About a new family of sequences

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July 22, 2016

Abstract

First we define a new kind of function over $\mathbb{N}$. For each $i \in \mathbb{N}$ we have an associated function, which will be called S_i . Then we define a new kind of sequence, to be made from the functions S_i . Finally, we will see that some of these sequences has a self-similarity feature.

1 Introduction

There are sequences that are of great value for mathematics, many sequences are also important for other sciences. For these reasons the study of sequences is very important. But we can not forget that creativity in mathematics is also important, and this is more true when we are dealing with sequences, it is not enough just to study sequences, we also need to create sequences. This work deals with a new kind of sequence, we will see that this kind of sequence has interesting properties, that is reason enough to make this work worthwhile.

For this work, we are considering $\mathbb{N} = \{1, 2, 3, \ldots\}$. Consider $t \in \mathbb{N}$, such that t has $n \in \mathbb{N}$ digits, so we can write $t = a_n a_{n-1} \ldots a_2 a_1$. We can divide t in blocks of i digits (except at most one block), from right to left, with $i \in \mathbb{N}$ and $i \leq n$. If i divides n, then all blocks has i digits, otherwise the last block will have fewer digits.

Below are some examples to get a better understanding. Consider the number 123456, we divide it into blocks of i digits. Also consider that each block will be bracketed. number of digits.

Blocks of 1 digit: [1][2][3][4][5][6]
Blocks of 2 digits:[12][34][56]
Blocks of 3 digits:[123][456]
Blocks of 4 digits:[12][3456]
Blocks of 5 digits:[1][23456]
Blocks of 6 digits:[123456]
Blocks of 7 digits:[123456]
⋮

Define the function $T : \mathbb{N} \to \mathbb{N}$ such that $T(m) =$ "Sum of digits of m" $\cdot\, m$. Consider the number t divided into blocks of i digits. Let $[a_k a_{k-1} \ldots a_{k-i+1}]$ be one of its blocks, then $T([a_k a_{k-1} \dots a_{k-i+1}]) = (a_k + a_{k-1} + \dots + a_{k-i+1}) \cdot (a_k a_{k-1} \dots a_{k-i+1})$. With this in mind, now we define the function $S_i : \mathbb{N} \to \mathbb{N}$ to be

$$S_i(t) = \sum T([a_k a_{k-1} \dots a_{k-i+1}]),$$

where the sum goes over all the blocks as we described before.

Taking our previous example $t = 123456$, we have the following.

$$S_1(123456) = T(1)+T(2)+T(3)+T(4)+T(5)+T(6) = 1\cdot1+2\cdot2+3\cdot3+4\cdot4+5\cdot5+6\cdot6$$
$$= 1+4+9+16+25+36 = 91$$

$$S_2(123456) = T(12)+T(34)+T(56) = (1+2)\cdot12+(3+4)\cdot34+(5+6)\cdot56 = 3\cdot12+7\cdot34+11\cdot56 = 890$$

$$S_3(123456) = T(123)+T(456) = (1+2+3)\cdot123+(4+5+6)\cdot456 = 6\cdot123+15\cdot456 = 7578$$

$$S_4(123456) = T(12)+T(3456) = (1+2)\cdot12+(3+4+5+6)\cdot3456 = 3\cdot12+18\cdot3456 = 62244$$

$$S_5(123456) = T(1)+T(23456) = 1\cdot1+(2+3+4+5+6)\cdot23456 = 1+20\cdot23456 = 469121$$

$$S_6(123456) = T(123456) = (1+2+3+4+5+6)\cdot123456 = 21\cdot123456 = 2592576$$

$$S_7(123456) = T(123456) = (1+2+3+4+5+6)\cdot123456 = 21\cdot123456 = 2592576$$

$$\vdots$$

We can also define $S_\infty(t)$, which of course means "Sum of digits of t" $\cdot\, t$. Note that this definition of $S_\infty(t)$ coincides with the definition of the sequence A057147, hence, this characterization represents a more general view.

2 Some Results

Theorem 1: If $0 \leq t \leq 9$, then $S_i(t) = t^2$, for all $i \in \mathbb{N}$.

Proof: If $1 \leq i \leq 9$, then it's clear that $S_i(t) = (t) \cdot t = t^2$.

