Independence and matching numbers of some token graphs

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Abstract

Let G be a simple graph of order n and let k be an integer such that $1 \leq k \leq n-1$. The k-token graph $F_k(G)$ of G, or the *k-th* symmetric power of G, is defined as the graph with vertex set all k-subsets of $V(G)$, where two vertices are adjacent in $F_k(G)$ whenever their symmetric difference is an edge of G. Here we study the independence and matching numbers of $F_k(G)$. We start by giving a tight lower bound for the matching number $\nu(F_k(G))$ of $F_k(G)$ for the case in which G has either a perfect matching or an almost perfect matching. Using this result, we estimate the independence number for a large class of bipartite k-token graphs, and determine the exact value of $\beta(F_2(K_{m,n})), \beta(F_2(C_n))$ and $\beta(F_k(G))$ for $G \in \{P_m, K_{1,m}, K_{m,m}, K_{m,m+1}\}$ and $1 \leq k \leq |G|-1$.

Keywords: Token graphs; Matchings; Independence number.
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1 Introduction

All the graphs under consideration in this paper are simple and finite. Let G be a graph of order n and let k be an integer such that $1 \leq k \leq n-1$. The *k-token graph* $F_k(G)$ of G is the graph whose vertices are all the k-subsets of $V(G)$ and two k-subsets are adjacent if their symmetric difference is an edge of G. In particular, observe that $F_1(G)$ and G are isomorphic, which, as usual, is denoted by $F_1(G) \simeq G$. Moreover, note also that $F_k(G) \simeq F_{n-k}(G)$ for any admissible k (i.e., $1 \leq k \leq n-1$). Often, throughout this paper, we simply write token graph instead of k-token graph.

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1.1 Token graphs

The origins of the notion of token graphs can be dated back at least to 2002 when Terry Rudolph [21] used $F_k(G)$ to study the graph isomorphism problem. In such a work Rudolph gave examples of non-isomorphic graphs G and H which are cospectral, but with $F_2(G)$ and $F_2(H)$ non-cospectral. He emphasized this fact by saying the following about the eigenvalues of $F_2(G)$: "then the eigenvalues of this larger matrix are a graph invariant, and in fact are a more powerful invariant than those of the original matrix G". Five years later, this notion was extended by Audenaert et al. in [4] to any integer k such that $1 \leq k \leq |G| - 1$, calling to $F_k(G)$ the *symmetric k-th power of G*. In [4] was proved that the 2-token graphs of strongly regular graphs with the same parameters are cospectral and some connections with generic exchange Hamiltonians in quantum mechanics were also discussed. Following Rudolph's study, Barghi and Ponomarenko [6] and Alzaga et al. [2] proved, independently, that for a given positive integer k there exists infinitely many pairs of non-isomorphic graphs with cospectral k-token graphs.

In 2012 Ruy Fabila-Monroy et al. [13] reintroduced, independently, the concept of k-token graphs "as a model in which k indistinguishable tokens move from vertex to vertex along the edges of a graph" and began the systematic study of the combinatorial parameters of $F_k(G)$. In particular, the investigation presented in [13] includes the study of connectivity, diameter, cliques, chromatic number, Hamiltonian paths, and Cartesian products of token graphs. Following this line of research, in 2015, Fabila-Monroy and three of the authors of this paper presented results about the planarity and the regularity of token graphs [10].

From the model of $F_k(G)$ proposed in [13] it is clear that the k-token graphs can be considered as part of several models of swapping in the literature [14, 26] that are part of reconfiguration problems (see e.g. [9, 19]). For instance, the pebble motion (PM) problem of determining if an arrangement A of $1 < k < |G|$ distinct pebbles numbered $1, \dots, k$ and placed on k distinct vertices of G can be transformed into another given arrangement B by moving the pebbles along edges of G provided that at any given time at most one pebble is traveling along an edge and each vertex of G contains at most one pebble, has been studied in [5] and [17] from the algorithmic point of view. Also, in such papers several applications of the PM problem have been mentioned, which include the management of memory in totally distributed computing systems and problems in robot motion planning. On the other hand, note that the PM problem is a variant of the problem of determining the diameter of $F_k(G)$ (the only difference is that in the PM problem, the pebbles or tokens are distiguishable).

The k-token graphs also are a generalization of Johnson graphs: if G is the complete graph of order n, then $F_k(G)$ is isomorphic to the Johnson graph $J(n, k)$. The Johnson graphs have been studied from several approaches, see for example [1, 20, 23]. In particular, the determination of the exact value of the independence number $\beta(J(n, k))$ of the Johnson graph, as far as we know, remains open in its generality, albeit it has been widely studied [7, 8, 12, 15, 18]. Possibly, the last effort to determine $\beta(J(n,k))$ was made by K. G. Mirajkar et al. in 2016 [18]. In such a work they presented an exact formula for $\beta(J(n,k))$, which is unfortunately wrong: the independence number of $J(7,3)$ is 7 because it is equal to the distance-4 constant wight code $A(7,4,3)$ [7], but the formula in [18] gives 6.

1.2 Main results

The graph parameters of interest in this paper are the independence number and the matching number. A set I of vertices in a graph G is an *independent set* if no two vertices in I are adjacent; a *maximal independent set* is an independent set such that it is not a proper subset of any independent set in G. The *independence number* $\beta(G)$ of G is the number of vertices in a largest independent set in G, and its computation is NP-hard [16].

On the other hand, recall that the *matching number* $\nu(G)$ of G is the number of edges of any largest matching in G. A matching M of G is called *perfect matching* (respectively *almost perfect matching*) if $|M| = n$ (respectively $|M| = n-1$). Note that $\nu(G) = |G|/2$ if and only if G has a perfect matching. Similarly, $\nu(G) = (|G|-1)/2$ if and only if G has an almost perfect matching. In this case, Jack Edmonds proved in 1965 that the matching number of a graph can be determined in polynomial time [11].

Our main results in this paper are Theorems 1.1, 1.2 and 1.3.

In our attempt to estimate the independence number of the token graphs of certain families of bipartite graphs, we meet the following natural question:

Question 1. If $\nu(G) = \lfloor |G|/2 \rfloor$, what can we say about $\nu(F_k(G))$?

In Theorem 1.1 we answer Question 1 by providing a lower bound for $\nu(F_k(G))$ and exhibiting some graphs for which such a bound is tight.

Theorem 1.1. *Let G be a graph of order n and let k be an integer with $1 \le k \le n-1$. If $\nu(G) = \lfloor n/2 \rfloor$, then*

(1) $\nu(F_k(G)) = \binom{n}{k}/2$, if n is even and k is odd.

(2) $\nu(F_k(G)) \ge \left(\binom{n}{k} - \binom{n/2}{k/2}\right)/2$, if n is even and k is even.

