Set-Valued Tableaux & Generalized Catalan Numbers

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Abstract

Standard set-valued Young tableaux are a generalization of standard Young tableaux in which cells may contain more than one integer, with the added conditions that every integer at position (i, j) must be smaller than every integer at positions $(i, j + 1)$ and $(i + 1, j)$. This paper explores combinatorial interpretations of standard set-valued Young tableaux that generalize the well-known relationship between standard Young tableaux of shape $\lambda = n^2$ and the Catalan numbers. Interpretations in terms of two-row standard set-valued Young tableaux are provided for both the k-Catalan numbers and the two-parameter Fuss-Catalan numbers (Raney numbers), generalizing earlier work by Heubach, Li and Mansour. We then draw a general bijection between classes of two-row standard set-valued Young tableaux and collections of two-dimensional lattice paths that lie weakly below a unique maximal path. This bijection specializes to give a new interpretations of rational Dyck paths (and by extension the rational Catalan numbers) as well as the solution to the so-called "generalized tennis ball problem".

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1 Introduction

For a non-increasing strong partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, a Young diagram Y of shape λ is a left-justified array of cells with exactly λ_i cells in its i^{th} row. If Y is a Young diagram of shape λ with $\sum_i \lambda_i = n$, a Young tableau of shape λ is an assignment of the integers $[n] = \{1, \ldots, n\}$ to the cells of Y such that every integer is used precisely once. A Young tableau in which integers increase from top-to-bottom down every column and increase from left-to-right across every row is said to be a standard Young tableau. We denote the set of all standard Young tableau of shape λ by $S(\lambda)$. For m-row rectangular shapes $\lambda = (n, \ldots, n)$ we use the abbreviated notation $S(n^m)$. For a comprehensive introduction to Young tableaux, see Fulton [8].

Let Y be a Young diagram of shape λ, and let $w = \{w_{i,j}\}$ be a collection of positive integers such that $\sum_{i,j} w_{i,j} = m$. A **set-valued tableau** of shape λ and weight w is an assignment of $[m]$ to the cells of Y such that every element of $[m]$ is used precisely once and the cell at position (i, j) receives a set $B_{i,j}$ of integers with $|B_{i,j}| = w_{i,j}$. A set-valued tableau is said to be a **standard set-valued Young tableau** if we additionally require that $\max(B_{i,j}) < \min(B_{i+1,j})$ and $\max(B_{i,j}) < \min(B_{i,j+1})$ for all indices i, j. In analogy with standard Young tableaux, we refer to these added conditions as "column-standardness" and "row-standardness". We denote the set of all standard set-valued Young tableaux of shape λ and weight w by $\mathbb{S}(\lambda, w)$. For set-valued tableaux in which cells have a fixed weight

w_i across each row, we adopt the shorthand notations $w = (w_1, w_2, \ldots)$ and $\mathbb{S}(\lambda, (w_1, w_2, \ldots))$. See Figure 1 for an example showing all six elements of $\mathbb{S}(\lambda, w)$ when $\lambda = (2, 2)$ and $w = (2, 2)$.

$$\begin{array}{|c|c|}\hline 1\,2 & 3\,4 \\ \hline 5\,6 & 7\,8 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1\,2 & 3\,5 \\ \hline 4\,6 & 7\,8 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1\,2 & 3\,6 \\ \hline 4\,5 & 7\,8 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1\,2 & 4\,5 \\ \hline 3\,6 & 7\,8 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1\,2 & 4\,6 \\ \hline 3\,5 & 7\,8 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1\,2 & 5\,6 \\ \hline 3\,4 & 7\,8 \\ \hline\end{array}$$

Figure 1: The set $\mathbb{S}(\lambda, w)$ for $\lambda = (2, 2)$ and $w = (2, 2)$

Set-valued tableaux were introduced by Buch [6] in his investigation of the K-theory of Grassmannians.[1] More directly influencing this paper is the work of Heubach, Li and Mansour [12], who argued that the cardinality of $\mathbb{S}(n^2, (k-1, 1))$ equalled the k-Catalan number C_n^k. For a more recent appearance of standard set-valued tableaux see Reiner, Tenner and Yong [19], who investigated so-called "barely set-valued tableaux" with a single non-unitary weight $w_{i,j} = 2$ (not necessarily located at a fixed position i, j). It should be noted that much of our notation is modeled after that of Heubach, Li and Mansour [12].

The primary goal of this paper is provide new combinatorial interpretations of generalized Catalan numbers in terms of two-row standard set-valued Young tableaux $\mathbb{S}(\lambda, w)$. In particular, we give a new interpretations of the Raney numbers (two-parameter Fuss-Catalan numbers) as set-valued tableaux that recover the k-Catalan interpretation of Heubach, Li and Mansour for an appropriate choice of parameters (Theorem 2.3). We then draw a bijection between two-row standard set-valued Young tableau with arbitrary weight and classes of two-dimensional integer lattice paths with "East" $E = (1, 0)$ and "North" $N = (0, 1)$ steps. In particular, $\mathbb{S}(n^2, w)$ with $w_{1,j} = a_j$ and $w_{2,j} = b_j$ are placed in bijection with all such lattice paths that lie weakly below the lattice path $P = E^{a_1}N^{b_1}E^{a_2}N^{b_2}\ldots$ (Theorem 3.2). This bijection is then applied to give new combinatorial interpretations of the rational Catalan numbers (Corollary 3.6) as well as the solution to the "(s,t)-tennis ball problem" of Merlini, Sprugnoli, and Verri [14] (Theorem 3.9). See Figure 2 for an overview of the various weights needed to achieve our desired interpretations.

k-Catalan numbers C_n^k for $\lambda = n^2$

$k-1$	$\ldots$	$k-1$
1	$\ldots$	1

Raney numbers $R_{k,r}(n)$ for $\lambda = (n+1)^2$

$k-1$	$\ldots$	$k-1$	$r-1$
1	$\ldots$	1	1

Rational Catalan numbers $C(a, b)$ for $\lambda = a^2$

1	1	$\ldots$	1
$\lfloor \frac{ab}{a} \rfloor - \lfloor \frac{(a-1)b}{a} \rfloor$	$\ldots$	$\lfloor \frac{2b}{a} \rfloor - \lfloor \frac{b}{a} \rfloor$	$\lfloor \frac{b}{a} \rfloor - \lfloor 0 \rfloor$

Solution to (s,t)-tennis ball problem for $\lambda = (n+1)^2$ (after n turns)

$s-t$	$\ldots$	$s-t$
t	$\ldots$	t

Figure 2: Weights w needed for $|\mathbb{S}(\lambda, w)|$ to yield various combinatorial interpretations.

