.3. *Let G be a graph with clique partition π. The vertex decomposable graph G^π has the property that* $\mathrm{Shed}(G^\pi)$ *is a dominating set.*

Proof. If $\pi = \{W_1, \ldots, W_t\}$, then the vertex set of G^π is $\{x_1, \ldots, x_n, w_1, \ldots, w_t\}$. Every vertex x_i belongs to some clique W_j. So, in G^π, the vertex x_i is adjacent to w_j. By construction, w_j is adjacent only to the vertices of W_j, and since W_j is a clique, w_j is a simplicial vertex. Thus by Lemma 2.9, $x_i \in N(w_j) \subseteq \mathrm{Shed}(G^\pi)$. Thus $\{x_1, \ldots, x_n\} \subseteq \mathrm{Shed}(G^\pi)$, and this subset forms a dominating set. □

4. Very well-covered graphs

We now show that all very well-covered vertex decomposable graphs satisfy Question 1.2. A well-covered graph is *very well-covered* if every maximal independent set has cardinality $|V|/2$. Vertex decomposable very well-covered graphs were first classified by Mahmoudi, Mousivand, Crupi, Rinaldo, Terai, and Yassemi [16]:

Theorem 4.1 ([16, Lemma 3.1 and Theorem 3.2])**.** *Let G be a very well-covered graph with $2h$ vertices. Then the following are equivalent:*

- (i) *G is vertex decomposable;*
- (ii) *There is a relabeling of the vertices $V = X \cup Y = \{x_1, \ldots, x_h\} \cup \{y_1, \ldots, y_h\}$ such that the following five conditions hold:*
  - (a) *X is a minimal vertex cover of G and Y is a maximal independent set of G;*
  - (b) *$\{x_1, y_1\}, \ldots, \{x_h, y_h\} \in E$;*
  - (c) *if $\{z_i, x_j\}, \{y_j, x_k\} \in E$, then $\{z_i, x_k\} \in E$ for distinct i, j, k and for $z_i \in \{x_i, y_i\}$;*
  - (d) *if $\{x_i, y_j\} \in E$, then $\{x_i, x_j\} \notin E$; and*
  - (e) *if $\{x_i, y_j\} \in E$, then $i \leq j$.*

Using the above structure result, we will now show the following result.

Theorem 4.2. *Let G be a very well-covered graph. If G is vertex decomposable, then* $\mathrm{Shed}(G)$ *is a dominating set.*

Proof. Suppose G is a very well-covered vertex decomposable graph. We can assume that the vertices have of G have been relabeled as $V = \{x_1, \ldots, x_h, y_1, \ldots, y_h\}$ so that the five conditions of Theorem 4.1 hold.

For each leaf $z \in V$, the unique neighbour of z is in $\mathrm{Shed}(G)$ by Lemma 2.9. So, if $S = \{N(z) \mid z \text{ is a leaf of } G\}$, then $S \subseteq \mathrm{Shed}(G)$. Note that $S \neq \emptyset$ because y_1 is a leaf because condition (a) indicates y_1 is not adjacent to any of the other y_j's, condition (b) means $\{y_1, x_1\} \in E$, and condition (e) implies $\{y_1, x_j\} \notin E$ for all $j = 2, \ldots, h$.

To finish the proof, it suffices to prove that S is a dominating set of G. Suppose not, that is, suppose there is a vertex w that is neither in S nor adjacent to a vertex in S (in particular, w is not a leaf). We now consider two cases.

Case 1: Suppose $w \in X$. In this case, $w = x_i$ for some i, $1 \leq i \leq h$. Furthermore, we can assume that i is maximal, that is, for any $i < j \leq h$, x_j is either in S or adjacent to something in S. Since x_i is adjacent to y_i, and w is not a leaf, there is another vertex adjacent to w.

Suppose w is adjacent to some vertex in X, say x_j with $j \neq i$. Both y_i and y_j are not leaves; otherwise x_i or x_j would belong to S. Thus y_i is adjacent to some x_p for $p < i$ and y_j is adjacent to some x_q with $q < j$ by condition (e). Now $p \neq q$, because if they were equal, we would have $\{x_i, x_j\}, \{y_j, x_p\} \in E$ implying that $\{x_i, x_p\}$ is an edge of G by condition (c). But this contradicts condition (d) since $\{x_p, y_i\} \in E$.

So we have $\{x_i, x_j\}, \{y_j, x_q\} \in E$, and then by condition (c), $\{x_i, x_q\} \in E$. (Note that $q \neq i$ because if $q = i$, then we would contradict condition (d).) Similarly, because $\{x_j, x_i\}, \{y_i, x_p\} \in E$, we have $\{x_j, x_p\} \in E$. Finally, because $\{x_q, x_i\}, \{y_i, x_p\} \in E$, we have $\{x_q, x_p\} \in E$ by condition (c).

Let $X_q = \{x_i, x_j, x_p, x_q\}$. Note that $x_q \notin S$ since $x_q \in N(w)$. This means that y_q is not a leaf. Hence there exists a vertex $x_{q_1} \in X$ adjacent to y_q, $q_1 < q$. By condition (d), $x_{q_1} \notin N(x_q)$ (and so $q_1 \notin \{p, i\}$). Also, $q_1 \neq j$ since $q_1 < q < j$. Thus $x_{q_1} \notin X_q$. Note also that by condition (c), $N(x_q) \subseteq N(x_{q_1})$ (and hence also $w \in N(x_{q_1})$).

Let $X_{q_1} = \{x_i, x_j, x_p, x_q, x_{q_1}\}$. Inductively, we can see that for each positive integer $n \geq 2$, $x_{q_{n-1}} \notin S$ and hence there exists a vertex $x_{q_n} \in X$, $q_n < q_{n-1}$, with x_{q_n} adjacent to $y_{q_{n-1}}$ such that $x_{q_n} \notin X_{q_{n-1}} = \{x_i, x_j, x_p, x_q, x_{q_1}, \ldots, x_{q_{n-1}}\}$ and $N(x_{q_{n-1}}) \subseteq N(x_{q_n})$. But this contradicts the fact that G is a finite graph. Therefore, w is not adjacent to any vertex in X.

