Groups With At Most Twelve Subgroups

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In [2] I showed that the finite groups with a specified number of subgroups can always be described as a finite list of similarity classes. Neil Sloane suggested that I submit the corresponding sequence, the number of similarity classes with n subgroups, to his Online Encyclopedia of Integer Sequences [4]. I thought this would be a quick calculation until I discovered that my "example" case in the paper ($n = 6$) was wrong (as noted in [3]). I realized that producing a reliable count required a full proof. This note contains the proof behind my computation of the first 12 terms of the sequence.

For completeness, we include a section of [2]:

> Now, we saw [above] that cyclic Sylow subgroups of G which are direct factors allow many non-isomorphic groups to have the same number of subgroups. [This lemma] tells us that these are precisely the cyclic Sylow subgroups which lie in the center of G. Consequently, if $p_1, \ldots, p_c$ are the primes which divide $|G|$ such that a Sylow p_i-subgroup is cyclic and central, then we can write
>
> $$G = P_1 \times P_2 \times \cdots \times P_c \times O_{\pi'}(G)$$
>
> where each P_i is a Sylow p_i-subgroup, the set $\pi = \{p_1, \ldots, p_c\}$ and $O_{\pi'}(G)$ is the largest normal subgroup of G with order not divisible by any prime in π. We will write $\widetilde{G} = O_{\pi'}(G)$. In other words, $\widetilde{G}$ is the unique subgroup of G left after factoring out the cyclic, central Sylow subgroups. It is the part of G that we can hope to restrict in terms of the number of subgroups. On the other hand, if we substituted a different prime (relatively prime to $|G|$) for any one of the p_i, it would not affect the number of subgroups of G. Thus, we are led to define the following equivalence relation (which we will call "similar" for this note).
>
> **Definition.** Let G and H be two finite groups. Write G as a product of cyclic central Sylow subgroups and $\widetilde{G}$ as above. Hence, $G = P_1 \times P_2 \times \cdots \times P_c \times \widetilde{G}$, and similarly, $H = Q_1 \times Q_2 \times \cdots \times Q_d \times \widetilde{H}$. We say that G is *similar* to H if, and only if, the following three conditions hold:
>
> 1. $\widetilde{G}$ is isomorphic to $\widetilde{H}$;
> 2. $c = d$;

3. $n_i = m_i$ for some reordering, where $|P_i| = p_i^{n_i}$ and $|Q_i| = q_i^{m_i}$.

From the comments before the definition, we see that if G is similar to H, then they will have the same number of subgroups. Also, note that the equivalence class of G is determined by $\widetilde{G}$ and the multiset $[n_1, \ldots, n_c]$. If we have a bound on the number of subgroups of G, then this will bound c and each n_i. Therefore, the only remaining hole in our theorem will be filled with the following lemma.

Lemma. *Given $k > 1$ there are a finite number of isomorphism classes of groups $G = \widetilde{G}$ (i.e. G has no cyclic, central Sylow subgroups) having at most k subgroups.*

We wish to identify all similarity classes of groups with at most 12 subgroups.
To begin, we will find all groups with $G = \widetilde{G}$ and at most 12 subgroups.

1 Abelian Groups

If G is abelian, then no Sylow p-subgroup can be cyclic. This means that if G is not a p-group, then it has at least $p+1$ subgroups of order p and $q+1$ subgroups of order q for some distinct primes p and q and so at least $(p+1)(q+1) \geq 12$ non-trivial subgroups. So, G must be a non-cyclic p-group.

If G contains $Z_p \times Z_p \times Z_p$ then it has at least $2p^2 + 2p + 4 \geq 16$ subgroups. Consequently, $G = Z_{p^r} \times Z_{p^s}$ with $r \geq s \geq 1$.

Now, $Z_{p^2} \times Z_{p^2}$ has $p^2 + 3p + 5 \geq 15$ subgroups, so we must have $s = 1$. Furthermore, $Z_{p^4} \times Z_p$ has $4p + 6 \geq 14$ subgroups, so $r \leq 3$.