Before proceeding to the central sections of the paper, we pause to prove one basic result about rectangular set-valued tableaux that we will repeatedly reference throughout. So let $\lambda = n^m$ be a rectangular tableau shape. For any weight $w = \{w_{i,j}\}$ we may define an "inverted weight" $w^{-1} = \{w_{n-i+1, m-j+1}\}$ that corresponds to the composition of a horizontal and a vertical reflection of individual cell weights.

Proposition 1.1. *For any rectangular shape $\lambda = n^m$ and any weight w, $|\mathbb{S}(\lambda, w)| = |\mathbb{S}(\lambda, w^{-1})|$.*

[1] Buch actually considered a semistandard generalization of set-valued tableaux where entries were only required to weakly increase from left-to-right across each row. We will not consider that generalization here.

Proof. Take any $T \in \mathbb{S}(\lambda, w)$, and assume $\sum_{i,j} w_{i,j} = k$. Relabel the entries of T according to the bijection $x \mapsto k - x + 1$. This produces a "reverse standard" set-valued tableaux where $\max(B_{i+1,j}) < \min(B_{i,j})$ and $\max(B_{i,j+1}) < \min(B_{i,j})$ for all i, j. Collectively reassigning the sets $B_{i,j}$ so that all elements of $B_{i,j}$ are moved to position $(n - i + 1, m - j + 1)$ then gives an element of $\mathbb{S}(\lambda, w^{-1})$. $\square$

Proposition 1.1 will prove especially useful for set-valued tableaux whose cells have a constant weight across each row. In this specialization, the proposition manifests as invariance under a vertical reflection of row weights:

Corollary 1.2. *Take any rectangular shape $\lambda = n^m$ and row-constant weight $w = (w_1, \ldots, w_m)$. Then $|\mathbb{S}(\lambda, (w_1, \ldots, w_m))| = |\mathbb{S}(\lambda, (w_m, \ldots, w_1))|$.*

2 Generalized Catalan Numbers and Set-Valued Tableaux

Recall that the Catalan numbers are a sequence of positive integers whose n^{th} entry ($n \geq 0$) is $C_n = \frac{1}{n+1}\binom{2n}{n}$. The Catalan numbers satisfy the recurrence $C_0 = 1$ and $C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}$ for all $n \geq 1$, from which one may derive the generating function $C(x) = \frac{1-\sqrt{1-4x}}{2x}$. Among the hundreds of combinatorial interpretations of the Catalan numbers compiled by Stanley [17] are that C_n equals the number standard Young tableaux of shape $\lambda = (n, n)$, the number of Dyck paths of lengths $2n$, and the number of rooted binary trees with $2n$ edges. Since we will utilize their generalizations in upcoming subsections, we quickly review the bijections between these combinatorial interpretations. For a significantly more in-depth treatment of these bijections, see Stanley [17].

So let $\mathcal{D}_n$ denote the set of all Dyck paths of length $2n$. Formally, $\mathcal{D}_n$ consists of all integer lattice paths from $(0, 0)$ to (n, n) that utilize only "East" $E = (1, 0)$ and "North" $N = (0, 1)$ steps and which stay weakly below the line $y = x$. That final condition is equivalent to saying that a path $P = \{v_0, \ldots, v_{2n}\}$ in $\mathcal{D}_n$ satisfies $x_i \geq y_i$ at every integer lattice point $v_i = (x_i, y_i)$. Elements of $\mathcal{D}_n$ are alternatively referred to 2-good paths or 2-Catalan paths. The most straightforward bijection between $\mathcal{D}_n$ and $S(n^2)$ associates $P = \{v_0, \ldots, v_{2n}\}$ with the unique row-standard tableau T such that i lies in the first row of T if v_i follows an East step of P and i lies in the second row of T if v_i follows a North step of P. The condition that P stays weakly below $y = x$ ensures that the resulting tableau T is column-standard. For an example of this bijection, see Figure 3

Similarly let $\mathcal{T}_n$ denote the set of all rooted binary trees with $2n$ total edges. Observe that elements of $\mathcal{T}_n$ have n internal vertices, each of which has a designated "left child" and "right child". One bijection between $\mathcal{T}_n$ and $S(n^2)$ is obtained by beginning at the root vertex of $G \in \mathcal{T}_n$ and tracing around G in the clockwise direction. Every time an edge is met for the first time (always on its left side) assign that edge the smallest available element of $[2n]$; if the edge is a left child place that integer in the first row of the associated tableau T, and if the edge is a right child place the integer in the second row of T. The fact that the number of left children encountered at any point never exceeds the number of right children encountered guarantees that T is column-standard. Once again see Figure 3 for an example of this bijection.

1	2	4	7
3	5	6	8

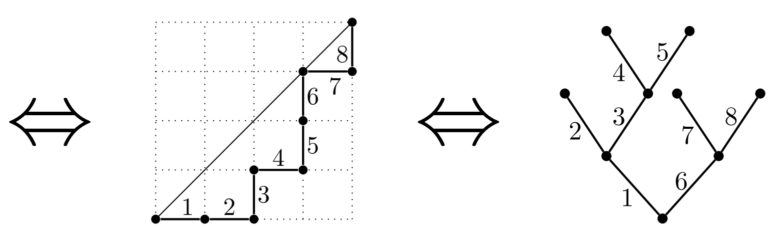

Figure 3: A standard Young tableau $T \in S(4^2)$, alongside the corresponding elements of $\mathcal{D}_4$ and $\mathcal{T}_4$.

2.1 k-Catalan numbers

The generalizations of the Catalan numbers that we will address are the k-Catalan numbers (one-parameter Fuss-Catalan numbers), the Raney numbers (two-parameter Fuss-Catalan numbers), and the rational Catalan numbers. We begin with a brief discussion of the k-Catalan numbers, and then give a new proof of the result from Heubach, Li and Mansour [12] that associates the k-Catalan numbers with two-row standard set-valued Young tableaux of row-constant weight $w = (k-1, 1)$.

For any $k \geq 1$, the k-Catalan numbers are given by $C_n^k = \frac{1}{kn+1}\binom{kn+1}{n} = \frac{1}{(k-1)n+1}\binom{kn}{n}$ for all $n \geq 0$. Notice that the k-Catalan numbers give the constant sequence $1, 1, \ldots$ when $k = 1$ and specialize to the usual Catalan numbers when $k = 2$. The k-Catalan numbers are alternatively defined by the recurrence of Equation 1, a derivation of which may be found in Hilton and Pedersen [11]. Here we use the standard notation of $\vdash N$ for an ordered (weak) partition of the positive integer N.

$$C_0^k = 1 \qquad\qquad C_n^k \;=\; \sum_{(i_1,\ldots,i_k)\vdash n-1} C_{i_1}^k \ldots C_{i_k}^k \quad \text{for all } n \geq 1 \tag{1}$$

See Hilton and Pedersen [11] or Heubach, Li and Mansour [12] for various combinatorial interpretations of the k-Catalan numbers. Relevant to our work are that C_n^k equals the number of k-good lattice paths of length kn, the number of rooted k-ary trees with kn edges, and the number of standard set-valued Young tableaux with shape $\lambda = (n, n)$ and row-constant weight $w = (1, k-1)$. Observe that Corollary 1.2 places the final set of objects in bijection with standard set-valued Young tableaux of shape $\lambda = (n, n)$ and weight $w = (k-1, 1)$, putting our results in alignment with those of Heubach, Li and Mansour [12]. Our weight is chosen to allow for a more straightforward bijection between standard set-valued Young tableaux and k-ary trees.

