Spin Multiplicities

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Abstract

The number of times spin s appears in the Kronecker product of n spin j representations is computed, and the large n asymptotic behavior of the result is obtained. Applications are briefly sketched.

Introduction

We present a derivation of the spin multiplicities that occur in n-fold tensor products of spin-j representations, $j^{\otimes n}$. We make use of group characters, properties of special functions, and asymptotic analysis of integrals. While previous derivations for some of our results are scattered throughout the literature, especially for specific values of j, we provide here a treatment that is self-contained, and valid for any j and for any n. We emphasize two types of novel features: patterns that arise when comparing different values of j, and asymptotic behavior for large n.

Our methods and results should be useful for various calculations. In particular, the asymptotic behavior that we obtain should be helpful in the analysis of statistical problems such as the determination of partition functions. In the last section some other applications are briefly discussed, including a problem of interest for quantum computing, namely, an estimation of the number of entangled states.

Basic Theory of Group Characters

The character $\chi(R)$ of a group representation R succinctly encodes considerable information about R, as is well-known [1]. For irreducible representations the characters are orthogonal,

$$\sum\!\!\!\!\!\!\int \mu \; \chi^* (R_1) \, \chi (R_2) = \delta_{R_1,R_2} \; , \tag{1}$$

where the sum or integral is over the group parameter space with an appropriate measure μ.

For a Kronecker product of n representations, the character is given by the product of the individual characters,

$$\chi (R_1 \otimes R_2 \otimes \cdots \otimes R_n) = \chi (R_1) \, \chi (R_2) \cdots \chi (R_n) \; , \tag{2}$$

from which follows an explicit expression for the number of times that a given representation R appears in the product (e.g., see [2] Chapter I §4.7). This multiplicity is

$$M (R; R_1, \cdots , R_n) = \sum\!\!\!\!\!\!\int \mu \; \chi^* (R) \, \chi (R_1) \, \chi (R_2) \cdots \chi (R_n) \; . \tag{3}$$

For real characters, this is totally symmetric in $\{R, R_1, \cdots , R_n\}$, and it immediately shows that the number of times R appears in the product $R_1 \otimes \cdots \otimes R_n$ is equal to the number of times the trivial or "singlet" representation appears in the product $R \otimes R_1 \otimes \cdots \otimes R_n$.

The $SU(2)$ Case

Consider now the Lie group $SU(2)$. In this case the irreducible representations are labeled by angular momentum or spin, j or s, the classes of the group are specified by the angle of rotation about an axis, θ, and the characters are Chebyshev polynomials of the second kind, $\chi_j(\theta) = U_{2j}(\cos(\theta/2))$. Explicitly, for either integer or semi-integer j,

$$\chi_j(\theta) = \frac{\sin((2j+1)\theta/2)}{\sin(\theta/2)} \ . \tag{4}$$

These characters are all real. Therefore the number of times that spin s appears in the product $j_1 \otimes \cdots \otimes j_n$ is

$$M(s, j_1, \cdots, j_n) = \frac{1}{\pi}\int_0^{2\pi} \chi_s(2\vartheta)\,\chi_{j_1}(2\vartheta)\cdots\chi_{j_n}(2\vartheta)\sin^2\vartheta \; d\vartheta \ , \tag{5}$$

where we have taken $\theta = 2\vartheta$ to avoid having half-angles appear in the invariant measure and the Chebyshev polynomials (e.g., see [2] Chapter III §8.1) thereby mapping the $SU(2)$ group manifold $0 \le \theta \le 4\pi$ to $0 \le \vartheta \le 2\pi$. To re-emphasize earlier remarks, we note that (5) is totally symmetric in $\{s, j_1, \cdots, j_n\}$ and valid if s or any of the js are integer or semi-integer, and we also note that $M(s, j_1, \cdots, j_n) = M(0, s, j_1, \cdots, j_n)$. In general $M(s, j_1, \cdots, j_n)$ will obviously reduce to a finite sum of integers through use of the Chebyshev product identity, $U_m U_n = \sum_{k=0}^{n} U_{m-n+2k}$ for $m \ge n$. Alternatively, the integral form (5) for the multiplicity always reduces to a finite sum of hypergeometric functions (e.g. see (17) and (18) to follow).