Theorem 2: Let $k, t \in \mathbb{N}$ such that $t = a_n a_{n-1} \dots a_1$ and k divides a_m for each $m \in \{1, 2, \dots, n\}$. Then, for all $i \in \mathbb{N}$, exists $b \in \mathbb{N}$ such that $S_i(t) = k^2 \cdot S_i(b)$.

Proof: Since k divides all the digits of t, for each digit a_m of t there is a natural b_m such that $a_m = k \cdot b_m$. Thus, we have that $t = a_n a_{n-1} \dots a_1 = k \cdot b_n \ k \cdot b_{n-1} \ k \cdot b_1$.

$$S_i(t) = S_i(k \cdot b_n \ k \cdot b_{n-1} \ k \cdot b_1) =$$

$$= S_i(k \cdot b_n \ k \cdot b_{n-1} \ \dots k \cdot b_{n-q}) + \dots + S_i(k \cdot b_i \ \dots k \cdot b_1) =$$

$$= (k \cdot b_n + k \cdot b_{n-1} + \dots + k \cdot b_{n-q}) \cdot (k \cdot b_n \ k \cdot b_{n-1} \dots k \cdot b_{n-q}) + \dots + (k \cdot b_i + \dots + k \cdot b_1) \cdot (k \cdot b_i \dots k \cdot b_1) =$$

$$= k(b_n + b_{n-1} + \dots + b_{n-q}) \cdot (k \cdot b_n \ k \cdot b_{n-1} \dots k \cdot b_{n-q}) + \dots + k(b_i + \dots + b_1) \cdot (k \cdot b_i \dots k \cdot b_1) =$$

$$= k\Big((b_n + b_{n-1} + \dots + b_{n-q}) \cdot (k \cdot b_n \ k \cdot b_{n-1} \dots k \cdot b_{n-q}) + \dots + (b_i + \dots +_b 1) \cdot (k \cdot b_i \dots k \cdot b_1)\Big).$$

Note that any number of the form $k \cdot b_x \ k \cdot b_{x-1} \dots k \cdot b_{x-y}$ can be written as

$$10^y \cdot k \cdot b_x + 10^{y-1} \cdot k \cdot b_{x-1} + \dots + 10^0 \cdot k \cdot b_{x-y}.$$

We can write the last equation as follows:

$$k\Big((b_n + b_{n-1} + \dots + bn - q) \cdot (10^q \cdot k \cdot b_n + 10^{q-1} \cdot k \cdot b_{n-1} + \dots + 10^0 \cdot k \cdot b_{n-q}) + \dots + (b_i + \dots + b_1) \cdot (10^i \cdot k \cdot b_i \dots 10^0 \cdot k \cdot b_1)\Big) =$$

$$= k\Big((b_n + b_{n-1} + \dots + b_{n-q}) \cdot k(10^q \cdot b_n + 10^{q-1} \cdot b_{n-1} + ... + 10^0 \cdot b_{n-q}) + \dots + (b_i + \dots + b_1) \cdot k(10^i \cdot b_i \dots 10^0 \cdot b_1)\Big) =$$

$$= k^2\Big((b_n + b_{n-1} + \dots + b_{n-q}) \cdot (10^q \cdot b_n + 10^{q-1} \cdot b_{n-1} + \dots + 10^0 \cdot b_{n-q}) + \dots + (b_i + \dots + b_1) \cdot (10^i \cdot b_i \dots 10^0 \cdot b_1)\Big) =$$

$$= k^2\Big((b_n + b_{n-1} + \dots + b_{n-q}) \cdot (b_n b_{n-1} \dots b_{n-q}) + \dots + (b_i + \dots + b_1) \cdot (b_i \dots b_1)\Big).$$

Let $b \in \mathbb{N}$ such that $b = b_n b_{n-1} \dots b_1$, the existence of b is guaranteed, because it is guaranteed the existence of each of b_m . Also note that the above equation is equal to $k^2 \cdot S_i(b)$, as we wanted to prove.

Corollary 2.1: If $0 \leq a \leq 9$, then for each $i \in \mathbb{N}$, $S_i(\underbrace{a \ a \dots a}_{n \ times}) = a^2 \cdot S_i(\underbrace{1 \ 1 \dots 1}_{n \ times})$.