A k-good path of length kn is an integer lattice path from $(0, 0)$ to $(n, (k-1)n)$ that utilizes only East $E = (1, 0)$ and North $N = (0, 1)$ steps and which stays weakly below the line $y = (k-1)x$. Rooted k-ary trees with kn total edges are a natural generalization of binary-trees whereby each of the n internal vertices now has k (ordered) children. If we define a p-star to be a rooted tree with p terminal edges atop a single base vertex, a rooted k-ary tree with kn edges is formed via the recursive placement of n total k-stars atop terminal edges. We denote the set of all k-good paths of length kn by $\mathcal{D}_n^k$ and the set of all rooted k-ary trees with kn edges by $\mathcal{T}_n^k$. For a standard proof of the fact that $|\mathcal{D}_n^k| = |\mathcal{T}_n^k| = C_n^k$, see Hilton and Pedersen [11].

Heubach, Li and Mansour [12] originally established $|\mathbb{S}(n^2, (k-1, 1))| = C_n^k$ by placing their set-valued tableaux in bijection with so-called k-ary paths of length kn. These k-ary paths are a slight modification of our k-good paths whose bijection with $\mathbb{S}(n^2, (k-1, 1))$ may be very directly modified into a bijection between k-good paths and the set-valued tableaux of $\mathbb{S}(n^2, (1, k-1))$. Proposition 2.1 presents this modified bijection between $\mathbb{S}(n^2, (1, k-1))$ and $\mathcal{D}_n^k$, as well as a bijection between $\mathbb{S}(n^2, (1, k-1))$ and $\mathcal{T}_n^k$. For an example illustrating the bijections of Proposition 2.1, see Figure 4.

Proposition 2.1. *For any $k \geq 1$ and $n \geq 0$, $|\mathbb{S}(n^2, (1, k-1))| = |\mathcal{D}_n^k| = |\mathcal{T}_n^k| = C_n^k$.*

Proof. We begin with a bijection between $\mathbb{S}(n^2, (1, k-1))$ and $\mathcal{D}_n^k$ that directly generalizes the bijection between $S(n^2)$ and $\mathcal{D}_n$. So take $P = \{v_0, \ldots, v_{kn}\}$ in $\mathcal{D}_n^k$. Use P to construct a row-standard set-valued tableaux T of weight $w = (1, k-1)$ such that i lies in the first row of T if v_i follows an East step and i lies in the second row of T if v_i follows a North step. The condition that P lies weakly below $y = (k-1)x$ ensures that the number of North steps never exceeds $k-1$ times the number of East steps at any point along P. This ensures that all $k-1$ integers at position $(2, j)$ in T are larger than the integer at position $(1, j)$, making T column-standard. Our construction is well-defined and injective, as $P \in \mathcal{D}_n^k$ is uniquely defined by its collection of East steps and $T \in \mathbb{S}(n^2, (1, k-1))$ is uniquely defined by the entries in its first row. The procedure is clearly reversible.

We now give a bijection between $\mathbb{S}(n^2,(1,k-1))$ and $\mathcal{T}_n^k$ that directly generalizes the bijection between $S(n^2)$ and $\mathcal{T}_n$. For any $G \in \mathcal{T}_n^k$, start at the root vertex of G and trace around the tree in the clockwise direction, enumerating the kn edges in the order at which they are first encountered. Construct a set-valued tableau T of weight $w = (1, k-1)$ by placing all integers corresponding to leftmost children of G in the first row of T, and then placing all remaining integers in the second row of T. As a leftmost child is always encountered before any remaining children of a fixed vertex, the integer at position $(1,j)$ of T is always smaller than all integers at position $(2,j)$ of T. As with our first bijection, our construction is clearly well-defined and injective. To see that this map is reversible, notice that G may be recovered from T by beginning with a "base" k-star and proceeding to enumerate the edges via "clockwise tracing" as above, placing another k-star atop edge i if and only if $i+1$ appears in the first row of T. $\square$

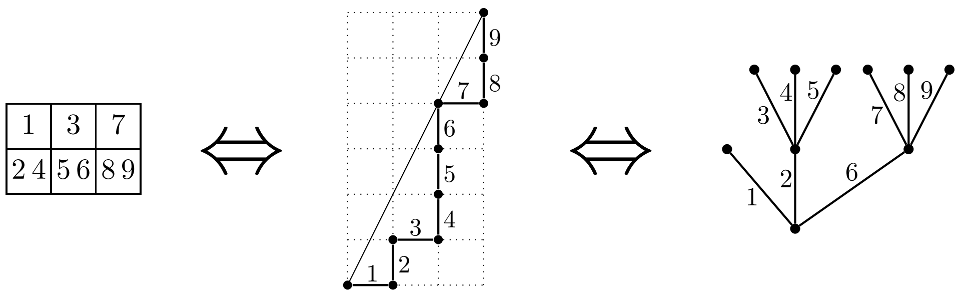

Figure 4: A standard set-valued Young tableau $T \in \mathbb{S}(3^2,(1,2))$, alongside the corresponding 3-good path of $\mathcal{D}_3^3$ and rooted ternary tree of $\mathcal{T}_3^3$.

2.2 Raney numbers

A further generalization of the Catalan numbers that have become prominent in recent decades are the Raney numbers, also known as the two-parameter Fuss-Catalan numbers. For any $k \geq 1$ and $r \geq 1$, the Raney numbers are given by $R_{k,r}(n) = \frac{r}{kn+r}\binom{kn+r}{n}$ for all $n \geq 0$.[2] The Raney numbers specialize to the k-Catalan numbers as $R_{k,1}(n) = C_n^k$ and thus to the original Catalan numbers as $R_{2,1}(n) = C_n$. The Raney numbers were introduced by Raney [16] in his study of functional composition patterns, and have more recently found use in noncommutative probability [15], the enumeration of planar embeddings [2], and (s,t)-core partitions [21].