In particular, for $j_1 = \cdots = j_n = j$, the n-fold product $j^{\otimes n}$ can yield spin s a number of times, as given by

$$M(s; n; j) = \frac{1}{\pi}\int_0^{2\pi} \sin((2s+1)\vartheta)\left(\frac{\sin((2j+1)\vartheta)}{\sin(\vartheta)}\right)^n \sin\vartheta \; d\vartheta \ , \tag{6}$$

for $s, j \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots\right\}$. Yet again, we note that $M(j; n; j) = M(0; n+1; j)$. Moreover, the symmetry of the integrand in (6) permits us to write

$$M(s; n; j) = \int_0^{2\pi} \frac{\exp(2is\vartheta)}{2\pi}\left(\frac{\sin((2j+1)\vartheta)}{\sin(\vartheta)}\right)^n d\vartheta - \int_0^{2\pi} \frac{\exp(2i(s+1)\vartheta)}{2\pi}\left(\frac{\sin((2j+1)\vartheta)}{\sin(\vartheta)}\right)^n d\vartheta \ . \tag{7}$$

Each integral in the last expression reduces to a simple residue,

$$\int_0^{2\pi} \frac{\exp(2is\vartheta)}{2\pi}\left(\frac{\sin((2j+1)\vartheta)}{\sin(\vartheta)}\right)^n d\vartheta = \frac{1}{2\pi i}\oint z^{2s}\left(\frac{z^{2j+1} - z^{-2j-1}}{z - z^{-1}}\right)^n \frac{dz}{z} = c_0(s, n, j) \ , \tag{8}$$

where c_k are the coefficients in the Laurent expansion of the integrand,

$$z^{2s}\left(\frac{z^{2j+1} - z^{-2j-1}}{z - z^{-1}}\right)^n = z^{2s}\left(\sum_{m=0}^{2j} z^{2(m-j)}\right)^n = \sum_{k=-2(jn-s)}^{2(jn+s)} z^k \, c_k(s, n, j) \ . \tag{9}$$

That is to say, c_0 is the coefficient of z^{-2s} (or of z^{+2s}) in the Laurent expansion of $(z^{-2j} + z^{-2j+2} + \cdots + z^{2j-2} + z^{2jn})$ [3], a coefficient that is easily obtained, e.g. using either Maple® or Mathematica®.

Explicit $SU(2)$ Results as Binomial Coefficients

So then, the multiplicity is always given by a difference,

$$M(s; n; j) = c_0(s, n, j) - c_0(s+1, n, j) \ , \tag{10}$$

where $2s$ is any integer such that $0 \le 2s \le 2nj$, and where $s = 0$ is always allowed when j is an integer but is only allowed for even n when j is a semi-integer. To be more explicit, the expansion of

$\left(z^{-2j}+z^{-2j+2}+\cdots+z^{2j-2}+z^{2j}\right)^n$ involves so-called "generalized binomial coefficients" (see Eqn(3) in [4]) which can be written as sums of products of the usual binomial coefficients. Eventually (see Lemma 6 in [5] and the Appendix in [6]) this leads to

$$c_0\left(s,n,j\right)=\sum_{k=0}^{\left\lfloor\frac{nj+s}{2j+1}\right\rfloor}\left(-1\right)^k\binom{n}{k}\binom{nj+s-\left(2j+1\right)k+n-1}{nj+s-\left(2j+1\right)k}\ . \tag{11}$$

For example, if $j=1/2$ the c_0s reduce to a single binomial coefficient [7].

$$c_0\left(s,n,1/2\right)=\binom{n}{n/2-s}\ ,\quad M\left(s;n;1/2\right)=\binom{n}{n/2-s}-\binom{n}{n/2-s-1}\ , \tag{12}$$

where $0\leq 2s\leq n$, with $s=0$ allowed only for even n.