Proof: Note that $(a \ a \dots a) = (a \cdot 1 \ a \cdot 1 \dots a \cdot 1)$. Using theorem 2, we get $S_i(a \cdot 1 \ a \cdot 1 \dots a \cdot 1) = a^2 \cdot S_i(1 \ 1 \ \dots 1)$, as desired.

Corollary 2.2: If $S_i(t)$ is a prime number, then there is not a number (greater than 1) that divides all the digits of t.

Proof: By Theorem 2, if all the digits of t are divisible by a number (greater than 1), then $S_i(t)$ can't be prime because it's the product of two numbers greater than 1.

Theorem 3: No function S_i is injective.

Proof: For every $i \in \mathbb{N}$, we have that $S_i(1) = 1$, we also have that $S_i(10^i) = S_i(\underbrace{100\ldots0}_{i+1\ digits}) = 1$. Therefore, S_i is not injective.

Theorem 4: Every function S_i is surjective.

Proof: Let $n, t \in \mathbb{N}$ such that n is arbitrary and $t = 1\underbrace{\underbrace{0\ 0\ \ldots1}_{i\ digits}\ldots\underbrace{0\ 0\ \ldots1}_{i\ digits}}_{n-1\ times}$. Then $S_i(t) = 1 + \underbrace{1 + 1 + \ldots + 1}_{n-1\ times} = n$. Therefore, S_i is surjective.

We can construct some special sequences from the functions S_i. For each $i \in \mathbb{N}$, we will be interested in the sequence $\mathcal{S}_i = (S_i(1), S_i(2), S_i(3), \ldots)$.

3 Charts

Below are some charts of some of these sequences. These charts clearly show that there is a kind of pattern. In particular, the charts of S_1 , S_2 and S_7 has some self-similarity features.

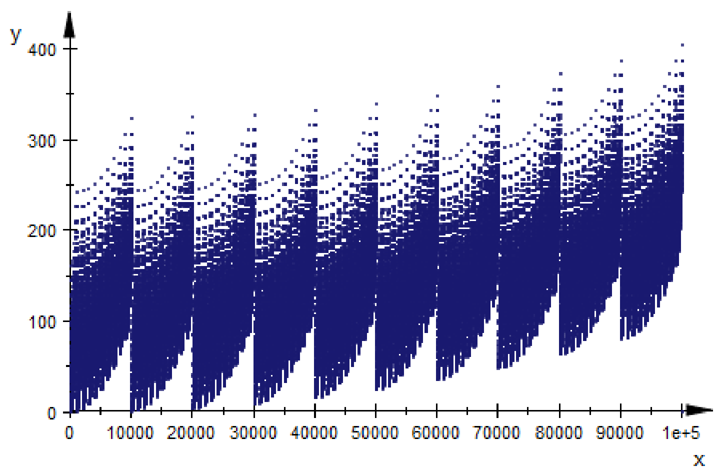

Fig. 1: values of $\mathcal{S}_1(x)$, with $x = 1 \ldots 10^5$.

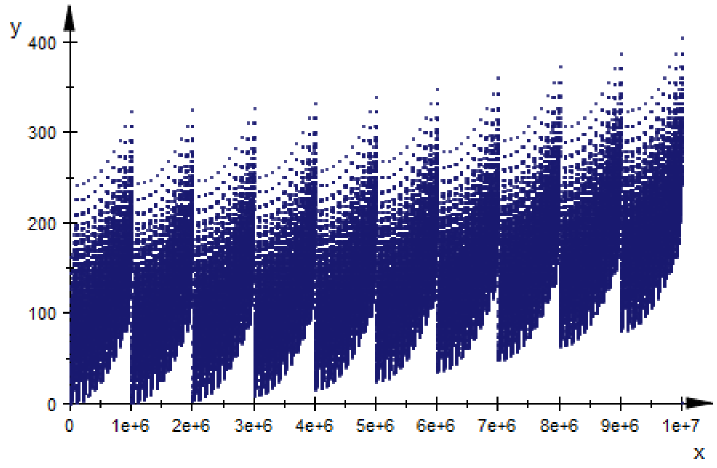

Fig. 2: values of $\mathcal{S}_1(x)$, with $x = 1 \ldots 10^7$.