Hilton and Pedersen [11] further related the Raney numbers to the k-Catalan numbers via Equation 2. This equation may be viewed as a generalization of the recurrence from Equation 1 when one notes that $R_{k,k}(n-1) = C_n^k$, a distinct identity from the "obvious" specialization of Equation 2 to the k-Catalan numbers as $R_{k,1}(n) = C_n^k$. The fact that $R_{k,k}(n-1) = R_{k,1}(n)$ is easily verified by hand.

$$R_{k,r}(n) = \sum_{(i_1,\ldots,i_r)\vdash n} C_{i_1}^k C_{i_2}^k \ldots C_{i_r}^k \tag{2}$$

Beagley and the author [2] utilized the decomposition of Equation 2 to give a combinatorial interpretation of $R_{k,r}(n)$ as the number of so-called "coral diagrams of type (k,r,n)". A coral diagram of type (k,r,n) is a rooted planar graph constructed from a single "base" r-star via the recursive placement of n total k-stars atop terminal edges of the graph.[3] This is equivalent to a rooted tree with

[2]Hilton and Pedersen [11] use the alternative notation $d_{qk}(p) = \frac{p-q}{pk-q}\binom{pk-q}{k-1}$ for their two-parameter generalization. Our two notations are related via the change of variables $R_{p,p-q}(k-1) = d_{qk}(p)$.

[3]Beagley and the author [2] originally replaced the base r-star with a base $(r+1)$-star in which the leftmost edge cannot be the attachment site of any k-stars. This clearly does not change the number of distinct graphs possible.

$kn + r$ edges that is a k-ary tree apart from its root vertex, which is an r-star. Via Equation 2, it is also useful to think of a coral diagram of type (k, r, n) as an ordered collection of r (possibly empty) k-ary trees, with one k-ary tree attached atop each edge of the base r-star. For an example of a coral diagram, see Figure 5.

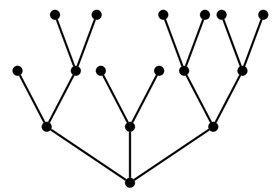

Figure 5: A coral diagram of type $(2, 3, 6)$, constructed from a base 3-star via the recursive placement of six total 2-stars.

In order to interpret the Raney numbers in terms of standard set-valued Young tableaux, we need to expand our focus beyond set-valued tableaux that have a fixed weight across each row. As the first step in obtaining this interpretation, notice that Equation 2 and Proposition 2.1 allow us to associate the Raney numbers with ordered tuples of set-valued tableaux with identical row-constant weights:

Proposition 2.2. *Fix $k, r \geq 1$, $n \geq 0$. Then $R_{k,r}(n)$ equals the number of ordered r-tuples $(T_1, \ldots, T_r)$ of standard set-valued Young tableaux such that $T_j \in \mathbb{S}(i_j^2, (1, k-1))$ and $i_1 + \ldots + i_r = n$.*

Our goal is to directly replace the ordered r-tuples of Proposition 2.2 with a single set-valued tableau of shape $\lambda = (n+1)^2$. This utilizes a technique that we refer to "horizontal tableaux concatenation", whereby the entries of the ordered r-tuple are continuously reindexed and a new column of weight $\widetilde{w} = (1, r-1)$ is added to the front of the resulting tableau. This additional column carries the information needed to recover the original partition of the tableau into r pieces.

So fix $n \geq 0$ and take any two-row rectangular shape $\lambda = (n+1)^2$. In order to ease notation, for any $k, r \geq 1$ we define the weight $w(k, r) = \{w_{i,j}\}$ by $w_{1,j} = 1$ for all $1 \leq j \leq n$, $w_{2,1} = r - 1$, and $w_{2,j} = k - 1$ for all $2 \leq j \leq n$. Notice that this is actually the inverted weight from what appears under the $R_{k,r}(n)$ label in Figure 2 of Section 1, but that those two weights yield sets of equivalent size by Proposition 1.1.

Theorem 2.3. *Take any $k, r \geq 1$, $n \geq 0$, and define the weight $w(k, r)$ as above. Then $R_{k,r}(n) = |\,\mathbb{S}((n+1)^2, w(k, r))|$.*

Proof. We construct a bijection between the r-tuples of Proposition 2.2 and $\mathbb{S}((n+1)^2, w(k, r))$ using the aforementioned method of horizontal concatenation. For an example illustrating both directions of our bijection, see Figure 6.

So take $(T_1, \ldots, T_r)$, where $T_j \in \mathbb{S}(i_j^2, (1, k-1))$ and $i_1 + \ldots + i_r = n$. Observe that a total of kn integers appear across the $2n$ cells of the T_j. Create a partially-filled Young diagram D of shape $\lambda_D = (n+r)^2$ by adding an empty column c_j in front of each T_j and then horizontally concatenating the resulting tableaux in the given order. Notice that this will result in multiple consecutive empty columns if any of the T_j are empty. Mark the top cell of column c_1 and the bottom cells of columns $c_2, \ldots, c_r$. This gives r markings in addition to the kn integers of D. Re-index these $kn + r$ items with $[kn + r]$ by working through D from left-to-right. Every time a marking is encountered, assign the marked cell the smallest available element of $[kn + r]$. When T_j is encountered, simultaneously replace the ki_j integers of T_j with the ki_j smallest available elements of $[kn + r]$, preserving the relative order of the entries within T_j.

This process gives a partially-filled set-valued Young tableau $\widetilde{D}$ that is row-standard and column-standard if you look past the empty cells. We "collapse" the entries of $\widetilde{D}$ off the interstitial columns

$c_2, \ldots, c_j$ (but not off c_1) by shifting all entries leftward until the cells corresponding to c_1 have weight $(1, r-1)$, the cells corresponding to the T_j have row-constant weight $(1, k-1)$, and the cells corresponding to $c_2, \ldots, c_j$ are empty. Deleting the columns corresponding to $c_2, \ldots, c_j$ then produces a set-valued tableau T of shape $\lambda = (n+1)^2$ and weight $w(k, r)$. T is obviously row-standard. To see that T is also column-standard, notice that first row entries of $\widetilde{D}$ that were originally associated with a particular T_j are shifted leftward by precisely $j-1$ cells as we pass from $\widetilde{D}$ to T, whereas second row entries in $\widetilde{D}$ that were originally associated with T_j are shifted leftward by at least $j-1$ cells as we pass from $\widetilde{D}$ to T. The latter observation follows from the fact that $r-1$ integers must eventually appear in the $(2,1)$ cell of T, and that there are $j-1 \leq r-1$ marked second-row cells to the left of the entries associated with T_j. As second row entries are shifted at least as far left as first row entries, the fact that $\widetilde{D}$ was column-standard implies that T is also column-standard.

