Note on level r consensus

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Abstract. We show that the hierarchy of level r consensus partially collapses. In particular, any profile $\pi \in \mathcal{P}$ that exhibits consensus of level $(K-1)!$ around $\succ_0$ in fact exhibits consensus of level 1 around $\succ_0$.

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The concept of level r consensus was introduced in [1] in the context of the metric approach in social choice theory. We will mainly use the notation and definitions of [1]. Let $A = \{1, 2, \ldots, K\}$ be a set of $K > 2$ alternatives and let $N = \{1, 2, \ldots, n\}$ be a set of individuals. Each linear order (i.e. complete, transitive and antisymmetric binary relation) on the set A is called a *preference relation.* The set of all preference relations is denoted by $\mathcal{P}$. The *inversion metric* is the function $d : \mathcal{P} \times \mathcal{P} \to \mathbb{R}$ defined by

$$d(\succ, \succ') = \frac{|(\succ \setminus \succ') \cup (\succ' \setminus \succ)|}{2}$$

(since all preference relations in $\mathcal{P}$ have the same cardinality we have also: $d(\succ, \succ') = |\succ \setminus \succ'| = |\succ' \setminus \succ|$).

Let $\succ_0$ be a preference relation in $\mathcal{P}$. A metric on $\mathcal{P}$ allows to determine which one of any two preference relations is closer to a third one. This comparison can be extended to equal-sized sets of preferences.

Definition 1 *Let C and C' be two disjoint nonempty subsets of $\mathcal{P}$ with the same cardinality, and let $\succ_0 \in \mathcal{P}$ be a preference relation on A. We say that C is at least as close to $\succ_0$ as C', denoted by $C \geq_{\succ_0} C'$, if there is a one-to-one function $\phi : C \to C'$ such that for all $\succ \in C$, $d(\succ, \succ_0) \leq d(\phi(\succ), \succ_0)$. We also say that C is closer than C' to $\succ_0$, denoted by $C >_{\succ_0} C'$, if there is a one to one function $\phi : C \to C'$ such that for all $\succ \in C$, $d(\succ, \succ_0) \leq d(\phi(\succ), \succ_0)$, with strict inequality for at least one $\succ \in C$.*

Using the concept of closeness the authors define the correspondence between preference profiles $\pi \in \mathcal{P}^n$ and preference relations $\succ \in \mathcal{P}$ depending on a natural parameter r called "*preference profile π exhibits consensus of level r around $\succ$*".

For any $\pi = (\succ_1, \succ_2, \ldots, \succ_n) \in \mathcal{P}^n$, $\succ \in \mathcal{P}$, and $C \subseteq \mathcal{P}$

$$\mu_\pi(\succ) = |\{i \in \mathbb{N} : \succ_i = \succ\}|, \ \ \mu_\pi(C) = |\{i \in \mathbb{N} : \succ_i \in C\}|$$

(obviously, $\mu_\pi(C) = \sum_{\succ \in C} \mu_\pi(\succ)$).

Definition 2 *Let $r \in \{1, 2, \ldots, \frac{K!}{2}\}$, and let $\succ_0 \in \mathcal{P}$. A preference profile $\pi \in \mathcal{P}^n$ exhibits consensus of level r around $\succ_0$ if*

1. *for all disjoint subsets C, C' of $\mathcal{P}$ with cardinality r, $C \geq_{\succ_0} C' \to \mu_\pi(C) \geq \mu_\pi(C')$*
2. *there are disjoint subsets C, C' of $\mathcal{P}$ with cardinality r, such that $C >_{\succ_0} C'$ and $\mu_\pi(C) > \mu_\pi(C')$.*

Proposition 1 of [1] states that the set of profiles that exhibit consensus of level $r+1$ around $\succ_0$ extends the set of profiles that exhibit consensus of level r around $\succ_0$. Thus, each preference relation $\succ_0$ determines the hierarchy of preference profiles.

Let a preference profile π exhibit consensus of level r around $\succ_0$. We call $\succ_0$ a *level r consensus relation of* π and simply *consensus relation of* π if $r = \frac{K!}{2}$ (the level $\frac{K!}{2}$ is the maximum level for which this concept is nontrivial).

A level r consensus relation $\succ_0$ of profile π may be considered as one of probable social binary relations on the profile π. Theorem 1 of [1] states that if n is odd, then each profile π have at most one consensus relation $\succ_0$ and the consensus relation $\succ_0$ coincides with the relation M_π assigned by the majority rule to π. This result gives an interesting sufficient condition for transitivity of M_π. Furthermore, regardless of parity of n, the $\succ_0$-largest element a_1 is a *Condorcet winner* on π.

