THE SEQUENCE OF MIDDLE DIVISORS IS UNBOUNDED

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ABSTRACT. The sequence of middle divisors is shown to be unbounded. For a given number n, $a_{n,0}$ is the number of divisors of n in between $\sqrt{n/2}$ and $\sqrt{2n}$. We explicitly construct a sequence of numbers $n(i)$ and a list of divisors in the interesting range, so that the length of the list goes to infinity as i increases.

Keywords: integer sequence, polynomial coefficient
2010 Mathematics subject classification: 11B83(primary), 11T55(secondary)

1. Introduction

In [1], Kassel and Reutenauer studies the zeta function of the Hilbert scheme of n points in the two-torus. The polynomial counting ideals of codimension n in the Laurent algebra in two variables turns out to have an interesting quotient, whose middle coefficient $a_{n,0}$ has a direct description:

$$a_{n,0} = \left| \{d \,:\, d|n \,,\, \frac{\sqrt{2n}}{2} < d \leq \sqrt{2n}\} \right|.$$

We follow the symbolism from [1], which the reader should also consult for more motivation. In a talk at the conference *Algebraic geometry and Mathematical Physics 2016*, in honour of A. Laudal's 80th birthday, Kassel discussed the results in [1] and asked whether the sequence $a_{n,0}$ is bounded or not. Evidently it grows very slowly. The sequence is included in the online encyclopedia of integer sequences as sequence A067742 [2].

In this short note, we will show that the sequence is unbounded. The idea is to choose n such that $\sqrt{n/2}$ is a divisor, and to multiply this divisor with a number slightly larger than one repeatedly, making sure that the product still divides n as long as it is smaller than $\sqrt{2n}$.

2. Unboundedness of the sequence

Theorem 2.1. *Let*

$$a_{n,0} = \left| \{d \,:\, d|n \,,\, \frac{\sqrt{2n}}{2} < d \leq \sqrt{2n}\} \right|.$$

Then

$$\limsup_{n\to\infty} a_{n,0} = \infty$$

More precisely, for any $i \geq 1$ *define* $s_{max} = \ln(2)/\ln(1+i^{-1})$ *and*

$$n(i) = 2(i+1)^{\lceil 2s_{max} \rceil} \cdot i^{2\lceil s_{max} \rceil}. \tag{1}$$

Then $\lim_{i\to\infty} a_{n(i),0} = \infty$.

Proof. With the choice of $n(i)$ from (1), we have that

$$\sqrt{n/2} = (i+1)^{\lceil s_{max} \rceil} \cdot i^{\lceil s_{max} \rceil},$$

a divisor of $n(i)$. For each $s = 1, 2, \ldots, \lfloor s_{max} \rfloor$, consider

$$d(s) = \sqrt{n/2} \left(\frac{i+1}{i}\right)^s = (i+1)^{\lceil s_{max} \rceil + s} \cdot i^{\lceil s_{max} \rceil - s}.$$

This divides $n(i)$ as long as $\lceil s_{max} \rceil + s \leq 2\lceil s_{max} \rceil$ and $\lceil s_{max} \rceil - s \geq 0$, which in both cases translates simply to $s \leq \lfloor s_{max} \rfloor$. Thus we have exhibited a number of divisors, so that

$$a_{n(i),0} \geq \lfloor s_{max} \rfloor.$$

Note also that s_{max} is chosen so that

$$\left(\frac{i+1}{i}\right)^{s_{max}} = 2.$$

Therefore all the $d(s)$ are in the interesting interval. Since

$$\lim_{i\to\infty} s_{max}(i) = \lim_{i\to\infty} \frac{\ln 2}{\ln(1+i^{-1})} = \infty$$

this proves the theorem. □

The sequence $n(i)$ grows very quickly whereas as the sequence $s_{max}(i)$ grows slowly. It is likely that the minimal n needed to find a given value for $a_{n,0}$ is a lot smaller than what is constructed in the proof.

Acknowledgements

Thanks are due to C. Kassel for telling me about this problem and encouraging me to write down the proof.

References

[1] C. Kassel and C. Reutenauer: The zeta function of the Hilbert scheme of n points on a two-dimensional torus, preprint, arXiv:1505.07229

[2] N. J. A. Sloane, editor, The On-Line Encyclopedia of Integer Sequences, published electronically at https://oeis.or, Sequence A067742

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