OCCURRENCE GRAPHS OF PATTERNS IN PERMUTATIONS

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Abstract. We define the *occurrence graph* $G_p(\pi)$ of a pattern p in a permutation π as the graph with the occurrences of p in π as vertices and edges between the vertices if the occurrences differ by exactly one element. We then study properties of these graphs. The main theorem in this paper is that every *hereditary property* of graphs gives rise to a *permutation class*.

Keywords: permutation patterns, graphs

1. Introduction

We define the *occurrence graph* $G_p(\pi)$ of a pattern p in a permutation π as the graph where each vertex represents an occurrences of p in π. Vertices share an edge if the occurrences they represent differ by exactly one element. We study properties of these graphs and show that every *hereditary property* of graphs gives rise to a *permutation class*.

The motivation for defining these graphs comes from the algorithm discussed in the proof of the *Simultaneous Shading Lemma* by Claesson, Tenner and Ulfarsson in [1]. The steps in that algorithm can be thought of as constructing a path in an occurrence graph, terminating at a desirable occurrence of a pattern.

2. Basic definitions

In this article we will be working with permutations and undirected, simple graphs. The reader does not need to have prior knowledge of either as we will define both.

Definition 2.1. A *graph* is an ordered pair $G = (V, E)$ where V is a set of *vertices* and E is a set of two element subsets of V. The elements $\{u, v\} \in E$ are called *edges* and connect the vertices. Two vertices u and v are *neighbors* if $\{u, v\} \in E$. The *degree* of a vertex v is the

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number of neighbors it has. A graph $G' = (V', E')$ is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq \big\{\{u, v\} \in E \colon u, v \in V'\big\}$.

The reader might have noticed that our definition of a graph excludes those with loops and multiple edges between vertices. We often write uv as shorthand for $\{u, v\}$ and in case of ambiguity we use $V(G)$ and $E(G)$ instead of V and E.

Definition 2.2. Let V' be a subset of V. The *induced subgraph* $G[V']$ is a subgraph of G with vertex set V' and edges $\{uv \in E \colon u, v \in V'\}$.

Two graphs G and H are *isomorphic* if there exist a bijection from $V(G)$ to $V(H)$ such that two vertices in G are neighbors if and only if the corresponding vertices (according to the bijection) in H are neighbors. We denote this with $G \cong H$.

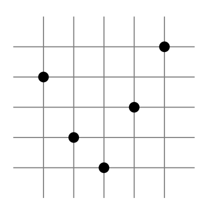
We let $[\![1, n]\!]$ denote the integer interval $\{1, \ldots, n\}$.

Definition 2.3. A *permutation of length* n is a bijective function $\sigma \colon [\![1, n]\!] \to [\![1, n]\!]$. We denote the permutation with $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$. The permutation $\mathrm{id}_n = 12\cdots n$ is the *identity permutation* of length n.

The *set of permutations* of length n is denoted by $\mathfrak{S}_n$. The set of all permutations is $\mathfrak{S} = \cup_{n=0}^{+\infty}\mathfrak{S}_n$. Note that $\mathfrak{S}_0 = \{\mathscr{E}\}$, where $\mathscr{E}$ is the empty permutation, and $\mathfrak{S}_1 = \{1\}$. There are $n!$ permutations of length n.

Definition 2.4. A *grid plot* or *grid representation* of a permutation $\pi \in \mathfrak{S}_n$ is the subset $\mathrm{Grid}(\pi) = \big\{(i, \pi(i)) : i \in [\![1, n]\!]\big\}$ of the Cartesian product $[\![1, n]\!]^2 = [\![1, n]\!] \times [\![1, n]\!]$.

Example 2.5. Let $\pi = 42135$. The grid representation of π is

Grid(42135) $=$ 

The central definition in the theory of permutation patterns is how permutations lie inside other (larger) permutations. Before we define that precisely we need a preliminary definition:

Definition 2.6. Let $a_1, \ldots, a_k$ be distinct integers. The *standardisation* of the string $a_1 \cdots a_k$ is the permutation $\sigma \in \mathfrak{S}_k$ such that $a_1 \cdots a_k$ is order isomorphic to $\sigma(1) \cdots \sigma(k)$. In other words, for every

$i \neq j$ we have $a_i < a_j$ if and only if $\sigma(i) < \sigma(j)$. We denote this with $\mathrm{st}(a_1 \cdots a_k) = \sigma$.

For example $\mathrm{st}(253) = 132$ and $\mathrm{st}(132) = 132$.

Definition 2.7. Let p be a permutation of length k. We say that the permutation $\pi \in \mathfrak{S}_n$ *contains* p if there exist indices $1 \leq i_1 < \cdots < i_k \leq n$ such that $\mathrm{st}\big(\pi(i_1) \cdots \pi(i_k)\big) = p$. The subsequence $\pi(i_1) \cdots \pi(i_k)$ is an *occurrence* of p in π with the *index set* $\{i_1, \dots, i_k\}$. The increasing sequence $i_1 \cdots i_k$ will be used to denote the order preserving injection $i \colon [\![1, k]\!] \to [\![1, n]\!], j \mapsto i_j$ which we call the *index injection* of p into π for this particular occurrence.

