

EULERIAN POLYNOMIALS, PERFECT MATCHINGS AND STIRLING PERMUTATIONS OF THE SECOND KIND

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ABSTRACT. In this paper, we first present combinatorial proofs of a kind of expansions of the Eulerian polynomials of types A and B , and then we introduce Stirling permutations of the second kind. In particular, we count Stirling permutations of the second kind by their cycle ascent plateaus, fixed points and cycles.

Keywords: Eulerian polynomials; Perfect matchings; Stirling permutations of the second kind; Stirling derangements

1. INTRODUCTION

Let \mathfrak{S}_n be the symmetric group on the set $[n] = \{1, 2, \dots, n\}$ and let $\pi = \pi(1)\pi(2) \cdots \pi(n) \in \mathfrak{S}_n$. Denote by \mathcal{B}_n the hyperoctahedral group of rank n . Elements π of \mathcal{B}_n are signed permutations of the set $\pm[n]$ such that $\pi(-i) = -\pi(i)$ for all i , where $\pm[n] = \{\pm 1, \pm 2, \dots, \pm n\}$. Let $\#S$ denote the cardinality of a set S . We define

$$\begin{aligned} \text{des}_A(\pi) &:= \#\{i \in \{1, 2, \dots, n-1\} \mid \pi(i) > \pi(i+1)\}, \\ \text{des}_B(\pi) &:= \#\{i \in \{0, 1, 2, \dots, n-1\} \mid \pi(i) > \pi(i+1)\}, \end{aligned}$$

where $\pi(0) = 0$. The Eulerian polynomials of types A and B are respectively defined by

$$\begin{aligned} A_n(x) &= \sum_{\pi \in \mathfrak{S}_n} x^{\text{des}_A(\pi)}, \\ B_n(x) &= \sum_{\pi \in \mathcal{B}_n} x^{\text{des}_B(\pi)}. \end{aligned}$$

There is a larger literature devoted to $A_n(x)$ and $B_n(x)$ (see, e.g., [5, 6, 12, 15, 20, 27] and references therein). Let $s = (s_1, s_2, \dots)$ be a sequence of positive integers. Let

$$I_n^{(s)} = \{(e_1, e_2, \dots, e_n) \in \mathbb{Z}^n \mid 0 \leq e_i < s_i\},$$

which known as the set of s -inversion sequences. The number of *ascents* of an s -inversion sequence $e = (e_1, e_2, \dots, e_n) \in I_n^{(s)}$ is defined by

$$\text{asc}(e) = \#\left\{i \in [n-1] : \frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}}\right\} \cup \{0 : \text{if } e_1 > 0\}.$$

Let $E_n^s(x) = \sum_{e \in I_n^s} x^{\text{asc}(e)}$. Following [27], we have

$$\begin{aligned} A_n(x) &= E_n^{(1, 2, \dots, n)}(x), \\ B_n(x) &= E_n^{(2, 4, \dots, 2n)}(x). \end{aligned}$$

Let $M_n(x)$ be a sequence of polynomials defined by

$$M(x, z) = \sum_{n \geq 0} M_n(x) \frac{z^n}{n!} = \sqrt{\frac{x-1}{x - e^{2z(x-1)}}}. \quad (1)$$

Combining (1) and an explicit formula of the Ehrhart polynomial of the s -lecture hall polytope, Savage and Viswanathan [28] proved that $M_n(x) = E_n^{(1,3,\dots,2n-1)}(x)$.

A *perfect matching* of $[2n]$ is a partition of $[2n]$ into n blocks of size 2. Denote by $N(n, k)$ the number of perfect matchings of $[2n]$ with the restriction that only k matching pairs have even larger entries. The numbers $N(n, k)$ satisfy the recurrence relation

$$N(n+1, k) = 2kN(n, k) + (2n - 2k + 3)N(n, k-1)$$

for $n, k \geq 1$, where $N(1, 1) = 1$ and $N(1, k) = 0$ for $k \geq 2$ or $k \leq 0$ (see [25, Proposition 1]). Let $N_n(x) = \sum_{k=1}^n N(n, k)x^k$. The first few of the polynomials $N_n(x)$ are

$$N_0(x) = 1, N_1(x) = x, N_2(x) = 2x + x^2, N_3(x) = 4x + 10x^2 + x^3.$$

The exponential generating function for $N_n(x)$ is given as follows (see [22, Eq. (25)]):

$$N(x, z) = \sum_{n \geq 0} N_n(x) \frac{z^n}{n!} = \sqrt{\frac{1-x}{1 - xe^{2z(1-x)}}}. \quad (2)$$

Combining (1) and (2), we get $M_n(x) = x^n N_n(\frac{1}{x})$ for $n \geq 0$.

Context-free grammar was introduced by Chen [7] and it is a powerful tool for studying exponential structures in combinatorics. We refer the reader to [9, 10, 13, 23] for further information. In particular, using [23, Theorem 10], it is easy to present a grammatical proof of the following result.

Proposition 1. *For $n \geq 0$, we have*

$$2^n x A_n(x) = \sum_{k=0}^n \binom{n}{k} N_k(x) N_{n-k}(x), \quad (3)$$

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} N_k(x) M_{n-k}(x). \quad (4)$$

Recall that the exponential generating function for $x A_n(x)$ is

$$A(x, z) = 1 + \sum_{n \geq 1} x A_n(x) \frac{t^n}{n!} = \frac{1-x}{1 - xe^{z(1-x)}}.$$

An equivalent formula of (3) is given as follows:

$$N^2(x, z) = A(x, 2z). \quad (5)$$

One purpose of this paper is to study the correspondence between permutations and pairs of perfect matchings. Motivated by (5), another purpose of this paper is to explore some cycle structure related to $N(x, z)$ or $M(x, z)$. This paper is organized as follows. In Section 2, we present a combinatorial proof of Proposition 1. In Section 3, we introduce the Stirling permutations of the second kind. In Section 4, we count Stirling permutations of the second kind by their cycle ascent plateaus, fixed points and cycles.

2. A COMBINATORIAL PROOF OF PROPOSITION 1

Let \mathcal{M}_{2n} be the set of perfect matchings of $[2n]$, and let $M \in \mathcal{M}_{2n}$. The *standard form* of M is a list of blocks $\{(i_1, j_1), (i_2, j_2), \dots, (i_n, j_n)\}$ such that $i_r < j_r$ for all $1 \leq r \leq n$ and $1 = i_1 < i_2 < \dots < i_n$. In the following discussion we always write M in standard form. Let $\text{el}(M)$ (resp. $\text{ol}(M)$) be the number of blocks of M with even larger (resp. odd larger) entries. Therefore, we have

$$N_n(x) = \sum_{M \in \mathcal{M}_{2n}} x^{\text{el}(M)},$$

$$M_n(x) = \sum_{M \in \mathcal{M}_{2n}} x^{\text{ol}(M)}.$$

For convenience, we call (i, j) a *marked block* (resp. an *unmarked block*) if j is even (resp. odd) and large than i .

2.1. Permutations and pairs of perfect matchings.

Let the entry $\pi(i)$ be called a *descent* (resp. an *ascent*) of π if $\pi(i) > \pi(i+1)$ (resp. $\pi(i) < \pi(i+1)$). By using the reverse map, it is evident that the ascent and descent statistics are equidistributed. Let $\text{asc}(\pi)$ be the number of ascents of π . Hence

$$A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{asc}(\pi)}. \quad (6)$$

Throughout this subsection, we shall always use (6) as the definition of $A_n(x)$.