Hence there exists a $j > i$ so that $w = x_i$ is adjacent to y_j. By condition (d), $\{x_i, x_j\} \notin E$. The vertex x_j is not a leaf (otherwise $w = x_i$ would be adjacent to $y_j \in N(x_j) \subseteq S$), so $N(x_j) \setminus \{y_j\} \neq \emptyset$. For any $u \in N(x_j) \setminus \{y_j\}$, since $\{u, x_j\}, \{y_j, x_i\} \in E$, condition (c) implies that $\{u, x_i\} \in E$. In other words, $N(x_j) \subseteq N(x_i)$. By our assumption on the maximality of i, x_j is either in S, or x_j is adjacent to a vertex in S. If x_j is adjacent to a vertex in S, then so is x_i since $N(x_j) \subseteq N(x_i)$ contradicting our choice of x_i. On the other hand, x_j cannot be in S since every neighbour of x_j is also a neighbour of x_i, that is, every neighbour has at least degree two. So w cannot be adjacent to any vertex in Y. Therefore $w \notin X$.

Case 2: Suppose $w \in Y$. Let i be the minimal index such that $w = y_i$ is not in S or adjacent to a vertex in S. Note that $i > 1$ since we already observed that $x_1 \in S$. By Lemma 2.9, y_i is not a leaf. So by (a) there is some x_j adjacent to y_i and $j < i$ by (e). By our choice of i, y_j is either in S or y_j is adjacent to something in S. If $y_j \in S$, then y_j is adjacent to a leaf x_k. By (e), $k \leq j$. Further, even though $\{x_j, y_j\} \in E$, x_j is not a leaf since x_j is also adjacent to y_i. Hence $k < j$. But since we have $\{x_k, y_j\}, \{x_j, y_i\} \in E$, we also have $\{x_k, y_i\}$ which means that x_j is not a leaf. So $y_j \notin S$ and hence y_j is adjacent to something in S. But then either $x_j \in S$, which means that y_i is adjacent to something in S, or y_j is adjacent to some $x_k \in S$ with $k < j$ by (e). But then x_k is also adjacent to y_i by condition (c). Therefore $w \notin Y$.

These two cases show that every vertex of G is either in S or adjacent to a vertex in S. Therefore S is a dominating set, and so $\mathrm{Shed}(G)$ is a dominating set. □

Corollary 4.3. *Suppose G is a bipartite graph. If G is vertex decomposable, then* $\mathrm{Shed}(G)$ *is a dominating set.*

Proof. By Lemma 2.3, we can assume that G is connected. Suppose that G is a bipartite graph with vertex partition $V = V_1 \cup V_2 = \{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_m\}$. The sets V_1 and V_2 are independent sets. In fact, they are maximal independent sets. Indeed, if V_1 is not maximal, there is a vertex $y_j \in V_2$ such that $V_1 \cup \{y_j\}$ is independent. But since y_j is not adjacent to any vertex of V_2, this means that y_j is not adjacent to any vertex, contradicting the fact that G is connected. The same proof works for V_2.

If G is vertex decomposable, then G must be well-covered, So, for any maximal independent set W, we have $|W| = |V_1| = |V_2| = n = |V|/2$. So G is very well-covered. Now apply Theorem 4.2. □

Remark 4.4. As we showed in the previous proof, the class of very well-covered graphs contains the family of well-covered bipartite graphs. Theorem 4.1 can be viewed as a generalization of results first proved about well-covered bipartite graphs. Herzog and Hibi gave a combinatorial classification of Cohen-Macaulay bipartite graphs in [13, Corollary 9.1.14] which prefigures the classification of Theorem 4.1. Van Tuyl [21] then showed that a bipartite graph is vertex decomposable if and only if it is Cohen-Macaulay.

5. Graphs with girth at least five

We now consider all vertex decomposable graphs with girth five or larger. These graphs were independently classified by Bıyıkoğlu and Civan [2] and Hoang, Minh, and Trung [15]. Both of these results relied on the classification of well-covered graphs with girth five or larger due to Finbow, Hartnell, and Nowakowski [9].

To state the required classification, we first review the relevant background. The *girth* of a graph G is the number of vertices of a smallest induced cycle of G. If G has no cycles, then we say G has infinite girth. A *pendant edge* is an edge that is incident to a leaf. A *matching* is a subset of edges of G that do not share any common endpoints. A matching is *perfect* if the set of vertices in the edges of the matching are all of the vertices.

An induced 5-cycle is said to be *basic* if it contains no adjacent vertices of degree three or larger. A graph G is in the class $\mathcal{PC}$ if V can be partitioned into subsets $V = P \cup C$ where P contains all the vertices incident with pendant edges and the pendant edges form a perfect matching of P, and where C contains the vertices of basic 5-cycles, and these basic 5-cycles form a partition of C.

We then have the following classification (see the cited papers for additional equivalent statements).

Theorem 5.1 ([2, 15])**.** *Let G be a connected graph of girth at least 5. If G is well-covered, then the following are equivalent:*

(*i*) *G is vertex decomposable;*
(*ii*) *G is a vertex or in the class $\mathcal{PC}$.*

We first prove a lemma.

Lemma 5.2. *Let B be a basic 5-cycle of a well-covered graph $G \in \mathcal{PC}$. If B has a vertex x adjacent to two vertices of B of degree two in G, then $x \in \mathrm{Shed}(G)$.*

Proof. Suppose $G \in \mathcal{PC}$ with partition $V(G) = C(G) \cup P(G)$. Let B be a basic 5-cycle of G. Suppose $E(B) = \{x, x_1\} \cup \{x_1, y_1\} \cup \{y_1, y_2\} \cup \{y_2, x_2\} \cup \{x_2, x\}$ with x_1 and x_2 both of degree 2 in G.

Let $H = G \setminus x$. We first show that H is well-covered. Let W be any maximal independent set of H; consequently, W is also an independent set of G. If W is not maximal in G, then $W \cup \{x\}$ is an independent set of G. In particular, W does not contain either x_1 or x_2. In H, x_1 and x_2 are leaves, and W contains at most one of y_1 and y_2. But if $y_1 \in W$, then $W \cup \{x_2\}$ is an independent set in H, contradicting our choice of W. Similarly, if $y_2 \in W$, then $W \cup \{x_1\}$ is an independent set. So, W must also be a maximal independent set of G. Because G is well-covered, all the maximal independent sets of H will have the same cardinality, that is, H is well-covered.