For small values of r, level r consensus relations $\succ_0$ of profile π have some interesting additional properties. Namely, the largest element a_1 with respect $\succ_0$ is selected by any scoring rule. A *scoring rule* is characterized by a non-increasing sequence $S = (S_1, S_2, \ldots, S_K)$ of non-negative real numbers for which $S_1 > S_K$. For $k = 1, 2, \ldots, K$, each individual with the preference relation $\succ$ assigns S_k points to the k-th alternative in the linear order $\succ$. The scoring rule associated with S is the function $V_S : \mathcal{P}^n \to 2^A$ whose value at any profile $\pi = \{\succ_1, \succ_2, \ldots, \succ_n\}$ is the set $V_S(\pi)$ of alternatives a with the maximum total score (i.e. with the maximum sum $\sum_{1 \leq i \leq K} S_{k_i}$ where k_i is the rank of a in $\succ_i$). Theorem 2 in [1] claims that if a preference profile π exhibits consensus of level $r \leq (K-1)!$ around $\succ_0$, then the $\succ_0$-largest element a_1 belongs to $V_S(\pi)$ for all scoring rules V_S.

However, the authors did not notice some combinatorial properties of the concepts introduced. We show that the hierarchy of preference profile partially collapses. In particular, any profile $\pi \in \mathcal{P}$ that exhibits consensus of level $(K-1)!$ around $\succ_0$ in fact exhibits consensus of level 1 around $\succ_0$. Thus, it would be desirable to slightly adjust the assumption of Theorem 2 of [1].

Theorem 1 *For any natural number $K > 2$ there is a natural number $c \leq \frac{K(K-1)}{4}$ such that for any natural numbers $n \geq 1$ and $r \in \{1, 2, \ldots, \frac{K!}{2} - c\}$, any preference profile $\pi \in \mathcal{P}^n$, and any linear order $\succ_0 \in \mathcal{P}$ the following conditions are equivalent*

1. *π exhibits consensus of level r around $\succ_0$*
2. *π exhibits consensus of level 1 around $\succ_0$.*

Proof. The implication $2 \to 1$ follows from Proposition 1 of [1]. We will prove the reverse implication. Let $\succ_0$ be a linear order in $\mathcal{P}$ and let

$$\mathcal{P}_k(\succ_0) = \{\succ \in \mathcal{P} : d(\succ, \succ_0) = k\}.$$

for any natural number k. Obviously, $|\mathcal{P}_k(\succ_0)|$ coincides with the number of permutations of $\{1, 2, \ldots, K\}$ with k inversions, i.e. with the *Mahonian number* $T(K, k)$ (sequence A008302 in OEIS, see [2]). The set $\mathcal{P}_{\frac{K(K-1)}{2}}$ contains exactly one element. We denote this element by $\overline{\succ}_0$: $\mathcal{P}_{\frac{K(K-1)}{2}} = \{\overline{\succ}_0\}$.

Let c' be the number of k for which $T(K, k)$ is odd:

$$c' = |\{k \in \mathbb{N} : T(K, k) \equiv 1 \pmod 2\}|.$$

So, $c' \leq \frac{K(K-1)}{2}$ because $\frac{K(K-1)}{2}$ is the maximum distance between the linear orders in $\mathcal{P}$. Moreover, c' is even because

$$\sum_{0 \leq k \leq \frac{K(K-1)}{2}} T(K, k) = K! \equiv 0 \pmod 2.$$

Let $c = \frac{c'}{2}$. Then the inequality $c \leq \frac{K(K-1)}{4}$ holds.

Definition 3 *For any natural number m a pair $(C_1, C_2) \in 2^{\mathcal{P}} \times 2^{\mathcal{P}}$ is called* m-balanced (around $\succ_0$) *iff*

1. $C_1 \cap C_2 = \varnothing$,
2. $|C_1| = |C_2| = m$,
3. $|C_1 \cap \mathcal{P}_k(\succ_0)| = |C_2 \cap \mathcal{P}_k(\succ_0)|$ *for any* $k = 0, 1, \ldots, \frac{K(K-1)}{2}$.