We now constructively define a set of decorated permutations on $[n]$ with some entries of permutations decorated with hats and circles, denoted by \mathcal{P}_n . Let $w = w_1 w_2 \dots w_n \in \mathcal{P}_n$. We say that w_i with a hat (resp. circle) if $w_i = \widehat{k}$ or $w_i = \widehat{\textcircled{k}}$ (resp. $w_i = \textcircled{k}$ or $w_i = \widehat{\textcircled{k}}$) for some $k \in [n]$. Start with $\mathcal{P}_1 = \{1, \widehat{1}\}$. Suppose we have get \mathcal{P}_{n-1} , where $n \geq 2$. Given $v = v_1 v_2 \dots v_{n-1} \in \mathcal{P}_{n-1}$. We now construct entries of \mathcal{P}_n by inserting $n, \textcircled{n}, \widehat{n}$ or $\widehat{\textcircled{n}}$ into v according the following rules:

- (r₁) We can only put n or \widehat{n} at the end of v ;
- (r₂) For $1 \leq i \leq n-1$, if v_i with no bar, then we can only put n or \textcircled{n} immediately before v_i ;
if v_i with a bar, then we can only put \widehat{n} or $\widehat{\textcircled{n}}$ immediately before v_i . In other words, if v_i with a hat (resp. with no hat), then we can only insert n with a hat (resp. with no hat) immediately before v_i .

It is clear that there are $2n$ elements in \mathcal{P}_n that can be generated from any $v \in \mathcal{P}_{n-1}$. By induction, we obtain $|\mathcal{P}_n| = 2n|\mathcal{P}_{n-1}| = 2^n n!$. Let $\varphi(w) = \varphi(w_1)\varphi(w_2)\dots\varphi(w_n)$ be a permutation of \mathfrak{S}_n obtained from $w \in \mathcal{P}_n$ by deleting the hats and circles of all w_i . For example, $\varphi(\widehat{3} \widehat{1} \textcircled{4} 2) = 3142$. Let $\mathcal{P}_n(\pi) = \{w \in \mathcal{P}_n : \varphi(w) = \pi\}$. Let $k\ell$ be a consecutive subword of $\pi \in \mathfrak{S}_n$. By using the above rules, we see that if $k < \ell$, then $k\ell$ can be decorated as follows:

$$k\ell, k\widehat{\ell}, \widehat{k}\ell, \widehat{k}\widehat{\ell};$$

If $k > \ell$, then $k\ell$ can be decorated as $k\ell, \textcircled{k}\ell, \widehat{k}\widehat{\ell}, \widehat{\textcircled{k}}\widehat{\ell}$. Therefore, $|\mathcal{P}_n(\pi)| = 2^n$ for any $\pi \in \mathfrak{S}_n$. It should be noted that $k\widehat{\ell}$ or $\widehat{k}\ell$ is a consecutive subword of $w \in \mathcal{P}_n$ if and only if $k < \ell$. Let the entry w_i be called an *ascent* (resp. a *descent*) of w if $\varphi(w_i) < \varphi(w_{i+1})$ ($\varphi(w_i) > \varphi(w_{i+1})$).

Also a conventional ascent is counted at the beginning of w . That is, we identify a decorated permutation $w = w_1 \cdots w_n$ with the word $w_0 w_1 \cdots w_n$, where $w_0 = 0$. Let $\text{asc}(w)$ be the number of ascents of w . Therefore, we obtain

$$2^n x A_n(x) = \sum_{w \in \mathcal{P}_n} x^{\text{asc}(w)}.$$

Example 2. *The following decorated permutations are generated from $\widehat{3} \widehat{14} 2$:*

$$\begin{aligned} & \widehat{3} \widehat{14} 25, \widehat{3} \widehat{14} 2 \widehat{5}, \widehat{3} \widehat{14} 5 2, \widehat{3} \widehat{14} \textcircled{5} 2, \widehat{3} \widehat{15} 42, \\ & \widehat{3} \widehat{1} \textcircled{5} 42, \widehat{3} \widehat{5} \widehat{14} 2, \widehat{3} \textcircled{5} \widehat{1} 4 2, \widehat{5} \widehat{3} \widehat{14} 2, \textcircled{5} \widehat{3} \widehat{14} 2. \end{aligned}$$

Example 3. *We have $\mathcal{P}_2 = \{12, \widehat{12}, \widehat{1} \widehat{2}, 21, \textcircled{2} 1, \widehat{2} \widehat{1}, \textcircled{2} \widehat{1}\}$.*

Let $I_{n,k}$ be the set of subsets of $[n]$ with cardinality k . Let $\text{Hat}(w)$ be the set of entries of w with hats and let $\text{hat}(w) = \#\text{Hat}(w)$. Let $\varphi(\text{Hat}(w))$ be a subset of $[n]$ obtained from $\text{Hat}(w)$ by deleting all hats and circles of all entries of $\text{Hat}(w)$. For example, if $w = \textcircled{5} \widehat{3} \widehat{14} 2$, then $\text{Hat}(w) = \{\widehat{1}, \widehat{3}, \textcircled{5}\}$ and $\varphi(\text{Hat}(w)) = \{1, 3, 5\}$. We define

$$\begin{aligned} \mathcal{PM}_{n,k} &= \{(S_1, S_2, I_{n,k}) : S_1 \in \mathcal{M}_{2k}, S_2 \in \mathcal{M}_{2n-2k}, I_{n,k} \in I_{n,k}\}, \\ \mathcal{P}_{n,k} &= \{w \in \mathcal{P}_n : \text{hat}(w) = k\}. \end{aligned}$$

In this subsection, we always assume that the weight of $w \in \mathcal{P}_{n,k}$ is $x^{\text{asc}(w)}$ and that of the pair of matchings (S_1, S_2) is $x^{\text{el}(S_1) + \text{el}(S_2)}$.

Now we start to construct a bijection, denoted by Φ , between $\mathcal{P}_{n,k}$ and $\mathcal{PM}_{n,k}$. When $n = 1$, set $\Phi(1) = (\emptyset, (12), \emptyset)$ and $\Phi(\widehat{1}) = ((12), \emptyset, \{1\})$. This gives a bijection between $\mathcal{P}_{1,k}$ and $\mathcal{PM}_{1,k}$. When $n = 2$, the bijection between $\mathcal{P}_{2,k}$ and $\mathcal{PM}_{2,k}$ is given as follows:

$$\begin{aligned} \Phi(12) &= (\emptyset, (12)(34), \emptyset), \quad \Phi(21) = (\emptyset, (13)(24), \emptyset) \\ \Phi(\textcircled{2} 1) &= (\emptyset, (14)(23), \emptyset), \quad \Phi(\widehat{1} \widehat{2}) = ((12), (12), \{1\}), \\ \Phi(\widehat{1} \widehat{2}) &= ((12), (12), \{2\}), \quad \Phi(\widehat{1} \widehat{2}) = ((12)(34), \emptyset, \{1, 2\}), \\ \Phi(\widehat{2} \widehat{1}) &= ((13)(24), \emptyset, \{1, 2\}), \quad \Phi(\textcircled{2} \widehat{1}) = ((14)(23), \emptyset, \{1, 2\}). \end{aligned}$$

Suppose Φ is a bijection between $\mathcal{P}_{m-1,k}$ and $\mathcal{PM}_{m-1,k}$ for all k , where $m \geq 3$. Assume that $w = w_1 w_2 \cdots w_{m-1} \in \mathcal{P}_{m-1,k}$, $\text{asc}(w) = i + j$ and $\text{Hat}(w) = \{w_{i_1}, w_{i_2}, \dots, w_{i_k}\}$. Let $\Phi(w) = (S_1, S_2, I_{m-1,k})$, where $S_1 \in \mathcal{M}_{2k}$, $S_2 \in \mathcal{M}_{2m-2k-2}$, $I_{m-1,k} = \varphi(\text{Hat}(w))$, $\text{el}(S_1) = i$, $\text{el}(S_2) = j$.