We now show $H \in \mathcal{PC}$. Removing x from G breaks the 5-cycle B, so we have $V(H) = (C(G) \setminus V(B)) \cup (P(G) \cup \{x_1, x_2, y_1, y_2\})$. The graph H has girth at least 5, since no new cycles are created by removing a vertex x from G. We claim that $H \in \mathcal{PC}$ and in particular, $C(H) = C(G) \setminus V(B)$ and $P(H) = P(G) \cup \{x_1, x_2, y_1, y_2\}$. The first equality is due to the fact that exactly one basic cycle was destroyed in G to create H and no new cycles were created. The second equality follows from the fact that x_1 and x_2 are leaves in H so $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are now part of a perfect matching on $(V \setminus C(H))$. Therefore $H \in \mathcal{PC}$.

Because H is well-covered and in $\mathcal{PC}$, Theorem 5.1 implies H is vertex decomposable, and consequently, $x \in \operatorname{Shed}(G)$. □

Theorem 5.3. *Let G be a graph with girth of at least five. If G is vertex decomposable, then* $\operatorname{Shed}(G)$ *is a dominating set.*

Proof. If G is vertex decomposable, by Theorem 5.1, G is either a single vertex or $G \in \mathcal{PC}$. Because the statement is vacuous for a single vertex, we can assume that $G \in \mathcal{PC}$.

Let $V = P \cup C$ be the corresponding partition of G. Let $x \in V$. We claim that x is either a shedding vertex of G or adjacent to a shedding vertex of G. With this claim, we can conclude that $\operatorname{Shed}(G)$ is a dominating set.

Suppose $x \in P$. Then x is either a leaf or adjacent to a leaf y. So by Lemma 2.9, x is a shedding vertex of G or adjacent to one.

Suppose $x \in C$. Then there is a basic 5-cycle B such that $x \in V(B)$. If x is adjacent to two vertices of degree two, then $x \in \operatorname{Shed}(G)$ by Lemma 5.2. So suppose that there exists $y \in V(B)$ adjacent to x such that y has degree greater at least three. Because B is a basic 5-cycle, y must be adjacent to two vertices of degree two. By Lemma 5.2, $y \in \operatorname{Shed}(G)$. Hence x is adjacent to a shedding vertex. Therefore every vertex in C is a shedding vertex of G or adjacent to one. □

6. Three new vertex decomposable graphs

In this section we will construct three infinite family of graphs. Each family will have the property that all members are vertex decomposable, but $\operatorname{Shed}(G)$ is **not** a dominating set, thus giving a negative answer to Question 1.2 in general.

6.1. **Construction 1.** Fix m integers $k_i \geq 2$, and suppose that $k_1 + \cdots + k_m = n$. We define $D_n(k_1, \ldots, k_m)$ to be the graph on the $5n$ vertices

$$V = X \cup Y \cup Z = \{x_1, \ldots, x_{2n}\} \cup \{y_1, \ldots, y_{2n}\} \cup \{z_1, \ldots, z_n\}$$

with the edge set given by the following conditions:

(i) the induced graph on Z is a complete graph K_n;
(ii) Y is an independent set, i.e., $G[Y] = \overline{K_{2n}}$, where $\overline{H}$ denotes the complement of the graph H;
(iii) the induced graph $G[X]$ is $K_{k_1,k_1} \sqcup \cdots \sqcup K_{k_m,k_m}$ where the vertices of $G[X]$ are labeled so that the i-th complete bipartite graph has bipartition

$$\{x_{2w+1}, x_{2w+3}, \ldots, x_{2(w+k_i)-1}\} \cup \{x_{2w+2}, x_{2w+4}, \ldots, x_{2(w+k_i)}\}$$

with $w = \sum_{\ell=1}^{i-1} k_\ell$ where $w = 0$ if $i = 1$;
(iv) $\{x_j, y_j\}$ are edges for $1 \leq j \leq 2n$; and
(v) $\{z_j, y_{2j}\}$ and $\{z_j, y_{2j-1}\}$ are edges for $1 \leq j \leq n$.

Roughly speaking, the graph $D_n(k_1, \ldots, k_m)$ is formed by "joining" m complete bipartite graphs to a complete graph K_n by first passing through an independent set of vertices Y. Going forward, it is useful to make the observation that the induced graph $G[X \cup Y]$ has a perfect matching given by the edges $\{x_j, y_j\}$ for $j = 1, \ldots, 2n$.

Example 6.1. To illustrate our construction, the graph of $D_5(2, 3)$ is given in Figure 2.

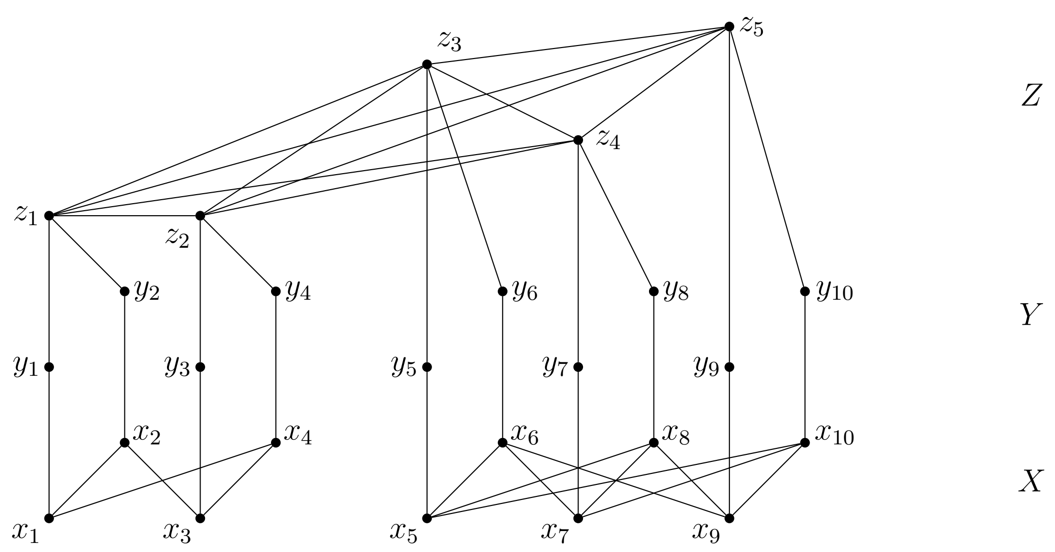

FIGURE 2. $D_5(2, 3)$.

We now show that the graphs $D_n(k_1, \ldots, k_m)$ are all well-covered. In what follows, we write $\alpha(G)$ to denote the cardinality of a maximal independent set in G.

