(Total) Domination in Prisms

Jernej Azarija [a,b] Michael A. Henning [c] Sandi Klavžar [a,b,d]

[a] Faculty of Mathematics and Physics, University of Ljubljana, Slovenia
jernej.azarija@gmail.com
sandi.klavzar@fmf.uni-lj.si

[b] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

[c] Department of Pure and Applied Mathematics
University of Johannesburg, South Africa
mahenning@uj.ac.za

[d] Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia

Abstract

With the aid of hypergraph transversals it is proved that $\gamma_t(Q_{n+1}) = 2\gamma(Q_n)$, where $\gamma_t(G)$ and $\gamma(G)$ denote the total domination number and the domination number of G, respectively, and Q_n is the n-dimensional hypercube. More generally, it is shown that if G is a bipartite graph, then $\gamma_t(G \,\square\, K_2) = 2\gamma(G)$. Further, we show that the bipartite condition is essential by constructing, for any $k \geq 1$, a (non-bipartite) graph G such that $\gamma_t(G \,\square\, K_2) = 2\gamma(G) - k$. Along the way several domination-type identities for hypercubes are also obtained.

Keywords: domination; total domination; hypercube; Cartesian product of graphs; covering codes; hypergraph transversal

AMS Subj. Class.: 05C69, 91C65, 05C76, 94B65

1 Introduction

Domination and total domination in graphs are very well studied in the literature, here we study these concepts in prisms of graphs, in particular in hypercubes. To determine the domination number γ of the n-dimensional hypercube Q_n, is a fundamental problem in coding theory, computer science, and of course in graph theory. In coding theory, the problem equivalent to the determination of $\gamma(Q_n)$ is to find the size of a minimal covering code of length n and covering radius 1. In computer science, different distribution type problems on interconnection networks can be modelled by domination invariants, where hypercubes in turn form a central model for interconnection networks.

To determine $\gamma(Q_n)$ turns out to be an intrinsically difficult problem. To date, exact values are only known for $n \leq 9$. These results are summarized in Table 1.

n	1	2	3	4	5	6	7	8	9	10
$\gamma(Q_n)$	1	2	2	4	7	12	16	32	62	107-120

Table 1: Domination numbers of hypercubes up to dimension 10

We have checked these values by formulating an integer linear program and solving it with CPLEX. The result $\gamma(Q_9) = 62$ due to Östergård and Blass [18] actually presented a breakthrough back in 2001. The value of $\gamma(Q_{10})$ is currently unknown, see [1] for the present best lower bound as given in Table 1 and [14] for the present best upper bound.

Total domination γ_t is, besides classical domination, among the most fundamental concepts in domination theory. It has in particular been extensively investigated on Cartesian product graphs (cf. [3, 10, 16]), which was in a great part motivated by the famous Vizing's conjecture [2]. Specifically, $\gamma_t(Q_n)$ was recently investigated in the thesis [20] under the notion of a *binary covering code of empty spheres of length n and radius* 1. In particular, values $\gamma_t(Q_n)$ for $n \leq 10$ were computed and some bounds established. These exact values intrigued us to wonder whether there exists some general relation between the domination number and the total domination number in hypercubes.

From our perspective it is utmost important that Q_n can be represented as the n^{th} power of K_2 with respect to the Cartesian product operation $\Box$, that is, $Q_1 = K_2$ and $Q_n = Q_{n-1} \Box K_2$ for $n \geq 2$. Our immediate aim in this paper is to prove that $\gamma_t(Q_{n+1}) = 2\gamma(Q_n)$ holds for all $n \geq 1$. For this purpose, we prove the following much more general result that the total domination of a bipartite prism of a graph G is equal to twice the domination number of G.

Theorem 1.1 *If G is a bipartite graph, then*

$$\gamma_t(G \Box K_2) = 2\gamma(G)\,.$$

Since Q_n, $n \geq 1$, is a bipartite graph, as a special case of Theorem 1.1 we note that $\gamma_t(Q_{n+1}) = 2\gamma(Q_n)$. Our second aim is to show that the bipartite condition in the statement of Theorem 1.1 is essential. For this purpose, we prove the following result.

Theorem 1.2 *For each integer $k \geq 1$, there exists a connected graph G_k satisfying*

$$\gamma_t(G_k \Box K_2) - 2\gamma(G_k) = k.$$

We proceed as follows. In the next section concepts used throughout the paper are introduced and known facts and results needed are recalled. In particular, the state of the art on $\gamma(Q_n)$ is surveyed. In Section 3, Theorem 1.1 is proved and several of its consequences listed. A proof of Theorem 1.2 is given in Section 4. We conclude the paper with some open problems. In particular we conjecture that the equality in Theorem 1.1 holds for almost all graphs.