Consider the case $n = m$. Let w' be a decorated permutation generated from w . We first distinguish two cases: If $w' = wm$, then let $\Phi(w') = (S_1, S_2(2m-2k-1, 2m-2k), I_{m,k})$, where $I_{m,k} = \varphi(\text{Hat}(w))$; If $w' = w\widehat{m}$, then let $\Phi(w') = (S_1(2k+1, 2k+2), S_2, I_{m,k+1})$, where $I_{m,k+1} = \varphi(\text{Hat}(w)) \cup \{m\}$.

Now let $\ell_1 \ell_2$ be a consecutive subword of w . Firstly, suppose that ℓ_2 with no hat. We say ℓ_2 is a *unhat-ascent-top* (resp. *unhat-descent-bottom*) if $\varphi(\ell_1) < \varphi(\ell_2)$ (resp. $\varphi(\ell_1) > \varphi(\ell_2)$). Consider the following two cases:

- (c₁) If $w' = \cdots \ell_1 m \ell_2$ (resp. $w' = \cdots \ell_1 \hat{\otimes} \ell_2 \cdots$) and ℓ_2 is the p th unhat-ascent-top of w , then let $\Phi(w') = (S_1, S'_2, I_{m,k})$, where $I_{m,k} = \varphi(\text{Hat}(w))$ and S'_2 is obtained from S_2 by replacing p th marked block (a, b) by two blocks $(a, 2m - 2k - 1), (b, 2m - 2k)$ (resp. $(a, 2m - 2k), (b, 2m - 2k - 1)$).
- (c₂) If $w' = \cdots \ell_1 m \ell_2 \cdots$ (resp. $w' = \cdots \ell_1 \hat{\otimes} \ell_2 \cdots$) and ℓ_2 is the p th unhat-descent-bottom of w , then let $\Phi(w') = (S_1, S'_2, I_{m,k})$, where $I_{m,k} = \varphi(\text{Hat}(w))$ and S'_2 is obtained from S_2 by replacing p th unmarked block (a, b) by two blocks $(a, 2m - 2k - 1), (b, 2m - 2k)$ (resp. $(a, 2m - 2k), (b, 2m - 2k - 1)$).

Secondly, suppose that ℓ_2 with a hat. We say ℓ_2 is a *hat-ascent-top* (resp. *hat-descent-bottom*) if $\varphi(\ell_1) < \varphi(\ell_2)$ (resp. $\varphi(\ell_1) > \varphi(\ell_2)$). Consider the following two cases:

- (c₁) If $w' = \cdots \ell_1 \hat{m} \ell_2 \cdots$ (resp. $w' = \cdots \ell_1 \hat{\otimes} \ell_2 \cdots$) and ℓ_2 is the p th hat-ascent-top of w , then let $\Phi_1(w') = (S'_1, S_2, I_{m,k+1})$, where $I_{m,k+1} = \varphi(\text{Hat}(w)) \cup \{m\}$ and S'_1 is obtained from S_1 by replacing the p th marked block (a, b) by two blocks $(a, 2k + 1), (b, 2k + 2)$ (resp. $(a, 2k + 2), (b, 2k + 1)$).
- (c₂) If $w' = \cdots \ell_1 \hat{m} \ell_2 \cdots$ (resp. $w' = \cdots \ell_1 \hat{\otimes} \ell_2 \cdots$) and ℓ_2 is the p th hat-descent-bottom of w , then let $\Phi(w') = (S'_1, S_2, I_{m,k+1})$, where $I_{m,k+1} = \varphi(\text{Hat}(w)) \cup \{m\}$ and S'_1 is obtained from S_1 by replacing the p th unmarked block (a, b) by two blocks $(a, 2k + 1), (b, 2k + 2)$ (resp. $(a, 2k + 2), (b, 2k + 1)$).

After the above step, we write the obtained perfect matchings in standard form. Suppose that $w \in \mathcal{P}_{n,k}$ and $\Phi(w) = (S_1, S_2, I_{n,k})$. Then $\text{asc}(w) = i + j$ if and only if $\text{el}(S_1) + \text{el}(S_2) = i + j$. By induction, we see that Φ is the desired bijection between $\mathcal{PM}_{n,k}$ to $\mathcal{P}_{n,k}$ for all k , which also gives a constructive proof of (3).

Example 4. Let $w = \hat{3} \hat{1}42\hat{\otimes}5 \in \mathcal{P}_{6,4}$. The correspondence between w and $\Phi(w)$ is built up as follows:

$$\begin{aligned} \hat{1} &\Leftrightarrow ((12), \emptyset, \{1\}); \\ \hat{1}2 &\Leftrightarrow ((12), (12), \{1\}); \\ \hat{3} \hat{1}2 &\Leftrightarrow ((13)(24), (12), \{1, 3\}); \\ \hat{3} \hat{1}42 &\Leftrightarrow ((13)(24), (13)(24), \{1, 3\}); \\ \hat{3} \hat{1}42\hat{5} &\Leftrightarrow ((13)(24)(5, 6), (13)(24), \{1, 3, 5\}); \\ \hat{3} \hat{1}42\hat{\otimes}5 &\Leftrightarrow ((13)(24)(5, 8)(6, 7), (13)(24), \{1, 3, 5, 6\}). \end{aligned}$$

2.2. Signed permutations and pairs of perfect matchings.

In this subsection, we shall write signed permutations of \mathcal{B}_n as $\pi = \pi(0)\pi(1)\pi(2)\cdots\pi(n)$, where some elements are associated with the minus sign and $\pi(0) = 0$. As usual, we denote by \bar{i} the negative element $-i$. For $\pi \in \mathcal{B}_n$, let $\text{RLMIN}(\pi) = \{\pi(i) : |\pi(i)| < |\pi(j)| \text{ for all } j > i\}$. For example, $\text{RLMIN}(\bar{3} \bar{1}42\bar{5}) = \{\bar{1}, 2, \bar{5}\}$. Let $\text{rlmin}(\pi) = \#\text{RLMIN}(\pi)$. It is clear that if $\pi \in \mathfrak{S}_n$, then $\text{rlmin}(\pi)$ is the number of *right-to-left minima* of π . Thus

$$\sum_{\pi \in \mathcal{B}_n} x^{\text{rlmin}(\pi)} = 2^n \sum_{\pi \in \mathfrak{S}_n} x^{\text{rlmin}(\pi)} = 2^n x(x+1)(x+2)\cdots(x+n-1) \quad \text{for } n \geq 1.$$

Definition 5. A block of π is a maximal subsequence of consecutive elements of π ending with $\pi(i) \in \text{RLMIN}(\pi)$ and not contain any other element of $\text{RLMIN}(\pi)$.

It is clear that any π has a unique decomposition as a sequence of its blocks. If $\text{rlmin}(\pi) = k$, then we write $\pi \mapsto B_1 B_2 \cdots B_k$, where B_i is i th block of π . A *bar-block* (resp. *unbar-block*) is a block ending with a negative (resp. positive) element. Let $\text{Bar}(\pi)$ be the union of elements of bar-blocks of π and let $\text{bar}(\pi) = \#\text{Bar}(\pi)$. We define a map θ by

$$\theta(\text{Bar}(\pi)) = \{|\pi(i)| : \pi(i) \in \text{Bar}(\pi)\}.$$

Set $\text{NBar}(\pi) = [n]/\text{Bar}(\pi)$. For example, if $\pi = \bar{3} \bar{1} 4 2 \bar{6} \bar{7} \bar{5}$, then $\pi \mapsto [\bar{3} \bar{1}][42][\bar{6} \bar{7} \bar{5}]$, $[\bar{3} \bar{1}]$ and $[\bar{6} \bar{7} \bar{5}]$ are bar-blocks of π , $\text{Bar}(\pi) = \{\bar{6}, \bar{5}, \bar{3}, \bar{1}, 7\}$, $\text{bar}(\pi) = 5$, $\theta(\text{Bar}(\pi)) = \{1, 3, 5, 6, 7\}$. and $\text{NBar}(\pi) = \{2, 4\}$.

