# SELF DUAL REFLEXIVE SIMPLICES WITH EULERIAN POLYNOMIALS

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Abstract. A lattice polytope $\mathcal{P}$ is called reflexive if its dual $\mathcal{P}^\vee$ is a lattice polytope. The property that $\mathcal{P}$ is unimodularly equivalent to $\mathcal{P}^\vee$ does not hold in general, and in fact there are few examples of such polytopes. In this note, we introduce a new reflexive simplex $Q_n$ which has this property. Additionally, we show that $\delta$-polynomalial of $Q_n$ is the Eulerian polynomial and show the existence of a regular, flag, unimodular triangulation.

Let $\mathcal{P} \subset \mathbb{R}^d$ be a $d$-dimensional lattice polytope, that is, a convex polytope all of whose vertices belong to $\mathbb{Z}^d$. Let $\mathrm{Vol}(\mathcal{P})$ denote the *normalized volume* of $\mathcal{P}$, which is $d!$ times the Euclidean volume (Lebesgue measure) of $\mathcal{P}$. For $k \in \mathbb{Z}_{>0}$, the *lattice point enumerator* $i(\mathcal{P}, k)$ counts the number of lattice points in $k\mathcal{P} = \{k\alpha : \alpha \in \mathcal{P}\}$, the $k$th dilation of $\mathcal{P}$. That is,

$$i(\mathcal{P}, k) = \#(k\mathcal{P} \cap \mathbb{Z}^d), \quad k \in \mathbb{Z}_{>0}.$$

Provided that $\mathcal{P}$ is a lattice polytope, it is known that $i(\mathcal{P}, k)$ is a polynomial in the variable $k$ of degree $d$ ([4]). The *Ehrhart Series* for $\mathcal{P}$, $\mathrm{Ehr}_\mathcal{P}(z)$, is the rational generating function

$$\mathrm{Ehr}_\mathcal{P}(z) = 1 + \sum_{k \geq 1} i(\mathcal{P}, k) z^k = \frac{\delta(\mathcal{P}, z)}{(1-z)^{d+1}}$$

where $\delta(\mathcal{P}, z) = 1 + \delta_1 z + \delta_2 z^2 + \cdots + \delta_d z^d$ is the *$\delta$-polynomial* of $\mathcal{P}$ (cf. [5, Chapter 9]). The $\delta$-polynomial is endowed with the following properties:

- $\delta_0 = 1$, $\delta_1 = i(\mathcal{P}, 1) - (d+1)$, and $\delta_d = \#(\mathcal{P} \setminus \partial\mathcal{P} \cap \mathbb{Z}^d)$;
- $\delta_i \geq 0$ for all $0 \leq i \leq d$ ([10]);
- If $\delta_d \neq 0$, then $\delta_1 \leq \delta_i$ for each $0 \leq i \leq d-1$ ([6]).

For proofs of the first three properties of the coefficients, the reader should consult [5, Chapter 9] or [2, Chapter 3]. The Ehrhart series and $\delta$-polynomials for polytopes have been studied extensively. For a detailed background on these topics, please refer to [2, 4, 5, 11].

Given two polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$ in $\mathbb{R}^d$, we say that $\mathcal{P}_1$ and $\mathcal{P}_2$ are *unimodularly equivalent* if there exists a unimodular matrix $U \in \mathbb{Z}^{d\times d}$ (i.e. $\det(U) = \pm 1$) and an integral vector $\boldsymbol{v} \in \mathbb{Z}^d$, such that $\mathcal{P}_2 = f_U(\mathcal{P}_1) + \boldsymbol{v}$, where $f_U$ is the linear transformation defined by $U$, i.e., $f_U(\mathbf{v}) = \mathbf{v}U$ for all $\mathbf{v} \in \mathbb{R}^d$. . We write $\mathcal{P}_1 \cong \mathcal{P}_2$ in the case of unimodular equivalence. It is clear that if $\mathcal{P}_1 \cong \mathcal{P}_2$, then $\delta(\mathcal{P}_1, z) = \delta(\mathcal{P}_2, z)$.

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We say that a lattice polytope $\mathcal{P}$ is *reflexive* if the origin is the unique interior lattice point of $\mathcal{P}$ and its dual polytope

$$\mathcal{P}^\vee = \left\{ y \in \mathbb{R}^d \ : \langle x, y \rangle \leq 1 \text{ for all } x \in \mathcal{P} \right\}$$

is a lattice polytope. Moreover, it follows from [7] that the following statements are equivalent:

- $\mathcal{P}$ is unimodularly equivalent to some reflexive polytope;
- $\delta(\mathcal{P}, z)$ is of degree $d$ and is symmetric, that is $\delta_i = \delta_{d-i}$ for $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$.

A polytope $\mathcal{P}$ is called *self dual* if $\mathcal{P}$ is unimodularly equivalent to its dual polytope $\mathcal{P}^\vee$. This is an extremely rare property in reflexive polytopes, especially for reflexive simplices. There are two families known self dual reflexive simplices. The first such family is given in [8] and the second family is given in [12]. A construction for self dual reflexive polytopes is given in [12], though these polytopes are not simplicial and hence not simplices. In this paper, we provide a new family of self dual reflexive simplices $Q_n$ with small volume.