The set of all index sets of p in π is the *occurrence set* of p in π, denoted with $V_p(\pi)$. If π does not contain p, then π *avoids* p. In this context the permutation p is called a (*classical permutation*) *pattern*.

Unless otherwise stated, we write the index set $\{i_1, \dots, i_n\}$ in ordered form, i.e., such that $i_1 < \cdots < i_n$, in accordance with how we write the index injection.

The set of all permutations that avoid p is $\mathrm{Av}(p)$. More generally for a set of patterns M we define

$$\mathrm{Av}(M) = \bigcap_{p \in M} \mathrm{Av}(p).$$

Example 2.8. The permutation 42135 contains five occurrences of the pattern 213, namely 425, 415, 435, 213 and 215. The occurrence set is

$$V_{213}(42135) = \big\{\{1,2,5\}, \{1,3,5\}, \{1,4,5\}, \{2,3,4\}, \{2,3,5\}\big\}.$$

The permutation 42135 avoids the pattern 132.

3. Occurrence graphs

We now formally define occurrence graphs.

Definition 3.1. For a pattern p of length k and a permutation π we define the *occurrence graph* $G_p(\pi)$ of p in π as follows:

- The set of vertices is $V_p(\pi)$, the occurrence set of p in π.
- uv is an edge in $G_p(\pi)$ if the vertices $u = \{u_1, \dots, u_k\}$ and $v = \{v_1, \dots, v_k\}$ in $V_p(\pi)$ differ by exactly one element, i.e., if

$$|u \setminus v| = |v \setminus u| = 1.$$

Example 3.2. In Example 2.8 we derived the occurrence set $V_{213}(42135)$. We compute the edges of $G_{213}(42135)$ by comparing the vertices two at a time to see if the sets differ by exactly one element. The graph is shown in Figure 1.

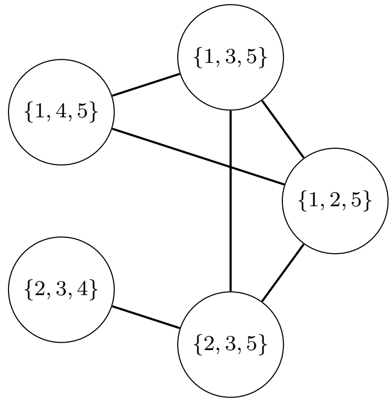


FIGURE 1. The occurrence graph $G_{213}(42135)$

Remark 3.3. For a permutation π of length n the graph $G_{\mathscr{E}}(\pi)$ is a graph with one vertex and no edges and $G_1(\pi)$ is a clique on n vertices.

Following the definition of these graphs there are several natural questions that arise. For example, for a fixed pattern p, which occurrence graphs $G_p(\pi)$ satisfy a given graph property, such as being connected or being a tree? Before we answer questions of this sort we consider a simpler question: What can be said about the graph $G_{12}(\mathrm{id}_n)$?

4. THE PATTERN $p = 12$ AND THE IDENTITY PERMUTATION

In this section we only consider the pattern $p = 12$ and let $n \geq 2$. For this choice of p and a fixed n the identity permutation $\pi = 1 \cdots n$ contains the most occurrences of p. Indeed, every set $\{i, j\}$ with $i \neq j$ is an index set of p in π. We can choose this pair in

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

different ways. Therefore, this is the size of the vertex set of $G = G_p(\pi)$.

Every vertex $u = \{i, j\}$ in G is connected to $n-2$ vertices $v = \{i, j'\}$, $j' \neq j$, and $n-2$ vertices $w = \{i', j\}$, $i' \neq i$. Thus, the degree of every vertex in G is $2(n-2)$. By summing this over the set of vertices and dividing by two we get the number of edges in G:

$$|E(G)| = \frac{n(n-1)(n-2)}{2} = 3\binom{n}{3}.$$

A triangle in G consists of three vertices u, v, w with edges uv, vw, wu. If $u = \{i, j\}$ (not neccessarily in ordered form) then we can assume v is

$\{j, k\}$. For this triplet to be a triangle w must connect to both u and v, and therefore w must either be the index set $\{i, k\}$ or $\{j, j'\}$ where $j' \neq i, k$. In the first case, we just need to choose three indices i, j, k. In the second case we start by choosing the common index k and then we choose the remaining indices. Thus the number of triangles in G is

$$\binom{n}{3} + n\binom{n-1}{3} = (n-2)\binom{n}{3}.$$

Example 4.1. The graph $G_{12}(12345)$ is pictured in Figure 2. It has 10 vertices, 30 edges, and 30 triangles. It also has 5 subgraphs isomorphic to K_4, one of them highlighted with bolder gray edges and gray vertices.

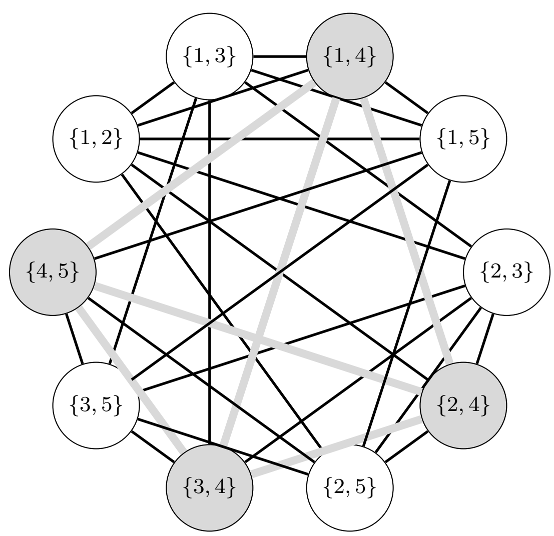


FIGURE 2. The graph $G_{12}(12345)$

The following proposition generalizes the observations above to larger cliques.