Let $\mathcal{B}_{n,k} = \{\pi \in \mathcal{B}_n : \text{bar}(\pi) = k\}$ and let

$$\mathcal{BM}_{n,k} = \{(T_1, T_2, I_{n,k}) : T_1 \in \mathcal{M}_{2k}, T_2 \in \mathcal{M}_{2n-2k}, I_{n,k} \in \mathcal{I}_{n,k}\},$$

where $\mathcal{I}_{n,k}$ is the set of subsets of $[n]$ with cardinality k . In this subsection, we always assume that the weight of $\pi \in \mathcal{B}_{n,k}$ is $x^{\text{des}_B(\pi)}$ and that of the pair of matchings (T_1, T_2) is $x^{\text{el}(T_1) + \text{ol}(T_2)}$.

Along the same lines as the proof of (3), we start to construct a bijection, denoted by Ψ , between $\mathcal{B}_{n,k}$ and $\mathcal{BM}_{n,k}$. When $n = 1$, set $\Psi(1) = (\emptyset, (12), \emptyset)$ and $\Psi(\bar{1}) = ((12), \emptyset, \{1\})$. This gives a bijection between $\mathcal{B}_{1,k}$ and $\mathcal{BM}_{1,k}$. When $n = 2$, the bijection Ψ between $\mathcal{B}_{2,k}$ and $\mathcal{BM}_{2,k}$ is given as follows:

$$\begin{aligned} \Psi(12) &= (\emptyset, (12)(34), \emptyset), & \Psi(21) &= (\emptyset, (13)(24), \emptyset) \\ \Psi(\bar{2}1) &= (\emptyset, (14)(23), \emptyset), & \Psi(\bar{1}2) &= ((12), (12), \{1\}), \\ \Psi(1\bar{2}) &= ((12), (12), \{2\}), & \Psi(\bar{1}\bar{2}) &= ((12)(34), \emptyset, \{1, 2\}), \\ \Psi(2\bar{1}) &= ((13)(24), \emptyset, \{1, 2\}), & \Psi(\bar{2}\bar{1}) &= ((14)(23), \emptyset, \{1, 2\}). \end{aligned}$$

Suppose Ψ is a bijection between $\mathcal{B}_{m-1,k}$ and $\mathcal{BM}_{m-1,k}$ for all k , where $m \geq 3$. Assume that $\pi = \pi(1)\pi(2) \cdots \pi(m-1) \in \mathcal{B}_{m-1,k}$, $\text{des}_B(\pi) = i + j$ and $\text{Bar}(\pi) = \{\pi(i_1), \pi(i_2), \dots, \pi(i_k)\}$. Let $\Psi(\pi) = (T_1, T_2, I_{m-1,k})$, where $T_1 \in \mathcal{M}_{2k}$, $T_2 \in \mathcal{M}_{2m-2k-2}$, $I_{m-1,k} = \theta(\text{Bar}(\pi))$, $\text{el}(T_1) = i$, $\text{ol}(T_2) = j$. Consider the case $n = m$. Let π' be obtained from π by inserting the entry m (resp. \bar{m}) into π . We first distinguish two cases: If $\pi' = \pi m$, then let

$$\Psi(\pi') = (T_1, T_2(2m-2k-1, 2m-2k), I_{m,k}),$$

where $I_{m,k} = \theta(\text{Bar}(\pi))$; If $\pi' = \pi \bar{m}$, then let $\Psi(\pi') = (T_1(2k+1, 2k+2), T_2, I_{m,k+1})$, where $I_{m,k+1} = \theta(\text{Bar}(\pi)) \cup \{m\}$.

For $0 \leq i \leq m-2$, consider the consecutive subword $\pi(i)\pi(i+1)$ of π . Firstly, suppose that $\pi(i+1) \in \text{NBar}(\pi)$. We say $\pi(i+1)$ is a *unbar-ascent-top* (resp. *unbar-descent-bottom*) if $\pi(i) < \pi(i+1)$ (resp. $\pi(i) > \pi(i+1)$). Consider the following two cases:

- (c₁) If $\pi' = \cdots \pi(i)m\pi(i+1) \cdots$ (resp. $\pi' = \cdots \pi(i)\bar{m}\pi(i+1) \cdots$) and $\pi(i+1)$ is the p th unbar-ascent-top of π , then let $\Psi(\pi') = (T_1, T'_2, I_{m,k})$, where T'_2 is obtained from T_2 by replacing the p th marked block (a, b) by two blocks $(a, 2m-2k-1), (b, 2m-2k)$ (resp. $(a, 2m-2k), (b, 2m-2k-1)$) and $I_{m,k} = \theta(\text{Bar}(\pi))$.

(c_2) If $\pi' = \cdots \pi(i)m\pi(i+1)\cdots$ (resp. $\pi' = \cdots \pi(i)\overline{m}\pi(i+1)\cdots$) and $\pi(i+1)$ is the p th unbar-descent-bottom of π , then let $\Psi(\pi') = (T_1, T'_2, I_{m,k})$, where T'_2 is obtained from T_2 by replacing the p th unmarked block (a, b) by two blocks $(a, 2m - 2k - 1), (b, 2m - 2k)$ (resp. $(a, 2m - 2k), (b, 2m - 2k - 1)$) and $I_{m,k} = \theta(\text{Bar}(\pi))$.

Secondly, suppose that $\pi(i+1) \in \text{Bar}(\pi)$. We say $\pi(i+1)$ is a *bar-ascent-top* (resp. *bar-descent-bottom*) if $\pi(i) < \pi(i+1)$ (resp. $\pi(i) > \pi(i+1)$). Consider the following two cases:

- (c_1) If $\pi' = \cdots \pi(i)m\pi(i+1)\cdots$ (resp. $\pi' = \cdots \pi(i)\overline{m}\pi(i+1)\cdots$) and $\pi(i+1)$ is the p th bar-ascent-top of π , then let $\Psi(\pi') = (T'_1, T_2, I_{m,k+1})$, where T'_1 is obtained from T_1 by replacing the p th unmarked block (a, b) by two blocks $(a, 2k + 1), (b, 2k + 2)$ (resp. $(a, 2k + 2), (b, 2k + 1)$) and $I_{m,k+1} = \theta(\text{Bar}(\pi)) \cup \{m\}$.
- (c_2) If $\pi' = \cdots \pi(i)m\pi(i+1)\cdots$ (resp. $\pi' = \cdots \pi(i)\overline{m}\pi(i+1)\cdots$) and $\pi(i+1)$ is the p th bar-descent-bottom of π , then let $\Psi(\pi') = (T'_1, T_2, I_{m,k+1})$, where T'_1 is obtained from T_1 by replacing the p th marked block (a, b) by two blocks $(a, 2k + 1), (b, 2k + 2)$ (resp. $(a, 2k + 2), (b, 2k + 1)$) and $I_{m,k+1} = \theta(\text{Bar}(\pi)) \cup \{m\}$.