Hence, the abelian groups with $G = \widetilde{G}$ and at most 12 subgroups are:
$r = 1$: ($p + 3$ subgroups) p = 2, 3, 5, 7
$r = 2$: ($2p + 4$ subgroups) p = 2, 3
$r = 3$: ($3p + 5$ subgroups) p = 2

2 Non-Abelian Groups

Now we consider non-abelian G. Let q be any prime dividing $|G|$. If Sylow q is not normal, there are at least $q+1$ Sylow q-subgroups. If Sylow q is normal, there is a q-complement. If the complement is normal, then the Sylow q cannot be cyclic (else $G \neq \widetilde{G}$) and so has at least $q+1$ maximal subgroups. Finally, if the complement is not normal, then we have at least q complements and 1 Sylow q. So, in every case, with the trival subgroup and G, the number of subgroups is at least $q+3$. This implies $q \leq 9$.

??? Can $|G|$ be div by all of 2, 3, 5, and 7 ??? As below, at least 2 must be normal and their product has a complement with 2 primes that can have at most 3 subgroups $\Rightarrow\Leftarrow$ impossible.

Suppose $|G|$ is divisible by 3 distinct primes p, q, r.

If none of the Sylows are normal, we have at least $(p+1)+(q+1)+(r+1) \geq 13$ subgroups, so at least one is normal. Call it an r-subgroup N. Complementing N, we have a two prime group H in G and every subgroup of H gives rise to at least two subgroups of G (itself and itself times N). So, H has at most 6 subgroups.

???Must we have $H = \widetilde{H}$??? NO: Z_6 on Z_7.

Lemma. *A group H divisible by 2 primes with at most 6 subgroups is either $Z_p \times Z_q$, $Z_p \times Z_{q^2}$, or S_3.*

Proof. If H is abelian, one sylow has 2 subgroups, the other 2 or 3 subgroups, so $H = Z_p \times Z_q$ or $Z_p \times Z_{q^2}$. If H has a non-abelian sylow, the sylow has at least 3 maximal subgroups, 1 prime order, 1 triv, 1 whole subgrp, plus the other sylow $\Rightarrow \geq 7$ subgrps, which is too many. So both sylows of H are abelian and so at least one is not normal. Assume WLOG $p < q$. Then there are at least p+1 non-normal sylows, one other sylow, the whole group, and trivial $\Rightarrow \geq p + 4$ subgroups. So we must have equality on our estimate which means, $p = 2$, sylow 2 is not normal, $p + 1$ divides $|G|$, so $q = 3$, sylow 3 is normal, sylows have no non-trivial subgroups $\Rightarrow H = S_3$. ☐

If H is abelian with cyclic sylows, then neither Sylow can act trivially on N or it would be central cyclic in G. Therefore N cannot normalize the sylows or H and so the number of conjugates of each is at least r. This gives at least $3r$ subgroups of G. Furthermore we have G, N, 1, and NH_p and NH_q, so at least $3r + 5$ subgroups. Consequently, we must have $r = 2$.

Our list of subgroups includes only N and 1 from subgroups of N. So, if N has at least 4 subgroups, then G will have at least 13 subgroups. Therefore, N is cyclic of order 2 or 4. But odd primes cannot act non-trivially on a cyclic group of order 2 or 4, so this situation is impossible.

Last, we consider the case of $H = S_3$. In this case, $r \geq 5$. If H acts trivially on N, then N cannot be cyclic and so has at least 8 subgroups. Since S_3 has 6 subgroups, G will have at least 48 subgroups. So, H acts non-trivially on N and a sylow 2, H_2, must also act non-trivally on N. In particular, N does not normalize H or H_2 and, as above, this gives us at least $2r$ subgroups of G. We also have G, N, and 1. Since r is at least 5, this implies G has more than 12 subgroups.

Consequently, $|G|$ must be divisible by at most two primes.

3 Groups with two primes in the order

Suppose $|G| = p^a q^b$, $p < q$ primes and G is non-abelian.

The argument for two primes is more complicated than I would like. I'll use indentation to organize assumptions (like in computer code).

If neither sylow is cyclic, then we have $(p+1)+(q+1)$ maximals in the sylows, the two sylows and the whole group and trivial. That is, the number of subgroups is at least $p + q + 6$. This can only be less than or equal to 12 when $p = 2, q = 3$. Furthermore, the close estimate means each sylow must be normal and prime squared order. That forces G to be abelian $\Rightarrow\Leftarrow$.

If sylow p is cyclic,

If sylow p is normal, then the sylow q cannot act non-trivially and so the sylow p is cyclic and central $\Rightarrow\Leftarrow$.

else sylow p is not normal, then the number of sylow p is $\geq q$.