Proposition 4.2. *For $n > 0$, the number of cliques of size $k > 3$ in $G_{12}(\mathrm{id}_n)$ is*

$$(k+1)\binom{n}{k+1} = n\binom{n-1}{k}.$$

Proof. The vertices $(a_1, b_1), (a_2, b_2), \dots, (a_k, b_k)$ in a clique of size $k > 3$ must have a common index, say $\ell = a_1 = a_2 = \cdots = a_k$, without loss of generality. The remaining indices $b_1, b_2, \dots, b_k$ can chosen as any subset of the other $n-1$ indices. This explains the right hand side of the equation in the proposition. $\square$

5. Hereditary properties of graphs

Intuitively one would think that if a pattern p is contained inside a larger pattern q, that either of the occurrence graphs $G_p(\pi)$ and $G_q(\pi)$ (for any permutation π) would be contained inside the other. But this is not the case as the following examples demonstrate.

Example 5.1.
(1) Let $p = 12$, $q = 231$ and $\pi = 3421$. The occurrence sets are $V_p(\pi) = \big\{\{1, 2\}\big\}$ and $V_q(\pi) = \big\{\{1, 2, 3\}, \{1, 2, 4\}\big\}$. The cardinality of the set $V_p(\pi)$ is smaller then the cardinality of $V_q(\pi)$.
(2) If on the other hand $p = 12$, $q = 123$ and $\pi = 123$ then the occurrence sets are $V_p(\pi) = \big\{\{1, 2\}, \{1, 3\}, \{2, 3\}\big\}$ and $V_q(\pi) = \big\{\{1, 2, 3\}\big\}$. The relative size of the occurrence sets are now changed.

However, for a fixed pattern p, we obtain an inclusion of occurrence graph in Proposition 5.4. First we need a lemma.

Lemma 5.2. *Let π and σ be two permutations. For every occurrence of π in σ the index injection induces an injection $\Phi_p\colon V_p(\pi) \to V_p(\sigma)$, for all patterns p.*

Proof. Let p, π, σ be permutations of length l, m, n respectively. Every $v = \{i_1, \dots, i_l\} \in V_p(\pi)$ is an index set of p in π with index injection i. Let j be an index injection for an index set $\{j_1, \dots, j_m\}$ of π in σ. It's easy to see that $u = \{j_{i_1}, \dots, j_{i_l}\}$ is an index set of p in σ because $j \circ i$ is an index injection of p into σ. Define $\Phi_p(v) = u$. □

Example 5.3. Let $p = 12$, $\pi = 132$ and $\sigma = 24153$. There are three occurrences of π in σ: 243, 253 and 153 with respective index injections 125, 145 and 345.

For a given index injection, say $i = 345$, we obtain the injection Φ_p by mapping every $\{v_1, v_2\} \in V_p(\pi)$ to $\{i_{v_1}, i_{v_2}\} \in V_p(\sigma)$. We calculate that Φ_p maps $\{1, 2\}$ to $\{i_1, i_2\} = \{3, 4\}$ and $\{1, 3\}$ to $\{i_1, i_3\} = \{3, 5\}$, see Figure 3.

Proposition 5.4. *Let p be a pattern and π, σ be two permutations. For every occurrence of π in σ the index injection induces an isomorphism of the occurrence graph $G_p(\pi)$ with a subgraph of $G_p(\sigma)$.*

Proof. From Lemma 5.2 we have the injection $\Phi_p\colon V_p(\pi) \to V_p(\sigma)$. We need to show that for every $uv \in E\left(G_p(\pi)\right)$ that $\Phi_p(u)\Phi_p(v) \in E\left(G_p(\sigma)\right)$. Let uv be an edge in $G_p(\pi)$, where $u = \{u_1, \dots, u_l\}$ and $v = \{v_1, \dots, v_l\}$. For every index injection j of π into σ, the vertices

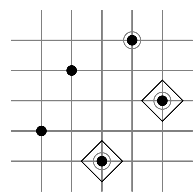


FIGURE 3. The occurrence of π in σ that is defined by the index injection $i = 345$ is highlighted with gray circles. The occurrence set $\{1,3\}$ of p in π is mapped with the injection Φ_p, induced by i, to the index set $\{3,5\}$ of p in σ, highlighted with black diamonds

u, v map to $\Phi_p(u) = \{j(u_1), \ldots, j(u_l)\}$, $\Phi_p(v) = \{j(v_1), \ldots, j(v_l)\}$ respectively. Since j is an injection there exists an edge between these two vertices in $G_p(\sigma)$. $\square$

Example 5.5. We will continue with Example 5.3 and show how the index injection $i = 345$ define subgraph of $G_p(\sigma)$ which is isomorphic to $G_p(\pi)$. The occurrence graph of p in π is a graph on two vertices $\{1,2\}$ and $\{1,3\}$ with an edge between them. The occurrence graph $G_p(\sigma)$ with the highlighted subgraph induced by i is shown in Figure 4.