After the above step, we write the obtained perfect matchings in standard form. Suppose that $\pi \in \mathcal{B}_{n,k}$ and $\Psi(\pi) = (T_1, T_2, I_{n,k})$. Then $\text{des}_B(\pi) = i + j$ if and only if $\text{el}(T_1) + \text{ol}(T_2) = i + j$. By induction, we see that Ψ is the desired bijection between $\mathcal{B}_{n,k}$ to $\mathcal{BM}_{n,k}$ for all k , which also gives a constructive proof of (4).

Example 6. Let $\pi = \overline{3} \overline{1}42\overline{6}7\overline{5}$. The correspondence between π and $\Psi(\pi)$ is built up as follows:

$$\begin{aligned} \overline{1} &\Leftrightarrow ((12), \emptyset, \{1\}); \\ \overline{1}2 &\Leftrightarrow ((12), (12), \{1\}); \\ \overline{3} \overline{1}2 &\Leftrightarrow ((14)(23), (12), \{1, 3\}); \\ \overline{3} \overline{1}42 &\Leftrightarrow ((14)(23), (13)(24), \{1, 3\}); \\ \overline{3} \overline{1}42\overline{5} &\Leftrightarrow ((14)(23)(5, 6), (13)(24), \{1, 3, 5\}); \\ \overline{3} \overline{1}42\overline{6} \overline{5} &\Leftrightarrow ((13)(24)(5, 8)(6, 7), (13)(24), \{1, 3, 5, 6\}); \\ \overline{3} \overline{1}42\overline{6}7\overline{5} &\Leftrightarrow ((13)(24)(5, 8)(6, 9)(7, 10), (13)(24), \{1, 3, 5, 6, 7\}). \end{aligned}$$

3. THE STRING PERMUTATIONS OF THE SECOND KIND

Stirling permutations were introduced by Gessel and Stanley [16]. Let $[n]_2 = \{1, 1, 2, 2, \dots, n, n\}$. A *Stirling permutation* of order n is a permutation of the multiset $[n]_2$ such that every element between the two occurrences of i is greater than i for each $i \in [n]$. For example, $\mathcal{Q}_2 = \{1122, 1221, 2211\}$. Let $\sigma_1\sigma_2\cdots\sigma_{2n} \in \mathcal{Q}_n$. An index i is a *descent* of σ if $\sigma_i > \sigma_{i+1}$ or $i = 2n$. Let $C(n, k)$ be the number of Stirling permutations of $[n]_2$ with k descents. Following [16, Eq. (6)], the numbers $C(n, k)$ satisfy the recurrence relation

$$C(n, k) = kC(n-1, k) + (2n-k)C(n-1, k-1) \quad (7)$$

for $n \geq 2$, with the initial conditions $C(1, 1) = 1$ and $C(1, 0) = 0$. The *second-order Eulerian polynomial* is defined by

$$C_n(x) = \sum_{k=1}^n C(n, k)x^k.$$

In recent years, there has been much work on Stirling permutations (see [2, 14, 18, 17, 24, 26]). In particular, Bóna [2] introduced the plateau statistic on Stirling permutations, and proved that descents and plateaus have the same distribution over \mathcal{Q}_n . Given $\sigma \in \mathcal{Q}_n$, the index i is called a *plateau* if $\sigma_i = \sigma_{i+1}$. We say that an index $i \in [2n - 1]$ is an *ascent plateau* if $\sigma_{i-1} < \sigma_i = \sigma_{i+1}$, where $\sigma_0 = 0$. Let $\text{ap}(\sigma)$ be the number of the ascent plateaus of σ . For example, $\text{ap}(\mathbf{221133}) = 2$. Very recently, we present a combinatorial proof of the following identity (see [25, Theorem 3]):

$$\sum_{\sigma \in \mathcal{Q}_n} x^{\text{ap}(\sigma)} = \sum_{M \in \mathcal{M}_{2n}} x^{\text{el}(M)}. \quad (8)$$

Motivated by (5) and (8), we shall introduce Stirling permutations of the second kind.

Let $[k]^n$ denote the set of words of length n in the alphabet $[k]$. For $\omega = \omega_1\omega_2 \cdots \omega_n \in [k]^n$, the reduction of ω , denoted by $\text{red}(\omega)$, is the unique word of length n obtained by replacing the i th smallest entry by i . For example, $\text{red}(33224547) = 22113435$.

Definition 7. A permutation σ of the multiset $[n]_2$ is a *Stirling permutation of the second kind of order n* whenever σ can be written as a nonempty disjoint union of its distinct cycles and σ has a standard cycle form satisfying the following conditions:

- (i) For each $i \in [n]$, the two copies of i appear in exactly one cycle;
 - (ii) Each cycle is written with one of its smallest entry first and the cycles are written in increasing order of their smallest entry;
 - (iii) The reduction of the word formed by all entries of each cycle is a Stirling permutation.
- In other words, if $(c_1, c_2, \dots, c_{2k})$ is a cycle of σ , then $\text{red}(c_1c_2 \cdots c_{2k}) \in \mathcal{Q}_k$.

Let \mathcal{Q}_n^2 denote the set of Stirling permutations of the second kind of order n . In the following discussion, we always write $\sigma \in \mathcal{Q}_n^2$ in standard cycle form.

Example 8.

$$\begin{aligned} \mathcal{Q}_1^2 &= \{(11)\}, \mathcal{Q}_2^2 = \{(11)(22), (1122), (1221)\}, \\ \mathcal{Q}_3^2 &= \{(11)(22)(33), (11)(2233), (11)(2332), (1133)(22), (1331)(22), (1122)(33), (112233), \\ &\quad (112332), (113322), (133122), (1221)(33), (122133), (122331), (123321), (133221)\}. \end{aligned}$$

Let $(c_1, c_2, \dots, c_{2k})$ be a cycle of σ . An entry c_i is called a *cycle plateau* (resp. *cycle ascent*) if $c_i = c_{i+1}$ (resp. $c_i < c_{i+1}$), where $1 \leq i < 2k$. Let $\text{cplat}(\pi)$ and $\text{casc}(\pi)$ be the number of cycle plateaus and cycle ascents of π , respectively. For example, $\text{cplat}(\mathbf{(1221)(33)}) = 2$ and $\text{casc}(\mathbf{(1221)(33)}) = 1$. Now we present a dual result of [2, Proposition 1].

Proposition 9. For $n \geq 1$, we have

$$C_n(x) = \sum_{\pi \in \mathcal{Q}_n^2} x^{\text{cplat}(\pi)} = \sum_{\pi \in \mathcal{Q}_n^2} x^{\text{casc}(\pi)+1}.$$

Proof. There are two ways in which a permutation $\sigma' \in \mathcal{Q}_n^2$ with k cycle plateaus can be obtained from a permutation $\sigma \in \mathcal{Q}_{n-1}^2$. If $\text{cplat}(\sigma) = k$, then we can put the two copies of n right after a cycle plateau of σ . This gives k possibilities. If $\text{cplat}(\sigma) = k - 1$, then we can append a new cycle (nn) right after σ or insert the two copies of n into any of the remaining $2n - 2 - (k - 1) = 2n - k - 1$ positions. This gives $2n - k$ possibilities. Comparing with (7), this completes the proof of $C_n(x) = \sum_{\pi \in \mathcal{Q}_n^2} x^{\text{cplat}(\pi)}$. Along the same lines, one can easily prove the assertion for cycle ascents. This completes the proof. \square

Let $(c_1, c_2, \dots, c_{2k})$ be a cycle of σ , where $k \geq 2$. An entry c_i is called a *cycle ascent plateau* if $c_{i-1} < c_i = c_{i+1}$, where $2 \leq i \leq 2k - 1$. Denote by $\text{cap}(\sigma)$ (resp. $\text{cyc}(\sigma)$) the number of cycle ascent plateaus (resp. cycles) of σ . For example, $\text{cap}((1221)(33)) = 1$. We define

$$Q_n(x, q) = \sum_{\sigma \in \mathcal{Q}_n^2} x^{\text{cap}(\sigma)} q^{\text{cyc}(\sigma)},$$

$$Q(x, q; z) = 1 + \sum_{n \geq 1} Q_n(x, q) \frac{z^n}{n!}.$$

Our main result of this section is the following.