Lemma 6.2. *Let $G = D_n(k_1, \ldots, k_m)$ be constructed as above. Then G is well-covered.*

Proof. Let $G = D_n(k_1, \dots, k_m)$. It suffices to show that every maximal independent set has the same cardinality.

We can partition V into n sets of five vertices, namely, $\{x_{2i-1}, x_{2i}, y_{2i-1}, y_{2i}, z_i\}$ for $1 \leq i \leq n$. The induced graph on each such set is a five cycle. Since $\alpha(C_5) = 2$, it follows that $\alpha(G) \leq 2n$. On the other hand, Y is a maximal independent set of vertices with $|Y| = 2n$, so $\alpha(G) = 2n$.

Let H be any maximal independent set with $|H| < 2n$. If $H \cap Z = \emptyset$, then because there are $2n$ edges of the form $\{x_j, y_j\}$, there exists an i such that neither x_i nor y_i belong to H. But then $H \cup \{y_i\}$ is an independent set since y_i is only adjacent to a vertex in Z and x_i. This contradicts the fact that H is a maximal independent set.

So, there exists a $z_i \in H \cap Z$. Because $G[Z]$ is a complete graph, $H \cap Z = \{z_i\}$. Thus each edge $\{x_j, y_j\}$ for $j \neq 2i$ or $2i-1$ has a vertex in H, otherwise $H \cup \{y_j\}$ is a larger independent set. Because $|H| \leq 2n-1$, we have already accounted for all the vertices in H. So, neither x_{2i} nor x_{2i-1} are in H. Hence x_{2i}, respectively x_{2i-1}, is adjacent to some vertex $x_l \in H$, respectively $x_k \in H$. Further, $x_{2i-1}, x_l, x_{2i}, x_k$ all belong to the same complete bipartite graph K_{k_r,k_r}. Then l must be odd since $2i$ is even and k must be even since $2i-1$ is odd. However, then x_k is adjacent to x_l, contradicting the fact that $x_k, x_l \in H$. Thus H cannot be a maximal independent set if $|H| < 2n$, and so every maximal independent set has cardinality $2n$. Therefore G is well-covered. □

We now show that any graph made via our construction is vertex decomposable, and furthermore, we determine its set of shedding vertices.

Theorem 6.3. *Let $G = D_n(k_1, \dots, k_m)$ be constructed as above. Then G is vertex decomposable and* $\mathrm{Shed}(G) = Z$.

Proof. Let $G = D_n(k_1, \dots, k_m)$. By Lemma 6.2, G is well-covered. We show that G is vertex decomposable by first working through four claims.

Claim 1: For each $i = 1, \dots, n$, $G_i = (((G \setminus z_1) \setminus z_2) \cdots \setminus z_i)$ is a well-covered graph.

Fix some $i \in \{1, \dots, n\}$. We will show that G_i is well-covered. Let H be any maximal independent set of G_i. Since $\{x_1, x_2\}, \dots, \{x_{2i-1}, x_{2i}\}$ are edges of G_i, for each $j = 1, \dots, i$, H contains at most one of x_{2j-1} and x_{2j}. Then H contains at least one of y_{2j-1} or y_{2j} for each $j = 1, \dots, i$, since H is maximal and y_{2j-1} and y_{2j} are leaves in G_i. But then H is also a maximal independent set of G since each vertex $z_1, \dots, z_i$ of G is adjacent to at least one vertex in H. Because G is well-covered, $|H| = \alpha(G)$. So G_i is also well-covered.

Claim 2: The graph G_n is vertex decomposable.

The graph G_n is the same as the induced graph $G[X \cup Y]$. So G_n is the graph of m disjoint graphs, where the j-th connected component is the complete bipartite graph K_{k_j,k_j} with whiskers at every vertex. One can use Theorem 2.4 to show that each connected component is vertex decomposable. Indeed, to apply Theorem 2.4, take $S = V(K_{k_j,k_j})$, and note that $K_{k_j,k_j} \cup W(S)$ is well-covered. So G_n is vertex decomposable by Lemma 2.3.

Claim 3: For each $i = 1, \dots, n$, $N_i = G_{i-1} \setminus N[z_i]$ is a well-covered graph.

For a fixed i, suppose that x_{2i-1} and x_{2i} appear in the complete bipartite graph K_{k_j,k_j}. Then the graph N_i consists of m disjoint graphs: $m-1$ of these graphs are the complete bipartite graphs with whiskers at every vertex, and the m-th graph is the graph K_{k_j,k_j} with whiskers at every vertex except x_{2i-1} and x_{2i}. Note that $m-1$ graphs are well-covered as was argued in Claim 2. The m-th graph is also well-covered: let $S = V(K_{k_j,k_j} \setminus \{x_{2i-1}, x_{2i}\})$ and apply Theorem 2.4 to $K_{k_j,k_j} \cup W(S)$. Therefore N_i is well-covered.

Claim 4: For each $i = 1, \dots, n$, N_i is vertex decomposable.

As shown in the previous proof, N_i is made up of m disjoint graphs, where each graph is either a complete bipartite graph with whiskers at every vertex, or a complete bipartite graph with whiskers at every vertex except at two adjacent vertices. It follows from Theorem 2.4 that in both cases, each disjoint graph is vertex decomposable. By Lemma 2.3, it then follows that N_i is vertex decomposable.

Thus we have established Claims 1–4. By definition, G is vertex decomposable if we can show that G_1 and N_1 are vertex decomposable. But G_1 is vertex decomposable if we can show that G_2 and N_2 are vertex decomposable. Continuing in this fashion, to show that G is vertex decomposable, it suffices to show that G_n and $N_1, \dots, N_n$ are all vertex decomposable. But this was shown in Claims 1–4. So G is vertex decomposable.

We next observe that $\mathrm{Shed}(G) = Z$. Note that to show G is vertex decomposable, we showed that $z_1 \in \mathrm{Shed}(G)$. By graph symmetry, $z_j \in \mathrm{Shed}(G)$ for any $z_j \in Z$. So $Z \subseteq \mathrm{Shed}(G)$.

Next, we show $Y \cap \mathrm{Shed}(G) = \emptyset$. Let $y \in Y$. After relabeling, assume that $y = y_{2n}$. Then $\{y_1, \dots, y_{2n-1}, x_{2n}\}$ and $\{z_1, x_1, y_3, \dots, y_{2n-2}, x_{2n-1}\}$ are maximal independent sets, in $G \setminus y$, of cardinality $2n$ and $2n-1$ respectively. Thus $G \setminus y$ is not well-covered and so $y \notin \mathrm{Shed}(G)$.