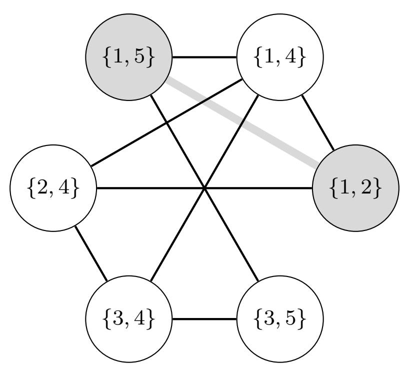


FIGURE 4. The graph $G_{12}(24153)$ with a highlighted subgraph isomorphic to $G_{12}(132)$

The next example shows that different occurrences of π in σ do not necessarily lead to different subgraphs of $G_p(\sigma)$.

Example 5.6. If $p = 12$, $\pi = 312$ and $\sigma = 3412$ there are two occurrences of π in σ. The index injections are $i = 134$ and $i' = 234$. However, as $(i_2, i_3) = (i'_2, i'_3)$ and the fact that $\{2,3\}$ is the only index set of p in π, we obtain the same injection Φ_p and therefore the same subgraph of $G_p(\sigma)$ for both index injections.

We call a property of a graph G *hereditary* if it is invariant under isomorphisms and for every subgraph of G the property also holds. For example the properties of being a forest, bipartite, planar or k-colorable are hereditary properties, while being a tree is not hereditary. A set of graphs defined by a hereditary property is a *hereditary class*.

Given c, a property of graphs, we define a set of permutations:

$$\mathscr{G}_{p,c} = \{\pi \in \mathfrak{S} \colon G_p(\pi) \text{ has property } c\}.$$

Theorem 5.7. *Let c be a hereditary property of graphs. For any pattern p the set $\mathscr{G}_{p,c}$ is a permutation class, i.e., there is a set of classical permutation patterns M such that*

$$\mathscr{G}_{p,c} = \mathrm{Av}(M).$$

Proof. Let σ be a permutation such that $G_p(\sigma)$ satisfies the hereditary property c and let π be a pattern in σ. By Lemma 5.4 the graph $G_p(\pi)$ is isomorphic to a subgraph of $G_p(\sigma)$ and thus inherits the property c. □

In the remainder of this section we consider two hereditary classes of graphs: bipartite graphs and forests. Recall that a non-empty simple graph on n vertices ($n > 0$) is a *tree* if and only if it is connected and has $n-1$ edges. An equivalent condition is that the graph has at least one vertex and no simple cycles (a simple cycle is a sequence of unique vertices $v_1, \ldots, v_k$ with edges $v_1v_2, \ldots, v_{k-1}v_k, v_kv_1$). A *forest* is a disjoint union of trees. The empty graph is a forest but not a tree. *Bipartite* graphs are graphs that can be colored with two colors in such a way that no edge joins two vertices with the same color. We note that every forest is a bipartite graph.

Table 1 shows experimental results, obtained using [3], on which occurrence graphs with respect to the patterns $p = 12$, $p = 123$, $p = 132$ are bipartite.

Table 1. Experimental results for bipartite occurrence graphs. Computed with permutations up to length 8

p	basis	Number seq.	OEIS
12	123, 1432, 3214	$1, 2, 5, 12, 26, 58, 131, 295$	A116716
123	1234, 12543, 14325, 32145	$1, 2, 6, 23, 100, 462, 2207, 10758$	
132	1432, 12354, 13254, 13452, 15234, 21354, 23154, 31254, 32154	$1, 2, 6, 23, 95, 394, 1679, 7358$	

In the following theorem we verify the statements in line 1 of Table 1. We leave the remainder of the table as conjectures.

Theorem 5.8. *Let c be the property of being bipartite and $p = 12$. Then*

$$\mathscr{G}_{p,c} = \mathrm{Av}\,(123, 1432, 3214)\,.$$

The OEIS sequence A116716 enumerates a symmetry of this permutation class.

The proof of this theorem relies on a proposition characterizing the cycles in the graphs under consideration.

Proposition 5.9. *If the graph $G_{12}(\pi)$ has a cycle of length $k > 4$ then it also has a cycle of length* 3.

Proof. Let $p = 12$ and let π be a permutation such that $G_p(\pi)$ contains a cycle of length $k > 4$. Label the vertices in the cycle $v_1, \ldots, v_k$ with $v_l = \{i_l, j_l\}$, $i_l < j_l$, for $l = 1, \ldots, k$.

The vertices v_1 and v_2 in the cycle have exactly one index in common. If $i_2 = j_1$ then the vertices v_1, v_2, $\{i_1, j_2\}$ form a triangle. So we can assume $i_1 = i_2$. If $j_1 < j_2$ and $\pi(j_1) < \pi(j_2)$ (or $j_1 > j_2$ and $\pi(j_1) > \pi(j_2)$) then $u = \{j_1, j_2\}$ is an occurrence of p in π, forming a triangle v_1, v_2, u. So either $j_1 > j_2$ and $\pi(j_1) < \pi(j_2)$ holds, or, without loss of generality (see Figure 5), $j_1 < j_2$ and $\pi(j_1) > \pi(j_2)$.