(3) $\nu(F_k(G)) \ge \left(\binom{n}{k} - \binom{(n-1)/2}{\lfloor k/2 \rfloor}\right)/2$, if n is odd.

Moreover, when G is a perfect matching (respectively, almost perfect matching), the bound (2) (respectively, (3)) is tight.

The proof of Theorem 1.1 is given in Section 2. Sections 3, 4, and 5 are mostly devoted to the determination of the exact value of the independence numbers of the token graphs of certain common families of graphs. Our main results in this direction are the following results.

Theorem 1.2. *If G is the complete bipartite graph $K_{m,n}$, then*

$$\beta(F_2(G)) = \max\{mn, \binom{m+n}{2} - mn\}.$$

In Section 3 we present some results which will be used in the proof of Theorem 1.2 and also help to determine some of the exact values of $\beta(F_k(G))$ for $G \in \{P_m, C_{2m}, K_{1,m}, K_{m,m}, K_{m,m+1}\}$ and $2 \le k \le |G|-2$. The proof of Theorem 1.2 is given in Section 4.

Theorem 1.3. *If p is a nonnegative integer and C_p is the cycle of length p, then $\beta(F_2(C_p)) = \lfloor p\lfloor p/2\rfloor/2\rfloor$.*

This formula for $\beta(F_2(C_p))_{p\ge 3}$ produces the sequence A189889 in OEIS [22], which counts the maximum number of non-attacking kings on an $p \times p$ toroidal board (see, e.g., [25, Theorem 11.1, p. 194]). In Section 5 we show Theorem 1.3.

2 Proof of Theorem 1.1

First we state a couple of lemmas.

Lemma 2.1. *Let G be a graph of order $n \ge 6$ and let k be an integer with $3 \le k \le n-3$. Let $e = [v,w]$ be an edge of G and let $H := G - \{v,w\}$. If N and L are matchings (possibly empty) of $F_k(H)$ and $F_{k-2}(H)$, respectively, then $F_k(G)$ has a matching of order $|N| + |L| + 2\binom{n-2}{k-1}$.*

Proof. Let $G, H, N, L, n, k, e, v, w$ be as in the statement of the lemma. Since N is also a matching of $F_k(G)$, it is enough to exhibit a matching N', disjoint from N, of order $|L| + 2\binom{n-2}{k-1}$. We construct N' as follows: the edge $[\{v\}\cup A, \{w\}\cup A]$, will belong to N', whenever $|A| = k-1$ and $\{v,w\}\cap A = \emptyset$. For every $[B, B'] \in L$, we add to N' the edge $[\{v,w\}\cup B, \{v,w\}\cup B']$. Then $N \cup N'$ is the required matching of $F_k(G)$. □

Lemma 2.2. *Let G be a perfect matching or an almost perfect matching of order $n \ge 3$. Then*

$$2\nu(F_2(G)) = \binom{n}{2} - \lfloor n/2\rfloor.$$

Proof. The case $n = 3$ is easy to check, so we may assume that $n \ge 4$. Let $m := \lfloor n/2\rfloor$ and $s := n - 2m$. Suppose that $\{A, B\}$ is a bipartition of G with $A = \{a_1, \dots, a_m\}$, $B = \{b_1, \dots, b_m, b_{m+s}\}$ and that $E(G) = \{[a_1, b_1], \dots, [a_m, b_m]\}$. Since $\{a_i, b_i\} \in V(F_2(G))$ is an isolated vertex for every $i \in \{1, \dots, m\}$, then

$$2\nu(F_2(G)) \le \binom{n}{2} - m.$$

Now, consider the following sets of edges in $F_2(G)$.

$$\begin{aligned} M_1 &= \{[\{a_i, a_j\}, \{a_j, b_i\}] \colon 1 \le i < j \le m\}, \\ M_2 &= \{[\{b_i, b_j\}, \{a_i, b_j\}] \colon 1 \le i < j \le m+s\}. \end{aligned}$$

It is easy to check that $M_1 \cup M_2$ is a set of $\binom{m}{2} + \binom{m+s}{2}$ independent edges. As $s \in \{0,1\}$ then $2\binom{m}{2} + 2\binom{m+s}{2} = \binom{2m+s}{2} - m$. Therefore $2\nu(F_2(G)) \ge \binom{2m+s}{2} - m$. □

Proof of Theorem 1.1. We proceed by induction on n and analyze the three cases separately.
Proof of (1). We claim that the assertion follows easily for $n = 2$ and $n = 4$. Indeed, (i) if $n = 2$, then the only possible value for k is 1, and in such a case $F_k(G) \simeq G$. Similarly, (ii) if $n = 4$, then $k = 1$ or $k = 3$ and in all these cases $F_k(G) \simeq G$.

For the inductive step, we suppose that there is an even $n \geq 6$ such that the assertion holds for all even m such that $0 < m < n$. In other words, we will assume that the assertion is true for any $F_k(G)$ whenever $m = |G| < n$, and k be an odd integer such that $1 \leq k \leq m - 1$.

Let G be a graph of order n containing a perfect matching M. Let $e = [v, w]$ be an edge of M and let $H := G - \{v, w\}$. Then $M - e$ is a perfect matching of H, and by induction, $F_{k-2}(H)$ and $F_k(H)$ have perfect matchings, say L and N, respectively. Then, by Lemma 2.1, $F_k(G)$ has a matching of order $\binom{n-2}{k-2} + \binom{n-2}{k} + 2\binom{n-2}{k-1} = \binom{n}{k}$.
Proof of (2). Assume that $n = 4$ and that M is a perfect matching of G. Then the only possible value for k in $F_k(G)$ is 2. By applying Lemma 2.2 to $F_2(M)$ and considering that $F_2(M)$ is a subgraph of $F_2(G)$, we have the base case for the induction:

$$2\nu(F_2(G)) \geq 2\nu(F_2(M)) \geq \binom{n}{2} - \binom{2}{1}.$$

For the inductive step, we suppose that there is an even $n \geq 6$ such that the assertion is true for any $F_k(H)$ whenever $m = |H| < n$, and k be an even integer such that $2 \leq k \leq m - 2$. Since the assertion holds for $k = 2$ by Lemma 2.2 and $F_2(G) \simeq F_{n-2}(G)$, we may assume that $4 \leq k \leq n - 4$.

Let $e = [v, w]$ be an edge of M and let $G' := G - \{v, w\}$. Then $M - e$ is a perfect matching of G', and by induction, $F_k(G')$ and $F_{k-2}(G')$ contain matchings N and L, respectively, such that $|N| \geq \binom{n-2}{k} - \binom{n/2-1}{k/2}$ and $|L| \geq \binom{n-2}{k-2} - \binom{n/2-1}{k/2-1}$.