Lemma 1 *Let $\succ_1, \succ_2 \in \mathcal{P} \setminus \{\succ_0, \overline{\succ}_0\}$ and $\succ_1 \neq \succ_2$. Then there is a $(\frac{K!}{2} - c)$-balanced pair (C_1, C_2) for which $\succ_1 \in C_1$ and $\succ_2 \in C_2$.*

Proof. Note that $T(K, k) \geq 2$ for any $k \in \{1, 2, \ldots, \frac{K(K-1)}{2} - 1\}$ (this follows, for example, from a recurrence formula for $T(K, k)$, see [2]). Using this fact, for each $k \in \{k \in \mathbb{N} : T(K, k) \equiv 1 \pmod 2\}$ choose a preference relation $\succ_{(k)} \in \mathcal{P}_k(\succ_0) \setminus \{\succ_1, \succ_2\}$. Let

$$\mathcal{P}'_k(\succ_0) = \begin{cases} \mathcal{P}_k(\succ_0) & \text{if } T(K, k) \equiv 0, \\ \mathcal{P}_k(\succ_0) \setminus \{\succ_{(k)}\} & \text{if } T(K, k) \equiv 1 \end{cases} \pmod 2.$$

For each $k \in \{1, \ldots, \frac{K(K-1)}{2} - 1\}$ choose a set $C_{(k)}$ with properties

1. $C_{(k)} \subseteq \mathcal{P}'_k(\succ_0)$,
2. $|C_{(k)}| = \frac{|\mathcal{P}'_k(\succ_0)|}{2}$,
3. $d(\succ_1, \succ_0) = k \to \succ_1 \in C_{(k)}$,
4. $\succ_2 \notin C_{(k)}$.

Let

$$C_1 = \bigcup_{1 \le k \le \frac{K(K-1)}{2}-1} C_{(k)} \text{ and } C_2 = \bigcup_{1 \le k \le \frac{K(K-1)}{2}-1} \mathcal{P}'_k(\succ_0) \setminus C_{(k)}.$$

Obviously, items 1– 3 of Definition 3 hold. Lemma 2 is proved.

Lemma 2 *For any natural number m and m-balanced pair (C_1, C_2) there is a one-to-one function $\phi : C_1 \to C_2$ satisfying*

$$d(\succ, \succ_0) = d(\phi(\succ), \succ_0)$$

for all $\succ \in C_1$.

Proof. By item 3 of Definition 3 for any $k = 0, 1, \ldots, \frac{K(K-1)}{2}$ there is a one-to-one mappings $\phi_k : C_1 \cap \mathcal{P}_k(\succ_0) \to C_2 \cap \mathcal{P}_k(\succ_0)$ (maybe empty if $C_1 \cap \mathcal{P}_k(\succ_0) = \varnothing$). Obviously, we can put $\phi = \bigcup_{0 \le i \le \frac{K(K-1)}{2}} \phi_k$. Lemma 3 is proved.

Corollary 1 *For any natural number m and m-balanced pair (C_1, C_2)*

$$C_1 \ge_{\succ_0} C_2 \text{ and } C_2 \ge_{\succ_0} C_1.$$

Proof. Let ϕ be a function from Lemma 2. Then

$$d(\succ, \succ_0) = d(\phi^{-1}(\succ), \succ_0)$$

for all $\succ \in C_2$, and it remains to recall Definition 1.

Let $\pi \in \mathcal{P}^n$ and let π exhibit consensus of level $r \in \{1, 2, \ldots, \frac{K!}{2} - c\}$ around $\succ_0$. By Proposition 1 of [1] π exhibits consensus of level $\frac{K!}{2} - c$ around $\succ_0$. Our next goal is to prove that item 1 of Definition 2 holds for the profile π and $r = 1$.

Lemma 3 *For any different $\succ_1, \succ_2 \in \mathcal{P}$*

$$d(\succ_1, \succ_0) \le d(\succ_2, \succ_0) \to \mu_\pi(\succ_1) \ge \mu_\pi(\succ_2).$$

Proof. Let $\succ_1, \succ_2 \in \mathcal{P}$, $\succ_1 \ne \succ_2$ and $d(\succ_1, \succ_0) \le d(\succ_2, \succ_0)$.

First, let $\{\succ_1, \succ_2\} \cap \{\succ_0, \overline{\succ_0}\} = \varnothing$. Consider a $(\frac{K!}{2} - c)$-balanced pair (C_1, C_2) for which $\succ_2 \in C_1$ and $\succ_1 \in C_2$, and a on-to-one function $\phi : C_1 \to C_2$ satisfying

$$d(\succ, \succ_0) = d(\phi(\succ), \succ_0)$$

for all $\succ \in C_1$. By Definition 2 and Corollary 3 we have

$$\mu_\pi(C_1) = \mu_\pi(C_2). \tag{1}$$

Let $C'_1 = (C_1 \setminus \{\succ_2\}) \cup \{\succ_1\}$ and $C'_2 = (C_2 \setminus \{\succ_1\}) \cup \{\succ_2\}$. Consider the function $\phi' : C'_1 \to C'_2$ defined by

$$\phi'(\succ) = \begin{cases} \succ_2 & \text{if } \succ = \succ_1, \\ \phi(\succ_2) & \text{if } \succ = \phi^{-1}(\succ_1) \ne \succ_2, \\ \phi(\succ) & \text{otherwise.} \end{cases}$$