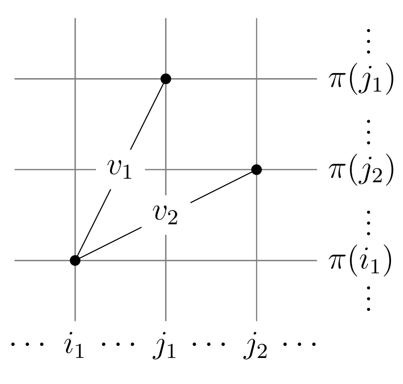


FIGURE 5. The vertices v_1 and v_2 (shown as line segments inside the permutation π) share the index i_1

Next we look at the edge v_2v_3 in the cycle. If the vertices have the index i_1 in common then v_1, v_2, v_3 forms a triangle in $G_p(\pi)$. So assume that v_2 and v_3 have the index j_2 in common with the conditions

$i_3 > i_1$ and $\pi(i_3) < \pi(i_1)$ (because else there are more vertices and edges forming a cycle of length 3 in $G_p(\pi)$). Continuing down this road we know that v_3v_4 is an edge with shared index i_3 and conditions $j_3 > i_3$ and $\pi(j_3) < \pi(j_1)$, see Figure 6, where we consider the case $i_3 > j_1$, and $\pi(j_3) < \pi(i_1)$.

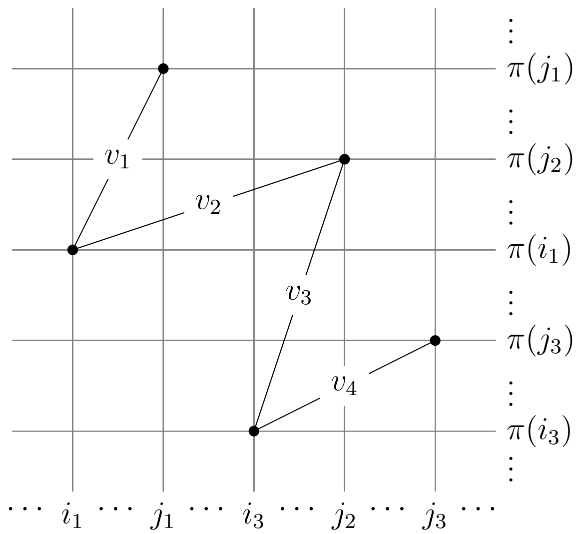


FIGURE 6. The vertices v_1, v_2, v_3, v_4

Graphically, it is quite obvious that we cannot extend the graphical line path in Figure 6 with more southwest-northeast line segments (a sequence of vertices $v_5, \ldots, v_k$) such that the extension closes the path into a cycle without adding more edges (line segments) between vertices that are not adjacent in the cycle and thus forming a cycle of length 3 in the occurrence graph. More precisely for an edge between v_k and v_1 to exist we must have $v_k = \{i_k, j_k\}$ with a non-empty intersection with v_1. Analyzing each of these cases completes the proof. □

Proof of Theorem 5.8. If π contains any of the patterns 123, 1432, 3214 then $G_p(\pi)$ contains a subgraph that is isomorphic to a triangle. So if $\pi \notin \mathrm{Av}\,(123, 1432, 3214)$ then $G_p(\pi)$ contains an odd cycle and is therefore not bipartite.

On the other hand, let π be a permutation such that $G_p(\pi)$ is not bipartite. Then the occurrence graph contains an odd cycle which by Proposition 5.9 implies the graph has a cycle of length 3. The indices corresponding to this cycle form a pattern of length 3 or 4 in π with occurrence graph that is a cycle of lengths 3. It is easy to see that the only permutations of these length whith occurrence graph a cycle of length 3 are 123, 1432 and 3214. Therefore π must contain at least one of the patterns. □

Table 2 considers occurrence graphs that are forests.

TABLE 2. Experimental results for occurrence graphs that are forests. Computed with permutations up to length 8

p	basis	Number seq.	OEIS
12	123, 1432, 2143, 3214	$1, 2, 5, 11, 24, 53, 117, 258$	A052980
123	1234, 12543, 13254, 14325, 21354, 21435, 32145	$1, 2, 6, 23, 97, 429, 1947, 8959$	
132	1432, 12354, 12453, 12534, 13254, 13452, 14523, 15234, 21354, 21453, 21534, 23154, 31254, 32154	$1, 2, 6, 23, 90, 359, 1481, 6260$	

In the following theorem we verify the statements in line 1 of Table 2. We leave the remainder of the table as conjectures.

Theorem 5.10. *Let c be the property of being a forest and $p = 12$. Then*

$$\mathscr{G}_{p,c} = \mathrm{Av}\,(123, 1432, 2143, 3214)\,.$$

Proof. If π contains the pattern 2143 then $G_p(\pi)$ contains a subgraph that is isomorphic to a cycle of length four, according to Lemma 5.4, because $G_p(2143)$ is a cycle of length four. If π contains any of the patterns 123, 1432, 3214 then its occurrence graph is bipartite by Theorem 5.8, and in particular not a forest.