Finally, we show that $X \cap \mathrm{Shed}(G) = \emptyset$. Again, we show that for any $x \in X$, the graph $G \setminus x$ is not well-covered. After relabeling, assume $x = x_1$. The set Y is an independent set of $G \setminus x$ of cardinality $2n$. Note that since $k_1 \geq 2$, the vertex x_3 is adjacent to x_2 and x_4. It follows that $L = \{z_1, x_3, y_4, \dots, y_{2n}\}$ is a maximal independent set of $G \setminus x$ with $2n-1$ vertices.

Thus $\mathrm{Shed}(G) = Z$, as desired. □

The graphs constructed in this subsection give us the first family of graphs that fail Question 1.2 since no vertex in X is adjacent to any vertex in Z.

Corollary 6.4. *Let $G = D_n(k_1, \dots, k_m)$ be constructed as above. Then* $\mathrm{Shed}(G)$ *is not a dominating set.*

6.2. **Construction 2.** Next we construct a graph $G = P_m$ with vertex set $V = X \cup Y \cup Z$ with $X = \{x_1, \dots, x_{2m}\}$, $Y = \{y_1, y_2\}$, and $Z = \{z_1, z_2, z_3\}$ and edge set given by the following conditions:

- (i) the induced subgraph $G[X]$ is the m-partite graph $K_{2,2,\dots,2}$, whose complement is the matching with edges $\{x_{2i-1}, x_{2i}\}$, $1 \leq i \leq m$;
- (ii) y_1 is adjacent to z_1 and each x_{2i-1}, $1 \leq i \leq m$;
- (iii) y_2 is adjacent to z_2 and each x_{2i} for $1 \leq i \leq m$; and
- (iv) the induced subgraph on Z is K_3.

Note that if we let $X_1 = \{x_1, x_3, \ldots, x_{2m-1}\} \cup \{y_1\}$ and $X_2 = \{x_2, x_4, \ldots, x_{2m}\} \cup \{y_2\}$, then $G[X_1]$ and $G[X_2]$ are both cliques isomorphic to K_{m+1}.

Example 6.5. In Figure 3 is the graph P_2, while in Figure 4 is the graph P_3.

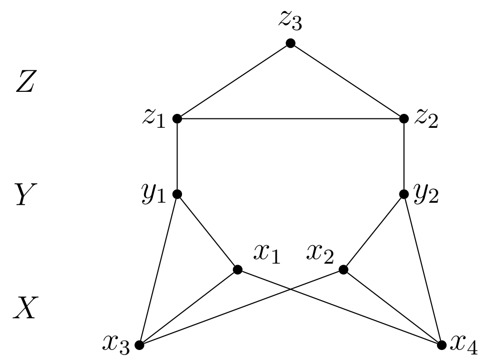

FIGURE 3. P_2.

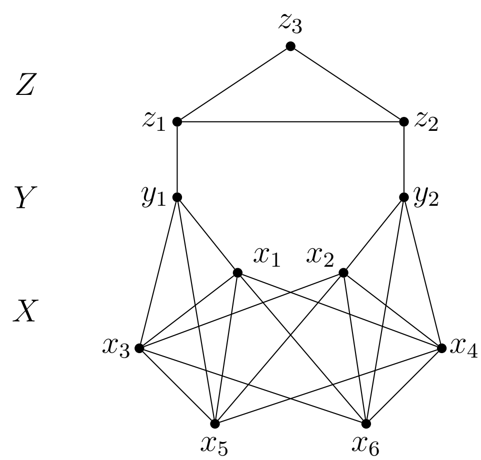

FIGURE 4. P_3.

Theorem 6.6. *The graph P_m is well-covered for $m \geq 2$.*

Proof. Note that we can partition the vertex set of $G = P_m$ into X_1, X_2 and Z. Further, $G[X_1]$, $G[X_2]$ and $G[Z]$ are all complete graphs. Hence, any maximal independent set will have cardinality 3 or less. Let H be an independent set of G. Suppose $Z \cap H = \emptyset$. Then $H \cup \{z_3\}$ is an independent set since z_3 is only adjacent to vertices in Z. Thus $Z \cap H \neq \emptyset$. Suppose $X_1 \cap H = \emptyset$. If y_2 is in H or $H \cap X_2 = \emptyset$, let $x = x_1$. Otherwise let $x = x_{2k-1}$ if x_{2k} is a vertex in H. Then $H \cup \{x\}$ is an independent set. Thus $X_1 \cap H \neq \emptyset$ and by symmetry $X_2 \cap H \neq \emptyset$.

Therefore all maximal independent sets of G must have cardinality 3, so P_m is well-covered. □

Lemma 6.7. *Given $m \geq 2$, if $G = P_m$, then $G[X \cup Y]$ is vertex decomposable.*

Proof. Since $G[X \cup Y]$ is a clique-whiskered graph, it is vertex decomposable by [6, Theorem 3.3]. □

Lemma 6.8. *Given $m \geq 2$, and $G = P_m$. Let $S = X \cup \{y_1\}$. Then $G[S]$ is vertex decomposable.*

Proof. Let $H = G[S]$. Note that y_1 is a simplicial vertex of H. Let x be a vertex adjacent to y_1. The graph $H \setminus N_H[x]$ is a single isolated vertex and hence is vertex decomposable.

Note that H is well-covered with $\alpha(H) = 2$. Thus $H \setminus x$ is well-covered by Lemma 2.8. Using Lemma 2.8 we can continue to remove vertices adjacent to y_1 while maintaining a well-covered graph until we obtain the graph with isolated vertex y_1 and complete graph on vertex set $X_2 \setminus y_2$. This resultant graph is a union of two complete graphs and hence is vertex decomposable by Lemma 2.3. Therefore $H \setminus x$ is vertex decomposable. Since $H \setminus N_H[x]$ is an isolated vertex, it is vertex decomposable. Therefore x is a shedding vertex of H and H is vertex decomposable. □

Given $\alpha = \alpha(G)$, define i_r to be the number of independent sets of G of cardinality r for $1 \leq r \leq \alpha$ with $i_0 = 1$. Define the h-vector $h_G = (h_0, h_1, \ldots, h_\alpha)$ by

$$h_k = \sum_{r=0}^{k} (-1)^{k-r} \binom{\alpha - r}{k - r} i_r .$$

A result of Villareal [21, Theorem 5.4.8] demonstrates that if a graph is Cohen-Macaulay, then the h-vector is a non-negative vector.[1] Since every vertex decomposable graph is Cohen-Macaulay, we have the following restatement of Villareal's result which we will use to limit the cardinality of $\mathrm{Shed}(P_m)$.