2 Preliminaries

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *order* of G is denoted by $n(G) = |V(G)|$. The *open neighborhood* of a vertex v in G is $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$.

For graphs G and H, the *Cartesian product* $G \,\Box\, H$ is the graph with vertex set $V(G) \times V(H)$ where vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$. If $(u, v) \in V(G \,\Box\, H)$, then the subgraph of $G \,\Box\, H$ induced by the vertices of the form (u, x), $x \in V(H)$, is isomorphic to H; it is called the *H-layer* (through (u, v)). Analogously G-layers are defined. The *prism* of a graph G is the graph $G \,\Box\, K_2$. Note that $G \,\Box\, K_2$ contains precisely two G-layers. Further, if G is a bipartite graph, then we call the prism $G \,\Box\, K_2$ the *bipartite prism* of G. As already mentioned in the introduction, Q_n is a (bipartite) prism because $Q_n = Q_{n-1} \,\Box\, K_2$.

A *dominating set* of a graph G is a set S of vertices of G such that every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S, while a *total dominating set* of G is a set S of vertices of G such that every vertex in $V(G)$ is adjacent to at least one vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G and the *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. We refer to the books [8, 12] for more information on the domination number and the total domination number, respectively.

The values $\gamma(Q_7) = 16$ and $\gamma(Q_8) = 32$ also follow from the following result which gives exact values for two infinite families of hypercubes.

Theorem 2.1 *If $k \geq 1$, then $\gamma(Q_{2^k-1}) = 2^{2^k-k-1}$ and $\gamma(Q_{2^k}) = 2^{2^k-k}$.*

The first assertion of Theorem 2.1 is based on the fact that hypercubes Q_{2^k-1} contain perfect codes, cf. [7]. Since the domination number of a graph with a perfect code is equal to the size of such a code, the assertion follows. Knowing the existence of such codes, by the divisibility condition one immediately infers that Q_n contains a perfect code if and only if $n = 2^k - 1$ for some $k \geq 1$. Lee [15, Theorem 3] further proved that this is equivalent to the fact that Q_n is a regular covering of the complete graph K_{n+1}. The second assertion of Theorem 2.1 is due to van Wee [23]. Related aspects of domination in hypercubes were investigated in [22].

A set S of vertices in G is a *paired-dominating set* if every vertex of G is adjacent to a vertex in S and the subgraph induced by S contains a perfect matching (not necessarily as an induced subgraph). The minimum cardinality of a paired-dominating set of G is the *paired-domination number* of G, denoted $\gamma_{\rm pr}(G)$. A survey on paired-domination in graphs can be found in [4]. By definition every paired-dominating set is a total dominating set, and every total dominating set is a dominating set. Hence we have the following result first observed by Haynes and Slater [9].

Observation 2.2 ([9]) *For every isolate-free graph G, $\gamma(G) \leq \gamma_t(G) \leq \gamma_{\rm pr}(G)$.*

A *total restrained dominating set* of G is a total dominating set S of G with the additional property that every vertex outside S has a neighbor outside S; that is, $G[V(G) \setminus S]$ contains no isolated vertex. The *total restrained domination number* of G, denoted $\gamma_{\rm tr}(G)$, is the minimum cardinality of a total restrained dominating set. The concept of total restrained domination in graphs was introduced by Telle and Proskurowksi [19] as a vertex partitioning problem. By definition every total restrained dominating set if a total dominating set, implying the following observation.

Observation 2.3 ([9]) *For every isolate-free graph G, $\gamma_t(G) \le \gamma_{\rm tr}(G)$.*

The *open neighborhood hypergraph*, abbreviated ONH, of G is the hypergraph H_G with vertex set $V(H_G) = V(G)$ and with edge set $E(H_G) = \{N_G(x) \mid x \in V(G)\}$ consisting of the open neighborhoods of vertices in G. The *closed neighborhood hypergraph*, abbreviated CNH, of G is the hypergraph H_G^c with vertex set $V(H_G^c) = V(G)$ and with edge set $E(H_G) = \{N_G[x] \mid x \in V(G)\}$ consisting of the closed neighborhoods of vertices in G.

A subset T of vertices in a hypergraph H is a *transversal* (also called *vertex cover* or *hitting set*) if T has a nonempty intersection with every edge of H. The *transversal number* $\tau(H)$ of H is the minimum size of a transversal in H. A transversal of size $\tau(H)$ is called a $\tau(H)$-set.