Theorem 10. *The polynomials $Q_n(x, q)$ satisfy the recurrence relation*

$$Q_{n+1}(x, q) = (q + 2nx)Q_n(x, q) + 2x(1 - x) \frac{\partial}{\partial x} Q_n(x, q) \quad (9)$$

for $n \geq 0$, with the initial condition $Q_0(x) = 1$. Moreover,

$$Q(x, q; z) = \left(\sqrt{\frac{x-1}{x - e^{2z}(x-1)}} \right)^q. \quad (10)$$

Proof. Given $\sigma \in \mathcal{Q}_n^2$. Let σ_i be an element of \mathcal{Q}_{n+1}^2 obtained from σ by inserting the two copies of $n + 1$, in the standard cycle decomposition of σ , right after $i \in [n]$ or as a new cycle $(n + 1, n + 1)$ if $i = n + 1$. It is clear that

$$\text{cyc}(\sigma_i) = \begin{cases} \text{cyc}(\sigma), & \text{if } i \in [n]; \\ \text{cyc}(\sigma) + 1, & \text{if } i = n + 1. \end{cases}$$

Therefore, we have

$$\begin{aligned} Q_{n+1}(x, q) &= \sum_{\pi \in \mathcal{Q}_{n+1}^2} x^{\text{cap}(\pi)} q^{\text{cyc}(\pi)} \\ &= \sum_{i=1}^{n+1} \sum_{\sigma_i \in \mathcal{Q}_n^2} x^{\text{cap}(\sigma_i)} q^{\text{cyc}(\sigma_i)} \\ &= \sum_{\sigma \in \mathcal{Q}_n^2} x^{\text{cap}(\sigma)} q^{\text{cyc}(\sigma)+1} + \sum_{i=1}^n \sum_{\sigma_i \in \mathcal{Q}_n^2} x^{\text{cap}(\sigma_i)} q^{\text{cyc}(\sigma_i)} \\ &= qQ_n(x, q) + \sum_{\sigma \in \mathcal{Q}_n^2} (2\text{cap}(\sigma)x^{\text{cap}(\sigma)} + (2n - 2\text{cap}(\sigma))x^{\text{cap}(\sigma)+1})q^{\text{cyc}(\sigma)} \end{aligned}$$

and (9) follows. By rewriting (9) in terms of the exponential generating function $Q(x, q; z)$, we have

$$(1 - 2xz) \frac{\partial}{\partial z} Q(x, q; z) = qQ(x, q; z) + 2x(1 - x) \frac{\partial}{\partial x} Q(x, q; z). \quad (11)$$

It is routine to check that the generating function

$$\tilde{Q}(x, q; z) = \left(\sqrt{\frac{x-1}{x - e^{2z(x-1)}}} \right)^q$$

satisfies (11). Also, this generating function gives $\tilde{Q}(x, q; 0) = 1$, $\tilde{Q}(x, 0; z) = 1$ and $\tilde{Q}(0, q; z) = e^{qz}$. Hence $Q(x, q; z) = \tilde{Q}(x, q; z)$. \square

Combining (1) and (10), we get $Q(x, q; z) = M^q(x, z)$. Thus $Q_n(x, 1) = M_n(x)$. Moreover, it follows from (9) that $Q_{n+1}(1, q) = (q + 2n)Q_n(1, q)$. So the following corollary is immediate.

Corollary 11. *For $n \geq 1$, we have*

$$\sum_{\sigma \in \mathcal{Q}_n^2} q^{\text{cyc}(\sigma)} = q(q+2) \cdots (q+2n-2).$$

We now introduce a statistic on \mathcal{Q}_n that is equidistributed with the cycle statistic on \mathcal{Q}_n^2 . Denote by $[i, j]$ the interval of all integers between i and j , where $i \leq j$. In particular, when $i = j$, we denote by $[i]$ the singleton interval. For $\sigma = \sigma_1 \sigma_2 \cdots \sigma_{2n} \in \mathcal{Q}_n$, if $\sigma_i > \sigma_{i+1}$, all of the different elements before σ_{i+1} appear a second time and all of these different entries construct an interval, then this interval is called a *descent interval* of σ , where $i \in [2n]$ and $\sigma_{2n+1} = 0$. For example, if $\sigma = 44223311$ and $\tau = 113322$, then the descent intervals of σ are $[4]$, $[2, 4]$ and $[1, 4]$, and that of τ is $[1, 3]$.

Let $\text{desi}(\sigma)$ be the number of descent intervals of σ . Define

$$L_n(q) = \sum_{\sigma \in \mathcal{Q}_n} q^{\text{desi}(\sigma)}.$$

Let $\sigma^{(i)} \in \mathcal{Q}_{n+1}$ be obtained from $\sigma \in \mathcal{Q}_n$ by inserting two copies of $n+1$ before σ_i . It is evident that

$$\text{desi}(\sigma^{(i)}) = \begin{cases} \text{desi}(\sigma) + 1, & \text{if } i = 1; \\ \text{desi}(\sigma), & \text{otherwise.} \end{cases}$$

Thus

$$L_{n+1}(q) = (q + 2n)L_n(q).$$

The following result is immediate.

Proposition 12. *For $n \geq 1$, we have*

$$\sum_{\sigma \in \mathcal{Q}_n} q^{\text{desi}(\sigma)} = \sum_{\sigma \in \mathcal{Q}_n^2} q^{\text{cyc}(\sigma)}.$$

Let $A_n(x) = \sum_{k=0}^{n-1} \langle n \rangle_k x^k$. The numbers $\langle n \rangle_k$ are called Eulerian numbers and satisfy the recurrence relation

$$\langle n \rangle_k = (k+1) \langle n-1 \rangle_k + (n-k) \langle n-1 \rangle_{k-1},$$

with initial conditions $\langle 0 \rangle = 1$ and $\langle k \rangle$ for $k \geq 1$ (see [3, 6] for instance). Hence

$$A_{n+1}(x) = (1 + nx)A_n(x) + x(1 - x)A'_n(x), \quad (12)$$

Let \mathcal{CQ}_n denote the set of Stirling permutations of \mathcal{Q}_n^2 with only one cycle, which can be named as the set of *cyclic Stirling permutations*. Define

$$Y_n(x) = \sum_{\sigma \in \mathcal{CQ}_n} x^{\text{cap}(\sigma)}.$$

Comparing (9) with (12), we get the following corollary.

Corollary 13. *For $n \geq 1$, we have*

$$Y_{n+1}(x) = 2^n x A_n(x).$$

4. THE DISTRIBUTION OF CYCLE ASCENT PLATEAUS AND FIXED POINTS ON \mathcal{Q}_n^2

Given $\sigma \in \mathcal{Q}_n^2$. Let the entry $k \in [n]$ be called a *fixed point* of σ if (kk) is a cycle of σ . The number of fixed points of σ is defined by

$$\text{fix}(\sigma) = \#\{k \in [n] : (kk) \text{ is a cycle of } \sigma\}.$$

For example, $\text{fix}((1133)(22)) = 1$. Define

$$P_n(x, y, q) = \sum_{\sigma \in \mathcal{Q}_n^2} x^{\text{cap}(\sigma)} y^{\text{fix}(\sigma)} q^{\text{cyc}(\sigma)},$$

$$P(x, y, q; z) = \sum_{n \geq 0} P_n(x, y, q) \frac{z^n}{n!}.$$

Now we present the main result of this section.