Lemma 6.9 ([21, Theorem 5.4.8])**.** *If G is a vertex decomposable graph, then h_G is a non-negative vector.*

Theorem 6.10. *For all $m \geq 2$, the graph P_m is vertex decomposable and a vertex $v \in \mathrm{Shed}(P_m)$ if and only if $v = z_1$ or z_2.*

Proof. We first show that if $v \notin \{z_1, z_2\}$ then $P_m \setminus v$ is not vertex decomposable.

Suppose that $v \in X$. By the symmetry of the graph, we can assume $v = x_1$. Then $\{y_1, y_2, z_3\}$ and $\{x_2, z_1\}$ are maximal independent sets of different cardinality in $P_m \setminus v$. Thus $P_m \setminus v$ is not well-covered and hence not vertex decomposable.

Next we consider a vertex in $v \in Y$. By symmetry, assume $v = y_1$. We will show that $P_m \setminus v$ is not vertex decomposable by showing that its h-vector has a negative entry. We first calculate the number i_r of independent sets of cardinality r in $P_m \setminus v$, for $1 \leq r \leq \alpha$. Note that $\alpha(P_m \setminus v) = 3$. There are $2m + 4$ vertices in $P_m \setminus v$ so $i_1 = 2m + 4$. An independent set of cardinality 2 can be of the form $\{y_2, x_i\}$, $\{y_2, z\}$ $\{z, x_i\}$ or $\{x_i, x_j\}$ for some $x_i, x_j \in X$ and $z \in Z$. There are m, 2, $6m$ and m such different independent sets respectively. Thus $i_2 = 8m + 2$. An independent set of cardinality 3 must have one

[1] Note that the f-vector $(f_0, f_1, \ldots, f_{\alpha-1})$ described in [21] is $(i_1, i_2, \ldots, i_\alpha)$ with $f_{-1} = 1$.

vertex in Z, one in X_2 and one in $X_1 \setminus y_1$ since these sets partition the vertex set, and induce complete subgraphs, of $P_m \setminus v$. There are m maximal independent sets containing z_2 and for each $z \in Z \setminus z_2$, there are $2m$ maximal independent sets containing z. Thus $i_3 = 5m$. Therefore $(i_0, i_1, i_2, i_3) = (1, 2m + 4, 8m + 2, 5m)$. But this implies that the h-vector has $h_3 = 1 - m$. Hence $h_3 < 0$ for $m > 1$ and by Lemma 6.9, $P_m \setminus v$ is not vertex decomposable. Thus no vertex in Y can be a shedding vertex of P_m if P_m is vertex decomposable.

Since $\{z_1, x_1, x_2\}$ and $\{y_1, y_2\}$ are maximal independent sets with different cardinalities in $P_m \setminus z_3$, $P_m \setminus z_3$ is not well-covered and hence not vertex decomposable.

Therefore, if $P_m \setminus v$ is vertex decomposable, then $v \in \{z_1, z_2\}$.

Now suppose $v = z_1$. We claim that $P_m \setminus v$ is vertex decomposable. The graph $P_m \setminus N_{P_m}[z_1]$ is the graph $G[S]$ described in Lemma 6.8 and so it is vertex decomposable and hence well-covered.

Next we claim that the graph $G = P_m \setminus z_1$ is well-covered. We can partition the vertices of G into the sets $Z \setminus z_1 \cup X_1 \cup X_2$. Since each part in the vertex partition induces a complete graph, we can construct an independent set of cardinality at most 3. Thus $\alpha(G) \leq 3$. Using an argument similar to Lemma 6.6, one can show that every maximal independent set of G is of cardinality 3 and hence $G = P_m \setminus z_1$ is well-covered.

We show that $G = P_m \setminus z_1$ is vertex decomposable by showing that z_2 is a shedding vertex of G. First $G \setminus N_{P_m}[z_2] = P_m \setminus N_{P_m}[z_2]$ since z_1 is adjacent to z_2, and $P_m \setminus N_{P_m}[z_2]$ is isomorphic to $P_m \setminus N_{P_m}[z_1]$. Thus $G \setminus N_{P_m}[z_2]$ is vertex decomposable. Next, $G \setminus z_2 = P_m \setminus \{z_1, z_2\}$ has an isolated vertex z_3 and a component described in Lemma 6.7 and so is vertex decomposable by Lemma 2.3.

Therefore $P_m \setminus N_{P_m}[z_1]$ and $P_m \setminus z_1$ are well-covered, so P_m is vertex decomposable and it follows that z_1 (and z_2 by symmetry) are shedding vertices of P_m. □

Corollary 6.11. *For all $m \geq 2$, the set* Shed(P_m) *is not a dominating set.*

Proof. Since each vertex in X is not adjacent to a shedding vertex of P_m, Shed(P_m) is not a dominating set. □

6.3. **Construction 3.** We finish this section by describing another family of vertex decomposable well-covered graphs whose set of shedding vertices fails to be a dominating set. Unlike the previous constructions, for the sake of brevity, we only sketch out the details of the proof.

Fix an integer $n \geq 1$. Let

$$\begin{aligned} X &= \{x_{1,1}, x_{1,2}\} \cup \{x_{2,1}, x_{2,2}\} \cup \ldots \cup \{x_{n,1}, x_{n,2}\}, \\ Y &= \{y_{1,1}, y_{1,2}, y_{1,3}\} \cup \{y_{2,1}, y_{2,2}, y_{2,3}\} \cup \ldots \cup \{y_{n,1}, y_{n,2}, y_{n,3}\}, \quad \text{and} \\ Z &= \{z_{1,1}, z_{1,2}, z_{1,3}\} \cup \ldots \cup \{z_{n,1}, z_{n,2}, z_{n,3}\}. \end{aligned}$$

We define the graph L_n to be the graph on $8n + 1$ vertices $V = X \cup Y \cup Z \cup \{w\}$. with the edge set given by the following conditions:

(*i*) for each $i = 1, \ldots, n$, the induced graph on $\{x_{i,1}, x_{i,2}, y_{i,1}, y_{i,2}, y_{i,3}\}$ is a 5-cycle with edges $\{y_{i,1}, y_{i,2}\}, \{y_{i,2}, y_{i,3}\}, \{y_{1,3}, x_{i,2}\}, \{x_{i,2}, x_{i,1}\}, \{x_{i,1}, y_{i,1}\}$;

(*ii*) $\{z_{i,1}, y_{i,1}\}, \{z_{i,2}, y_{i,2}\}$, and $\{z_{i,3}, y_{i,3}\}$ are edges for $i = 1, \ldots, n$, forming a matching between Y and Z; and

(*iii*) the induced graph on $Z \cup \{w\}$ is the complete graph K_{3n+1}.

