On Two OEIS Conjectures

Jeremy M. Dover

Abstract

In [2], Stephan enumerates a number of conjectures regarding integer sequences contained in Sloane's On-line Encyclopedia of Integer Sequences [1]. In this paper, we prove two of these conjectures.

1 Proof of Conjecture 110

In [2] Stephan formulates the following conjecture:

> Define $a_n = |\{(i,j) : 0 \leq i,j < n \text{ and } i \text{ AND } j > 0\}|$, where AND is the bitwise and operator. Then a_n is the sequence given by the recursions $a_{2n} = 3a_n + n^2$, $a_{2n+1} = a_n + 2a_{n+1} + n^2 - 1$, with initial conditions $a_0 = a_1 = 0$.

Truth for the initial conditions is easy to calculate. In what follows, we define $S_n = \{(i,j) : 0 \leq i,j < n \text{ and } i \text{ AND } j > 0\}$ from which we have $a_n = |S_n|$.

To see that $a_{2n} = 3a_n + n^2$, we partition the set S_{2n} into four parts, EE_{2n}, EO_{2n}, OE_{2n} and OO_{2n} where $EE_{2n} = \{(x,y) \in S_{2n} : x,y \text{ even}\}$ and the other sets are defined analogously. We count the number of elements in each set.

Since the AND of two odd numbers is always at least 1, OO_{2n} consists of all pairs of odd numbers (x,y) with $0 \leq x,y < n$, of which there are n^2 possibilities. To count the number of elements in EE_{2n}, let $(2i,2j)$ be an element of EE_{2n} which forces $0 \leq i,j < n$. We know $2i$ AND $2j$ is nonzero, and since the low order bit of both numbers is zero, we must have i AND j nonzero. Thus for each $(2i,2j) \in EE_{2n}$ we have the corresponding pair $(i,j) \in S_n$. Moreover for any pair $(i,j) \in S_n$ it is easy to see that $(2i,2j) \in S_{2n}$. Therefore $|EE_{2n}| = |S_n| = a_n$.

As with the previous case we can write each element of OE_{2n} as $(2i+1,2j)$ for $0 \leq i,j < n$ where $2i+1$ AND $2j$ is nonzero. Since the low order bit of $2j$ is zero, we must have i AND j nonzero, thus $(i,j) \in S_n$. Then for any $(i,j) \in S_n$ we have $(2i+1,2j) \in S_{2n}$, showing that $|OE_{2n}| = a_n$. Since $|EO_{2n}|$ is obviously equal to $|OE_{2n}|$, we find that $a_{2n} = 3a_n + n^2$ as conjectured.

To show $a_{2n+1} = a_n + 2a_{n+1} + n^2 - 1$ we use the same sort of analysis, but one of the counts is more tricky. Using the same definitions for OO, et. al., the exact same argument as above shows that $|OO_{2n+1}| = n^2$. For EE_{2n+1} we again look at elements $(2i, 2j) \in S_{2n+1}$ and still must have i AND j nonzero, but now $0 \leq i, j < n+1$ since either i and/or j can be n. Thus $|EE_{2n+1}| = a_{n+1}$, as opposed to a_n for the previous case.

Counting $|OE_{2n+1}|$ is more tricky in this case. Again we have that the elements of OE_{2n+1} are the pairs $(2i+1, 2j)$ such that i AND j is nonzero, but in this case $0 \leq i < n$, while $0 \leq j < n+1$. Thus $|OE_{2n+1}|$ is the number of elements of $(i, j) \in S_{n+1}$ for which $i \neq n$. To determine this number we provide a different partition of S_{n+1} as:

$$\begin{aligned} S_{n+1} &= \{(i,j) \in S_{n+1} : i, j < n\} \cup \{(n, j) \in S_{n+1} : j < n\} \cup \\ &\quad \{(i, n) \in S_{n+1} : i < n\} \cup \{(n, n)\} \end{aligned}$$

In this expression, the set $\{(i, j) \in S_{n+1} : i, j < n\}$ is exactly S_n. If we let x be the cardinality of $\{(n, j) \in S_{n+1} : j < n\}$, then by taking cardinalities of each partition element we have

$$a_{n+1} = a_n + 2x + 1$$

Therefore, $x = \frac{1}{2}(a_{n+1} - a_n - 1)$ and we have $|OE_{2n+1}| = a_n + \frac{1}{2}(a_{n+1} - a_n - 1) = \frac{1}{2}(a_{n+1} + a_n - 1)$. Since EO_{2n+1} has the same cardinality we finally have

$$\begin{aligned} a_{2n+1} &= n^2 + a_{n+1} + \frac{1}{2}(a_{n+1} + a_n - 1) + \frac{1}{2}(a_{n+1} + a_n - 1) \\ &= a_n + 2a_{n+1} + n^2 - 1 \end{aligned}$$

which finishes the conjecture.

