Proof of a Conjecture of Z.-W. Sun on Trigonometric Series*

Brian Y. Sun† and J. X. Meng
College of Mathematics and System Science,
Xinjiang University, Urumqi 830046, P.R.China

Abstract. Recently, Z. W. Sun introduced a sequence $(S_n)_{n\geq 0}$, where $S_n = \frac{\binom{6n}{3n}\binom{3n}{n}}{2(2n+1)\binom{2n}{n}}$, and found one congruence and two convergent series on S_n by `Mathematica`. Furthermore, he proposed some related conjectures. In this paper, we first give analytic proofs of his two convergent series and then confirm one of his conjectures by invoking series expansions of $\sin(t\arcsin(x))$ and $\cos(t\arcsin(x))$.

Key words: Divisibility; Congruences; Trigonometric series; Convergent series

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1 Introduction

Throughout the paper, we let $\mathbb{R}$, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ and $\mathbb{N}^+ = \mathbb{N}\backslash\{0\}$ denote the set of real numbers, natural numbers and positive integer numbers, respectively.

Recently, Z. W. Sun [6] considered the divisibilities of products and sums concerned binomial coefficients and central binomial coefficients. He also studied the divisibility of $\binom{6n}{3n}\binom{3n}{n}$ and obtained the following result,

$$2(2n+1)\binom{2n}{n}\Big|\binom{6n}{3n}\binom{3n}{n} \text{ for all } n \in \mathbb{N}^+, \tag{1.1}$$

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†Corresponding author. Email: brianys1984@126.com.

where $\binom{m}{n} = \frac{m!}{n!(m-n)!}$ is a binomial coefficient. We call $\binom{2n}{n}$ the central binomial coefficient (cf. [4, A000984]). The binomial coefficients $\binom{6n}{3n}$ is the sequence [4, A066802] and $\binom{3n}{n}$ is the sequence [4, A005809].

According to (1.1), Z. W. Sun [6] introduced the following integer sequence [4, A176898].

$$S_n = \frac{\binom{6n}{3n}\binom{3n}{n}}{2(2n+1)\binom{2n}{n}}, \text{ for } n \in \mathbb{N}^+.$$

Here we list the values of $S_1, S_2, \ldots, S_8$ as follows:

5, 231, 14568, 1062347, 84021990,
7012604550, 607892634420, 54200780036595.

Z. W. Sun [6] proved that for any odd prime p,

$$S_p \equiv 15 - 30p + 60p^2 \pmod{p^3}.$$

Additionally, Guo [3] studied the sequence and proved that $3S_n \equiv 0 \pmod{2n+3}$, positively answering a question of Z. W. Sun [6]. By setting $S_0 = \frac{1}{2}$ and employing `Mathematica`, Z. W. Sun also obtained

$$\sum_{n\geq 0} S_n x^n = \frac{\sin(\frac{2}{3}\arcsin(6\sqrt{3x}))}{8\sqrt{3x}} \left(0 < x \leq \frac{1}{108}\right), \tag{1.2}$$

and

$$\sum_{n\geq 0} \frac{S_n}{(2n+3)108^n} = \frac{27\sqrt{3}}{256}. \tag{1.3}$$

Particularly, setting $x = 1/108$ in (1.2), we derive

$$\sum_{n=0}^{\infty} \frac{S_n}{108^n} = \frac{3\sqrt{3}}{8}. \tag{1.4}$$

Moreover, he proposed the following conjecture.

Conjecture 1.1 *There exist positive integers $T_1, T_2, \ldots$ such that*

$$\sum_{k=0}^{\infty} S_k x^{2k+1} + \frac{1}{24} - \sum_{k=1}^{\infty} T_k x^{2k} = \frac{\cos(\frac{2}{3}\arccos(6\sqrt{3}x))}{12} \tag{1.5}$$

for all real x with $|x| \leq \frac{1}{6\sqrt{3}}$. Also, $T_p \equiv -2 \pmod{p}$ for any prime p.