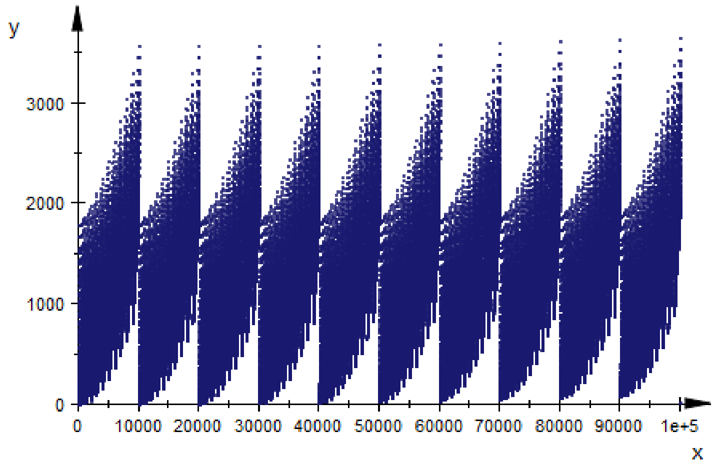

Fig. 3: values of $\mathcal{S}_2(x)$, with $x = 1 \ldots 10^5$.

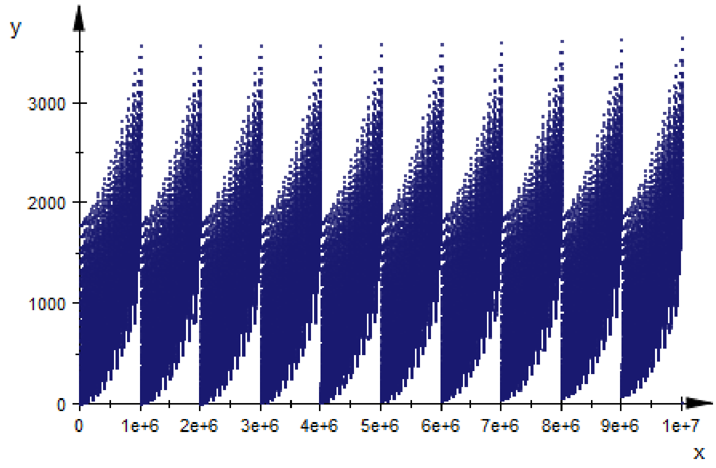

Fig. 4: values of $\mathcal{S}_2(x)$, with $x = 1 \ldots 10^7$.

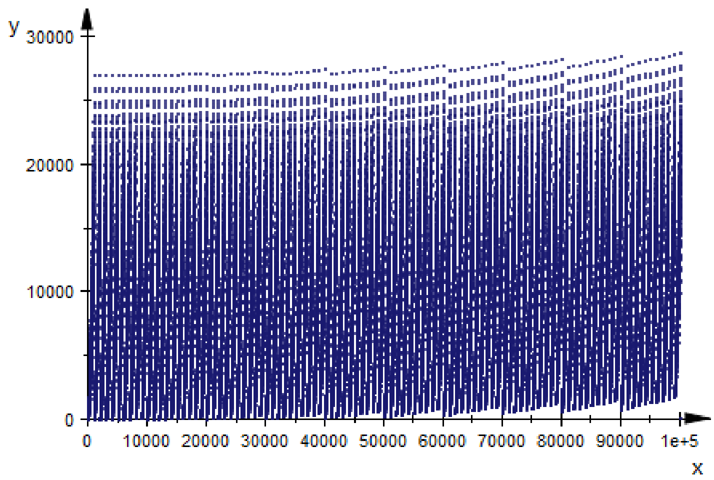

Fig. 5: values of $\mathcal{S}_3(x)$, with $x = 1 \ldots 10^5$.