Theorem 14. *For $n \geq 1$, the polynomials $P_n(x, y, q)$ satisfy the recurrence relation*

$$P_{n+1}(x, y, q) = qyP_n(x, y, q) + qx \sum_{k=0}^{n-1} \binom{n}{k} P_k(x, y, q) 2^{n-k} A_{n-k}(x), \quad (13)$$

with the initial conditions $P_0(x, y, q) = 1, P_1(x, y, q) = yq$. Moreover,

$$P_{n+1}(x, y, q) = (2nx + qy)P_n(x, y, q) + 2x(1 - x) \frac{\partial}{\partial x} P_n(x, y, q) + 2x(1 - y) \frac{\partial}{\partial y} P_n(x, y, q). \quad (14)$$

Furthermore,

$$P(x, y, q; z) = e^{qz(y-1)} Q(x, q; z). \quad (15)$$

In the following, we shall prove Theorem 14 by using context-free grammars. For an alphabet A , let $\mathbb{Q}[[A]]$ be the rational commutative ring of formal power series in monomials formed from letters in A . Following [7], a context-free grammar over A is a function $G : A \rightarrow \mathbb{Q}[[A]]$ that replace a letter in A by a formal function over A . The formal derivative D is a linear operator defined with respect to a context-free grammar G . More precisely, the derivative $D = D_G : \mathbb{Q}[[A]] \rightarrow \mathbb{Q}[[A]]$ is defined as follows: for $x \in A$, we have $D(x) = G(x)$; for a monomial u in $\mathbb{Q}[[A]]$, $D(u)$ is defined so that D is a derivation, and for a general element $q \in \mathbb{Q}[[A]]$, $D(q)$ is defined by linearity.

Lemma 15. *If $A = \{a, b, c, d\}$ and $G = \{a \rightarrow qab^2, b \rightarrow b^{-1}c^2d^2, c \rightarrow cd^2, d \rightarrow c^2d\}$, then*

$$D^n(a) = a \sum_{\sigma \in \mathcal{Q}_n^2} q^{\text{cyc}(\sigma)} b^{2\text{fix}(\sigma)} c^{2\text{cap}(\sigma)} d^{2n-2\text{fix}(\sigma)-2\text{cap}(\sigma)}. \quad (16)$$

Proof. Let $\mathcal{Q}_n^2(i, j, k) = \{\sigma \in \mathcal{Q}_n^2 : \text{cyc}(\sigma) = i, \text{fix}(\sigma) = j, \text{cap}(\sigma) = k\}$. Given $\sigma \in \mathcal{Q}_n^2(i, j, k)$. We now introduce a labeling scheme for σ :

- (i) Put a superscript label a at the end of σ and a superscript q before each cycle of σ ;
- (ii) If k is a fixed point of σ , then we put a superscript label b right after each k ;
- (iii) Put superscript labels c immediately before and right after each cycle ascent plateau;
- (iv) In each of the remaining positions except the first position of each cycle, we put a superscript label d .

When $n = 1$, we have $\mathcal{Q}_1^2(1, 1, 0) = \{q(1^b1^b)^a\}$. When $n = 2$, we have $\mathcal{Q}_2^2(2, 2, 0) = \{q(1^b1^b)^q(2^b2^b)^a\}$ and $\mathcal{Q}_2^2(1, 0, 1) = \{q(1^d1^c2^c2^d)^a, q(1^c2^c2^d1^d)^a\}$. Let $n = m$. Suppose we get all labeled permutations in $\mathcal{Q}_m^2(i, j, k)$ for all i, j, k , where $m \geq 2$. We consider the case $n = m + 1$. Let $\sigma' \in \mathcal{Q}_{m+1}^2$ be obtained from $\sigma \in \mathcal{Q}_m^2(i, j, k)$ by inserting two copies of the entry $m + 1$ into σ . Now we construct a correspondence, denoted by ϑ , between σ and σ' . Consider the following cases:

- (c₁) If the two copies of $m + 1$ are put at the end of σ as a new cycle $((m + 1)(m + 1))$, then we leave all labels of σ unchanged except the last cycle. In this case, the correspondence ϑ is defined by

$$\sigma = \dots(\dots)^a \xleftarrow{\vartheta} \sigma' = \dots(\dots)^q((m + 1)^b(m + 1)^b)^a,$$

which corresponds to the operation $a \rightarrow qab^2$. Moreover, $\sigma' \in \mathcal{Q}_{m+1}^2(i + 1, j + 1, k)$.

- (c₂) If the two copies of $m + 1$ are inserted to a position of σ with label b , then ϑ corresponds to the operation $b \rightarrow b^{-1}c^2d^2$. In this case, $\sigma' \in \mathcal{Q}_{m+1}^2(i, j - 1, k + 1)$.
- (c₃) If the two copies of $m + 1$ are inserted to a position of σ with label c , then ϑ corresponds to the operation $c \rightarrow cd^2$. In this case, $\sigma' \in \mathcal{Q}_{m+1}^2(i, j, k)$.
- (c₄) If the two copies of $m + 1$ are inserted to a position of σ with label d , then ϑ corresponds to the operation $c \rightarrow c^2d$. In this case, $\sigma' \in \mathcal{Q}_{m+1}^2(i, j, k + 1)$.

By induction, we see that ϑ is the desired correspondence between permutations in \mathcal{Q}_m^2 and \mathcal{Q}_{m+1}^2 , which also gives a constructive proof of (16). \square

Lemma 16. *If $A = \{b, c, d\}$ and $G = \{b \rightarrow b^{-1}c^2d^2, c \rightarrow cd^2, d \rightarrow c^2d\}$, then*

$$D^n(b^2) = 2^n \sum_{k=0}^{n-1} \binom{n}{k} c^{2k+2} d^{2n-2k} = 2^n d^{2n} c^2 A_n \left(\frac{c^2}{d^2} \right) \text{ for } n \geq 1.$$

Proof. Note that $D(b^2) = 2c^2d^2$. Hence $D^n(b^2) = 2D^{n-1}(c^2d^2)$ for $n \geq 1$. Note that $D(c^2d^2) = 2(c^2d^4 + c^4d^2)$. Assume that

$$D^n(b^2) = 2^n \sum_{k=0}^{n-1} F(n, k) c^{2n-2k} d^{2k+2}.$$

Since

$$D(D^n(b^2)) = 2^{n+1} \sum_{k=0}^{n-1} (n-k)F(n,k)c^{2n-2k}d^{2k+4} + 2^{n+1} \sum_{k=0}^{n-1} (k+1)F(n,k)c^{2n-k+2}d^{2k+2},$$

there follows

$$F(n+1, k) = (k+1)F(n, k) + (n-k+1)F(n, k-1).$$

We see that the coefficients $F(n, k)$ satisfy the same recurrence relation and initial conditions as $\langle n \rangle_k$, so they agree. \square

Proof of Theorem 14:

By Lemma 15 and Lemma 16, we get

$$\begin{aligned} D^{n+1}(a) &= qD^n(ab^2) \\ &= q \sum_{k=0}^n \binom{n}{k} D^k(a)D^{n-k}(b^2) \\ &= qb^2D^n(a) + q \sum_{k=0}^{n-1} \binom{n}{k} D^k(a)2^{n-k}d^{2n-2k}c^2A_{n-k} \left(\frac{c^2}{d^2} \right). \end{aligned}$$