On the other hand, let π be a permutation such that $G_p(\pi)$ is not a forest. Then the occurrence graph contains a cycle. Proposition 5.9 implies that the graph must have length either 3 or 4. But it's easy to see that the only permutations with occurrence graphs that are cycles of length 3 or 4 are $123, 1432, 2143, 3214$. Therefore π must contain at least one of the patterns. □

6. Non-hereditary properties of graphs

This section is devoted to graph properties that are not hereditary. Thus Theorem 5.7 does not guarantee that the permutations whose occurrence graphs satisfy the property form a pattern class. Experimental results in Table 3 seem to suggest that some properties still give rise to permutation classes.

TABLE 3. $p = 12$. Computed with permutations up to length 8

Property	basis	Number seq.	OEIS
connected	see Figure 7	$1, 2, 6, 23, 111, 660, 4656, 37745$	
chordal	1234, 1243, 1324, 2134, 2143	$1, 2, 6, 19, 61, 196, 630, 2025$	
clique	1234, 1243, 1324, 1342, 1423, 2134, 2143, 2314, 2413, 3124, 3142, 3412	$1, 2, 6, 12, 20, 30, 42, 56$	A002378[1]
tree	very large non-classical basis	$0, 1, 4, 9, 16, 25, 36, 49$	A000290

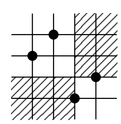


FIGURE 7. Mesh pattern classifying connected occurrence graphs with respect to $p = 12$

Theorem 6.1. *Let c be the property of being connected and $p = 12$. Then*

$$\mathscr{G}_{p,c} = \mathrm{Av}\,(m)\,,$$

where m is the mesh pattern in Figure 7. The generating function for the enumeration of these permutations is

$$\frac{F(x) - x}{(1 - x)^2} + \frac{1}{1 - x}$$

where $F(x) = 1 - 1/\sum k!x^k$ is the generating function for the skew-indecomposable permutations, see e.g., Comtet [2, p. 261]

Proof. If the graph $G_p(\pi)$ is disconnected then π has two occurrences of $p = 12$ in distinct skew-components, A and B, which we can take to be consecutive in the skew-decomposition of $\pi = \cdots \ominus A \ominus B \ominus \cdots$. Let ab be any occurrence of 12 in A. Let u be the highest point in B and v be the leftmost point in B. Then $abuv$ is an occurrence of the mesh pattern. It is clear that an occurrence $abuv$ of the mesh pattern will correspond to two vertices ab, uv in the occurrence graph, and the shadings ensure that there is no path between them.

The enumeration follows from the fact that these permutations must have no, or exactly one, skew-component of size greater than 1. The first case is counted by $1/(1-x)$ while the second case is counted by $(F(x)-x)/(1-x)^2$. □

Note that our software suggests a very large non-classical basis for the permutations with a tree as an occurrence graph. We omit displaying this basis here. However, since a graph is a tree if and only if it is a non-empty connected forest we obtain:

Corollary 6.2. *Let c be the property of being a tree and $p = 12$. Then*

$$\mathscr{G}_{p,c} = \mathrm{Av}\,(123, 1432, 2143, 3214, m) \setminus \mathrm{Av}(12),$$

where m is the mesh pattern in Figure 7.

Proof. This follows from Theorems 5.10 and 6.1. We must remove the decreasing permutations since they all have empty occurrence graphs. □

We end with proving the enumeration for the permutations in the corollary above. The proof is a rather tedious, but simple, induction proof.

Theorem 6.3. *The number of permutations of length n in $\mathscr{G}_{12,tree}$ is $(n-1)^2$.*

7. Future work

We expect the conjectures in lines 2 and 3 in Tables 1 and 2 to follow from an analysis of the cycle structure of occurrence graphs with respect to the patterns 123 and 132, similar to what we did in Proposition 5.9 for the pattern 12.

Other natural hereditary graph properties to consider would be k-colorable graphs, for $k > 2$, as these are supersets of bipartite graphs. Also planar graphs, which lie between forests and 4-colorable graphs.

It might also be interesting to consider the intersection $\bigcap_{p \in M} \mathscr{G}_{p,c}$ where M is some set of patterns, perhaps all.

We would like to note that Jason Smith [4] independently defined occurrence graphs and used them to prove a result on the shellability of a large class of intervals of permutations.

Appendix A. Proof of Theorem 6.3

We start by introducing a new notation.

Definition A.1. Let $\pi \in \mathfrak{S}_n$ and k be an integer such that $1 \leq k \leq n+1$. The *k-prefix of* π is the permutation $\pi' \in \mathfrak{S}_{n+1}$ defined by $\pi'(1) = k$ and

$$\pi'(i+1) = \begin{cases} \pi(i) & \text{if } \pi(i) < k, \\ \pi(i)+1 & \text{if } \pi(i) \geq k \end{cases}$$

for $i = 1, \ldots, n$. We denote π' with $k \succ \pi$. In a similar way we define the *k-postfix of* π as the permutation $\pi \prec k$ in $\mathfrak{S}_{n+1}$.