The map $(T_1, \ldots, T_r) \mapsto T$ is clearly well-defined. To show that it is bijective we outline a well-defined inverse. Given $T \in \mathbb{S}((n+1)^2, w(k, r))$, collectively shift all entries in the second row rightward so that the $(2,1)$ cell is empty and all remaining cells in the second row contain precisely $k-1$ entries (there will be an overflow of $r-1$ elements at the end of the second row that are not yet assigned a cell). Then proceed through the second-row from left-to-right and identify the smallest integer c that violates column-standardness. Insert a new, partially-filled column at the position of c whose top cell is empty and whose bottom cell contains c. Then re-allocate the remaining entries of the second row so that $k-1$ entries appear in each cell to the right of c, and repeat the above procedure until $r-1$ new columns have been added. The end result of this procedure is identical to the partially-filled tableau $\widetilde{D}$ from above. This follows from the fact that the second-row entries of the interstitial columns $c_2, \ldots, c_r$ are necessarily smaller than all entries in the "block" corresponding to T_j and hence would violate column-standarness if moved even one cell to their right. $\square$

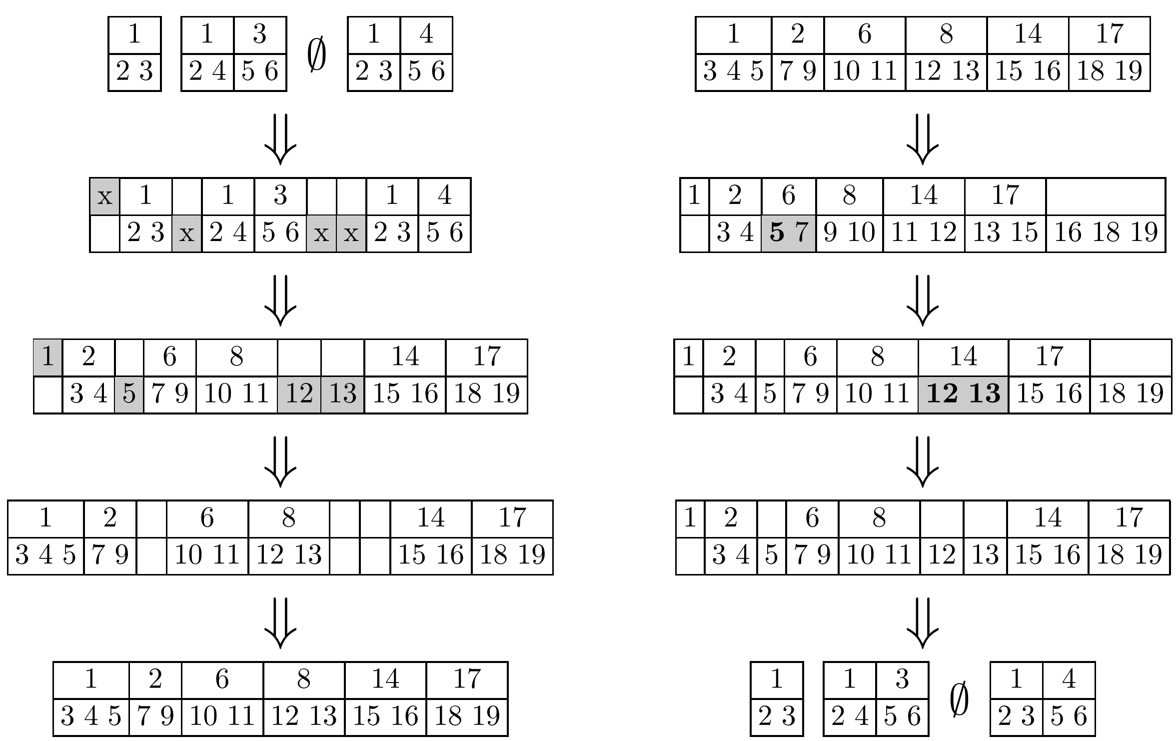

Figure 6: Transforming a tuple $(T_1, \ldots, T_r)$ of set-valued tableaux with weight $w = (1, k-1)$ into a single set-valued tableau of weight $w(k, r)$ via "horizontal tableaux concatenation" (left), and the inverse procedure (right).

Notice that our interpretation of $R_{k,r}(n)$ as the cardinality of $\mathbb{S}((n+1)^2, w(k,r))$ immediately recovers the k-Catalan specialization $R_{k,k}(n-1) = C_n^k$ via an application of Proposition 2.1. Also notice the special meaning of Theorem 2.3 as it applies to the extreme case of $r = 1$, as set-valued tableaux of weight $w(k,1)$ have an empty cell at position $(2,1)$. As demonstrated in Figure 7, one may construct a bijection from $\mathbb{S}((n+1)^2, w(k,1))$ to $\mathbb{S}(n^2, w(k,k))$ by deleting the first column of $T \in \mathbb{S}((n+1)^2, w(k,1))$ and re-indexing the remaining nk entries of T by $x \mapsto (x-1)$. This bijection directly corresponds to the aforementioned Raney number identity $R_{k,1}(n) = R_{k,k}(n-1) = C_n^k$.

1	2	4	6
	3 5	7 8	9 10

$\Longleftrightarrow$

1	3	5
2 4	6 7	8 9

Figure 7: The identity $R_{k,1}(n) = R_{k,k}(n-1) = C_n^k$ via a bijection on the associated set-valued tableaux.

We close this subsection by briefly proposing a bijection between the tableaux $\mathbb{S}((n+1)^2, w(k,r))$ of Theorem 2.3 and coral diagrams of type (k,r,n), further generalizing the k-Catalan bijection of Proposition 2.1. So take a coral diagram G of type (k,r,n) and enumerate the edges of G via clockwise tracing, beginning at the root vertex of the base r-star. Integers associated with leftmost children are placed in the top row of the corresponding tableaux T, while all other integers are placed in the bottom row. Here we merely enforce the added condition that the $(2,1)$ cell of T receives $r-1$ integers and all other cells in the second row receive $k-1$ integers. Notice that the $r-1$ integers at position $(2,1)$ need not necessarily correspond to the $r-1$ non-leftmost children of the base r-star. That this map is well-defined and bijective should follow from analogous reasoning to the proof of Proposition 2.1. See Figure 8 for an example of this bijection.

1	3	6	11
2 4 5	7 8	9 10	12 13

$\Longleftrightarrow$

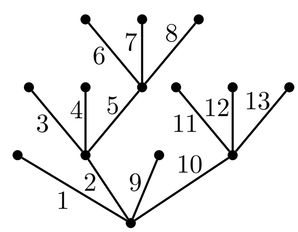

Figure 8: A standard set-valued Young tableau $T \in \mathbb{S}((3+1)^2, w(2,3))$, and the corresponding coral diagram of type $(3,4,3)$.

3 Set-Valued Tableaux & Two-Dimensional Lattice Paths

In this section, we derive further results involving set-valued tableaux by generalizing the bijection between tableaux and lattice paths from Proposition 2.1. This requires a consideration of all N-E lattice paths between two fixed points. By an N-E lattice path of shape $\eta = (a,b)$ we mean any integer lattice path from $(0,0)$ to (a,b) that uses only $E = (1,0)$ and $N = (0,1)$ steps. Denote the set of all N-E lattice paths of shape η by $\mathcal{P}_\eta$.