2 Proof of Conjecture 115

In [2] Stephan formulates the following conjecture:

> Define the sequence a_n by $a_1 = 1$ and $a_n = M_n + m_n$, where $M_n = \max_{1 \leq i < n}(a_i + a_{n-i})$ and $m_n = \min_{1 \leq i < n}(a_i + a_{n-i})$. Let further b_n be the number of binary partitions of $2n$ into powers of 2 (number of *binary partitions*). Then

$$m_n = \frac{3}{2}b_{n-1} - 1,\ M_n = n + \sum_{k=1}^{n-1} m_n,\ a_n = M_{n+1} - 1$$

We prove this conjecture through a series of short induction proofs. For convenience we define $m_1 = 1$.

Proposition 2.1. *The sequence a_n is strictly increasing, and thus positive for all $n \geq 1$. Moreover, each of the sequences M_n and m_n is also strictly increasing and positive for all $n \geq 2$.*

Proof. It suffices to show that $a_n > a_{n-1}$, $M_n > M_{n-1}$, and $m_n > m_{n-1}$ for all $n \geq 3$, and we proceed by induction on n. By definition $a_1 = 1$ and a_2 is easily computed to be 4, proving the base for our induction, as well as that $a_2 > a_1$.

To complete our induction step, we assume the result is true for all a_i with $i < n$, and consider a_n. By definition M_n and m_n are the maximum and minimum, respectively, of the set $S_n = \{a_i + a_{n-i} : 1 \leq i < n\}$. By our induction hypothesis each of the a_i's is positive, implying that M_n and m_n are positive as well. Now $a_n = M_n + m_n > M_n$. Since $a_{n-1} + a_1 = a_{n-1} + 1$ is in the set S, we must have $M_n \geq a_{n-1} + 1$. Thus $a_n > M_n \geq a_{n-1} + 1$, showing that the sequence a_n is increasing.

To show that M_n is increasing, suppose that the maximum value in S_n is given by the element $a_i + a_{n-i}$. Then S_{n+1} contains the element $a_i + a_{n+1-i}$ which is strictly greater than $a_i + a_{n-i}$ since a_i is increasing. Thus the maximum value in S_{n+1} must be larger than the greatest value in S_n, proving M_n is increasing.

A similar argument shows m_n is increasing. Now let $a_i + a_{n+1-i}$ be the smallest element in S_{n+1}, and thus equal to m_{n+1}. Then S_n must contain the element $a_i + a_{n-i}$ which is strictly less than m_{n+1}. Then we have $m_n \leq a_i + a_{n-i} < m_{n+1}$, showing m_n is increasing. □

Proposition 2.2. $M_{n+1} = a_n + 1$ *for all* $n \geq 1$.

Proof. We again proceed by induction on n, noting that the base case $M_2 = a_1 + 1 = 2$ is easily calculated. For our strong induction hypothesis assume $M_{i+1} = a_i + 1$ for all $1 \leq i < n$, and attempt to prove the result $M_{n+1} = a_n + 1$.

By the definition of a_n we have $a_n = M_n + m_n$, from which $a_n + 1 = M_n + m_n + 1$. Using our induction hypothesis we know $M_n = a_{n-1} + 1$

from which we obtain $a_n + 1 = a_{n-1} + m_n + 2$. Iterating this procedure of alternately applying the definition of a_i and the induction hypothesis gives:

$$a_n + 1 = a_i + \sum_{j=i+1}^{n} m_i + (n - i + 1) \tag{1}$$

for all $1 \leq i < n$.

To show $M_{n+1} = a_n + 1$ we must show that $a_n + 1 \geq a_i + a_{n+1-i}$ for all $2 \leq i < n$. By symmetry in the indices of the sequence this is equivalent to showing this result for all $\lceil \frac{n+1}{2} \rceil \leq i < n$.

Since a_i is positive and increasing for all $i \geq 1$, $m_n > a_i$ for all $1 \leq i \leq \lceil \frac{n}{2} \rceil$ as every element in $S_n = \{a_i + a_{n-i} : 1 \leq i < n\}$ has a summand a_j for which $j \geq \lceil \frac{n}{2} \rceil$.

From Equation 1 we know that $a_n + 1 = a_i + \sum_{j=i+1}^{n} m_j + (n-i+1) > a_i + m_n$ for all $1 \leq i < n$. This implies $a_n + 1 > a_i + a_{n+1-i}$ for all $\lfloor \frac{n}{2} \rfloor + 1 \leq i < n$. As $\lfloor \frac{n}{2} \rfloor + 1 = \lceil \frac{n+1}{2} \rceil$ for all positive integers n, this proves the claim. □

Corollary 2.3. $M_n = \sum_{k=1}^{n-1} m_k + n$ *for all* $n \geq 1$.

Proof. This follows immediately from Equation 1 with $i = 1$. □

The conjectured value for m_n is somewhat trickier, and requires the following intermediate result.