Taking $c^2 = x, b^2 = y$ and $d^2 = 1$ in both sides of the above identity, we immediately get (13). Set $S_n(i, j, k) = \#\mathcal{Q}_n^2(i, j, k)$. The following recurrence relation follows easily from the proof of Lemma 15:

$$S_{n+1}(i, j, k) = S_n(i-1, j-1, k) + 2(j+1)S_n(i, j+1, k-1) + 2kS_n(i, j, k) + 2(n-j-k+1)S_n(i, j, k-1).$$

Multiplying both sides of the last recurrence relation by $q^i y^j x^k$ and summing for all i, j, k , we immediately get (14). Note that

$$\begin{aligned} P_n(x, y, q) &= \sum_{i=0}^n \binom{n}{i} (yq - q)^i \sum_{\sigma \in \mathcal{Q}_n^2} x^{\text{cap}(\sigma)} q^{\text{cyc}(\pi)} \\ &= \sum_{i=0}^n \binom{n}{i} (yq - q)^i Q_{n-i}(x, q). \end{aligned}$$

Thus $P(x, y, q; z) = e^{qz(y-1)}Q(x, q; z)$. This completes the proof of Theorem 14.

Given $\sigma \in \mathcal{Q}_n^2$. We say that σ is a *Stirling derangement* if σ has no fixed points. Let \mathcal{DQ}_n be the set of Stirling derangements of \mathcal{Q}_n^2 . Let $R_{n,k}(x, q)$ be the coefficient of y^k in $P_n(x, y, q)$. Note that $R_{n,0}$ is the corresponding enumerative polynomials on \mathcal{DQ}_n . Set $R_n(x, q) = R_{n,0}(x, q)$. Note that

$$\begin{aligned} R_{n,k}(x, q) &= \sum_{\substack{\sigma \in \mathcal{Q}_n^2 \\ \text{fix}(\sigma)=k}} x^{\text{cap}(\sigma)} q^{\text{cyc}(\pi)} \\ &= \binom{n}{k} q^k \sum_{\sigma \in \mathcal{DQ}_{n-k}} x^{\text{cap}(\sigma)} q^{\text{cyc}(\pi)} \\ &= \binom{n}{k} q^k R_{n-k}(x, q). \end{aligned}$$

Comparing the coefficients of both sides of (14), we get the following result.

Theorem 17. For $n \geq 1$, the polynomials $R_n(x, q)$ satisfy the recurrence relation

$$R_{n+1}(x, q) = 2nxR_n(x, q) + 2x(1-x)\frac{\partial}{\partial x}R_n(x, q) + 2nxqR_{n-1}(x, q), \quad (17)$$

with the initial conditions $R_1(x, q) = 0, R_2(x, q) = 2qx, R_3(x, q) = 4qx(1+x)$.

Let $q_n = \#\mathcal{D}\mathcal{Q}_n$. Then the following corollary is immediate.

Corollary 18. For $n \geq 1$, the numbers q_n satisfy the recurrence relation $q_{n+1} = 2n(q_n + q_{n-1})$, with the initial conditions $q_0 = 1, q_1 = 0$ and $q_2 = 2$.

Note that $\#\mathcal{Q}_n^2 = \mathcal{Q}_n = (2n-1)!!$. Then

$$\sum_{n \geq 0} \#\mathcal{Q}_n^2 \frac{z^n}{n!} = \frac{1}{\sqrt{1-2z}}.$$

Thus

$$\sum_{n \geq 0} q_n \frac{z^n}{n!} = \frac{e^{-z}}{\sqrt{1-2z}},$$

which can be easily proved by using the *exponential formula* (see [3, Theorem 3.50]). It should be noted that q_{n+1} is also the number of minimal number of 1-factors in a $2n$ -connected graph having at least one 1-factor (see [1]). It would be interesting to study the relationship between Stirling permutations of the second kind and $2n$ -connected graphs.

In recent years, there has been much work on derangements polynomials of Coxeter groups (see [8, 11, 19, 20, 31] for instance). For each $\pi \in \mathfrak{S}_n$, an index i is called *excedance* (resp. *anti-excedance*) if $\pi(i) > i$ (resp. $\pi(i) < i$). Let $\text{exc}(\pi)$ be the number of excedances of π . A permutation $\pi \in \mathfrak{S}_n$ is a *derangement* if $\pi(i) \neq i$ for any $i \in [n]$. Let \mathcal{D}_n denote the set of derangements of \mathfrak{S}_n . The *derangements polynomial* is defined by

$$d_n(x) = \sum_{\pi \in \mathcal{D}_n} x^{\text{exc}(\pi)}.$$

Brenti [4, Proposition 5] derived that

$$d(x, z) = \sum_{n \geq 0} d_n(x) \frac{z^n}{n!} = \frac{1-x}{e^{xz} - xe^z}. \quad (18)$$

The *Stirling derangement polynomial* is defined by

$$R_n(x) = \sum_{\sigma \in \mathcal{D}\mathcal{Q}_n} x^{\text{cap}(\sigma)}.$$

Let

$$S(x, z) = \sum_{n \geq 0} R_n(x) \frac{z^n}{n!}.$$

Taking $y = 0$ and $q = 1$ in (15), we have

$$S(x, z) = \sqrt{\frac{x-1}{xe^{2z} - e^{2xz}}}. \quad (19)$$

Thus

$$S^2(x, z) = d(x, 2z),$$

which is a dual result of (5). Equivalently,

$$2^n d_n(x) = \sum_{k=0}^n \binom{n}{k} R_k(x) R_{n-k}(x).$$

Moreover, combining (17), (19) and [21, Corollary 2.4], we immediately get the following result.

Proposition 19. *For $n \geq 2$, the polynomial $R_n(x)$ is symmetric and has only simple real zeros.*

Let $i^2 = \sqrt{-1}$. From (19), putting $x = -1$, we deduce the expression

$$S(-1, z) = \sqrt{\frac{2}{e^{2z} + e^{-2z}}} = \sqrt{\sec(2iz)}. \quad (20)$$

Note that $\sec(z)$ is an even function. Therefore, for $n \geq 1$, we have

$$\sum_{\sigma \in \mathcal{DQ}_n} (-1)^{\text{cap}(\sigma)} = \begin{cases} 0, & \text{if } n = 2k - 1; \\ (-1)^k h_k, & \text{if } n = 2k, \end{cases}$$

where the number h_n is defined by the following series expansion:

$$\sqrt{\sec(2iz)} = \sum_{n \geq 0} (-1)^n h_n \frac{z^{2n}}{(2n)!}.$$

The first few of the numbers h_n are $h_0 = 1, h_1 = 2, h_2 = 28, h_3 = 1112, h_4 = 87568$. It should be noted that the numbers h_n also count permutations of \mathfrak{S}_{4n} having the following properties:

- (a) The permutation can be written as a product of disjoint cycles with only two elements;
- (b) For $i \in [2n]$, indices $2i - 1$ and $2i$ are either both excedances or both anti-excedances.

For example, when $n = 1$, there are only two permutations having the desired properties: $(1, 3)(2, 4)$ and $(14)(23)$. This kind of permutations was introduced by Sukumar and Hodges [30].

5. CONCLUDING REMARKS

A natural generalization of Stirling permutations is k -Stirling permutations. Let j^i denote the i copies of j , where $i, j \geq 1$. We call a permutation of the multiset $\{1^k, 2^k, \dots, n^k\}$ a k -Stirling permutation of order n if for each i , $1 \leq i \leq n$, all entries between the two occurrences of i are at least i . One can introduce k -Stirling permutations of the second kind along the same line as in Definition 7.

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