Fix a two-row tableaux shape $\lambda = n^2$. For any ordered pair of n-tuples $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_n)$, where the a_j, b_j are non-negative integers, there exists a valid weight $W_\beta^\alpha = \{w_{i,j}\}$ such that $w_{1,j} = a_j$ and $w_{2,j} = b_j$ for all j. Assuming $\sum_j a_j = a$ and $\sum_j b_j = b$, for every tableau $T \in \mathbb{S}(\lambda, W_\beta^\alpha)$ there is a unique N-E lattice path $P_T = \{v_0, \ldots, v_{a+b}\}$ of shape $\eta = (a,b)$ such that v_i follows an East step if i lies in the first row of T and v_i follows a North step if i lies in the second row of T. In this situation, we say that P_T is the lattice path induced by T. One fundamental observation to make about the relationship between T and P_T is that a point (x_i, y_i) lies on P_T if and only if x_i elements of $[i]$ lie in the first row of T and y_i elements of $[i]$ lie in the second row of T.

The map $\psi^\alpha_\beta(T) = P_T$ defines an injection from $\mathbb{S}(\lambda, W^\alpha_\beta)$ to $\mathcal{P}_\eta$, but its image is obviously dependent upon the choice of weight W^α_β. For row-constant weights $\alpha = (1, \ldots, 1), \beta = (k-1, \ldots, k-1)$, Proposition 2.1 identifies this image with k-good paths $\mathcal{D}^k_n \subset \mathcal{P}_\eta$ of shape $\eta = (n, n(k-1))$, but in general the image of ψ^α_β need not be so simple.

In order to succinctly characterize the image of $\psi|^\alpha_\beta$ for arbitrary weight W^α_β, we introduce a partial order on $\mathcal{P}_\eta$. So take paths $P_1, P_2 \in \mathcal{P}_\eta$, whose integer lattice points we specify using the abbreviated notation $P_1 = \{(x_i, y_i)\}$, $P_2 = \{(\widetilde{x}_i, \widetilde{y}_i)\}$. We define $P_1 > P_2$ if and only if $y_i \geq \widetilde{y}_i$ for all i. This is equivalent to saying that $P_1 > P_2$ if and only if P_1 lies weakly above P_2 across $0 \leq x \leq a$. Observe that the resulting poset is isomorphic to Young's lattice via the map that takes P_1 to the Young diagram lying above the conjugate path $\bar{P}_1 = \{(y_i, x_i)\}$.

Now take any weight W^α_β. There exists a unique tableau $T_{max} \in \mathbb{S}(n^2, W^\alpha_\beta)$ such that, for all $1 \leq j \leq (n-1)$, every integer in the $(j+1)^{st}$ column of T_{max} is larger than every integer in the j^{th} column of T_{max}. If all entries of α and β are nonzero, the path $P_{T_{max}} \in \mathcal{P}_\eta$ associated with T_{max} features precisely n horizontal runs (of respective lengths $a_1, \ldots, a_n$) alternating with precisely n vertical runs (of respective lengths $b_1, \ldots, b_n$). If α or β contains at least one zero entry, $P_{T_{max}}$ will contain fewer than n runs of at least one type and some runs may correspond to multiple cells of T_{max} (corresponding to when the interstitial "run" has length zero). In both cases we have $P_{T_{max}} = E^{a_1}N^{b_1}E^{a_2}N^{b_2}\ldots E^{a_n}N^{b_n}$. See Figure 9 for an example of T_{max} and $P_{T_{max}}$ when α and β are nonzero.

1 2	4 5 6
3	7 8

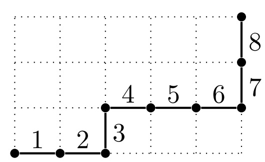

Figure 9: The "maximal" column-standard tableau T_{max} and the associated "maximal" N-E lattice path $P_{T_{max}}$ for the weight W^α_β with $\alpha = (2, 3), \beta = (1, 2)$.

The path $P_{T_{max}}$ is significant because, under certain modest conditions on W^α_β, the order ideal $I = \{P \in \mathcal{P}_\eta | P \leq P_{T_{max}}\}$ that it generates in the poset $\mathcal{P}_\eta$ will correspond to the image of $\psi|^\alpha_\beta$. Before proceeding to that result, we require a lemma showing that the order ideal generated by arbitrary $P \in \mathcal{P}_\eta$ is contained in ψ^α_β if and only if P itself is in ψ^α_β:

Lemma 3.1. *Fix $\lambda = n^2$, $\eta = (a, b)$, and W^α_β, and take $P_1, P_2 \in \mathcal{P}_\eta$ such that $P_1 > P_2$. If $P_1 \in \text{im}(\psi^\alpha_\beta)$, then $P_2 \in \text{im}(\psi^\alpha_\beta)$.*

Proof. We prove the proposition for when P_1 directly covers P_2 in $\mathcal{P}_\eta$. This corresponds to the situation where P_2 may be obtained from P_1 by replacing a single NE subsequence of edges in P_1 with an EN subsequence at the same position. Assume that this $NE \to EN$ replacement occurs at the i and $i+1$ steps of both P_1 and P_2. As $P_1 \in \text{im}(\psi^\alpha_\beta)$, there exists $T_1 \in \mathbb{S}(\lambda, W^\alpha_\beta)$ that induces P_1. By definition, i appears in the second row of T_1 and $i+1$ appears in the first row of T_1. As T_1 is standard, this also implies that i appears in a more leftward column of T_1 than does $i+1$. Now define T_2 to be the tableau of weight W^α_β that results from flipping the positions of i of $i+1$ in T_1. As i and $i+1$ are consecutive integers, and since i originally appeared left of $i+1$ in T_1, T_2 is row-standard and column-standard. By construction, $\psi^\alpha_\beta(T_2) = P_2$. $\square$

Lemma 3.1 applies to all possible weights W^α_β, but applying the result toward a characterization of $\text{im}(\psi^\alpha_\beta)$ we need to exclude certain "degenerate" weights that do need easily translate to results about N-E lattice paths. We say that W^α_β is a **reduced (two-row) weight** if there does not exist an index i such that b_i and a_{i+1} are both zero. Notice that this condition is not symmetric with respect to α and β: if a_i and b_{i+1} are both zero, the weight may still be reduced. Obviously, all W^α_β lacking a zero weight cell qualify as reduced weights.

Theorem 3.2. *Set $\lambda = n^2$ and $\eta = (a,b)$, and take any reduced weight W_β^α such that $\sum_i a_i = a$ and $\sum_i b_i = b$. If $P_{T_{max}}$ is defined as above, then $\mathbb{S}(\lambda, W_\beta^\alpha)$ is in bijection with the order ideal of N-E lattice paths $I = \{P \in \mathcal{P}_\eta \mid P \leq P_{T_{max}}\}$*

Proof. We have already argued that $\psi_\beta^\alpha(T) = P_T$ is injective. As $P_{T_{max}} \in \mathrm{im}(\psi_\beta^\alpha)$, from Lemma 3.1 we know that $I = \{P \in \mathcal{P}_\eta \mid P \leq P_{T_{max}}\}$ is contained in $\mathrm{im}(\psi_\beta^\alpha)$.