Example A.2. Let $\pi = 42135$ and $k = 2$. Visually, if we draw the grid representation of π, we are putting the new number k to the left on the x-axis and raising all the numbers $\geq k$ on the y-axis by one. Thus we have $2 \succ 42135 = 253146$. See Figure 8

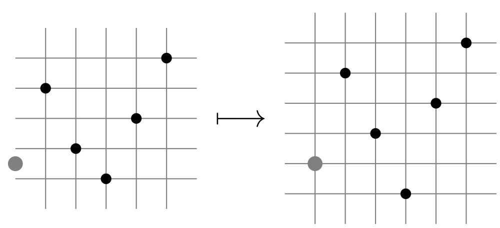


Figure 8. The 2-prefix of 42135 is 253146

We note that for every permutation $\pi' \in \mathfrak{S}_{n+1}$ there is one and only one pair (k, π) such that $\pi' = k \succ \pi$. We let $k = \pi'(1)$ and $\pi = \mathrm{st}(\pi'(2) \cdots \pi'(n+1))$.

Proof of 6.3. Let $p = 12$. We start by considering three base-cases.

For $n = 1$ the occurrence graph is the empty graph. For $n = 2$ we get two occurrence graphs: $G_p(12)$ is a single node graph and $G_p(21)$ is the empty graph. For $n = 3$ we have $3! = 6$ different permutations π. Of those we calculate that 132, 213, 231 and 312 result in connected occurrence graphs on one or two nodes but $G_p(123)$ is a triangle and $G_p(321)$ is the empty graph.

We have thus showed that the claimed enumeration is true for $n = 1, 2, 3$.

For the inductive step we assume $n \geq 4$ and let π be a permutation of length n. We look at four different cases of k to construct $\pi' = k \succ \pi$. We let x, y and z be the indices of $n-1$, n and $n+1$ in π' respectively.

(I) $\underline{k \leq n-2}$: The index sets $\{1, x\}$, $\{1, y\}$ and $\{1, z\}$ of p in π' all share exactly one common element and thus form a triangle in $G_p(\pi')$. Therefore there are no permutations π resulting in the occurrence graph $G_p(\pi')$ being a tree.

(II) $\underline{k = n-1}$: Let $T(n+1)$ denote the number of permutations π' of length $n+1$ with $\pi'(1) = n-1$ such that $G_p(\pi')$ is a tree. Note that $T(1) = T(2) = 0$, $T(3) = 1$ and $T(4) = 2$. In order to obtain a formula for T we need to look at a few subcases:

i) If $y < z$ then $\{1, y\}$, $\{1, z\}$ and $\{y, z\}$ form an triangle in $G_p(\pi')$, see Figure 9. Independent of the permutation π, the graph $G_p(\pi')$ is not a tree.

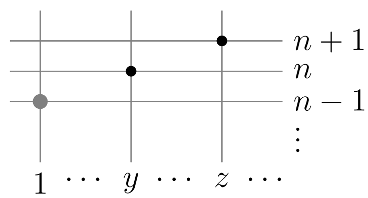


FIGURE 9. $k = n-1$ and $y < z$

ii) Assume $y > z$ and $z \neq 2$, see Figure 10. Then $\pi'(2) < n-1$ and $\{1, z\}$, $\{2, z\}$, $\{2, y\}$ and $\{1, y\}$ form a cycle of length 4 in $G_p(\pi')$, resulting in it not being a tree.

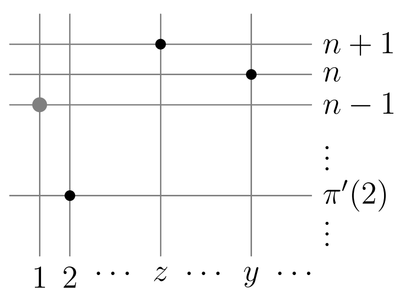


FIGURE 10. $k = n-1$, $y > z$ and $z \neq 2$

iii) Now lets assume $y > z$ and $z = 2$, see Figure 11.
If $y \geq 5$ then the vertices $\{1, y\}$, $\{3, y\}$ and $\{4, y\}$ form a cycle in $G_p(\pi')$.
If $y = 3$ then $\{1, 2\}$ and $\{1, 3\}$ will be an isolated path component in $G_p(\pi')$, making $\pi' = (n-1)(n+1)n(n-2)\cdots 1$ the only permutation such that the occurrence graph $G_p(\pi')$ is a tree.

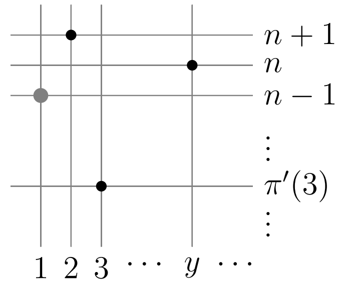


FIGURE 11. $k = n - 1$, $y > z$ and $z = 2$

Now fix $y = 4$ and lets look at some subsubcases for the value of $\pi'(3)$.

a) If $\pi'(3) \leq n-4$ then $\pi'(3)n$, $\pi'(3)(n-2)$ and $\pi'(3)(n-3)$ are all occurrences of p in π', with the respective index sets forming an triangle in $G_p(\pi')$.
b) If $\pi'(3) = n-2$ then $\pi' = (n-1)(n+1)(n-2)n(n-3)\cdots 1$ is the only permutation resulting in $G_p(\pi')$ being a tree.
c) If $\pi'(3) = n-3$ we look at Figure 12 where the permutation π' is shown.