In this paper, we first deduce formulas (1.2) and (1.3) by utilizing a series expansion in [5]. Besides, we confirm Conjecture 1.1 , i.e.,

Theorem 1.2 *Conjecture 1.1 is true.*

2 Trigonometric Series

Before proving Theorem 1.2, we need some formulas and series expansions on trigonometric functions. We first note that

$$\arcsin(x) + \arccos(x) = \frac{\pi}{2}, \tag{2.1}$$

where $x \in [-1, 1]$. What's more, the basic fact

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \tag{2.2}$$

is needed.

Here we also use two trigonometric series in [5, Ex. 44, p. 51], namely,

$$\sin(t\arcsin(x)) = \sum_{n\geq 0}(-1)^n t\left(\prod_{i=0}^{n-1}(t^2 - (2i+1)^2)\right)\frac{x^{2n+1}}{(2n+1)!}, \tag{2.3}$$

and

$$\cos(t\arcsin(x)) = \sum_{n\geq 0}(-1)^n\left(\prod_{i=0}^{n-1}(t^2 - (2i)^2)\right)\frac{x^{2n}}{(2n)!}. \tag{2.4}$$

With the formula (2.3) in hand, it is not difficult to derive the identities (1.2) and (1.3).

To begin with, we consider the uniform convergence of series (1.2). Due to Stirling [2], we hold the following approximate formula, which was called Stirling's formula,

$$\Gamma(\alpha) \approx \left(\frac{\alpha - 1}{e}\right)^{\alpha-1}\sqrt{2\pi(\alpha - 1)}, \text{ as } \alpha \to \infty, \tag{2.5}$$

where $\Gamma(\alpha)$ is the gamma function and is defined by

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}e^{-x}dx \text{ for } \alpha > 0.$$

For more information on this formula, one can consult [2]. By Stirling's formula (2.5), we have

$$\rho = \overline{\lim_{n\to\infty}}\sqrt[n]{S_n} = 108, \text{ as } n \to \infty. \tag{2.6}$$

Thus by Cauchy-Hadamard's theorem and (2.6), the series $\sum_{n=0}^{\infty} S_n x^n$ is uniformly convergent for $0 < x < \frac{1}{108}$. So, before we make operations on $\sum_{n=0}^{\infty} S_n x^n$, we designate $x \in (0, 1/108)$ in order that all the following operations are well defined.

Theorem 2.1 *For all $a, b \in \mathbb{R}$ and $bx \in [-1, 1]$, we have*

$$\int x \sin(a \arcsin(bx)) dx = \frac{\frac{\sin((a-2)\arcsin(bx))}{a-2} - \frac{\sin((a+2)\arcsin(bx))}{a+2}}{4b^2} + C,$$

where C is any constant.

Now we are in a position to consider formulas (1.2) and (1.3).

(i) As to (1.2), let $t = 2/3$ and $x = 6\sqrt{3x}$ in (2.3), we obtain

$$\begin{aligned} \sin(\frac{2}{3}\arcsin(6\sqrt{3x})) &= \sum_{n\geq 0} \frac{2}{3} \left(\prod_{i=0}^{n-1} ((2i+1)^2 - \frac{4}{9}) \right) \frac{(6\sqrt{3x})^{2n+1}}{(2n+1)!} \\ &= 4\sqrt{3x} \sum_{n\geq 0} \frac{12^n}{(2n+1)!} \left(\prod_{i=0}^{n-1} (6i+1)(6i+5) \right) x^n \\ &= 8\sqrt{3x} \sum_{n\geq 0} S_n x^n, \end{aligned} \tag{2.7}$$

which is nothing but (1.2).