If sylow q not cyclic, we have at least $q+1$ maximals and so at least $2q+4$ subgroups. Again the estimate forces sylow q to be $Z_3 \times Z_3$ and $p = 2$, at least 10 subgroups. Furthermore, we cannot have 9 sylow 2's, so there is a central subgroup of order 3. Counting again, we have whole, trivial, one 9, four 3's, three sylow 2's, three sylow 2 times central 3 $\Rightarrow \geq 13$ subgroups. $\Rightarrow\Leftarrow$

else sylow q is cyclic, then G must be supersolvable since both sylows are cyclic and it follows the sylow q must be normal. Sylow p must act non-trivially on q, so $(p, q) = (2, 3), (2, 5), (2, 7)$, or $(3, 7)$.
The only case where the action is not that of an element of order p is when $p = 2, q = 5$ when Z_4 acts faithfully on Z_5. In this case, G will have a quotient group of order 20 isomorphic to Z_4 acting faithfully on Z_5. Since this group has 14 subgroups, we do not need to consider such G. Therefore, we can assume the sylow p acts as an element of order p. The sylow p fixes only the identity in sylow q (only possible non-triv action). This means the sylow p is self-normalizing and so the number of sylow p subgroups is q^b. Now the subgroup of order p^{a-1} centralizes the sylow q and so is the intersection of the sylow p's. Hence the non-triv p-subgroups are order $p, \ldots, p^{a-1}$, and q^b subgroups order p^a. There are b non-triv q-subgroups and $(a-1)\cdot b$ proper abelian subgroups divisible by pq. Furthermore, for each non-triv, proper q-subgroup, order q^c, we can multiply that by the sylow p's to get subgroups of order $p^a q^c$. There will be q^c different sylow p's giving the same subgroup of order $p^a q^c$ and so a total of q^{b-c} such subgroups. Consequently, the number of (non-abelian) proper subgroups divisible by $p^a q$ will be $q + q^2 + \cdots + q^{b-1} = (q^b - q)/(q-1)$. Including trivial and whole group, the number of subgroups is:

$$(a - 1 + q^b) + b + (a-1)b + \frac{q^b - q}{q-1} + 2 = q^b + \frac{q^b - q}{q-1} + a(b+1) + 1.$$

The corresponding values are shown in the following tables.

$q = 3$

a \ b	1	2
1	6	16
2	8	19
3	10	22
4	12	25
5	14	28

$q = 5$

a \ b	1	2
1	8	34
2	10	37
3	12	40
4	14	43

$q = 7$

a \ b	1	2
1	10	60
2	12	63
3	14	66

Since the $q = 7$ table applies to both $p = 2$ and $p = 3$ we see there are 11 groups with at most 12 subgroups in this case.

else sylow p is not cyclic, and so sylow q must be cyclic and not central.

If neither sylow is normal, then number of sylow p is at least q and the number of sylow q is at least $q+1$. Including trivial and whole, we have at least $2q+3$ subgroups, which means $q < 5$ and so we have $q = 3, p = 2$. We have at least three sylow 2's and at least four sylow 3's, triv, whole, and one sylow 2 will contain three subgroups of order 2. However, that's already 12 subgroups and we have more order 2 subgroups in the other sylow 2's. $\Rightarrow\Leftarrow$

If sylow p is normal, then it cannot be cyclic, else it would be central. Furthermore, $Z_p \times Z_p \times Z_p$ has too many subgroups, so the sylow p must be a two-generator group. So, the sylow p has $p+1$ maximal subgroups, there are at least $q+1$ sylow q's and with sylow p itself, triv, and whole, we have at least $p+q+5$ subgroups. If $p \geq 3$, this is too many, so we must have $p = 2$. If $q \geq 5$, then our count gives at least 12 subgroups. The only way to avoid going over 12 would be to have the sylow p be $Z_2 \times Z_2$, sylow q be Z_5. But Z_5 has no non-trivial action on $Z_2 \times Z_2$, and so we are left only with the case $p = 2, q = 3$, and at least 10 subgroups. Now, the alternating group A_4 satisfies these conditions and has 10 subgroups. If sylow 3 has order larger than 3, then there will be a central 3-subgroup whose product with the various 2-subgroups will give more than 12 subgroups. So, the sylow 3 must be order 3. If sylow 2 has order larger than 4, then the frattini subgroup of the sylow 2 and the frattini subgroup times the various sylow 3-subgroups give more than 12 subgroups. So, A_4 is the only group in this case.