Then, by Lemma 2.1, $F_k(G)$ has a matching of order

$$\binom{n-2}{k} - \binom{n/2-1}{k/2} + \binom{n-2}{k-2} - \binom{n/2-1}{k/2-1} + 2\binom{n-2}{k-1} = \binom{n}{k} - \binom{n/2}{k/2}.$$

Finally, suppose that G is a perfect matching (i.e., G is a set of $n/2$ independent edges). Note that if M is a set of $k/2$ edges of G, then $V(M)$ is an isolated vertex of $F_k(G)$. Then $F_k(G)$ has exactly $\binom{n/2}{k/2}$ isolated vertices, and hence $\nu(F_k(G)) \leq \left(\binom{n}{k} - \binom{n/2}{k/2}\right)/2$.
Proof of (3). We take $n = 3$ and $n = 5$ as the base case. If $n = 3$, then the only possible values for k are 1 and 2, and in such cases $F_k(G) \simeq G$. For $n = 5$, we have that $1, 2, 3$ and 4 are the admissible values of k. Since $F_k(G) \simeq G$ for $k = 1, 4$; and $F_2(G) \simeq F_3(G)$, then, in order to establish the base case of the induction, we need only check the assertion for $F_2(G)$.

Let M be an almost perfect matching of G. By applying Lemma 2.2 to $F_2(M)$ and considering that $F_2(M)$ is a subgraph of $F_2(G)$, we have the base case:

$$2\nu(F_2(G)) \geq 2\nu(F_2(M)) = \binom{3}{2} - 1 = 2.$$

For the inductive step, we suppose that there is an odd $n \geq 7$ such that the assertion holds for all odd m such that $2 < m < n$. In other words, we will assume that the assertion is true for any $F_k(G)$ whenever $m = |G| < n$, and k be an integer such that $1 \leq k \leq m - 1$.

Let G be a graph of order n containing an almost perfect matching M. Since $F_1(G) \simeq F_{n-1}(G) \simeq G$, the assertion is trivial for $k = 1$ and $k = n - 1$. So we assume that $3 \leq k \leq n - 3$. Let $e = [v, w]$ be an edge of M and let $H := G - \{v, w\}$. Then $M - e$ is an almost perfect matching of H, and by induction, $F_k(H)$ and $F_{k-2}(H)$ contain matchings N and L, respectively, such that $|N| \geq \binom{n-2}{k} - \binom{(n-3)/2}{\lfloor k/2 \rfloor}$ and $|L| \geq \binom{n-2}{k-2} - \binom{(n-3)/2}{\lfloor (k-2)/2 \rfloor}$. Then, by Lemma 2.1, $F_k(G)$ has a matching of order

$$\binom{n-2}{k} - \binom{(n-3)/2}{\lfloor k/2 \rfloor} + \binom{n-2}{k-2} - \binom{(n-3)/2}{\lfloor (k-2)/2 \rfloor} + 2\binom{n-2}{k-1} = \binom{n}{k} - \binom{(n-1)/2}{\lfloor k/2 \rfloor}.$$

Now suppose that G is an almost perfect matching (i.e., G is a set of $(n-1)/2$ independent edges plus an isolated vertex v). Let M be a set of $\lfloor k/2 \rfloor$ edges of G. Note that if k is even (respectively odd), then $V(M)$ (respectively $V(M) \cup \{v\}$) is an isolated vertex of $F_k(G)$. Then $F_k(G)$ has exactly $\binom{(n-1)/2}{\lfloor k/2 \rfloor}$ isolated vertices, and hence

$$2\nu(F_2(G)) \leq \binom{n}{k} - \binom{(n-1)/2}{\lfloor k/2 \rfloor}.$$

□

The converse of Theorem 1.1 (1) is false in general. For example, $F_3(K_{1,5})$ has a perfect matching (the set of red edges in Figure 1) but $K_{1,5}$ does not have it. On the other hand, it is easy to find counterexamples to Theorem 1.1 (1) when we relax some of its hypothesis. For example, (a) if we assume the existence of an almost perfect matching instead of a perfect matching, then the path $G = P_5$ and $k = 3$ provide a counterexample (Figure 2). Similarly, (b) if n is not odd, then the graph M consisting of two independent edges, and $k = 2$ do.

Next corollary states that $2\nu(F_k(G)) \to |V(F_k(G))|$ when $|G| \to \infty$.

Corollary 2.3. *Let G be a graph of order n. If $\nu(G) = \lfloor n/2 \rfloor$, then*

(1) $\nu(F_k(G)) \geq \frac{\binom{n}{k} - \binom{n/2}{k/2}}{2\binom{n}{k}} \geq \left(1 - \left(\frac{k}{n}\right)^{k/2}\right)/2$, *for n even.*

(2) $\nu(F_k(G)) \geq \frac{\binom{n}{k} - \binom{(n-1)/2}{\lfloor k/2 \rfloor}}{2\binom{n}{k}} \geq \left(1 - \left(\frac{k}{n}\right)^{\lfloor k/2 \rfloor}\right)/2$, *for n odd.*

3 Estimation of $\beta(F_k(G))$ for G bipartite

This section is devoted to the study of the independence number of the k-token graphs of some common bipartite graphs. As we will see, most of the results stated in this section will be used, directly or indirectly, in the proof of Theorem 1.2.

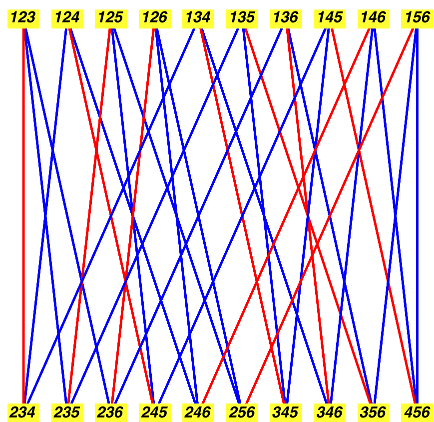

Figure 1: A perfect matching (the red edges) in the 3-token graph of $K_{1,5}$.

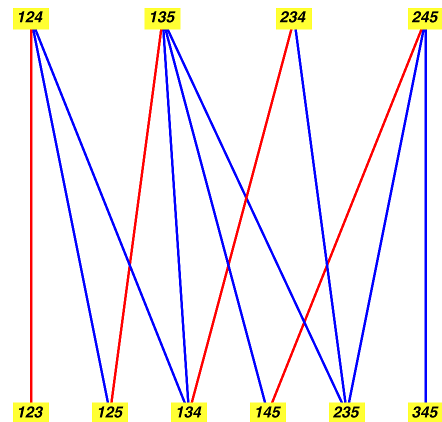

Figure 2: The 3-token graph of P_5.