The transversal number of the ONH of a graph is precisely the total domination number of the graph, while the transversal number of the CNH of a graph is precisely the domination number of the graph. We state this formally as follows.

Observation 2.4 *If G is a graph, then $\gamma_t(G) = \tau(H_G)$ and $\gamma(G) = \tau(H_G^c)$.*

We shall also need the following result from [11] (see also [12]).

Theorem 2.5 ([11]) *The ONH of a connected bipartite graph consists of two components (which are induced by the two partite sets of the graph), while the ONH of a connected graph that is not bipartite is connected.*

3 Proof of Theorem 1.1 and its Consequences

In this section, we first present a proof of Theorem 1.1. Recall its statement.

Theorem 1.1 *If G is a bipartite graph, then $\gamma_t(G \,\Box\, K_2) = 2\gamma(G)$.*

Proof. Note first that $K_1 \,\Box\, K_2 = K_2$, hence the assertion of the theorem holds for $G = K_1$. Since we can apply the result to each component of the bipartite graph G, we may assume that G is connected. Hence in the rest of the proof let G be a connected bipartite graph of order at least 2.

Let G_1 and G_2 be the G-layers of $G \,\Box\, K_2$, and let $V_i = V(G_i)$ for $i \in [2]$. For notational convenience, for each vertex v in G_1 we denote the corresponding vertex in

G_2 that is adjacent to v in $G \,\square\, K_2$ by v'. Thus, the set $\cup_{v \in V_1}\{vv'\}$ of edges between V_1 and V_2 in $G \,\square\, K_2$ forms a perfect matching in $G \,\square\, K_2$.

Since G is a bipartite graph, $G \,\square\, K_2$ is bipartite as well. Let X and Y be the partite sets of $G \,\square\, K_2$. If $w \in \{v, v'\}$ for some vertex $v \in V_1$, then we define the *complement* of the vertex w to be the vertex $\overline{w} \in \{v, v'\} \setminus \{w\}$. We note that if $w \in V_{3-i}$, then $\overline{w} \in V_i$ for $i \in [2]$. Further, we note that w and $\overline{w}$ belong to different partite sets of $G \,\square\, K_2$.

Let H be the ONH of $G \,\square\, K_2$. By Theorem 2.5, H consists of two components that are induced by the two partite sets, X and Y, of G. Let H_X and H_Y be the two components of H, where $V(H_X) = X$ and $V(H_Y) = Y$. We note that each edge in H_X and H_Y corresponds to the open neighborhood of some vertex in Y and some vertex in X, respectively, in G. For each vertex w in $G \,\square\, K_2$, let e_w be the associated hyperedge in H; that is, $e_w = N_G(w)$.

We proceed further with the following series of claims.

Claim 3.1 *The hypergraphs H_X and H_Y are isomorphic.*

Proof. Let $f\colon X \to Y$ be the function that assigns to each vertex $x \in X$ the vertex $\overline{x} \in Y$. Then, f is a bijection between the vertex set of H_X and H_Y. Suppose that e_X is an edge of H_X. Thus, $e_X = e_w$ for some vertex $w \in Y$. The function f maps the edge e_X to the edge e_Y. We show that e_Y is precisely the edge in H_Y associated with the vertex $\overline{w} \in X$.

Suppose first that $w \in V_1$. In this case, $\overline{w} = w'$. Let w have degree $k+1$ in $G \,\square\, K_2$, for some $k \ge 1$. Thus, w is adjacent in $G \,\square\, K_2$ to k vertices in V_1, say to $w_1, w_2, \ldots, w_k$, and to one vertex in V_2, namely the vertex w'. Since $w \in Y$ and $G \,\square\, K_2$ is bipartite, we note that $\{w_1, w_2, \ldots, w_k\} \subseteq V_1 \cap X$ and that $w' \in V_2 \cap X$. Further, the edge $e_X = e_w = \{w_1, w_2, \ldots, w_k, w'\} \in E(H_X)$. Since $f(w_i) = w'_i$ for $i \in [k]$ and $f(w') = w$, the function f maps the edge e_X to the edge $e_Y = \{w'_1, w'_2, \ldots, w'_k, w\}$. We note that $\{w'_1, w'_2, \ldots, w'_k, w\} \subseteq Y$, and that e_Y is precisely the edge in H_Y associated with the vertex $\overline{w} \in X$.