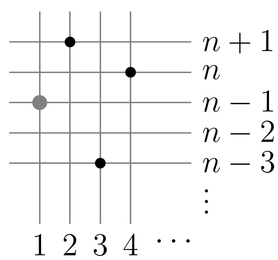


FIGURE 12. $k = n - 1$, $y = 4$ and $z = 2$

The permutation $\sigma = \mathrm{st}(\pi'(3)\cdots\pi'(n+1))$ is just like π' in the case $k = n-1$ and $z = 2$, only the length of σ is $n-1$. Because $\{1, 2\}$ is a vertex in $G_p(\sigma)$ the occurrence graph of p in σ is not the empty graph. Thus it is easy to see that $G_p(\pi')$ is a tree if and only if $G_p(\sigma)$ is a tree, and according to the aforementioned case there are $T(n-1)$ such permutations σ.

Summing up the subsubcases there are a total of $1 + 1 + T(n-1)$ permutations π' making the occurrence graph a tree, i.e., $T(n+1) = 2 + T(n-1)$. Because $T(4) = 2$ and $T(3) = 1$ we deduce that $T(n+1) = n-1$.

The whole case $k = n-1$ gives us that there are $n-1$ permutation π' such that $G_p(\pi')$ is a tree.

(III) $\underline{k = n}$: We need to examine three subcases:

i) If $z \geq 4$ then $\{1, z\}$, $\{2, z\}$, $\{3, z\}$ are all index sets of p in π', forming a triangle in $G_p(\pi')$.

ii) If $z = 3$, then $\{1, 2\}$ is an index set of p in π making the occurrence graph $G_p(\pi)$ non-empty, see Figure 13.

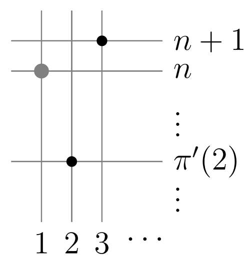


FIGURE 13. $k = n$ and $z = 3$

If $\pi'(2) \leq n-3$ then $\pi'(2)(n+1)$, $\pi'(2)(n-1)$ and $\pi'(2)(n-2)$ are all occcurrences of p in π', resulting in $G_p(\pi')$ having a triangle.
If $\pi'(2) = n-1$ then $\{1, 3\}$ and $\{2, 3\}$ is an isolated path component in $G_p(\pi')$ and $\pi' = n(n-1)(n+1)(n-2)\cdots 1$ is the only permutation such that the occurrence graph is a tree.
We therefore assume $\pi'(2) = n-2$, see Figure 14.

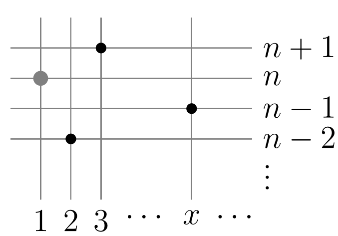


FIGURE 14. $k = n$, $z = 3$ and $\pi'(2) = n-2$

Let $\sigma = \mathrm{st}(\pi'(2)\cdots\pi'(n+1))$. Note that the occurrence graphs $G_p(\pi')$ and $G_p(\sigma)$ are the same except the former has an the extra vertex $\{1, 2\}$ and an edge connecting it to a graph corresponding to $G_p(\sigma)$. Therefore, $G_p(\pi')$ is a tree if and only if $G_p(\sigma)$ is a tree.

Note that $\sigma(1) = n-2$ and $\sigma(2) = n$ and therefore σ is like π' in the case $k = n-1$ and $z = 2$ as in Figure 11, only of length n instead of $n+1$. By the same reasoning as in that case the number of permutations σ (and therefore π') such that $G_p(\pi')$ is a tree is $T(n) = n-2$.

iii) If $z = 2$, then $\{1,2\}$ is an isolated vertex in $G_p(\pi')$, see Figure 15. The occurrence graph of p in π' is a tree if and only if $G_p(\pi)$ is the empty graph which is true if and only if π is the decreasing permutation. Therefore there is only one permutation $\pi' = n(n+1)(n-1)\dots 1$ such that $G_p(\pi')$ is a tree.

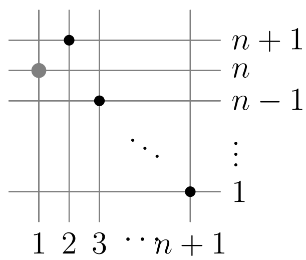


FIGURE 15. $k = n$ and $z = 2$

To sum up the the case $k = n$ there are $1 + (n-2) + 1 = n$ permutation π' such that $G_p(\pi')$ is a tree.

(IV) $\underline{k = n+1}$: Every occurrence $\pi(i)\pi(j)$ of p in π is also an occurrence of p in π', but with index set $\{i+1, j+1\}$ instead of $\{i, j\}$. There are no more occurrences of p in π' because $\pi'(1) = n+1 > \pi'(j')$ for every $j' > 1$ so $\pi'(1)\pi'(j')$ is not an occurrence of p for any $j' > 1$.

This means that $G_{12}(\pi') \cong G_{12}(\pi)$ so by the induction hypothesis we obtain that there are $(n-1)^2$ permutations π' such that the occurrence graph is a tree for this value of k.

To sum up the four instances there is a total of $0+(n-1)+n+(n-1)^2 = n^2$ permutations π' such that $G_p(\pi')$ is a tree. $\square$

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