A Lattice of Multiplicities

One may visualize $M\left(s;n;j\right)$ as a 3-dimensional semi-infinite lattice of points $\left(s;n;j\right)$ with integer multiplicities appropriately assigned to each lattice point. There are many straight lines on this lattice such that the multiplicities are polynomial in the line parameterization. For example, along some of the lattice diagonals,

$$M\left(n;n;1\right)=1\ ,\quad M\left(n-1;n;1\right)=n-1\ ,\quad M\left(n-2;n;1\right)=\tfrac{1}{2}\,n\left(n-1\right)\ . \tag{13}$$

These are, respectively, the number of ways the highest possible spin (i.e. $s=n$), the 2nd highest spin ($s=n-1$), and the 3rd highest spin ($s=n-2$) occur in the Kronecker product of n vector (i.e. $s=1$) representations. The form for the number of spins farther below the maximum $s=n$, that occur in products of n vectors, is

$$M\left(n-\left(2k+2\right);n;1\right)=\tfrac{1}{(2k+2)!}\ n\left(n-1\right)\left(n-2\right)\cdots\left(n-k\right)\times p_{k+1}\left(n\right)\ , \tag{14a}$$

$$M\left(n-\left(2k+3\right);n;1\right)=\tfrac{1}{(2k+3)!}\ n\left(n-1\right)\left(n-2\right)\cdots\left(n-k\right)\times q_{k+2}\left(n\right)\ , \tag{14b}$$

for $k=0,1,2,3,\cdots$, where p_{k+1} and q_{k+2} are polynomials in n of order $k+1$ and $k+2$, as follows.

$$p_{k+1}\left(n\right)=n^{k+1}+\tfrac{1}{2}\left(k+1\right)\left(5k-2\right)n^k+\tfrac{1}{24}\left(k\right)\left(k+1\right)\left(75k^2-205k-134\right)n^{k-1}+\cdots\ , \tag{15a}$$

$$q_{k+2}\left(n\right)=n^{k+2}+\tfrac{1}{2}\left(k\right)\left(5k+7\right)n^{k+1}+\tfrac{1}{24}\left(k+1\right)\left(75k^3-85k^2-410k-168\right)n^k+\cdots\ . \tag{15b}$$

As an exercise, the reader may verify the complete polynomials for orders 2, 3, 4, and 5.

$$\begin{gathered}p_2\left(n\right)=n^2+3n-22\ ,\quad q_2\left(n\right)=n^2-7\ ,\\ p_3\left(n\right)=n^3+12n^2-61n-192\ ,\quad q_3\left(n\right)=n^3+6n^2-49n+6\ ,\\ p_4\left(n\right)=n^4+26n^3-37n^2-1622n+120\ ,\quad q_4\left(n\right)=n^4+17n^3-91n^2-587n+1200\ ,\\ p_5\left(n\right)=n^5+45n^4+205n^3-5565n^2-17486n+48720\ ,\quad q_5\left(n\right)=n^5+33n^4-23n^3-3393n^2+2542n+21000\ .\end{gathered} \tag{16}$$

At the time of writing, the authors have not managed to identify the p_k and q_k polynomial sequences with any that were previously studied.

Tabulating Some Examples

For more explicit examples, we tabulate the number of singlets that appear in products $j^{\otimes n}$ for $j=1,\cdots,9$ and for $n=1,\cdots,10$. The Table entries below were obtained just by evaluation of the integrals in (6) for $s=0$.

M (0; n; j)	j = 1	j = 2	j = 3	j = 4	j = 5	j = 6	j = 7	j = 8	j = 9
n = 1	0	0	0	0	0	0	0	0	0
n = 2	1	1	1	1	1	1	1	1	1
n = 3	1	1	1	1	1	1	1	1	1
n = 4	3	5	7	9	11	13	15	17	19
n = 5	6	16	31	51	76	106	141	181	226
n = 6	15	65	175	369	671	1105	1695	2465	3439
n = 7	36	260	981	2661	5916	11 516	20 385	33 601	52 396
n = 8	91	1085	5719	19 929	54 131	124 501	254 255	474 929	827 659
n = 9	232	4600	33 922	151 936	504 316	1370 692	3229 675	6836 887	13 315 996
n = 10	603	19 845	204 687	1178 289	4779 291	15 349 893	41 729 535	100 110 977	217 915 579