Suppose next that $w \in V_2$. In this case, $w = v'$ for some vertex $v \in V_1$. Thus, $\overline{w} = v$. Let w have degree $k+1$ in $G \,\square\, K_2$, for some $k \ge 1$. Thus, the vertex v' is adjacent in $G \,\square\, K_2$ to k vertices in V_2, say to $v'_1, v'_2, \ldots, v'_k$, and to one vertex in V_1, namely the vertex v. Since $v' \in Y$ and $G \,\square\, K_2$ is bipartite, we note that $\{v'_1, v'_2, \ldots, v'_k\} \subseteq V_2 \cap X$ and that $v \in V_1 \cap X$. Further, the edge $e_X = e_w = \{v'_1, v'_2, \ldots, v'_k, v\} \in E(H_X)$. The function f maps the edge e_X to the edge $e_Y = \{v_1, v_2, \ldots, v_k, v'\}$. We note that $\{v_1, v_2, \ldots, v_k, v'\} \subseteq Y$, and that e_Y is precisely the edge in H_Y associated with the vertex $\overline{w} \in X$.

Suppose that e_Y is an edge of H_Y and $e_Y = e_w$ for some vertex $w \in X$. If the function f maps the edge e_Y to e_X, then analogously as before, e_X is precisely the edge in H_X associated with the vertex $\overline{w} \in Y$. Thus, the bijective function f preserves adjacency, implying that H_X and H_Y are isomorphic. $(\Box)$

Claim 3.2 $\gamma_t(G \,\square\, K_2) = 2\tau(H_X)$.

Proof. By Observation 2.4, $\gamma_t(G \,\square\, K_2) = \tau(H) = \tau(H_X) + \tau(H_Y)$. By Claim 3.1, $\tau(H_X) = \tau(H_Y)$, and so $\gamma_t(G \,\square\, K_2) = 2\tau(H_X)$. $(\Box)$

Claim 3.3 $\gamma_t(G \,\Box\, K_2) \le 2\gamma(G)$.

Proof. Let D be a minimum dominating set in G, and let D_1 and D_2 be the copies of G in G-layers G_1 and G_2, respectively. Clearly, $v \in D_1$ if and only if $v' \in D_2$. The set $D_1 \cup D_2$ is a total dominating set of $G \,\Box\, K_2$, and so $\gamma_t(G \,\Box\, K_2) \le |D_1 \cup D_2| = 2|D| = 2\gamma(G)$. $_{(\Box)}$

Claim 3.4 $\gamma(G) \le \tau(H_X)$.

Proof. Let H^c be the CNH of G. By Observation 2.4, $\gamma(G) = \tau(H^c)$. We show that $\tau(H^c) \le \tau(H_X)$. Let T_X be a minimum transversal in H_X, and so $|T_X| = \tau(H_X)$. We now define the set T_X^c as follows. For each vertex $v \in T_X$, we add v to T_X^c if $v \in V_1$, otherwise if add $\overline{v}$ to T_X^c if $v \in V_2$. We show that T_X^c is a transversal in H^c. Let e be an arbitrary edge in H^c. Thus, $e = N_G[w]$ for some vertex w in G. We may assume that the vertices of G_1 are named as in the graph G, and so $G_1 = G$. In particular, $w \in V_1$. Thus, $\overline{w} = w' \in V_2$.

Suppose that $w \in Y$. In this case, the edge $e_w = N_G(w) = (e \setminus \{w\}) \cup \{\overline{w}\}$ is an edge of H_X and is therefore covered by some vertex, say z, of T_X. If $z = \overline{w}$, then noting that $\overline{w} \in V_2$, the vertex $w \in T_X^c$, and the edge e is therefore covered by a vertex in T_X^c, namely the vertex w. If $z \ne \overline{w}$, then z is a vertex in e_w different from $\overline{w}$. However, $e_w \setminus \{\overline{w}\} = e \setminus \{w\} \subset V_1$, implying that the vertex $z \in V_1$ and therefore $z \in T_X^c$. The edge e is therefore covered by a vertex in T_X^c, namely the vertex z. Thus, if $w \in Y$, then the edge e is covered by a vertex in T_X^c.