3.1 Notation and auxiliary results

Let G be a bipartite graph with bipartition $\{B, R\}$. Let $m := |B| \geq 1, n := |R| \geq 1$, and let k be an integer such that $1 \leq k \leq m + n - 1$. Let

$$\mathcal{R} := \{A \subset V(G) \colon |A| = k, |R \cap A| \text{ is odd}\},$$

and let

$$\mathcal{B} := \{A \subset V(G) \colon |A| = k, |R \cap A| \text{ is even}\}.$$

From Proposition 12 in [13] we know that $F_k(G)$ is a bipartite graph. It is not difficult to check that $\{\mathcal{R}, \mathcal{B}\}$ is a bipartition of $F_k(G)$. Without loss of generality we can assume that $m \leq n$.

Remark 3.1. *Unless otherwise stated, from now on we will assume that $G, B, R, \mathcal{B}, \mathcal{R}, m, n$ and k are as above.*

Recall that a *matching of B into R* is a matching M in G such that every vertex in B is incident with an edge in M [3]. Now we recall the classical Hall's Theorem.

Theorem 3.2. *The bipartite graph G has a matching of B into R if and only if $|N(S)| \geq |S|$ for every $S \subseteq B$.*

Lemma 3.3. *If there exists a matching of B into R, then $\beta(G) = |R|$.*

Proof. Since G contains a matching M of B into R, it follows that $|R| \geq |B|$. Then $\beta(G) \geq |R|$, because R is an independent set of G.

Now we show that $\beta(G) \leq |R|$. Let X be any independent set of G. If $X \subseteq B$ or $X \subseteq R$ we are done. So we may assume that $B' := X \cap B \neq \emptyset$ and that $R' := X \cap R \neq \emptyset$. Let M' be the set of edges in M that have one endvertex in B', and let R'' be the set of endvertices of M' in R. Thus $V(M') = B' \cup R''$, and $|B'| = |R''|$. Since X is an independent set, then $R' \cap R'' = \emptyset$, and hence $R' \cup R''$ is also an independent set of G with $|R' \cup R''| = |X|$. $\square$

Proposition 3.4. *Let $G, B, R, \mathcal{B}, \mathcal{R}, m, n$ and k as in Remark 3.1. Then*

$$\beta\big(F_k(G)\big) \geq \max\left\{r, \binom{n+m}{k} - r\right\},$$

where

$$r = \sum_{i=1}^{\lceil k/2 \rceil} \binom{n}{2i-1}\binom{m}{k-2i+1}.$$

Proof. For $i = 1, \ldots, \lceil k/2 \rceil$, let $\mathcal{R}_i$ be the subset of $\mathcal{R}$ defined by

$$\mathcal{R}_i := \{A \subset V(G) \colon |A| = k, |R \cap A| = 2i-1\}.$$

Since $|\mathcal{R}_i| = \binom{n}{2i-1}\binom{m}{k-2i+1}$, the desired result it follows by observing that $|\mathcal{R}| = r$ and $|\mathcal{B}| = \binom{n+m}{k} - r$. $\square$

The bound for $\beta(F_k(G))$ given in Proposition 3.4 is not always attained: for instance, it is not difficult to see that the graph G in Figure 3 has $\beta(F_2(G)) = 12$ and $\max\{|\mathcal{R}|, |\mathcal{B}|\} = 11$. Note that $F_2(G)$, shown in Figure 4, does not satisfy Hall's condition for $\mathcal{A} = \{13, 14, 15, 16, 17, 23\}$, i.e., $|N(\mathcal{A})| < |\mathcal{A}|$.

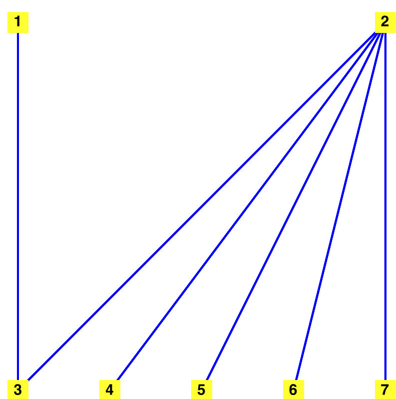

Figure 3: A bipartite graph G with bipartition $\{B, R\}$, and $|B| = 2, |R| = 5$.

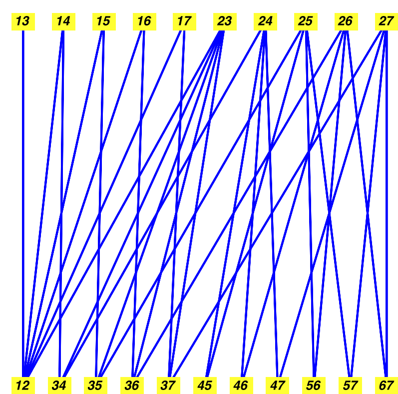

Figure 4: This is $F_2(G)$ for the graph G on the left. Note that $\{13, 14, 15, 16, 17, 23, 45, 46, 47, 56, 57, 67\}$ is an independent set.

Proposition 3.5. *If $k = 2$, then $|\mathcal{B}| \geq |\mathcal{R}|$ if and only if $n - m \geq \frac{1+\sqrt{1+8m}}{2}$.*

Proof. From the hypothesis $k = 2$ and Proposition 3.4 it follows that $F_2(G)$ has bipartition $\{\mathcal{R}, \mathcal{B}\}$ with $|\mathcal{R}| = (m+s)m$ and $|\mathcal{B}| = \binom{2m+s}{2} - (m+s)m$, where $s := n - m$. Thus $|\mathcal{B}| - |\mathcal{R}| = \binom{s}{2} - m$. This equality implies that $|\mathcal{B}| \geq |\mathcal{R}|$ if and only if $s^2 - s - 2m \geq 0$. The result it follows by solving the last inequality for s, and considering that $s \geq 0$. $\square$

3.2 Exact values for $\beta(F_k(G))$ for some G bipartite

Our aim in this subsection is to determine the exact independence number of the k-token graphs of some common bipartite graphs.

Next result is a consequence of our Theorem 1.1 (1) and Corollary 1.3 in [24] which in turn is a consequence of a Theorem of König and a Theorem of Gallai.

Theorem 3.6. *If G has a perfect matching and k is odd, then $\beta(F_k(G)) = \binom{m+n}{k}/2$.*

Proof. From Theorem 1.1 (1) it follows that $F_k(G)$ has a perfect matching. This and the fact that $F_k(G)$ is a bipartite graph implies that $\beta(F_k(G)) = \binom{m+n}{k}/2$. □

Corollary 3.7. *For $G \in \{P_{2n}, C_{2n}, K_{n,n}\}$ and k odd, $\beta(F_k(G)) = \binom{2n}{k}/2$.*

We noted that for $0 \le m < n$, $T(n,m) := \binom{2n}{2m+1}/2$ is a formula for the sequence A091044 in the "On-line Encyclopedia of Integer Sequences" (OEIS) [22], and so Corollary 3.7 provides a new interpretation for such a sequence.