We now define a family of reflexive simplices which are self dual. For $n \geq 2$, let $Q_n$ denote the $n-1$ dimensional simplex with $\mathcal{V}$-representation

$$Q_n := \text{conv} \begin{bmatrix} 1 & 1-n & 0 & 0 & \cdots & 0 \\ 1 & 1 & 2-n & 0 & \cdots & 0 \\ 1 & 1 & 1 & 3-n & \cdots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & -1 \end{bmatrix}$$

where we use the convention that the $x_{n-1}$ coordinate is given by the first row and the $x_1$ is given by the last row. We will adopt for simplicity the notation $C_i$ where $i \in \{0, \cdots, n-1\}$ for each column (vertex) such that $Q_n = \text{conv}[C_0 \ C_1 \ \cdots \ C_{n-1}]$.

We have the following theorem.

**Theorem 1.** *For $n \geq 2$, we have $Q_n \cong Q_n^\vee$.*

It behooves us to introduce an $\mathcal{H}$-representation for the simplex to compute its dual polytope. We now give such a representation.

**Proposition 2.** *For $n \geq 2$, $Q_n$ has the $\mathcal{H}$-representation*

$$Q_n = \left\{ \begin{matrix} x \in \mathbb{R}^{n-1} & : & kx_k - \sum_{i=1}^{k-1} x_i \leq 1 \ , \quad 1 \leq k \leq n-1 \\ & & -\sum_{i=1}^{n-1} x_i \leq 1 \end{matrix} \right\}.$$

*Proof.* It is sufficient to show that the vertices of $Q_n$ each satisfy precisely $n-1$ of the halfspace inequalities with equality and satisfies the other inequalty strictly. Let $f_k(\boldsymbol{x}) = kx_k - \sum_{i=1}^{k-1} x_i$, and $f_n(\boldsymbol{x}) = -\sum_{i=1}^{n-1} x_i$. For a vertex $C_j$, we have that $f_k(C_j) = 1$ for all $k \neq n-j$. This follows, because if $k < n-j$, we have $f_k(C_j) = (k)(1) - \sum_{i=1}^{k-1} 1 = 1$, if $k > n-j$ with $k \neq n$, we have $f_k(C_j) = (n-j) - \sum_{i=1}^{n-j-1} 1 = 1$, and if $k = n > n-j$, we have $f_n(C_j) = -(j-n) - \sum_{i=1}^{n-j-1} 1 = 1$. In the case of $k = n-j$, $f_{n-j}(C_j) = -(n-j)^2 - (n-1-j) < 1$ if $j \neq 0$ and $j \neq n-1$. For $j = 0$ we have $f_n(C_0) = 1-n < 1$ and for $j = n-1$, we have $f_1(C_{n-1}) = -1 < 1$. Thus, we have the correct $\mathcal{H}$-representation. □

By [5, Corollary 35.3], and Proposition 2, it is clear that $Q_n^\vee = -Q_n$. Therefore, we have shown Theorem 1.

*Remark* 3. We should note that $\mathrm{Vol}(Q_n) = n!$. For $n \geq 4$, it is immediate that these polytopes are different than previously known self dual reflexive simplices given in [8, 12].

Moreover, the self dual reflexive simplex of $Q_n$ has an interesting $\delta$-polynomial and a special triangulation.

**Theorem 4.** *Let $n \geq 2$.*
(i) *We have $\delta(Q_n, z) = A_n(z)$, where $A_n(z)$ is the Eulerian polynomial.*
(ii) *$Q_n$ has a regular, flag, unimodular triangulation.*

*Proof.* For a lattice polytope $\mathcal{P} \subset \mathbb{R}^d$, we set $\mathrm{Pyr}(\mathcal{P}) = \mathrm{conv}(\mathcal{P} \times \{0\}, (0, \cdots, 0, 1)) \subset \mathbb{R}^{d+1}$. Then it is well-known that $\delta(\mathrm{Pyr}(\mathcal{P}), z) = \delta(\mathcal{P}, z)$ (cf. [2, Section 2.4]) and $\mathcal{P}$ has a regular, flag, unimodular triangulation if and only if $\mathrm{Pyr}(\mathcal{P})$ has a regular, flag, unimodular triangulation (cf. [3, Section 4.2]).

Let $R_n$ denote the $n$ dimensional simplex with $\mathcal{V}$-representation

$$R_n := \mathrm{conv}\begin{bmatrix} 0 & n & n & n & \cdots & n \\ 0 & 0 & n-1 & n-1 & \cdots & n-1 \\ 0 & 0 & 0 & n-2 & \cdots & n-2 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \subset \mathbb{R}^n.$$

This polytope $R_n$ is called a *lecture hall polytope*. Notice that $\mathrm{Pyr}(Q_n)$ is unimodularly equivalent to $R_n$. Let $\widetilde{R_n}$ be the polytope defined from $R_n$ by removing the $(n+1)$th column and $n$th row, let $\mathcal{U}_n$ denote the $(n-1)\times(n-1)$ upper triangular matrix defined by $(\mathcal{U}_n)_{ij} = 1$ if $i \leq j$ and $(\mathcal{U}_n)_{ij} = 0$ otherwise, let $\mathbf{1}$ denote the all ones vector, and let $\mathbf{0}$ denote the all zeros vector. Then we have

$$Q_n \cong -f_{\mathcal{U}_n}(Q_n - \mathbf{1}) = \widetilde{R_n}.$$

Hence it follows that

$$\mathrm{Pyr}(Q_n) \cong \mathrm{Pyr}\left(\widetilde{R_n}\right) \cong R_n.$$

It is known that for $n \geq 2$, $\delta(R_n, z) = A_n(z)$ ([9]) and $R_n$ has a regular, flag, unimodular triangulation ([1]). Therefore, the assertion follows. □

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