Suppose that $w \in X$. We now consider the vertex $\overline{w} \in V_2$. We note that $\overline{w} \in Y$ and that the edge $e_{\overline{w}} = N_G(\overline{w})$ is an edge of H_X. Further, the edge $e_{\overline{w}}$ contains the vertex $w \in V_1$ and all other vertices in $e_{\overline{w}}$ belong to the set V_2. Further, if u is a vertex in the edge e, then either $u = w$, in which case u also belongs to the edge $e_{\overline{w}}$, or $u \ne w$, in which case u' belongs to the edge $e_{\overline{w}}$. Since the edge $e_{\overline{w}}$ is an edge of H_X, it is covered by some vertex, say z, of T_X. If $z = w$, then noting that $w \in V_1$, the vertex $w \in T_X^c$, and the edge e is therefore covered by a vertex in T_X^c, namely the vertex w. If $z \ne \overline{w}$, then z is a vertex in $e_{\overline{w}}$ different from w. Thus, $z = u'$ for some vertex $u \in V_1$. Since $u' \in V_2$, the vertex $u \in T_X^c$. As observed earlier, u belongs to the edge e, implying that the edge e is covered by a vertex in T_X^c, namely the vertex u. Thus, if $w \in X$, then the edge e is covered by a vertex in T_X^c.

Thus, whenever $w \in X$ or $w \in Y$, the edge e is covered by a vertex in T_X^c. Since e is an arbitrary edge of H^c, this implies that T_X^c is a transversal of H^c, and therefore that $\tau(H^c) \le |T_X^c| = |T_X| = \tau(H_X)$. $_{(\Box)}$

We now return to the proof of Theorem 1.1 one final time. By Claims 3.2, 3.3, and 3.4, the following holds.

$$2\tau(H_X) \overset{\text{Claim 3.2}}{=} \gamma_t(G \,\Box\, K_2) \overset{\text{Claim 3.3}}{\le} 2\gamma(G) \overset{\text{Claim 3.4}}{\le} 2\tau(H_X)\,.$$

Consequently, we must have equality throughout the above inequality chain. In particular, $\gamma_t(G \,\Box\, K_2) = 2\gamma(G)$. This completes the proof of Theorem 1.1. $\Box$

As an immediate consequence of Theorem 1.1 we state that the problems of determining the domination number and the total domination number of hypercubes are equivalent in the following sense:

Corollary 3.5 *If $n \geq 1$, then $\gamma_t(Q_{n+1}) = 2\gamma(Q_n)$.*

Combining Corollary 3.5 with Theorem 2.1 we also deduce the following result:

Corollary 3.6 *If $k \geq 1$, then $\gamma_t(Q_{2^k+1}) = 2^{2^k-k+1}$ and $\gamma_t(Q_{2^k}) = 2^{2^k-k}$.*

While the first assertion of Corollary 3.6 appears to be new, the second assertion goes back to Johnson [13], see also [21, Theorem 1(b)].

As another consequence of Theorem 1.1, we have the following result.

Corollary 3.7 *If G is a bipartite graph, then*

$$\gamma_t(G \,\Box\, K_2) = \gamma_{\rm pr}(G \,\Box\, K_2) = \gamma_{\rm tr}(G \,\Box\, K_2)\,.$$

Proof. As shown in the proof of Claim 3.3 in Theorem 1.1, if D_1 is a minimum dominating set in G_1, and $D_2 = \{v' \mid v \in D_1\}$, then the set $D^* = D_1 \cup D_2$ is a total dominating set of $G \,\Box\, K_2$. We note that D^* is also a paired-dominating set of $G \,\Box\, K_2$. Further, $|D^*| = 2\gamma(G)$. By Observation 2.2 and Theorem 1.1, this implies that

$$\gamma_t(G \,\Box\, K_2) \leq \gamma_{\rm pr}(G \,\Box\, K_2) \leq |D^*| = 2\gamma(G) = \gamma_t(G \,\Box\, K_2)\,.$$

Consequently, we must have equality throughout the above inequality chain. In particular, $\gamma_t(G \,\Box\, K_2) = \gamma_{\rm pr}(G \,\Box\, K_2)$. We note that D^* is also a total restrained dominating set of $G \,\Box\, K_2$. Thus, by Observation 2.3, $\gamma_t(G \,\Box\, K_2) \leq \gamma_{\rm tr}(G \,\Box\, K_2) \leq |D^*| = 2\gamma(G) = \gamma_t(G \,\Box\, K_2)$, implying that $\gamma_t(G \,\Box\, K_2) = \gamma_{\rm tr}(G \,\Box\, K_2)$. $\Box$

As a special case of Theorem 1.1 and Corollary 3.7, we have the following result.

Corollary 3.8 *If $n \geq 1$, then $\gamma_t(Q_n) = \gamma_{\rm pr}(Q_n) = \gamma_{\rm tr}(Q_n)$.*

4 Proof of Theorem 1.2

In this section, we consider general prisms and show that the bipartite condition in the statement of Theorem 1.1 is essential. First we recall the trivial lower bound on the total domination number of a graph in terms of the maximum degree of the graph: If G is a graph of order n and maximum degree Δ with no isolated vertex, then $\gamma_t(G) \geq n/\Delta$, cf. [12, Theorem 2.11].