For all $\succ\in C_1'$ we have $d(\succ,\succ_0)\le d(\phi'(\succ),\succ_0)$, so $C_1'\ge_{\succ_0} C_2'$ by Definition 1. Hence, by Definition 2

$$\mu_\pi(C_1')\ge\mu_\pi(C_2'). \tag{2}$$

Since $(\forall C\subseteq\mathcal{P})\,\mu_\pi(C)=\sum_{\succ\in C}\mu_\pi(\succ)$, we have

$$\mu_\pi(C_1')=\mu_\pi(C_1)-\mu_\pi(\succ_2)+\mu_\pi(\succ_1)\text{ and }\mu_\pi(C_2')=\mu_\pi(C_2)-\mu_\pi(\succ_1)+\mu_\pi(\succ_2). \tag{3}$$

Then by (1), (2) and (3)

$$\mu_\pi(\succ_1)-\mu_\pi(\succ_2)\ge\mu_\pi(\succ_2)-\mu_\pi(\succ_1),$$

and, finally,

$$\mu_\pi(\succ_1)\ge\mu_\pi(\succ_2).$$

For further discussion, note that this implies

$$d(\succ_1,\succ_0)=d(\succ_2,\succ_0)\to\mu_\pi(\succ_1)=\mu_\pi(\succ_2). \tag{4}$$

for all different $\succ_1,\succ_2\in\mathcal{P}$.

Consider the remaining cases.

Let $\succ_1=\succ_0$ and $\succ_2\ne\overline{\succ}_0$. Then denote $C_1''=(C_1\setminus\{\succ_2\})\cup\{\succ_0\}$ and $C_2''=(C_1\setminus\{\phi(\succ_2)\})\cup\{\succ_2\}$. Consider the function $\phi'':C_1''\to C_2$ defined by

$$\phi''(\succ)=\begin{cases}\succ_2 & \text{if }\succ=\succ_0,\\ \phi(\succ) & \text{otherwise.}\end{cases}$$

For all $\succ\in C_1''$ we have $d(\succ,\succ_0)\le d(\phi''(\succ),\succ_0)$ and, further, $C_1''\ge_{\succ_0} C_2''$. Reasoning as before we have

$$\mu_\pi(\succ_0)-\mu_\pi(\succ_2)\ge\mu_\pi(\succ_2)-\mu_\pi(\phi(\succ_2)).$$

Since $d(\succ_2,\succ_0)=d(\phi(\succ_2),\succ_0)$, we have $\mu_\pi(\succ_2)=\mu_\pi(\phi(\succ_2))$ by (4). Finally,

$$\mu_\pi(\succ_0)\ge\mu_\pi(\succ_2).$$

In the case $\succ_2=\overline{\succ}_0$ and $\succ_1\ne\succ_0$, the arguments are similar.

In the latter case $\succ_1=\succ_0$ and $\succ_2=\overline{\succ}_0$. We can choose a preference relation $\succ^*\in\mathcal{P}\setminus\{\succ_0,\overline{\succ}_0\}$. According to the above, we have

$$\mu_\pi(\succ_1)\ge\mu_\pi(\succ^*)\ge\mu_\pi(\succ_2).$$

Lemma 3 is proved.

To prove the theorem it remains to show that item 2 of Definition 2 holds for the profile π and $r=1$. Assume $\mu_\pi(\overline{\succ}_0)=\varnothing$. Then, for every preference relation $\succ$ of profile π we have

$$d(\succ,\succ_0)>d(\overline{\succ}_0,\succ_0)\text{ and }\mu_\pi(\succ)>\mu_\pi(\overline{\succ}_0).$$

In the opposite case, assume that item 2 of Definition 2 is not hold for the profile π and $r = 1$. Then by Lemma 3 the profile π contains the same number of all linear orders in $\mathcal{P}$. Thus, π does not exhibit consensus of any level, a contradiction.

Theorem 1 is proved.

Corollary 2 *Let profile π exhibit consensus of level $(K-1)!$ around $\succ_0$. Then π exhibits consensus of level 1 around $\succ_0$.*

Proof. Let $K \geq 4$. Then it suffices to prove the inequality

$$(K-1)! \leq \frac{K!}{2} - \frac{K(K-1)}{4}.$$

This is easily by induction. For $K = 3$ we can use the sufficiency of inequality

$$(K-1)! \leq \frac{K!}{2} - \frac{|\{k : T(K,k) = 1 \pmod 2\}|}{2}$$

(for $K = 3$ we have $|\{k : T(3,k) = 1 \pmod 2\}| = 2$).

References

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