To show that $\mathrm{im}(\psi_\beta^\alpha)$ contains I, take $P = \{(\widetilde{x}_i, \widetilde{y}_i)\}$ in $\mathcal{P}_\eta$ such that $P \nleq P_{T_{max}}$ and let T be the associated set-valued tableau of weight W_β^α. We need to show that T cannot be column-standard. As $P \nleq P_{T_{max}}$, there exists a smallest index i such that $\widetilde{y}_i > y_i$. Notice that the lattice point $(\widetilde{x}_i, \widetilde{y}_i)$ of P must follow a North step, whereas the corresponding lattice point (x_i, y_i) of $P_{T_{max}}$ must follow a East step. This means that all elements of $[i-1]$ lie in the same cells of T and T_{max}, whereas i lies in the first row of T_{max} but in the second row of T_{max}. As W_β^α is a reduced weight, for some fixed column j the construction of T_{max} implies that i must lie in the $(1,j)$ cell of T_{max} but in the $(2,j)$ cell of T. It follows that there must be some integer $k < i$ that lies above i in the $(1,j)$ cell of T. Thus T cannot lie in $\mathbb{S}(\lambda, W_\beta^\alpha)$, and $\mathrm{im}(\psi_\beta^\alpha)$ can consist only of those $P \in \mathcal{P}_\eta$ such that $P \leq P_{T_{max}}$. □

Pause to observe that Theorem 3.2 begins by fixing a reduced weight W_β^α. Alternatively beginning with a lattice path $P \in \mathcal{P}_\eta$, there always exists a weight W_β^α such that $\mathbb{S}(\lambda, W_\beta^\alpha)$ is in bijection with all N-E lattice paths weakly below P, but this choice of W_β^α need not be unique (even among reduced weights) if α or β contains zero entries.

Example 3.3. *For $\lambda = n^2$ and row-constant weight $w = (1, k-1)$, we have maximal lattice path $P_{T_{max}} = (EN^{k-1})^n$ with $\eta = (n, (k-1)n)$. N-E lattice paths lying weakly below $P_{T_{max}}$ are in bijection with N-E lattice paths lying weakly below the line $y = (k-1)x$, recovering the bijection of Proposition 2.1 between k-good paths $\mathcal{D}_n^k$ and $\mathbb{S}(n^2, (1, k-1))$*

Example 3.4. *For $\eta = (a,b)$, the poset $\mathcal{P}_\eta$ has unique greatest element $P_M = N^b E^a$ and unique least element $P_m = E^a N^b$. Choice of reduced weights corresponding to P_M and P_m are show below.*

$$P_M : \begin{array}{|c|c|} \hline 0 & a \\ \hline b & 0 \\ \hline \end{array} \qquad\qquad P_m : \begin{array}{|c|} \hline a \\ \hline b \\ \hline \end{array}$$

For P_M there exist $\binom{a+b}{a} = |\mathcal{P}_\eta|$ standard set-valued Young tableaux of the given weight, and for P_m there exists precisely one such tableaux, as expected.

3.1 Set-Valued Tableaux, Rational Dyck Paths & Rational Catalan Numbers

Rational Dyck paths are a generalization of k-good lattice paths to N-E lattice paths that lie weakly below some line of rational slope $y = \frac{b}{a}x$, with the added assumption that $\gcd(a,b) = 1$. We use (a,b)-Dyck path to refer to such a path that starts at $(0,0)$ and ends at (a,b). Work with rational Dyck paths dates back to Grossman [10] and Bizley [3], with Bizley proving that the number of (a,b)-Dyck paths equals $\frac{1}{a+b}\binom{a+b}{a}$. Observe the symmetry of this result with respect to a and b. Much more recently, these ideas were expanded upon by Armstrong, Rhoades and Williams [1] in the guise of rational Catalan numbers. For relatively prime positive integers $a < b$, the rational Catalan number $C(a,b) = \frac{1}{a+b}\binom{a+b}{a}$ is defined to equal the number of (a,b)-Dyck paths. Armstrong, Rhoades and Williams [1] went on to provide additional combinatorial interpretations of the $C(a,b)$ in terms of "rational noncrossing matchings" and polygon dissections, introduced a "rational associahedron", and provided rational generalizations of the Narayana and Kirkman numbers. For later work with the rational Catalan numbers, see Bodnar and Rhoades [4].

Notice that, as a and b are relatively prime, an (a,b)-Dyck path only meets the line $y = \frac{b}{a}x$ at $(0,0)$ and (a,b). If it is also the case that $b = 1 \bmod (a)$, there cannot exist $\frac{c}{d} \in \mathbb{Q}$ such that $\frac{b-1}{a} < \frac{c}{d} < \frac{b}{a}$

unless $d > a$. With the exception of the final N step, which connects $(a, b-1)$ to (a, b), this implies that a (a, b)-Dyck path with $b = 1 \bmod (a)$ also lies weakly below the line $y = \frac{b-1}{a}x$. This sets up a bijection between (a, b)-Dyck paths with $b = 1 \bmod (a)$ and N-E lattice paths from $(0, 0)$ to $(a, b-1)$ that lie weakly below $y = \frac{b-1}{a}x$, a correspondence that is significant in that it allows us to apply results about rational Dyck paths to a somewhat wider collections of N-E lattice paths. In particular, it allows us to recover the k-Catalan numbers (for any $k \geq 2$, $n \geq 1$) via Equation 3:

$$C(n, (k-1)n+1) \;=\; \frac{1}{kn+1}\binom{kn+1}{n} \;=\; C_n^k \tag{3}$$

In order to relate the rational Catalan numbers to standard set-valued Young tableaux, by Theorem 3.2 we merely need to find a unique maximal lattice path $P \in \mathcal{P}_\eta$ among all (a, b)-Dyck paths. This is actually relatively easy, although the formula proves to be somewhat convoluted:

Proposition 3.5. *Fix $\eta = (a, b)$ with $\gcd(a, b) = 1$. Define $P_{(a,b)} \in \mathcal{P}_\eta$ as $P_{(a,b)} = EN^{c_1} \dots EN^{c_a}$, where $c_i = \lfloor \frac{bi}{a} \rfloor - \lfloor \frac{b(i-1)}{a} \rfloor$. Then $P \in \mathcal{P}_\eta$ is an (a, b)-Dyck path if and only if $P \leq P_{(a,b)}$.*

Proof. Observe that $\sum_{i=1}^{k} c_i = \lfloor \frac{bk}{a} \rfloor$ for all $1 \leq k \leq a$. This implies that, for all $1 \leq k \leq a$, $P_{(a,b)}$ has a Northwest corner at the first integer lattice point below the intersection of $y = \frac{b}{a}x$ with $x = k$. The result immediately follows. ∎

A direct application of Theorem 3.5 to Proposition 3.5 gives the following combinatorial interpretation of $C(a, b)$ for all positive integers a, b with $\gcd(a, b) = 1$. This is equivalent to the interpretation presented in Figure 2 via Corollary 1.2.