Nonpolynomial Columns

The columns of the Table are *not* expressible as polynomials in n, for any fixed j, but they may be written as sums of hypergeometric or rational functions of n. For example, the first two columns may be written as

$$M\left(0;n;1\right)=3^{n}\sum_{k=0}^{n}\binom{n}{k}\binom{2k+1}{k+1}\left(-\frac{1}{3}\right)^{k}=\frac{4^{n}\,\Gamma\left(\frac{1}{2}+n\right)}{\sqrt{\pi}\,\Gamma\left(2+n\right)}\,{}_{2}F_{1}\left(-n,-1-n;\frac{1}{2}-n;\frac{1}{4}\right)\ , \tag{17}$$

$$M\left(0;n;2\right)=\frac{1}{2}\sum_{k=0}^{n}\frac{\left(-6\right)^{k}\ \Gamma\left(\frac{1}{2}+k\right)}{\Gamma\left(1+\frac{k}{2}\right)\Gamma\left(\frac{3}{2}+\frac{k}{2}\right)}\binom{n}{k}\,{}_{3}F_{2}\left(\frac{1}{4}+\frac{k}{2},\frac{3}{4}+\frac{k}{2},k-n;1+\frac{k}{2},\frac{3}{2}+\frac{k}{2};-16\right)\ . \tag{18}$$

To obtain these and other multiplicities as hypergeometric functions, for integer s and j, it is useful to change variables, to $t=\cos^{2}\vartheta$, so that (6) becomes

$$M\left(s;n;j\right)=\frac{2}{\pi}\ 4^{s+nj}\int_{0}^{1}\left(\prod_{k=1}^{s}\left(t-r_{k}\left(s\right)\right)\right)\left(\prod_{l=1}^{j}\left(t-r_{l}\left(j\right)\right)\right)^{n}\sqrt{\frac{1-t}{t}}\ dt\ . \tag{19}$$

The products here involve the known roots $r_{l}\left(j\right)$ of the Chebyshev polynomials. For integer j,

$$U_{2j}\left(\cos\left(\vartheta\right)\right)=4^{j}\prod_{l=1}^{j}\left(t-r_{l}\left(j\right)\right)\ ,\quad t=\cos^{2}\vartheta\ ,\quad r_{l}\left(j\right)=\cos^{2}\left(\frac{l\pi}{2j+1}\right)\ , \tag{20}$$

while for semi-integer j, for comparison to the integer case,

$$U_{2j}\left(\cos\left(\vartheta\right)\right)=4^{j}\sqrt{t}\ \prod_{l=1}^{j-\frac{1}{2}}\left(t-\ r_{l}\left(j\right)\right)\ , \tag{21}$$

with the usual convention that the empty product is 1.

The columns of the Table should be compared to the multiplicities of integer spins that appear in the product of $2m$ spin 1/2 representations. These are well-known to be given by the Catalan triangle [8],

$$M\left(s;2m;1/2\right)=\frac{\left(1+2s\right)\left(2m\right)!}{\left(m-s\right)!\left(m+s+1\right)!}\ , \tag{22}$$

as follows from (12). As an aside, it is perhaps not so well-known that multiplicities of all $SU\left(N\right)$ representations occurring in the product of n fundamental N-dimensional representations are given by N-dimensional Catalan structures [9, 10].

Be that as it may, this aside suggests an alternate route to obtain and to re-express some of the above results, especially for $j = 1$, a route that *retraces* [pun intended] many of the logical steps. This other route uses the explicit formula [10] for products of fundamental triplets of the group $SU(3)$ and the "tensor embedding" $SU(3) \supset SU(2)$ (where the triplet of $SU(3)$ is identified with the $s = 1$ vector representation) to deduce the number of $s = 0$ singlets appearing in the product of n vector representations of $SU(2)$, namely,

$$M(0;n;1) = (-1)^n \; {}_2F_1\left(-n, \tfrac{1}{2}; 2; 4\right) \ . \tag{23}$$

This is in exact agreement with the seemingly different result (17). Combining this with the elementary recursion relation that follows from $\overrightarrow{s} \otimes \overrightarrow{1} = \overrightarrow{s+1} \oplus \overrightarrow{s} \oplus \overrightarrow{s-1}$, namely,

$$M(s;n;1) = M(s+1;n-1;1) + M(s;n-1;1) + M(s-1;n-1;1) \ , \tag{24}$$

one then obtains $M(s;n;1)$ as a sum of Gauss hypergeometric functions. Relations between contiguous functions then simplify the result to a single hypergeometric function,

$$M(s;n;1) = (-1)^{n+s} \binom{n}{s} \; {}_2F_1\left(s-n, s+\frac{1}{2}; 2+2s; 4\right) \ . \tag{25}$$

Finally, the standard integral representation for ${}_2F_1$ eventually leads to the same integral form for $M(s;n;1)$ as given by (6) for $j = 1$.