Proposition 4.1 *If $k \geq 1$, then $\gamma_t(C_{6k+1} \,\Box\, K_2) = 2\gamma(C_{6k+1}) - 1$.*

Proof. Let $G \cong C_{6k+1}$ for some integer $k \ge 1$. Then, $\gamma(G) = \lceil n(G)/3 \rceil = 2k+1$. We show that $\gamma_t(G \,\square\, K_2) = 4k+1$. Let G_1 and G_2 be the G-layers of $G \,\square\, K_2$, where G_1 is the cycle $u_1 u_2 \ldots u_{6k+1} u_1$ and G_2 is the cycle $v_1 v_2 \ldots v_{6k+1} v_1$, and where $u_i v_i \in E(G)$. The set

$$S = \left(\bigcup_{i=0}^{k-1} \{u_{6i+1}, u_{6i+2}, v_{6k+4}, v_{6k+5}\} \right) \cup \{u_{6k+1}\}$$

is a total dominating set of $G \,\square\, K_2$, implying that $\gamma_t(G \,\square\, K_2) \le |S| = 4k+1$. Conversely, since $G \,\square\, K_2$ is a cubic graph of order $12k+2$, the trivial lower bound on the total domination number of $G \,\square\, K_2$ is given by $\gamma_t(G \,\square\, K_2) \ge (12k+2)/3$, implying that $\gamma_t(G \,\square\, K_2) \ge 4k+1$. Consequently, $\gamma_t(G \,\square\, K_2) = 4k+1$. As observed earlier, $\gamma(G) = 2k+1$. Therefore, $\gamma_t(G \,\square\, K_2) = 2\gamma(G) - 1$. $\square$

We show next that there are connected, non-bipartite graphs G for which the difference $\gamma_t(G \,\square\, K_2) - 2\gamma(G)$ can be arbitrarily large. Recall the statement of Theorem 1.2.

Theorem 1.2 *For each integer $k \ge 1$, there exists a connected graph G_k satisfying*

$$\gamma_t(G_k \,\square\, K_2) - 2\gamma(G_k) = k.$$

Proof. For $k = 1$, let $G_1 \cong C_7$. By Proposition 4.1, $\gamma_t(G_1 \,\square\, K_2) = 2\gamma(G_1) - 1$. Hence, we assume in what follows that $k \ge 2$. For $i \in [k]$, let F_i be the 5-cycle $v_{5(i-1)+1} v_{5(i-1)+2} v_{5(i-1)+4} v_{5(i-1)+5} v_{5(i-1)+3} v_{5(i-1)+1}$. Let G_k be obtained from the disjoint union of the cycles $F_1, \ldots, F_k$ by adding the edges $v_{5j} v_{5j+1}$ for $j \in [k-1]$. By construction, G_k is a connected graph of order k. The following two claims determine the domination number of G_k and total domination numbers of the prism $G_k \,\square\, K_2$.

Claim A *For $k \ge 2$, $\gamma(G_k) = 2k$.*

Proof. Every dominating set of G_k contains at least two vertices from $V(F_i)$ in order to dominate the vertices in $V(F_i)$ for each $i \in [k]$, and so $\gamma(G_k) \ge 2k$. Conversely, every set consisting of two non-adjacent vertices from each set $V(F_i)$ forms a dominating set of G_k, and so $\gamma(G_k) \le 2k$. Consequently, $\gamma(G_k) = 2k$. $(\square)$

Claim B *For $k \ge 2$, $\gamma_t(G_k \,\square\, K_2) = 3k$.*

Proof. Let G_k^1 and G_k^2 be the two copies of the graph G_k in the prism $G_k \,\square\, K_2$, where the vertex in G_k^1 and G_k^2 corresponding to the vertex v_j in G_k is labeled x_j and y_j, respectively, for $j \in [5k]$. Thus, the set $\cup_{j=1}^{5k} \{x_j y_j\}$ of edges between $V(G_k^1)$ and $V(G_k^2)$ in $G_k \,\square\, K_2$ forms a perfect matching in $G_k \,\square\, K_2$. For $i \in [k]$, let

$$V_i = \bigcup_{j=1}^{5} \{x_{5(i-1)+j}, y_{5(i-1)+j}\}.$$

When $k = 6$, the prism $G_k \,\square\, K_2$ is illustrated in Figure 1, where the vertices in V_1 are labelled. Let S be an arbitrary total dominating set of $G_k \,\square\, K_2$. For $i \in [k]$, let $S_i = S \cap V_i$. For $i \in [k]$, let

$$X_i = \bigcup_{j=2}^{4} \{x_{5(i-1)+j}\} \quad \text{and} \quad Y_i = \bigcup_{j=2}^{4} \{y_{5(i-1)+j}\}$$