As we will see, most of the results in the rest of the section exhibit families of graphs for which the bound for $\beta(F_k(G))$ given in Proposition 3.4 is attained.

Proposition 3.8. *Let $G = K_{m,n}$, with $m = 1$ and $n \ge 1$ (i.e., G is the star of order $n+1$). Then*

$$\beta(F_k(K_{1,n})) = \begin{cases} \binom{n}{k} & k \le (n+1)/2, \\ \binom{n}{k-1} & k > (n+1)/2. \end{cases}$$

Proof. Let $V(G) = \{v_1, \dots, v_{n+1}\}$ and let v_1 be the central vertex of G. Since $\beta(G) = n$, the assertion holds for $k \in \{1, n\}$. So we assume that $2 \le k \le n-1$. In this proof we take $R = \{v_1\}$ and $B = \{v_2, \dots, v_{n+1}\}$. Thus, the bipartition $\{\mathcal{R}, \mathcal{B}\}$ of $F_k(G)$ is given by $\mathcal{R} = \{A \in V(F_k(G)) : v_1 \in A\}$ and $\mathcal{B} = V(F_k(G)) \setminus \mathcal{R}$. Thus $|\mathcal{R}| = \binom{n}{k-1}$ and $|\mathcal{B}| = \binom{n}{k}$. Note that $F_k(G)$ is biregular: $d(A) = n+1-k$ for every $A \in \mathcal{R}$ and $d(B) = k$ for every $B \in \mathcal{B}$.

Suppose that $k \le (n+1)/2$. Then $|\mathcal{B}| \ge |\mathcal{R}|$. Now we show that $|N(\mathcal{A})| \ge |\mathcal{A}|$ for any $\mathcal{A} \subseteq \mathcal{R}$. Since $N(\mathcal{A}) = \cup_{A \in \mathcal{A}} N(A) \subseteq \mathcal{B}$ and every vertex of $\mathcal{B}$ has degree k, then every vertex in $N(\mathcal{A})$ appears at most k times in the disjoint union $\biguplus_{A \in \mathcal{A}} N(A)$. Therefore $|N(\mathcal{A})| \ge (n+1-k)|\mathcal{A}|/k \ge |\mathcal{A}|$, because $n+1-k \ge k$. From Hall's Theorem and Lemma 3.3 we have $\beta(F_k(G)) = |\mathcal{B}| = \binom{n}{k}$, as desired.

The case $k > (n+1)/2$ can be verified by a totally analogous argument. □

The number $\beta(F_2(K_{1,n-1}))$ is equal to $A000217(n-2)$, for $n \ge 4$, where $A000217$ is the sequence of triangular numbers [22].

Proposition 3.9. *Let $G, B, R, \mathcal{B}, \mathcal{R}$ and k be as in Remark 3.1. If $\beta(F_k(G))$ is equal to $\max\{|\mathcal{R}|, |\mathcal{B}|\}$ and G' is a bipartite supergraph of G with bipartition $\{R, B\}$, then $\beta(F_k(G')) = \max\{|\mathcal{R}|, |\mathcal{B}|\}$.*

Proof. The equality $V(G) = V(G')$ implies $V(F_k(G)) = V(F_k(G'))$ and $E(F_k(G)) \subseteq E(F_k(G')$. From Proposition 3.4 it follows $\beta(F_k(G')) \geq \max\{|\mathcal{R}|, |\mathcal{B}|\}$. On the other hand, since every independent set of $F_k(G')$ is an independent set of $F_k(G)$, we have $\beta(F_k(G')) \leq \beta(F_k(G)) = \max\{|\mathcal{R}|, |\mathcal{B}|\}$. ◻

Theorem 3.10. *If G' is a bipartite supergraph of G with bipartition $\{R, B\}$, and G has either a perfect matching or an almost perfect matching, then $\beta(F_k(G')) = \max\{|\mathcal{R}|, |\mathcal{B}|\}$.*

Proof. In view of Proposition 3.9, it is enough to show that if G is either a perfect matching or an almost perfect matching, then $\beta(F_k(G)) = \max\{|\mathcal{R}|, |\mathcal{B}|\}$.

Suppose that G is a perfect matching. Then $n = m$ and $|G| = 2m$. If k is odd, then, by Theorem 1.1 (1), $F_k(G)$ has a perfect matching. This fact together with Lemma 3.3 imply $\beta(F_k(G)) = \max\{|\mathcal{R}|, |\mathcal{B}|\}$. For k even, Theorem 1.1 (2) implies: (i) that the set $\mathcal{S}$ of isolated vertices of $F_k(G)$ has exactly $\binom{m}{k/2}$ elements (see the last paragraph of the proof of Theorem 1.1 (2)), and (ii) the existence of a matching M of $F_k(G)$ such that $V(M) = V(F_k(G)) \setminus \mathcal{S}$. Now, from the definition of $\mathcal{R}$ it follows that if $k/2$ is odd, then $\mathcal{S} \subseteq \mathcal{R}$, and if $k/2$ is even, then $\mathcal{S} \subseteq \mathcal{B}$. Therefore, we have that either M is a matching of $\mathcal{R}$ into $\mathcal{B}$ or M is a matching of $\mathcal{B}$ into $\mathcal{R}$. In any case, Lemma 3.3 implies $\beta(F_k(G)) = \max\{\mathcal{R}, \mathcal{B}\}$.

Now suppose that G is an almost perfect matching. Then $|E(G)| = |B| = m = n - 1$ and G has exactly an isolated vertex in R, say u. From Theorem 1.1 (3) it follows: (i) that the set $\mathcal{S}$ of isolated vertices of $F_k(G)$ has exactly $\binom{m}{\lfloor k/2 \rfloor}$ elements (see the last paragraph of the proof of Theorem 1.1 (3)), and (ii) the existence of a matching M of $F_k(G)$ such that $V(M) = V(F_k(G)) \setminus \mathcal{S}$. Again, it is easy to see that either $\mathcal{S} \subseteq \mathcal{R}$ or $\mathcal{S} \subseteq \mathcal{B}$. Then either M is a matching of $\mathcal{R}$ into $\mathcal{B}$ or M is a matching of $\mathcal{B}$ into $\mathcal{R}$. In any case, Lemma 3.3 implies $\beta(F_k(G)) = \max\{|\mathcal{R}|, |\mathcal{B}|\}$. ◻

Our next result is an immediate consequence of Proposition 3.4 and Theorem 3.10.