Polynomial Rows

In contrast to the columns, the rows of the Table *are* expressible as polynomials in j for any fixed n. Starting with $n = 3$, the entries in the nth row of the Table are polynomials in j of order $n - 3$. The fourth row is obviously just the dimension of the spin j representation, and the fifth row is less obviously $1 + \frac{5}{2} c_j$, where c_j is the quadratic $su(2)$ Casimir for spin j. In fact, based on the numbers displayed above and some modest extensions of the Table, the row entries are seen to be of the form $poly_{(n-3)/2}(c_j)$ for odd $n \geq 3$ and $poly_{(n-4)/2}(c_j) \times d_j$ for even $n \geq 4$, where $poly_k(c)$ is a polynomial in c of order k. For the last eight rows of the Table these polynomials are given by:

$$\begin{array}{llll}
\mathsf{n}=3 & 1 & \mathsf{n}=4 & d_j \\
\mathsf{n}=5 & 1+\frac{5}{2}c_j & \mathsf{n}=6 & (1+2c_j)\, d_j \\
\mathsf{n}=7 & 1+\frac{14}{3}c_j+\frac{77}{12}c_j^2 & \mathsf{n}=8 & \left(1+4c_j+\frac{16}{3}c_j^2\right) d_j \\
\mathsf{n}=9 & 1+\frac{27}{4}c_j+\frac{73}{4}c_j^2+\frac{289}{16}c_j^3 & \mathsf{n}=10 & \left(1+6c_j+\frac{143}{9}c_j^2+\frac{140}{9}c_j^3\right) d_j
\end{array} \tag{26}$$

$$\text{where } d_j = 1 + 2j \ , \quad \text{and} \quad c_j = j(1+j) \ . \tag{27}$$

Thus the ten rows of the Table may be effortlessly extended to arbitrarily large j. Moreover, to obtain the polynomial that gives any row for $n > 10$, for arbitrary values of j, it is only necessary to evaluate $M(0;n;j)$ for $1 \leq j \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. Once again, at the time of writing, the authors have not managed to identify this polynomial sequence with any that were previously studied.

Asymptotic Behavior

Finally, consider the extension of the columns of the Table to arbitrarily large n, or more generally, consider the asymptotic behavior of $M(s;n;j)$ as $n \to \infty$ for fixed s and j. This behavior can be determined in a straightforward way, for any s and j, by a careful asymptotic analysis of the integral in (6). Such $n \to \infty$ behavior may be of interest in various statistical problems.

The simplest illustration is $M(0; n; 1/2)$ for even n. For this particular case, (22) and Stirling's approximation, $n! \underset{n\to\infty}{\sim} \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, give directly the main term in the asymptotic behavior,

$$M(0; 2m; 1/2) \underset{m\to\infty}{\sim} \frac{4^m}{m^{3/2}\sqrt{\pi}}\left(1 + O\left(\frac{1}{m}\right)\right) . \tag{28}$$

On the other hand, upon setting $t = \cos^2\vartheta$ the integral (6) has a form like that in (19), namely,

$$M(0; 2m; 1/2) = \frac{2}{\pi}\, 4^m \int_0^1 t^m \sqrt{\frac{1-t}{t}}\, dt = \frac{2}{\pi}\, 4^m B\left(m + \frac{1}{2}, \frac{3}{2}\right) . \tag{29}$$

The t integral is just a beta function, $B\left(m+\frac{1}{2},\frac{3}{2}\right) = \Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)/\Gamma(m+2)$, which leads back to exactly (22) for $s = 0$. But rather than using Stirling's approximation, it is more instructive to determine the asymptotic behavior directly from the integral (29) using Watson's lemma. Thus

$$M(0; 2m; 1/2) \underset{m\to\infty}{\sim} 2\sqrt{2}\, \frac{2^{2m}}{(2m)^{3/2}\sqrt{\pi}}\left(1 - \frac{9}{8m} + O\left(\frac{1}{m^2}\right)\right) . \tag{30}$$

Naively it might be expected that the leading asymptotic behavior (28) follows from a heuristic, saddle-point-Gaussian evaluation of the integration in (29). Unfortunately, that expectation is not fulfilled. The correct m dependence is obtained for M, but with an incorrect overall coefficient. To obtain the correct coefficient, a more careful analysis of the asymptotic behavior is needed, as provided by Watson's lemma.