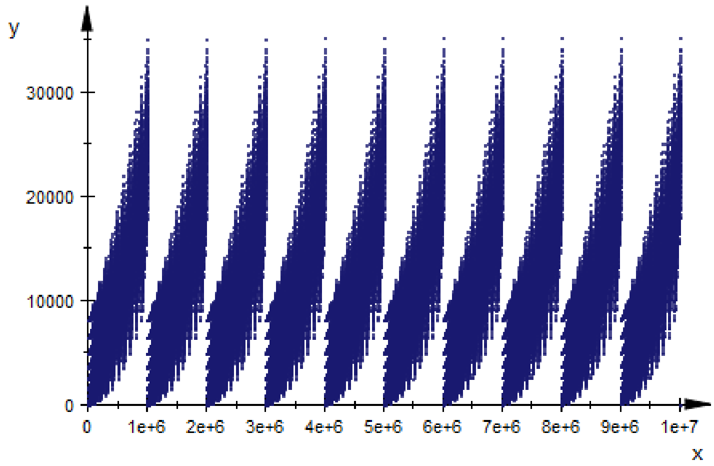

Fig. 6: values of $\mathcal{S}_3(x)$, with $x = 1 \dots 10^7$.

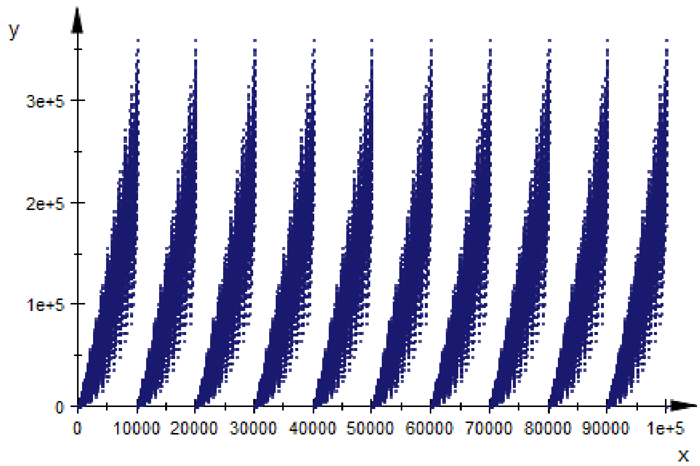

Fig. 7: values of $\mathcal{S}_4(x)$, with $x = 1 \dots 10^5$.

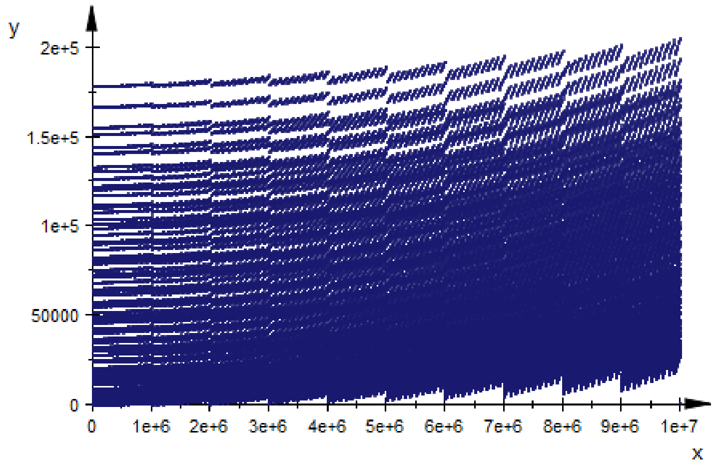

Fig. 8: values of $\mathcal{S}_4(x)$, with $x = 1 \dots 10^7$.

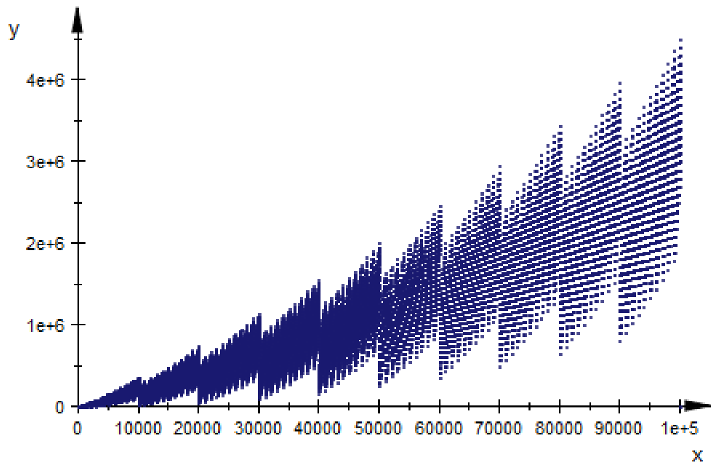

Fig. 9: values of $\mathcal{S}_5(x)$, with $x = 1 \dots 10^5$.