Example 6.12. Graph L_1 is given in Figure 5 and L_2 in Figure 6.

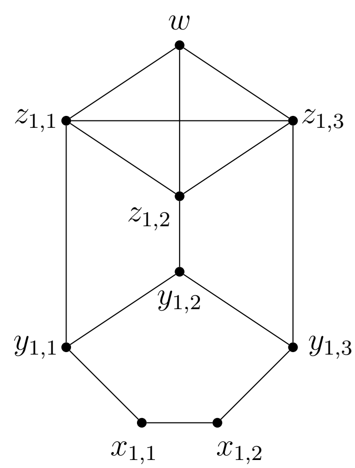

FIGURE 5. The graph L_1.

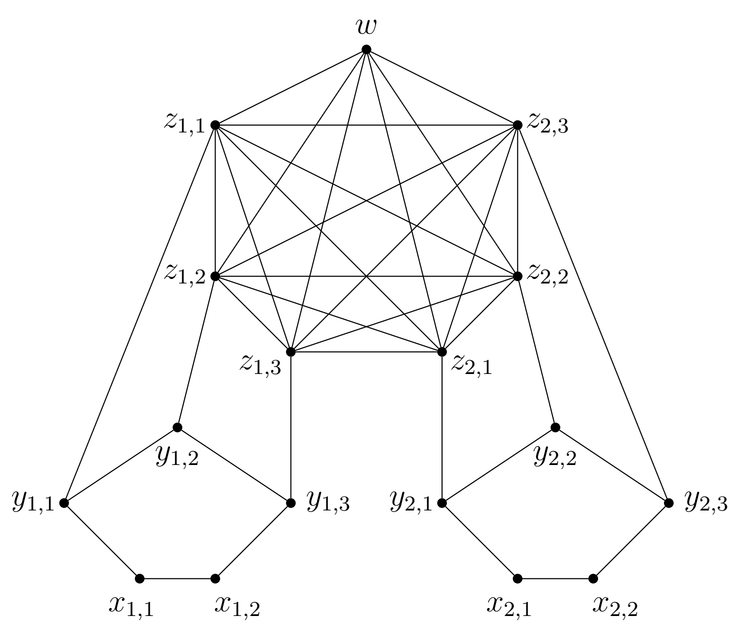

FIGURE 6. L_2.

We then have the following theorem, whose proof we only sketch.

Theorem 6.13. *For any integer $n \geq 1$, the graph L_n is vertex-decomposable, but* $\mathrm{Shed}(L_n)$ *is not a dominating set.*

Proof. Suppose $G = L_n$. To show that G is well-covered, show that every maximal independent set has cardinality $2n+1$.

To show that G is vertex decomposable, one can do induction on n. For $n = 1$, one can show that G is vertex decomposable directly. For $n \geq 2$, let $G_1 = G \setminus z_{n,1}$, $G_2 = G_1 \setminus z_{n,2}$ and $G_3 = G_2 \setminus z_{n,3}$. Furthermore, let $N_1 = G \setminus N[z_{n,1}]$, $N_2 = G_1 \setminus N[z_{n,2}]$, and $N_3 = G_2 \setminus N[z_{n,3}]$.

First show that all of the graphs G_1, G_2, G_3, N_1, N_2 and N_3 are well-covered. Then we note that $N_1 = N_2 = N_3$ consist of n connected components, where $(n-1)$ of these components are five cycles, and the last is the path of four vertices. All of these components are vertex decomposable, thus so is N_i. The graph G_3 consists of two components, L_{n-1} and a five cycle. By induction, these graphs are vertex decomposable. Using these facts, we can show that G is vertex decomposable.

Note to show that G is vertex decomposable, we show that $Z \subseteq \mathrm{Shed}(G)$. The next step of the proof is to show that $X \cap \mathrm{Shed}(G) = \emptyset$ and $Y \cap \mathrm{Shed}(G) = \emptyset$ by showing that if we remove any vertex $v \in X \cup Y$, then $G \setminus v$ is not well-covered. This shows that $\mathrm{Shed}(G)$ is not a dominating set since the vertices of X are only adjacent to vertices in Y, but no vertex of Y belongs to $\mathrm{Shed}(G)$. □

7. Graph expansions

In this section we briefly describe a way to extend any vertex decomposable graph whose shedding set is not a dominating set, to build a larger graph with the same property by adding one vertex at a time. The technique involves 'duplicating' a vertex in the shedding set.

Theorem 7.1. *Suppose G is a vertex decomposable graph and* $\mathrm{Shed}(G)$ *is not a dominating set. For any $x \in \mathrm{Shed}(G)$, let H be the graph with $V(H) = V(G) \cup \{x'\}$ and $E(H) = E(G) \cup \{\{x', y\} \mid y \in N[x]\}$. Then H is vertex decomposable and* $\mathrm{Shed}(H)$ *is not a dominating set.*

To prove Theorem 7.1, we use a result of [17]. First we define a graph expansion. Let G be a graph on the vertex set $\{x_1, \ldots, x_n\}$ and let $(s_1, \ldots, s_n)$ be an n-tuple of positive integers. The *graph expansion of* G, denoted $G^{(s_1,\ldots,s_n)}$, is the graph on the vertex set

$$\{x_{1,1}, \ldots, x_{1,s_1}\} \cup \{x_{2,1}, \ldots, x_{2,s_2}\} \cup \ldots \cup \{x_{n,1}, \ldots, x_{n,s_n}\}$$

with edge set $\{\{x_{i,j}, x_{k,l}\} \mid \{x_i, x_k\} \in E(G) \text{ or } i = k\}$. Moradi and Khosh-Ahang [17, Theorem 2.7] showed that vertex decomposability is invariant under graph expansion, that is, G is vertex decomposable if and only if $G^{(s_1,\ldots,s_n)}$ is vertex decomposable.