Similarly, for large n the number of singlets occurring in the product of n spin 1 representations behaves as

$$M(0; n; 1) \underset{n\to\infty}{\sim} \frac{3\sqrt{3}}{8}\, \frac{3^n}{n^{3/2}\sqrt{\pi}}\left(1 - \frac{21}{16n} + O\left(\frac{1}{n^2}\right)\right) , \tag{31}$$

and the number of singlets in the product of n spin 2 representations behaves as

$$M(0; n; 2) \underset{n\to\infty}{\sim} \frac{1}{8}\, \frac{5^n}{n^{3/2}\sqrt{\pi}}\left(1 - \frac{15}{16n} + O\left(\frac{1}{n^2}\right)\right) . \tag{32}$$

In general, the number of spin s representations occurring in the product of n spin j representations for large n has asymptotic behavior [11]

$$M(s; n; j) \underset{n\to\infty}{\sim} (1+2s)\left(\frac{3}{2j(j+1)}\right)^{3/2} \frac{(1+2j)^n}{n^{3/2}\sqrt{\pi}}\left(1 - \frac{3}{4n} - \frac{9}{8n}\frac{1}{j(j+1)} - \frac{3}{2n}\frac{s(s+1)}{j(j+1)} + O\left(\frac{1}{n^2}\right)\right) . \tag{33}$$

This is correct for either integer or semi-integer s or j, although of course n must be (odd) even to obtain (semi-)integer s from products of semi-integer j, and only integer s are produced by integer j. Asymptotically then, for integer j,

$$M(j; n; j)/M(0; n; j) = M(0; n+1; j)/M(0; n; j) \underset{n\to\infty}{\sim} 1 + 2j + O\left(\frac{1}{n}\right) . \tag{34}$$

Remarkably, this behavior is approximately seen in the Table, with errors $\lessapprox 10\%$. On the other hand, for semi-integer j and even n,

$$M(j; n+1; j)/M(0; n; j) = M(0; n+2; j)/M(0; n; j) \underset{n\to\infty}{\sim} (1+2j)^2 + O\left(\frac{1}{n}\right) . \tag{35}$$

For integer j, the result in (33) follows directly, albeit tediously, from an application of Watson's lemma to (19) after switching to exponential variables. In that case the overall coefficient in (33) arises as a simple algebraic function of the Chebyshev roots in (20), namely, $1/\left(\sum_{l=1}^{j}\frac{1}{1-r_l(j)}\right)^{3/2}$. This then reduces to the Casimir-dependent expression in (33) by virtue of the integer j identity,

$$\sum_{l=1}^{j}\frac{1}{1-r_l\left(j\right)}=\frac{2}{3}\,j\left(j+1\right)\ . \tag{36}$$

Similar statements apply when j is semi-integer leading again to (33). For semi-integer j the relevant identity is

$$\frac{1}{2}+\sum_{l=1}^{j-1/2}\frac{1}{1-r_l\left(j\right)}=\frac{2}{3}\,j\left(j+1\right)\ , \tag{37}$$

with the usual convention that the empty sum is 0.

All-Order Extensions of the Asymptotics

The asymptotic behavior given by (33) is useful for fixed s and j in the limit as $n\to\infty$. If the resulting spin s produced by the n-fold product is also allowed to become large in the limit, e.g. $s=O\left(\sqrt{n}\right)$, then (33) is *not* useful. However, in that particular case it is possible to use renomalization group methods [12] to sum the series of terms involving powers of $\frac{1}{n}\frac{s(s+1)}{j(j+1)}$ to obtain an exponential, and hence an improved approximation. The result is

$$M\left(s;n;j\right)\underset{n\to\infty}{\sim}\left(1+2s\right)\left(\frac{3}{2j\left(j+1\right)}\right)^{3/2}\frac{\left(1+2j\right)^n}{n^{3/2}\sqrt{\pi}}\;e^{-\frac{3}{2n}\frac{s(s+1)}{j(j+1)}}\;\left(1-\frac{3}{4n}-\frac{9}{8n}\frac{1}{j\left(j+1\right)}+O\left(\frac{1}{n^2}\right)\right)\ . \tag{38}$$

For large n this last expression gives an excellent approximation out to values of s of order $\sqrt{n}$ and beyond. Moreover, the peak in the distribution of spins s produced by the product of n spin js is given for large n by

$$s_{\text{mult}}\underset{n\to\infty}{\sim}\sqrt{nj\left(j+1\right)/3}\ . \tag{39}$$

This follows from the exact result (22) for spin 1/2, or from (38) for any j. Alternatively, for specific j the direct numerical evaluation of either (6) or (10) verifies (39) upon taking n large, say, $n\approx 10^4$.