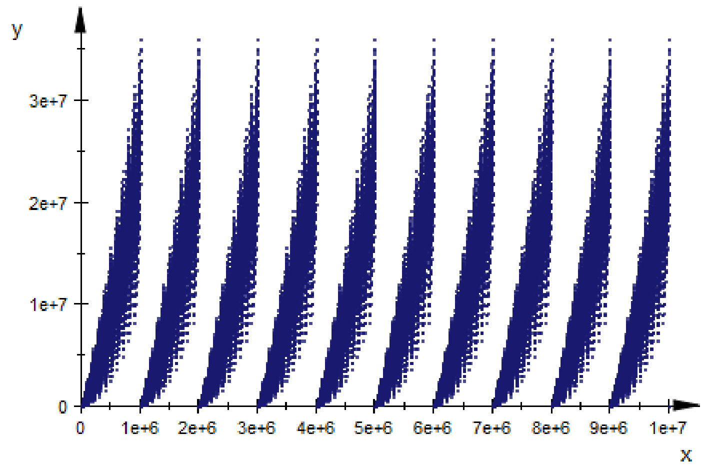

Fig. 10: values of $\mathcal{S}_6(x)$, with $x = 1 \ldots 10^7$.

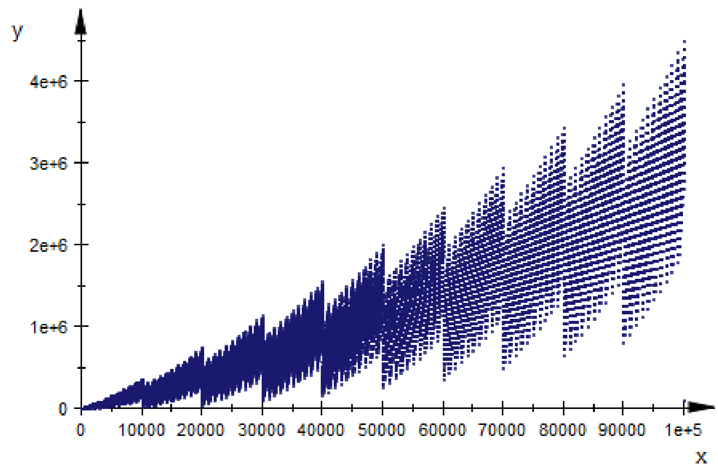

Fig. 11: values of $\mathcal{S}_7(x)$, with $x = 1 \ldots 10^5$.

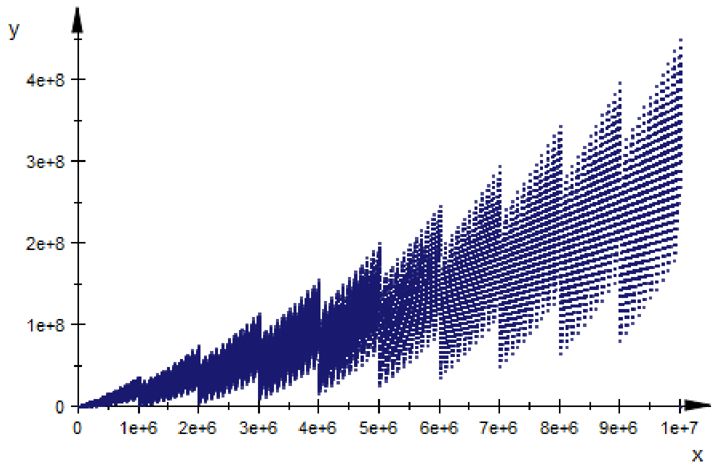

Fig. 12: values of $\mathcal{S}_7(x)$, with $x = 1 \dots 10^7$.

References

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[2] Elon L. Lima. Curso de Analise Volume 1, IMPA, 2009.

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