Corollary 3.11. *Let t be a positive integer. If $G \in \{P_t, K_{t,t}, K_{t,t+1}\}$ and k is an integer such that $1 \leq k \leq |G| - 1$, then*

$$\beta(F_k(G)) = \max\{r, \binom{p}{k} - r\},$$

where $p := |G|$ and $r := \sum_{i=1}^{\lceil k/2 \rceil} \binom{\lceil p/2 \rceil}{2i-1} \binom{\lfloor p/2 \rfloor}{k-2i+1}$.

It is a routine exercise to check that the sequence $\{\beta(F_2(P_t))\}_{t \geq 0}$ coincides with A002620 in the OEIS [22].

The following conjecture has been motivated by the results of Corollary 3.11 for $\beta(K_{t,t})$ and $\beta(K_{t,t+1})$, and experimental results. Our aim in the next section is to show Conjecture 3.12 for $k = 2$.

Conjecture 3.12. *If G is a complete bipartite graph with partition $\{R, B\}$, then $\beta(F_k(G)) = \max\{|\mathcal{R}|, |\mathcal{B}|\}$.*

4 Proof of Theorem 1.2

First of all we need some additional notation and a couple of technical lemmas.

For t a nonnegative integer, we use $[t]$ to denote the set $\{1,\ldots,t\}$, and for X a finite set, we use C_2^X to denote the set of all 2-sets of X. If $s_0 := \frac{1+\sqrt{1+8m}}{2}$ we have that $2 \le s < s_0$ if and only if $m > \binom{s}{2}$ and hence we can define an injective function from $C_2^{[s]}$ to $[m]$.

Lemma 4.1. *Let G be a bipartite graph with bipartition $\{R,B\}$, where $B=\{b_1,\ldots,b_m\}$, $R=\{r_1,\ldots,r_m,\ldots,r_{m+s}\}$, and with edges as follows: (i) $b_i \sim r_i$ for every $i\in[m]$, (ii) if $2\le s<s_0$, $b_{\phi(\{i,j\})}\sim r_{m+i}$ and $b_{\phi(\{i,j\})}\sim r_{m+j}$, for any $\{i,j\}\in C_2^{[s]}$, where ϕ is a fixed injective function from $C_2^{[s]}$ to $[m]$. Then $\beta(F_2(G))=|\mathcal{R}|$.*

Proof. If $m=1$, then $s_0=2$ and either $s=0$ or $s=1$. If $s=0$, we are done, since $|\mathcal{R}|=1$ and $F_2(G)\simeq K_1$. Similarly, if $s=1$, then $B=\{b_1\}$, $R=\{r_1,r_2\}$ and $E(G)=\{[b_1,r_1]\}$. Note that in this case $F_2(G)\simeq G$, $\beta(G)=2$ and $|\mathcal{R}|=2$, as desired. So we may assume that $m\ge 2$, and hence that $s_0>2$.

As $s<s_0$ and Proposition 3.5 we have $|\mathcal{R}|>|\mathcal{B}|$. Thus, by Lemma 3.3 and Hall's Theorem, it is enough to show that $|N(X)|\ge|X|$ for all $X\subseteq\mathcal{B}$.

Since $k=2$, $\mathcal{R}$ and $\mathcal{B}$ are given by

$$\begin{aligned}\mathcal{R} &= \{\{b,r\}\colon b\in B, r\in R\}\,,\\ \mathcal{B} &= C_2^B\cup C_2^R.\end{aligned}$$

Note that the pair $\mathcal{B}_2,\mathcal{B}_3$ with

$$\begin{aligned}\mathcal{B}_2 &:= \{\{r_i,r_j\}\colon 1\le i\le m, i<j\le m+s\},\\ \mathcal{B}_3 &:= \{\{r_{m+i},r_{m+j}\}\colon \{i,j\}\in C_2^{[s]}\}\end{aligned}$$

form a partition of C_2^R. Then $\{\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3\}$, with $\mathcal{B}_1:=C_2^B$, is a partition of $\mathcal{B}$. For $X\subseteq\mathcal{B}$, let $X_q:=X\cap\mathcal{B}_q$ for $q=1,2,3$, and let

$$\begin{aligned}X_1' &:= \{\{b_i,r_j\}\colon \{b_i,b_j\}\in X_1, 1\le j<i\le m\}\,,\\ X_2' &:= \{\{b_i,r_j\}\colon \{r_i,r_j\}\in X_2, 1\le i\le m, \text{ and } i<j\le m+s\}\,, \textit{and}\\ X_3' &:= \left\{\{b_{\phi(\{i,j\})},r_{m+j}\}\colon \{r_{m+i},r_{m+j}\}\in X_3, 1\le i<j\le s\right\}.\end{aligned}$$

Note that $X_1'\cap X_2'=\emptyset$ and $X_1'\cap X_3'=\emptyset$. From (i) and (ii) it follows that $X_q'\subseteq N(X_q)$ for every q, and hence $X_1'\cup X_2'\cup X_3'\subseteq N(X)$. Also note that $|X_q'|=|X_q|$ for every q (for $q=3$ take into account that ϕ is injective). If $s\in\{0,1\}$, then $X_3=\emptyset$ and $|N(X)|\ge|X_1'\cup X_2'|\ge|X|$. Similarly, if $s\ge 2$ and $X_2'\cap X_3'=\emptyset$, then $|N(X)|\ge|X_1'\cup X_2'\cup X_3'|\ge|X|$. Finally, suppose that $s\ge 2$ and $X_2'\cap X_3'\ne\emptyset$. Let

$$X_{2,3} := \left\{\{b_{\phi(\{i,j\})},r_{\phi(\{i,j\})}\}\colon \{b_{\phi(\{i,j\})},r_{m+j}\}\in X_2'\cap X_3'\right\}.$$

First we show that $X_{2,3}\subseteq N(X_2)$. Let $\{b_{\phi(\{i,j\})},r_{\phi(\{i,j\})}\}\in X_{2,3}$, then $\{b_{\phi(\{i,j\})},r_{m+j}\}\in X_2'$ and $\{r_{\phi(\{i,j\})},r_{m+j}\}\in X_2$ by definition of X_2'. The result follows because

$$\left[\{b_{\phi(\{i,j\})},r_{\phi(\{i,j\})}\},\{r_{\phi(\{i,j\})},r_{m+j}\}\right]\in E\left(F_2(G)\right).$$

It is clear that $X_{2,3} \cap X'_q = \emptyset$ for every q. Since ϕ is injective, the equality $\{b_{\phi(\{i,j\})}, r_{\phi(\{i,j\})}\} = \{b_{\phi(\{l,m\})}, r_{\phi(\{l,m\})}\}$, with $i < j$ and $l < m$, implies that $i = l$ and $j = m$, and hence $|X_{2,3}| = |X'_2 \cap X'_3|$. Therefore, by the inclusion-exclusion principle we have

$$|N(X)| \geq |X'_1 \cup X'_2 \cup X'_3 \cup X_{2,3}| = |X'_1| + |X'_2| + |X'_3| + |X_{2,3}| - |X'_2 \cap X'_3| = |X|.$$

□

If $s \geq s_0$ then $m \leq \binom{s}{2}$ and hence we can define an injective function from $[m]$ to $D := \{(i,j) \colon 1 \leq i < j \leq s\}$.