Perhaps some further insight is provided by the asymptotic behavior of the continuous function that gives the *normalized number of states with a given total spin*, s, as obtained from (38). This is

$$\frac{\left(1+2s\right)M\left(s;n;j\right)}{\left(1+2j\right)^n}\;dj\underset{n\to\infty}{\sim}\left(1-\frac{3}{4n}\left(1+\frac{1}{s\left(s+1\right)}\right)+O\left(\frac{1}{n^2}\right)\right)\;P\left(x\right)\;dx\ , \tag{40}$$

where, with a suitable choice of the variable x, $P\left(x\right)$ is the normalized chi-squared probability distribution function for *three* degrees of freedom:

$$x\equiv\frac{3\left(1+2j\right)^2}{8ns\left(s+1\right)}\ ,\quad P\left(x\right)=\frac{2}{\sqrt{\pi}}\;\sqrt{x}\;e^{-x}\ ,\quad \int_0^\infty P\left(x\right)dx=1\ . \tag{41}$$

In retrospect, this may not be a total surprise since the underlying rotation group may be parameterized by *three* Euler angles. Note that this last asymptotic form is correctly normalized to give the total number of states as $n\to\infty$, i.e.

$$\lim_{n\to\infty}\frac{1}{\sqrt{\pi}}\int_0^{nj}\left(1+2s\right)^2\;\left(\frac{3}{2nj\left(j+1\right)}\right)^{3/2}\;e^{-\frac{3}{2n}\frac{s(s+1)}{j(j+1)}}\;ds=1\ . \tag{42}$$

Also note that the expression for the number of states, (40), has a maximum at spin

$$s_{\text{state}}\underset{n\to\infty}{\sim}\sqrt{2}\;s_{\text{mult}}\underset{n\to\infty}{\sim}\sqrt{2nj\left(j+1\right)/3}\ . \tag{43}$$

Some Applications

In closing, we stress that spin multiplicities play useful roles in a wide range of fields, too numerous to present in detail here. But we briefly sketch a few applications of the results described above.

Some of the $SU(2)$ results for $s = 0$ have been used for decades in elasticity theory [13] and in quantum chemistry [14], as well as in nuclear physics, as is evident from the literature we have cited upon recognizing that the number of isotropic rank-n tensors in three dimensions is just $M(0;n;1)$.

The theory of group characters has been widely used in lattice gauge theory calculations for a long time [15, 16] and continues to play an important role in various strong coupling calculations [17]. Characters are also indispensible to determine the spin content of various string theories [18].

More generally, generic representation composition results continually find new uses. Recent examples include frustration and entanglement entropy for spin chains, with possible applications to black hole physics [19].

Multiplicities such as those in the Table have also attracted some recent attention in the field of quantum computing, ultimately with implications for cryptography. In particular, there are so-called "entanglement witness" (EW) operators that allow the detection of entangled states [20, 21, 22]. By knowing the degeneracy of the EW eigenstates for an n-particle state, one can determine the fraction of all states for which entanglement is "decidable" — a fraction that is especially of interest in the limit of large n. For systems of n spin j particles, with the EW operator taken to be the Casimir of the total spin, this fraction of decidable states [22] is denoted $f_j(n)$. In this case, from the asymptotic expression given above in (40), one readily obtains the exact result

$$\lim_{n\to\infty} f_j(n) = f_j(\infty) = \operatorname{erf}\left(\sqrt{\frac{3/2}{s+1}}\right) - \sqrt{\frac{6/\pi}{s+1}}\ \exp\left(-\frac{3/2}{s+1}\right)\ , \tag{44}$$

where $\operatorname{erf}(x) = 2\int_0^x \exp\left(-s^2\right) ds/\sqrt{\pi}$ is the conventional error function.

Many other statistical applications of spin multiplicities for large n have been proposed in a recent, independent investigation of this subject [23].

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