Proposition 2.4. $m_n = a_{\lfloor \frac{n}{2} \rfloor} + a_{\lceil \frac{n}{2} \rceil}$ *for all* $n \geq 2$.

Proof. By the definition of m_n, m_n is the smallest element of $S_n = \{a_i + a_{n-i} : 1 \leq i < n\}$. Using the results of Proposition 2.2 and Corollary 2.3, we can calculate:

$$\begin{aligned} a_i + a_{n-i} &= M_{i+1} - 1 + M_{n+1-i} - 1 \\ &= \sum_{k=1}^{i} m_k + (i+1) + \sum_{k=1}^{n-i} m_k + (n + 1 - i) - 2 \\ &= n + \sum_{k=1}^{i} m_k + \sum_{k=1}^{n-i} m_k \end{aligned}$$

Denoting $d_i = a_i + a_{n-i}$ and noting that $d_i = d_{n-i}$, we need to show that $d_{\lfloor \frac{n}{2} \rfloor} \leq d_i$ for all $1 \leq i < \lfloor \frac{n}{2} \rfloor$. To do this, we will show that d_i is a decreasing sequence in the interval $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ by looking at the differences $d_i - d_{i+1}$ for $1 \leq i < \lfloor \frac{n}{2} \rfloor$ and showing they are all positive.

To this end we calculate

$$\begin{aligned} d_i - d_{i+1} &= n + \sum_{k=1}^{i} m_k + \sum_{k=1}^{n-i} m_k - \left[n + \sum_{k=1}^{i+1} m_k + \sum_{k=1}^{n-(i+1)} m_k \right] \\ &= m_{n-i} - m_{i+1} \end{aligned}$$

By Proposition 2.1, m_n is an increasing sequence, and $n - i > i + 1$ for all $1 \leq i < \lfloor \frac{n}{2} \rfloor$, which implies that $d_i - d_{i+1}$ is positive for all $1 \leq i < \lfloor \frac{n}{2} \rfloor$, proving the result. □

Prior to proving the final piece of the conjecture, we need the following result from Sloane [1]. The sequence b_n, denoted A000123 by Sloane, where b_n is the number of partitions of $2n$ into powers of 2 satisfies the recursion $b_n = b_{n-1} + b_{\lfloor \frac{n}{2} \rfloor}$, for all $n \geq 2$ with initial condition $b_1 = 2$.

Proposition 2.5. *$m_n = \frac{3}{2} b_{n-1} - 1$ for all $n \geq 2$.*

Proof. To show that this equation holds, we prove that the sequence $c_n = \frac{2}{3}(m_{n+1} + 1)$ satisfies the recursion and initial conditions for b_n. Rather than use the form $b_n = b_{n-1} + b_{\lfloor \frac{n}{2} \rfloor}$ for the recursion, we will instead use $b_n - b_{n-1} = b_{\lfloor \frac{n}{2} \rfloor}$, being easier for computations.

Using Proposition 2.4, we have

$$c_n - c_{n-1} = \frac{2}{3}\left(a_{\lfloor \frac{n+1}{2} \rfloor} + a_{\lceil \frac{n+1}{2} \rceil} + 1 \right) - \frac{2}{3}\left(a_{\lfloor \frac{n}{2} \rfloor} + a_{\lceil \frac{n}{2} \rceil} + 1 \right)$$

Noting that $\lceil \frac{n}{2} \rceil = \lfloor \frac{n+1}{2} \rfloor$ and $\lceil \frac{n+1}{2} \rceil = \lfloor \frac{n}{2} \rfloor + 1$ for all integers $n \geq 1$, this reduces to

$$c_n - c_{n-1} = \frac{2}{3}\left(a_{\lfloor \frac{n}{2} \rfloor + 1} - a_{\lfloor \frac{n}{2} \rfloor} \right)$$

Now using the formula for a_i given in Proposition 2.2 and Corollary 2.3, we obtain

$$c_n - c_{n-1} = \frac{2}{3}\left(\lfloor \frac{n}{2} \rfloor + 2 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor + 1} m_k - \left[\lfloor \frac{n}{2} \rfloor + 1 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} m_k \right] \right)$$

which reduces to

$$\begin{aligned} c_n - c_{n-1} &= \frac{2}{3}\left(m_{\lfloor \frac{n}{2}+1 \rfloor} + 1\right) \\ &= c_{\lfloor \frac{n}{2} \rfloor} \end{aligned}$$

Therefore $c_n = \frac{2}{3}(m_n + 1)$ satsfies the same recursion as b_n, and $c_1 = \frac{2}{3}(m_2 + 1) = 2 = b_1$, implying that $b_n = \frac{2}{3}(m_n + 1)$ for all $n \geq 2$, from which we obtain the result. □

References

[1] N.J.A. Sloane, editor, The On-Line Encyclopedia of Integer Sequences, published electronically at `https://oeis.org`

[2] Ralf Stephan. Prove or Disprove 100 Conjectures from the OES, arXiv:math/0409509v4 [math.CO]