Lemma 4.2. *Let G be a bipartite graph with bipartition $\{R, B\}$, where $B = \{b_1, \ldots, b_m\}$, $R = \{r_1, \ldots, r_m, \ldots, r_{m+s}\}$, $s \geq s_0$, and with edges as follows: (i) $b_i \sim r_i$, for every $i \in [m]$, and (ii) $b_i \sim r_{m+i_1}$ and $b_i \sim r_{m+i_2}$, for any $i \in [m]$, where $(i_1, i_2) = \phi(i)$, with ϕ a fixed injective function from $[m]$ to D. Then $\beta(F_2(G)) = |\mathcal{B}|$.*

Proof. We proceed similarly as in Lemma 4.1. As $s \geq s_0$ and by Proposition 3.5 we have $|\mathcal{B}| \geq |\mathcal{R}|$. Thus, by Lemma 3.3 and Hall's Theorem, it is enough to show that $|N(X)| \geq |X|$ for all $X \subseteq \mathcal{R}$. Note that $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_3$ with

$$\begin{aligned}
\mathcal{R}_1 &:= \{\{b_i, r_j\} \colon 1 \leq j < i \leq m\}, \\
\mathcal{R}_2 &:= \{\{b_i, r_j\} \colon 1 \leq i \leq m, i < j \leq m+s\} \text{ and} \\
\mathcal{R}_3 &:= \{\{b_i, r_i\} \colon 1 \leq i \leq m\}
\end{aligned}$$

form a partition of $\mathcal{R}$. For $X \subseteq \mathcal{R}$, let $X_q := X \cap \mathcal{R}_q$ for $q \in \{1, 2, 3\}$, and let

$$\begin{aligned}
X'_1 &:= \{\{b_i, b_j\} \colon \{b_i, r_j\} \in X_1\}, \\
X'_2 &:= \{\{r_i, r_j\} \colon \{b_i, r_j\} \in X_2, 1 \leq i < j \leq m\}, \\
X''_2 &:= \{\{r_{m+i_1}, r_{m+i_2}\} \colon \{b_i, r_{m+i_1}\} \in X_2, 1 \leq i_1 < i_2 \leq s, \phi(i) = (i_1, i_2)\}, \\
X'_3 &:= \{\{r_i, r_{m+i_1}\} \colon \{b_i, r_i\} \in X_3, \phi(i) = (i_1, i_2)\}.
\end{aligned}$$

From (i) and (ii) it follows that $X'_1 \cup X'_2 \cup X''_2 \cup X'_3 \subseteq N(X) \subseteq \mathcal{B}$. Note that X'_1, X'_2, X''_2 and X'_3 are pairwise disjoint. Since $|X'_q| = |X_q|$ for $q \in \{1, 3\}$ and $|X_2| = |X'_2 \cup X''_2|$ (because ϕ is injective), then, by the inclusion-exclusion principle:

$$N(X) \geq |X'_1| + |X'_2 \cup X''_2| + |X'_3| = |X_1| + |X_2| + |X_3| = |X|.$$

□

Proof of Theorem 1.2. From Proposition 3.9 we know that $\beta(F_2(G')) = \max\{|\mathcal{R}|, |\mathcal{B}|\}$ for any bipartite supergraph G' of graph G in Lemma 4.1 or in Lemma 4.2 with bipartition $\{R, B\}$. □

5 Proof of Theorem 1.3

Again, we first need to give some preliminar results and notation. We start by stating recursive inequalities for $\beta(F_k(G))$.

Lemma 5.1. *Let G be a graph of order n. For $2 \leq k \leq n-1$, we have*

$$\max_{v\in V(G)} \{\beta\left(F_{k-1}(G-v)\right) + \beta\left(F_k\left(G-N[v]\right)\right)\} \leq \beta\big(F_k(G)\big) \leq \frac{1}{k} \sum_{v\in V(G)} \beta\big(F_{k-1}(G-v)\big). \tag{1}$$

Proof. We begin by proving the right inequality of (1). Let $\mathcal{I}$ be an independent set of vertices in $F_k(G)$ with maximum cardinality. For $v \in V(G)$, let $\mathcal{I}_v$ be the set formed by all the elements of $\mathcal{I}$ containing v. Since every vertex of $\mathcal{I}$ is a k-set of $V(G)$, then $k|\mathcal{I}| = \sum_{v\in V(G)} |\mathcal{I}_v|$. Furthermore, note that the collection $\{A \setminus \{v\}\colon\ A \in \mathcal{I}_v\}$ is an independent set of $F_{k-1}(G-v)$, and so $|\mathcal{I}_v| \leq \beta\big(F_{k-1}(G-v)\big)$ for every $v \in V(G)$. The desired inequality it follows from previous relations and the fact that $\beta\big(F_k(G)\big) = |\mathcal{I}|$.

Now we show the right inequality. For $v \in V(G)$, let $\mathcal{I}_{\neg v}$ (respectively $\mathcal{J}_{\neg v}$) be an independent set in $F_{k-1}(G-v)$ (respectively $F_k(G-N[v])$) with maximum cardinality. Then $|\mathcal{I}_{\neg v}| = \beta(F_{k-1}(G-v))$ and $|\mathcal{J}_{\neg v}| = \beta(F_k(G-N[v]))$. Let $\mathcal{I}_v$ be the collection of sets $\{A \cup \{v\}\colon\ A \in \mathcal{I}_{\neg v}\}$. From the construction of I_v and $J_{\neg v}$ it is easy to see that $\mathcal{I}_v \cap \mathcal{J}_{\neg v} = \emptyset$, and that $\mathcal{I}_v \cup \mathcal{J}_{\neg v}$ form an independent set of $F_k(G)$. Since the last two statements hold for every $v \in V(G)$, the required inequality follows. $\square$

Remark 5.2. *The bounds for $\beta(F_k(G))$ in (1) are best possible: for instance, the left (respectively right) hand side of (1) is reached when $G \simeq K_{1,3}$ and $k = 2$ (respectively $G \simeq K_n$ and $k = 2$).*

Corollary 5.3. *Let G be a vertex-transitive graph of order n and let w be any vertex in G. For $2 \leq k \leq n-2$, we have*

$$\beta\big(F_k(G)\big) \leq \min\left\{\frac{n}{k}\,\beta\big(F_{k-1}(G-w)\big), \frac{n}{n-k}\,\beta\big(F_k(G-w)\big)\right\}.$$