(ii) For (1.3), let

$$f(x) = \sum_{n=0}^{\infty} \frac{S_n}{2n+3} x^{2n+3}. \tag{2.8}$$

Thanks to (i), we get

$$f'(x) = \sum_{n=0}^{\infty} S_n x^{2n+2} = \frac{x \sin(\frac{2}{3}\arcsin(6x\sqrt{3}))}{8\sqrt{3}}.$$

From Theorem 2.1, it follows that

$$\begin{aligned} f(x) &= \frac{1}{8\sqrt{3}} \int_0^x t \sin(\frac{2}{3}\arcsin(6t\sqrt{3})) dt \\ &= \frac{\frac{3}{4}\sin\left(\frac{4}{3}\sin^{-1}\left(6\sqrt{3}x\right)\right) - \frac{3}{8}\sin\left(\frac{8}{3}\sin^{-1}\left(6\sqrt{3}x\right)\right)}{3456\sqrt{3}}, \end{aligned}$$

which indicates

$$f(1/6\sqrt{3}) = \frac{1}{6144}. \tag{2.9}$$

Therefore, we can deduce identity (1.3) by invoking (2.8) and (2.9) .

In addition, combining formulas (2.1) and (2.2), it is not difficult to obtain the following result,

Proposition 2.2 *For all $t \in \mathbb{R}$ and $x \in [-1, 1]$, we have*

$$\sin(\frac{\pi t}{2})\sin(t\arcsin(x)) + \cos(\frac{\pi t}{2})\cos(t\arcsin(x)) = \cos(t\arccos(x)).$$

With above results, we are ready to prove Theorem 1.2.

3 Proof of the Theorem 1.2

We shall give two lemmas before giving the proof of Theorem 1.2. The first lemma is Fermat's simple theorem [1].

Lemma 3.1 *If a is any integer prime to m, and if m is prime, then* $a^{m-1} \equiv 1 \pmod{m}$.

Lemma 3.2 *For any prime p, we have*

$$\frac{1}{p}\binom{3p-2}{p-1} \equiv -2 \pmod{p}.$$

Proof. By applying

$$\begin{aligned}\frac{1}{p}\binom{3p-2}{p-1} &= \frac{(3p-2)!}{p!(2p-1)!} \\ &= 2\prod_{j=1}^{p-2}\left(\frac{3p}{j+1} - 1\right)\end{aligned}$$

and

$$\begin{aligned}\prod_{j=1}^{p-2}\left(\frac{3p}{j+1} - 1\right) &\equiv (-1)^{p-2} \\ &= -1 \pmod{p},\end{aligned}$$

the conclusion can be derived at once.

Now we can prove Theorem 1.2.

Proof of Theorem 1.2. Firstly, we need to point out that (1.5) can be rewritten as follows:

$$\cos(\frac{2}{3}\arccos(6x\sqrt{3})) = \sin(\frac{\pi}{3})\sin(\frac{2}{3}\arcsin(6x\sqrt{3})) + \cos(\frac{\pi}{3})(1-24\sum_{k=1}^{\infty} T_k x^{2k}).$$

By Proposition 2.2, for $k = 1, 2, 3, \ldots$, if we can find T_k such that

$$1 - 24\sum_{k=1}^{\infty} T_k x^{2k} = \cos(\frac{2}{3}\arcsin(6x\sqrt{3})),$$

then we can prove the first part of Conjecture 1.1.

By (2.4), we see that

$$\begin{aligned}
\cos(\frac{2}{3}\arcsin(6x\sqrt{3})) &= -\sum_{n\geq 0}\frac{4}{9^n}\left(\prod_{i=1}^{n-1}(6i+2)(6i-2)\right)\frac{(6x\sqrt{3})^{2n}}{(2n)!} \\
&= -\sum_{n\geq 0}\frac{16^n}{3n-1}\binom{3n}{n}x^{2n} \\
&= 1 - \sum_{n\geq 1}\frac{16^n}{3n-1}\binom{3n}{n}x^{2n}.
\end{aligned}$$

For integers $n \geq 1$, if we set $T_n = \frac{16^n}{24(3n-1)}\binom{3n}{n}$, thus obtaining the desired sequence $(T_n)_{n\geq 1}$ for Conjecture 1.1.

It is clear that

$$\frac{16^n}{24(3n-1)}\binom{3n}{n} = 16^{n-1}\left(2\binom{3n-2}{n-1} - \binom{3n-2}{n}\right)$$

is an integer. One can refer to [7] for details. In view of Lemmas 3.1 and 3.2, we can get $T_p \equiv -2 \pmod{p}$ for any prime p. This completes the proof of Theorem 1.2.

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