If sylow q is normal, we are already assuming the sylow q is cyclic. Furthermore, the sylow p is not normal and not cyclic. So we have at least q sylow p's and the sylow p's have at least $p+1$ maximal subgroups.

If two sylow p's have a common maximal subgroup, M, then the normalizer of M will contain at least two sylow p-subgroups of G and so must be divisible by q. Thus the intersection of the normalizer with the sylow q-subgroup is a non-trivial central q subgroup. Therefore, our subgroups include q sylow p's, q sylow p's times the central q, at least $p+1$ maximal subgroups of sylow p, each of those $p+1$ times the central q, the sylow q, the central q, triv, and whole for at least $2q + 2p + 6 \geq 16$ subgroups. $\Rightarrow\Leftarrow$

else no shared maximal subgroups, which means we have at least $q(p+1)$ maximal subgroups of sylow p's altogether. Therefore we have at least these $q(p+1)$ subgroups, q sylow p-subgroups, one sylow q, triv, and whole. That is, $qp + 2q + 3 \geq 15$ subgroups. $\Rightarrow\Leftarrow$

Hence, the non-abelian groups with $G = \widetilde{G}$, $|G|$ divisible by at least two primes, and at most 12 subgroups are:

$Z_p \ltimes Z_q$: ($q+3$ subgroups) $(p, q) =$ (2, 3), (2, 5), (2, 7), or (3, 7)
$Z_{p^2} \ltimes Z_q$ with action of order p: ($q+5$ subgroups) $(p, q) =$ (2, 3), (2, 5), (2, 7), or (3, 7)
$Z_8 \ltimes Z_q$ with action of order 2: ($q+7$ subgroups) $q = 3$ or 5
$Z_{16} \ltimes Z_3$ with action of order 2: (12 subgroups)
A_4: (10 subgroups)

4 p-Groups

Finally, we have the case of G a non-abelian p-group, $|G| = p^n$.

The group $Z_p \times Z_p \times Z_p$ has at least 16 subgroups, so G must be a two-generator group. Consequently, we have G, $p{+}1$ maximals, the Frattini subgroup, and triv $\Rightarrow \geq p{+}4$ subgroups.

When p is odd we will have at least $p + 1$ order p, one of which might be the Frattini, so we'd have at least $2p + 4 \Rightarrow p = 2$ or 3.

Assume $|G'| = p$ and $Z(G)$ cyclic. Then $G' \subset Z(G)$ and for $x, y \in G$, $[x^p, y] = [x, y]^p = 1$ so x^p is central and $\Phi(G) \subset Z(G)$. Now $|G : \Phi(G)| = p^2$ and so we must have $\Phi(G) = Z(G)$ is cyclic order p^{n-2}.

If G has a single subgroup of order p, then $p = 2$ and G is generalized quaternion. Now, each generalized quaternion 2-group contains the next smaller as a subgroup. So, we can just check: Q_8 has 6 subgroups, Q_{16} has 11 subgroups, and Q_{32} has 20. That is, G must be Q_8 or Q_{16}.

Otherwise, G has more than one subgroup of order p and so we can choose $S \subset G$ with S order p and not in the center. Then $M = SZ(G)$ is an abelian subgroup isomorphic to $Z_p \times Z_{p^{n-2}}$. From the abelian case above, we see that the number of subgroups of M is given in the following table.

		n		
		3	4	5
p	2	5	8	11
	3	6	10	14

In addition to the subgroups in M, G also has p other maximal subgroups, and G itself. So, increasing each table entry by $p + 1$ we see the only possibilities for at most 12 subgroups are $|G| = 8$, 16, or 27.

By brute force check we find for order 8 the dihedral group D_8 with 10 subgroups, for order 16 a group with presentation $\langle a, b | a^2, b^8, b^a = b^5 \rangle$ having 11 subgroups, and for order 27 the extraspecial group of exponent 9 with 10 subgroups.

Thus we have found five groups with $|G'| = p$ and cyclic center. Now any non-abelian finite p-group will have such a group as a homomorphic image.