In order to totally dominate the vertices in the set X_i, we note that $|S_i| \ge 2$ for all $i \in [k]$. Suppose that $|S_i| = 2$ for some $i \in [k]$. If both vertices in S_i belong to the same copy of G_k, say to G_k^2, then at least one vertex in X_i is not totally dominated by S. If the vertices in S_i belong to different copies of G_k, then at least two vertices in $X_i \cup Y_i$ are not totally dominated by S. Both cases produce a contradiction, implying that $|S_i| \ge 3$. Hence,

$$|S| = \sum_{i=1}^{k} |S_i| \ge 3k.$$

Since S is an arbitrary total dominating set of $G_k \,\square\, K_2$, this implies that $\gamma_t(G_k \,\square\, K_2) \ge 3k$. To prove the converse, let

$$X = \bigcup_{i=1}^{\lfloor k/2 \rfloor} \{x_{10(i-1)+1}, x_{10(i-1)+2}, x_{10i}\} \quad \text{and} \quad Y = \bigcup_{i=1}^{\lfloor k/2 \rfloor} \{y_{10(i-1)+5}, y_{10(i-1)+6}, y_{10(i-1)+7}\}.$$

If k is even, let

$$D = (X \cup Y \cup \{x_{5k-1}\}) \setminus \{y_{5k-3}\}.$$

If k is odd, let

$$D = X \cup Y \cup \{x_{5k-4}, y_{5k-1}, y_{5k}\}.$$

For $k = 6$, the set D is illustrated by the darkened vertices in Figure 1. In both cases, D is a total dominating set of $G_k \,\square\, K_2$, and $|D \cap V_i| = 3$ for each $i \in [k]$, implying that

$$\gamma_t(G_k \,\square\, K_2) \le |D| = \sum_{i=1}^{k} |D \cap V_i| = 3k.$$

Consequently, $\gamma_t(G_k \,\square\, K_2) = 3k$. ($\Box$)

By Claim A and Claim B, for $k \ge 2$, $\gamma(G_k) = 2k$ and $\gamma_t(G_k \,\square\, K_2) = 3k$. This completes the proof of Theorem 1.2. $\Box$

5 Concluding Remarks

Let us say that a graph G is *γ_t-prism perfect* if $\gamma_t(G \,\square\, K_2) = 2\gamma(G)$. We have seen that all bipartite graphs are γ_t-prism perfect. It would certainly be interesting to characterize γ_t-prism perfect graphs in general, but this appears to be a challenging

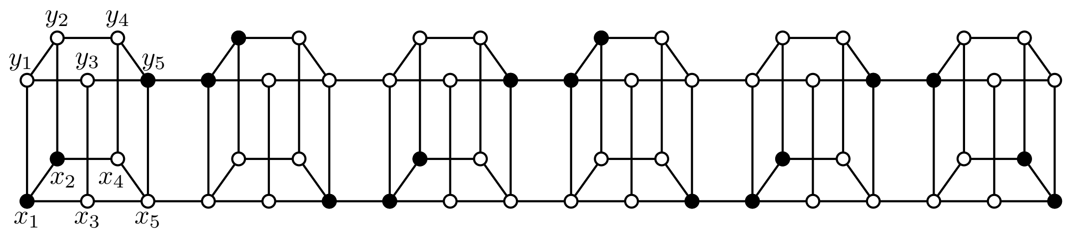

Figure 1: The prism $G_6 \,\square\, K_2$

problem. Instead, one could try to characterize γ_t-prism perfect graphs within some interesting families of graphs, say triangle-free graphs.

A computation shows that among the 11.117 connected graphs of order 8, precisely 297 graphs are not γ_t-prism perfect. Similarly, there are 79.638 graphs that are not γ_t-prism perfect among the 11.716.571 connected graphs of order 9. These computations led us to conjecture the following conjecture.

Conjecture 5.1 *Almost all graphs are γ_t-prism perfect.*

With respect to the conjecture we refer to [5] for the investigation of the behavior of the domination number in random graphs.

Motivated by the construction presented in the proof of Theorem 1.2 we wonder whether the following lower bound on the total domination number of prisms holds true. If so, then the construction implies that the bound is sharp.