Proof. Suppose G is a vertex decomposable graph with $V = \{x_1, \ldots, x_n\}$ and $\mathrm{Shed}(G)$ is not a dominating set of G. Suppose $x \in \mathrm{Shed}(G)$ and H is a graph with $V(H) = V \cup \{x'\}$ and $E(H) = E(G) \cup \{\{x', y\} \mid y \in N[x]\}$. Without loss of generality, assume $x = x_1$. Note that $H = G^{(2,1,\ldots,1)}$ and hence H is vertex decomposable since vertex decomposability is preserved under graph expansion.

Observe that $x, x' \in \mathrm{Shed}(H)$, since $H \setminus x$ and $H \setminus x'$ are both isomorphic to G and G is vertex decomposable.

Suppose $y \in V$ but $y \notin \text{Shed}(G)$. We claim $y \notin \text{Shed}(H)$. Suppose $y \in \text{Shed}(H)$. Then $H \setminus y$ is vertex decomposable. Note that $H \setminus y$ is a graph expansion of $(H \setminus y) \setminus x'$ and hence $(H \setminus y) \setminus x'$ is vertex decomposable. Now, $(H \setminus y) \setminus x'$ is isomorphic to $G \setminus y$, so $G \setminus y$ is vertex decomposable. But this contradicts the fact that $G \setminus y$ is not vertex decomposable if $y \notin \text{Shed}(G)$. Thus $y \notin \text{Shed}(H)$.

In particular, $\text{Shed}(H) \setminus \{x'\} \subseteq \text{Shed}(G)$. It follows that $\text{Shed}(H)$ is not a dominating set of H since a dominating set of H that includes both x and x' would essentially be a dominating set of G (since having both x and x' in a dominating set is redundant). □

It may be worth noting that it is also possible to construct vertex decomposable graphs which satisfy Question 1.2 via graph expansion. As observed in the proof above, the vertex x that gets duplicated as well as its duplicate x' are both in the set of shedding vertices in the graph expansion. It follows that if every vertex is duplicated at least once on a vertex decomposable graph, the resulting graph will be vertex decomposable with every vertex in its shedding set. Consequently, many graph expansions satisfy Question 1.2:

Theorem 7.2. *If G is any vertex decomposable graph and $s_i \geq 2$ for $1 \leq i \leq n$, then $G^{(s_1,s_2,\ldots,s_n)}$ is vertex decomposable and* $\text{Shed}\left(G^{(s_1,s_2,\ldots,s_n)}\right)$ *is a dominating set.*

8. Computational Results regarding Vertex Decomposability

In this final section, we summarize some of our computational observations while studying Question 1.2. We used *Macaulay2* [11] and the packages `EdgeIdeals` [10], `Nauty` [4], and `SimplicialDecomposability` [5] for our computations.

For all connected graphs on 10 or less vertices, we checked whether the graph was (a) well-covered, (b) Cohen-Macaulay, (c) vertex decomposable, and (d) if the graph was vertex decomposable, whether the graph satisfied Question 1.2. Table 1 summarizes our findings. The first column is the number of vertices, while the second column is the number of connected graphs on n vertices, and the third column is the number of well-covered graphs on n vertices. The second column is sequence A001349 in the OEIS, and the third column is sequence A2226525 in the OEIS [20].

A graph G is Cohen-Macaulay if the ring $R/I(G)$ is a Cohen-Macaulay ring, where $I(G)$ denotes the edge ideal of G. It is known that if G is vertex decomposable, then G is Cohen-Macaulay. As part of this computer experiment, we also counted the number of Cohen-Macaulay graphs. The fourth and fifth columns count the number of Cohen-Macaulay graphs, respectively, the number of vertex decomposable graphs. Our computations imply the following result:

Observation 8.1. Let G be a graph with 10 or fewer vertices. Then G is Cohen-Macaulay if and only if G is vertex decomposable.

It is not true that all graphs that are Cohen-Macaulay are vertex decomposable (see, e.g., [8] for a graph on 16 vertices that is Cohen-Macaulay, but not vertex decomposable). However, we currently do not know the smallest such example. Our computations reveal that the minimal such example has at least 11 vertices.

The last column counts the number of vertex decomposable graphs that do not satisfy Question 1.2. Among the 17 graphs on 9 vertices that fail Question 1.2, we found that the graph P_2 (see Figure 3) has the least number of edges.

Although this paper has focused on Question 1.2, we return to the conjecture that inspired this question. Specifically, our computational results imply the following result.

Observation 8.2. Conjecture 1.1 is true for all Cohen-Macaulay graphs on eight or less vertices. However, the graph P_2 on nine vertices and 13 vertices is the minimal counterexample to Conjecture 1.1.

Proof. Let G be any Cohen-Macaulay graph and let

$$S = \{x \in V \mid G \setminus x \text{ is a Cohen-Macaulay graph}\}.$$

If G is also vertex decomposable and if $x \in \mathrm{Shed}(G)$, then $G \setminus x$ is vertex decomposable, so $G \setminus x$ is Cohen-Macaulay. So, we always have $\mathrm{Shed}(G) \subseteq S$.

If G is a Cohen-Macaulay graph on eight or less vertices, it is also vertex decomposable by Remark 8.1. Also, our computations imply that $\mathrm{Shed}(G)$ is a dominating set for all such graphs and hence S is also a dominating set.

In our proof Theorem 6.10, we showed that P_2 is a vertex decomposable graph. Furthermore, for every vertex $x \in V(P_2) \setminus \mathrm{Shed}(P_2)$, the graph $P_2 \setminus x$ is either not well-covered (and thus not Cohen-Macaulay) or not Cohen-Macaulay. So, $\mathrm{Shed}(P_2) = S$, and thus P_2 does not satisfy Conjecture 1.1 by Corollary 6.11. The minimality in our statement follows via our computations. □

Vertices	Connected Graphs	Well-Covered	Cohen-Macaulay	Vertex Decomposable	Fail Ques. 1.2
1	1	1	1	1	0
2	1	1	1	1	0
3	2	1	1	1	0
4	6	3	2	2	0
5	21	6	5	5	0
6	112	27	20	20	0
7	853	108	82	82	0
8	11117	788	565	565	0
9	261080	9035	5688	5688	17
10	11716571	196928	102039	102039	942

TABLE 1. Number of well-covered, Cohen-Macaulay, and vertex decomposable graphs

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