Proof. Since G is vertex-transitive, then $\beta(F_{k-1}(G-w)) = \beta(F_{k-1}(G-u))$ for any $u \in V(G)$. From this and Theorem 5.1 it follows that

$$\beta(F_k(G)) \leq \frac{n}{k}\beta(F_{k-1}(G-w)).$$

In a similar way we can deduce that

$$\beta(F_{n-k}(G)) \leq \frac{n}{n-k}\beta(F_{n-k-1}(G-w)).$$

The desired inequality it follows from the previous inequality and considering that $F_k(G) \simeq F_{n-k}(G)$, and that $F_k(G-w) \simeq F_{(n-1)-k}(G-w)$. $\square$

Applying Lemma 5.1 and Corollary 5.3 to $G \simeq C_n$ and $G \simeq K_n$, we have the following corollary (we remark that equation (3) is in fact a theorem of Johnson [15]):

Corollary 5.4. *For $2 \le k \le n-2$ we have*

$$\beta\big(F_{k-1}(P_{n-1})\big)+\beta\big(F_k(P_{n-3})\big) \le \beta\big(F_k(C_n)\big) \le \min\left\{\frac{n}{k}\,\beta\big(F_{k-1}(P_{n-1})\big), \frac{n}{n-k}\,\beta\big(F_k(P_{n-1})\big)\right\} \tag{2}$$

and

$$\beta\big(J(n-1,k-1)\big) \le \beta\big(J(n,k)\big) \le \min\left\{\frac{n}{k}\,\beta\big(J(n-1,k-1)\big), \frac{n}{n-k}\,\beta\big(J(n-1,k)\big)\right\}. \tag{3}$$

Since the exact value of $\beta(F_2(C_p))$ for p even is given by Corollary 3.7 and the case $p=1$ is trivial, for the rest of this section, we assume that $p \ge 3$ is an odd integer.

Let $V(C_p) := \{1,\dots,p\}$ and $E(C_p) := \{[i,i+1] \colon i = 1,\dots,p-1\} \cup \{[p,1]\}$. If $A, B \subseteq V(C_p)$, we say that A and B are *linked* in C_p if and only if C_p contains an edge $[a,b]$ such that $a \in A$ and $b \in B$. We use $A \approx B$ to denote that A and B are linked in C_p. Recall that for $A, B \in V(F_2(C_p))$, $A \sim B$ if and only if either $A \triangle B = \{t, t+1\}$ for $1 \le t \le p-1$, or $A \triangle B = \{1,p\}$. For $i = 1,\dots,p-1$, let $L_i := \{\{j, p-(i-j)\} : 1 \le j \le i\} \subseteq V(F_2(C_p))$ (see Figure 5). Each assertion in the following observation it follows easily from the definition of L_i.

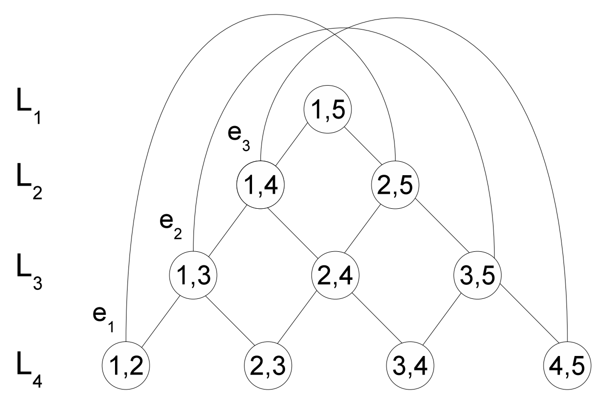

Figure 5: Here is shown $F_2(C_5)$. Note that $F_2(P_5) \simeq F_2(C_5) \setminus \{e_1,e_2,e_3\}$ and that $L_1 = \{\{1,5\}\},\dots,L_4 = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\}\}$.

Observation 5.5. *For p and L_i as above, we have that:*

1. *$L_i \approx L_{i+1}$ for $i \in \{1,\dots,p-2\}$.*

2. *$L_i \approx L_{p-i+1}$ and the underlying edge is of the form $[\{1,i\},\{i,p\}]$, for $i \in \{2,\dots,p-1\}$.*

3. *All the links between members of $\{L_1,\dots,L_{p-1}\}$ are given by (1)-(2).*

Proof of Theorem 1.3. It is easy to see that $\beta(F_2(C_3)) = 1$. Thus we only need to prove the case when $p = 2t+1$ for $t \geq 2$. First, we show that $\beta(F_2(C_p)) \leq \lfloor p\lfloor p/2\rfloor/2\rfloor$.

By Corollary 5.4 and Corollary 3.11, we have that

$$\beta(F_2(C_p)) \leq \min\left\{p/2\lceil (p-1)/2\rceil, p/(p-2)\lfloor (p-1)^2/4\rfloor\right\}.$$

and hence

$$\beta(F_2(C_p)) \leq (p/2)\lceil (p-1)/2\rceil = pt/2.$$

Thus $\beta(F_2(C_p)) \leq \lfloor pt/2\rfloor = \lfloor p\lfloor p/2\rfloor/2\rfloor$, because $t = \lfloor p/2\rfloor$.

Now we show that $\beta(F_2(C_p)) \geq \lfloor p\lfloor p/2\rfloor/2\rfloor$. Note that L_q is an independent set of $F_2(C_p)$ whenever $q \neq t+1$. We split the rest of the proof depending on whether t is odd or even.

Case 2.1. t is odd. From Observation 5.5 it follows that

$$\{L_1, L_3, \ldots, L_t, L_{t+3}, L_{t+5}, \ldots, L_{p-1}\}$$

is a collection of pairwise non-linked independent sets. Then,

$$I = L_1 \cup L_3 \cup \cdots \cup L_t \cup L_{t+3} \cup L_{t+5} \cup \cdots \cup L_{p-1}$$

is an independent set in $F_2(C_p)$. But

$$|I| = (1+3+\cdots+t) + ((t+3)+(t+5)+\cdots+(p-1)) = \frac{1}{2}(tp-1) = \lfloor p\lfloor p/2\rfloor/2\rfloor.$$

Case 2.2. t is even. Similarly, $\{L_1, L_3, \ldots, L_{t-1}, L_{t+2}, L_{t+4}, \ldots, L_{p-1}\}$ is a collection of pairwise non-linked independent sets, and hence

$$I = L_1 \cup L_3 \cup \cdots \cup L_{t-1} \cup L_{t+2} \cup L_{t+4} \cup \cdots \cup L_{p-1}$$

is an independent set in $F_2(C_p)$. In this case we have that

$$|I| = (1+3+\cdots+t-1) + ((t+2)+(t+4)+\cdots+p-1) = \frac{1}{2}tp = \lfloor p\lfloor p/2\rfloor/2\rfloor. \quad \square$$

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