Problem 5.2 *Is it true that for any graph G, $\gamma_t(G \,\square\, K_2) \geq \frac{3}{2}\gamma(G)$?*

One may be tempted to try to extend the presented results to additional Cartesian product graphs. Clearly, $\gamma(P_3) = 2$ and an easy computation gives $\gamma_t(P_3 \square K_3) = \gamma_t(P_3 \square P_3) = 4$. Similarly $\gamma_t(P_3 \square K_4) = 4$ and $\gamma_t(P_3 \square P_4) = 6$, indicating that our result cannot be extended by a matter of parity. Moreover for all listed Cartesian products we were able to find pairs of bipartite graphs with the same domination number so that the total domination number of the respective Cartesian product differs. These examples give a strong evidence that the identity of Theorem 1.1 cannot be generalized in "obvious" directions.

Acknowledgements

Research of M.A.H. is supported in part by the South African National Research Foundation and the University of Johannesburg. J.A. and S.K. are supported by the Ministry of Science of Slovenia under the grants P1-0297 and N1-0043.

References

[1] R. Bertolo, P.R.J. Östergård, W.D. Weakley, An updated table of binary/ternary mixed covering codes, J. Combin. Des. 12 (2004) 157–176.

[2] B. Brešar, P. Dorbec, W. Goddard, B.L. Hartnell, M.A. Henning, S. Klavžar, D.F. Rall, Vizing's conjecture: a survey and recent results, J. Graph Theory 69 (2012) 46–76.

[3] K. Choudhary, S. Margulies, I.V. Hicks, A note on total and paired domination of Cartesian product graphs, Electron. J. Combin. 20 (2013), no. 3, Paper 25, 9 pp.

[4] W. J. Desormeaux, M. A. Henning, Paired domination in graphs: A survey and recent results, Utilitas Mathematica 94 (2014), 101–166.

[5] R. Glebov, A. Liebenau, T. Szabó, On the concentration of the domination number of the random graph, SIAM J. Discrete Math. 29 (2015) 1186–1206.

[6] R. Hammack, W. Imrich, S. Klavžar, Handbook of Product Graphs. Second Edition, CRC Press, Boca Raton, FL, 2011.

[7] F. Harary, M. Livingston, Independent domination in hypercubes, Appl. Math. Lett. 6 (1993) 27–28.

[8] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, Inc., New York, 1998.

[9] T.W. Haynes, P.J. Slater, Paired domination in graphs, Networks 32 (1998) 199–206.

[10] M.A. Henning, D.F. Rall, On the total domination number of Cartesian products of graphs, Graphs Combin. 21 (2005) 63–69.

[11] M.A. Henning, A. Yeo, Hypergraphs with large transversal number and with edge sizes at least three, J. Graph Theory 59 (2008) 326–348.

[12] M.A. Henning, A. Yeo, Total Domination in Graphs, Springer Monographs in Mathematics, 2013.

[13] S.M. Johnson, A new lower bound for coverings by rook domains, Utilitas Math. 1 (1972) 121–140.

[14] G. Kéri, P.R.J. Östergård, Bounds for covering codes over large alphabets, Des. Codes Cryptogr. 137 (2005) 45–60.

[15] J. Lee, Independent perfect domination sets in Cayley graphs, J. Graph Theory 37 (2001) 213–219.

[16] Y. Lu, X. Hou, Total domination in the Cartesian product of a graph and K_2 or C_n, Util. Math. 83 (2010) 313–322.

[17] Y.S. Kwon, J. Lee, Perfect domination sets in Cayley graphs, Discrete Appl. Math. 162 (2014) 259–263.

[18] P.R.J. Östergård, U. Blass, On the size of optimal binary codes of length 9 and covering radius 1, IEEE Trans. Inform. Theory 47 (2001) 2556–2557.

[19] J. Telle, A. Proskurowski, Algorithms for vertex partitioning problems on partial k-trees, SIAM J. Discrete Math. 10 (1997) 529–550.

[20] K.P.F. Verstraten, A Generalization of the Football Pool Problem, Master thesis, Tilburg University, 2014, https://oeis.org/A238305/a238305.pdf.

[21] W.D. Weakley, Optimal binary covering codes of length 2^j, J. Combin. Des. 14 (2006) 1–13.

[22] P.M. Weichsel, Dominating sets in n-cubes, J. Graph Theory 18 (1994) 479–488.

[23] G.J.M. van Wee, Improved sphere bounds on the covering radius of codes, IEEE Trans. Inform. Theory 34 (1988) 237–245.