Note that no generalized quaternion group can have a smaller generalized quaternion group as a homomorphic image. One easy way to see this is to note that $Q_n/Z(Q_n) = D_{n/2}$ and any non-abelian image of a dihedral group will have more than one involution. We will use this fact several times below.

Suppose $|G| = 3^5$ and choose K maximal such that $\overline{G} = G/K$ is non-abelian. It follows that $\overline{G}$ will have $|\overline{G}'| = 3$ and $Z(\overline{G})$ cyclic. If $|K| \leq 3$, it follows from above that $\overline{G}$, and so G, will have more than 12 subgroups. If $|K| = 3^2$, then $|\overline{G}| = 3^3$ and so must be the extraspecial group of exponent 3^2. Thus, $\overline{G}$ has 10 subgroups. The only way we could have at most 12 subgroups in G is if K is cyclic and every subgroup either contains K or is order 3 or 1 in K. However, this implies that G has only one subgroup of order 3, which is impossible. Consequently, no non-abelian 3-group of order at least 3^5 can have at most 12 subgroups. We know from above that there is only one such group of order 3^3 and a computer check shows that there are no examples of order 3^4 (these groups have at least 14 subgroups).

Now suppose $|G| = 2^6$ and choose K maximal such that $\overline{G} = G/K$ is non-abelian. As above, $\overline{G}$ will have $|\overline{G}'| = 2$ and $Z(\overline{G})$ cyclic. If $|K| \leq 2$, it follows from above that $\overline{G}$, and so

G, will have more than 12 subgroups. If $|K| = 2^2$, then $|\overline{G}| = 2^4$ and so we see from above that $\overline{G}$ is one of two groups each of which have 11 subgroups. Since K has at least 2 proper subgroups, G must have 13 or more subgroups. Finally, consider $|K| = 2^3$. Then $\overline{G}$ must be D_8 or Q_8 with 10 or 6 subgroups respectively. Since K has at least 3 proper subgroups, $\overline{G}$ cannot be D_8 and so must be Q_8 with 6 subgroups. Even if all of the subgroups of G either contained K or were contained in K, then K would have to have at most 7 subgroups (6 proper). However, the only groups of order 2^3 with 7 or fewer subgroups are Z_8 and Q_8. Since each of these have a single subgroup of order 2, our assumption would force G to be generalized quaternion, which it clearly is not. Thus we see that no non-abelian 2-group of order at least 2^6 can have number of subgroups less than or equal to 12. A computer check shows there are no examples of order 2^5 (these groups have at least 14 subgroups) and only the two groups mentioned above for order 2^4.

Hence, the non-abelian p-groups with $G = \widetilde{G}$ and at most 12 subgroups are:

D_8: (10 subgroups)

Q_8: (6 subgroups)

Q_{16}: (11 subgroups)

$Z_2 \ltimes Z_8$: (11 subgroups)

E_{27} extraspecial order 27, exponent 9: (10 subgroups)

5 Conclusion

Collecting all of our results, we have:

n	Groups with $G = \widetilde{G}$ and n subgroups
1	Trivial group
2	
3	
4	
5	$Z_2 \times Z_2$
6	$Z_3 \times Z_3$, S_3, Q_8
7	
8	$Z_2 \times Z_4$, $Z_5 \times Z_5$, D_{10}, $Z_4 \ltimes Z_3$
9	
10	$Z_3 \times Z_9$, $Z_7 \times Z_7$, D_{14}, A_4, $Z_3 \ltimes Z_7$, $Z_4 \ltimes Z_5$, $Z_8 \ltimes Z_3$, E_{27}
11	$Z_2 \times Z_8$, Q_{16}, $Z_2 \ltimes Z_8$
12	$Z_4 \ltimes Z_7$, $Z_9 \ltimes Z_7$, $Z_8 \ltimes Z_5$, $Z_{16} \ltimes Z_3$

In particular the sequence of the number of groups with $G = \widetilde{G}$ and n subgroups would be:

$$1, 0, 0, 0, 1, 3, 0, 4, 0, 8, 3, 4, \ldots$$

Forming the direct product with a coprime, cyclic group of order p^k will multiply the number of subgroups by $k + 1$. Thus we find groups with n subgroups corresponding to various factorizations of n.

n	Similarity class representatives with n subgroups
1	Trivial group
2	Z_2
3	Z_{2^2}
4	Z_{2^3}, $Z_2 \times Z_3$
5	Z_{2^4}, $Z_2 \times Z_2$
6	Z_{2^5}, $Z_2 \times Z_{3^2}$, $Z_3 \times Z_3$, S_3, Q_8
7	Z_{2^6}
8	Z_{2^7}, $Z_2 \times Z_{3^3}$, $Z_2 \times Z_3 \times Z_5$, $Z_2 \times Z_4$, $Z_5 \times Z_5$, D_{10}, $Z_4 \ltimes Z_3$
9	Z_{2^8}, $Z_{2^2} \times Z_{3^2}$
10	Z_{2^9}, $Z_2 \times Z_{3^4}$, $Z_2 \times Z_2 \times Z_3$, $Z_3 \times Z_9$, $Z_7 \times Z_7$, D_{14}, A_4, $Z_3 \ltimes Z_7$, $Z_4 \ltimes Z_5$, $Z_8 \ltimes Z_3$, E_{27}
11	$Z_{2^{10}}$, $Z_2 \times Z_8$, Q_{16}, $Z_2 \ltimes Z_8$
12	$Z_{2^{11}}$, $Z_2 \times Z_{3^5}$, $Z_3 \times Z_3 \times Z_2$, $S_3 \times Z_5$, $Q_8 \times Z_3$, $Z_{2^2} \times Z_{3^3}$, $Z_2 \times Z_3 \times Z_{5^2}$, $Z_4 \ltimes Z_7$, $Z_9 \ltimes Z_7$, $Z_8 \ltimes Z_5$, $Z_{16} \ltimes Z_3$

In the following version of the previous table we represent classes using p, q, r to represent primes which do not occur anywhere else in the group order. This makes it a bit easier to recognize the infinite classes.

n	Similarity classes with n subgroups
1	Trivial group
2	Z_p
3	Z_{p^2}
4	Z_{p^3}, $Z_p \times Z_q$
5	Z_{p^4}, $Z_2 \times Z_2$
6	Z_{p^5}, $Z_p \times Z_{q^2}$, $Z_3 \times Z_3$, S_3, Q_8
7	Z_{p^6}
8	Z_{p^7}, $Z_p \times Z_{q^3}$, $Z_p \times Z_q \times Z_r$, $Z_2 \times Z_4$, $Z_5 \times Z_5$, D_{10}, $Z_4 \ltimes Z_3$
9	Z_{p^8}, $Z_{p^2} \times Z_{q^2}$
10	Z_{p^9}, $Z_p \times Z_{q^4}$, $Z_2 \times Z_2 \times Z_p$, $Z_3 \times Z_9$, $Z_7 \times Z_7$, D_{14}, A_4, $Z_3 \ltimes Z_7$, $Z_4 \ltimes Z_5$, $Z_8 \ltimes Z_3$, E_{27}
11	$Z_{p^{10}}$, $Z_2 \times Z_8$, Q_{16}, $Z_2 \ltimes Z_8$
12	$Z_{p^{11}}$, $Z_p \times Z_{q^5}$, $Z_3 \times Z_3 \times Z_p$, $S_3 \times Z_p$, $Q_8 \times Z_p$, $Z_{p^2} \times Z_{q^3}$, $Z_p \times Z_q \times Z_{r^2}$, $Z_4 \ltimes Z_7$, $Z_9 \ltimes Z_7$, $Z_8 \ltimes Z_5$, $Z_{16} \ltimes Z_3$

In conclusion, the sequence of number of similarity classes with a given number of subgroups begins:

$$1, 1, 1, 2, 2, 5, 1, 7, 2, 11, 4, 11, \ldots$$

Note: In [1], Miller lists the groups with specified number of subgroups where the number of subgroups runs from 1 through 9. His lists agree with ours.

References

[1] G.A. Miller, Groups having a small number of subgroups, *Proc. Natl. Acad. Sci. U S A*, vol. 25 (1939) 367–371.

[2] M.C. Slattery, On a property motivated by groups with a specified number of subgroups, *Amer. Math. Monthly*, vol. 123 (2016) 78–81.

[3] M.C. Slattery, Editor's End Notes, *Amer. Math. Monthly*, vol. 123 (2016) 515.

[4] N.J.A. Sloane, The on-line encyclopedia of integer sequences, `http://oeis.org`.