Corollary 3.6. *Take $(a, b) \in \mathbb{N}$ such that $\gcd(a, b) = 1$. Then $C(a, b) = |\,\mathbb{S}(a^2, w)|$ for the weight $w = \{w_{i,j}\}$ with $w_{1,j} = 1$ and $w_{2,j} = \lfloor \frac{bi}{a} \rfloor - \lfloor \frac{b(i-1)}{a} \rfloor$ for all $1 \leq j \leq a$.*

Example 3.7. $C(7, 9) = 715$. *By Corollary 3.6, $|\,\mathbb{S}(7^2, w)| = C(7, 9) = 715$ for the weight w below:*

$$w: \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 2 & 1 & 1 & 2 \\ \hline \end{array}$$

Example 3.8. *If $(a, b) = (n, (k-1)n+1)$, then $\lfloor \frac{bi}{a} \rfloor = \lfloor (k-1)i + \frac{i}{n} \rfloor = (k-1)i + \lfloor \frac{i}{n} \rfloor$ for all i and*

$$\lfloor \frac{bi}{a} \rfloor - \lfloor \frac{b(i-1)}{a} \rfloor \;=\; \begin{cases} k-1 & \text{if } \; 1 \leq i \leq n-1 \\ k & \text{if } \; i = n \end{cases}$$

By Corollary 3.6, $C_n^k = |\,\mathbb{S}(n^2, w)|$ for $w_{1,j} = 1$, $w_{2,j} = k-1$ if $1 \leq j \leq n-1$, and $w_{2,n} = k$. As we are merely adding an entry to the lower-rightmost cell, we clearly have $|\,\mathbb{S}(n^2, w)| = |\,\mathbb{S}(n^2, (1, k-1))|$, verifying the k-Catalan identity of Proposition 2.1.

3.2 Set-Valued Tableaux & the (s, t) Tennis Ball Problem

For a final application of Theorem 3.2, we present a set-valued tableaux interpretation of the solution to the so-called generalized tennis ball problem. The original version of the tennis ball problem was introduced by Tymoczko and Henle in their logic textbook [20] and subsequently formalized by Mallows and Shapiro as the "problem of balls on the lawn" [13]. The version of Mallows and Shapiro begins with $2n$ tennis balls, numbered $1, 2, \dots, 2n$, in a room of your house and proceeds through n turns. For the first turn, you take the balls numbers 1 and 2 and randomly throw one of them out of your window onto your lawn. For the second turn, the balls numbered 3 and 4 are added to the remaining ball from turn one, and you randomly throw one of those three balls out of your window onto your lawn. This process continues through to the n^{th} step, where you add the balls numbers $2n-1$ and $2n$ to the $n-1$ balls remaining from the previous steps, and randomly throw one of those $n+1$ balls onto your lawn. This results in precisely n balls on your lawn and n in your house. The tennis-ball problem then asks how many different sets of balls are possible on your lawn after n steps (irrespective of the order in which they appeared)?

Independent from the work of Mallows and Shapiro, Grimaldi and Moser [9] proved that the number of tennis ball arrangements after n turns was in fact the Catalan number C_{n+1}. This elegant result sparked renewed interest in the tennis-ball problem and invited several natural generalizations. One such generalization was the s-tennis ball problem of Merlini, Sprugnoli and Verri [14], whereby s new balls added at the beginning of each turn. Merlini, Sprugnoli and Verri used generating trees to associate the number of resulting arrangements after n turns with the number of s-ary trees in $\mathcal{T}^s_{n+1}$ and hence with the s-Catalan number C^s_{n+1}. In an appendix to that same paper, Merlini, Sprugnoli, and Verri introduced the even more generalized (s,t)-tennis ball problem, whereby s new balls are added and then t balls are thrown out the window during each turn. A solution to the (s,t)-tennis ball problem was computed numerically for the specific case of $s=4, t=2$. Notice that the original tennis ball problem corresponds to $s=2, t=1$.

More germane to the this paper are the techniques of Bonin, de Mier and Noy [5], which were also used by de Mier and Noy [7]. In their development of a generating function for the solution to the (s,t)-tennis ball problem, they noted that the number of arrangements after n turns equaled the number of N-E lattice paths from $(0,0)$ to $((s-t)n, tn)$ that never go above the path $P=(N^tE^{s-t})^n$. In light of Theorem 3.2, this allows us to quickly conclude the following:

Theorem 3.9. *The solution to the (s,t)-tennis ball problem after n steps equals the number of standard set-valued Young tableaux of shape $\lambda=(n+1)^2$ and row-constant weight $w=(s-t,t)$.*

Proof. For $\lambda=(n+1)^2$ and $w=(s-t,t)$, from Theorem 3.2 we know that $|\mathbb{S}(\lambda,w)|$ equals the number of N-E lattice paths from $(0,0)$ to $((n+1)(s-t),(n+1)t)$ that lie weakly below $P=(E^{s-t}N^t)^{n+1}$. Eliminating the first horizontal run E^{s-t} and the final vertical run N^t of P clearly doesn't change the number of valid lattice paths. Thus the number of set-valued tableaux in question equals the number of N-E lattice paths from $(0,0)$ to $(n(s-t),nt)$ that lie weakly below $\widetilde{P}=(N^tE^{s-t})^n$. The result of Bonin, de Mier and Noy [5] then gives the desired result. □

Comparing Theorem 3.9 with Proposition 2.1 in the case of $t=1$, we directly recover the result of Merlini, Sprungnoli, and Verri [14] that associates the solution to the s-tennis ball problem with C^s_{n+1}.

We close this section by addressing one final generalization of the tennis ball problem that also appears in the work of de Mier and Noy [7]. So let $\vec{s}=\{s_i\}$ and $\vec{t}=\{t_i\}$ be sequences of positive integers such that $t_i<s_i$ for all i, and define the $(\vec{s},\vec{t})$-tennis ball problem to be the "non-constant" generalization of the tennis ball problem wherein s_i new balls are added and t_i balls are thrown out the window during the i^{th} turn. If $A=\sum_{i=1}^n s_i$ and $B=\sum_{i=1}^n t_i$, after n turns precisely A balls have been used and B of those balls have been thrown onto the lawn. de Mier and Noy [7] note that the number of possible arrangements after n turns of this process is equal to the number of N-E lattice paths from $(0,0)$ to $(A-B,B)$ that lie weakly below $P=N^{t_1}E^{s_1-t_1}N^{t_2}E^{s_2-t_2}\ldots$. Via equivalent reasoning to Theorem 3.9, we may conclude with the following combinatorial interpretation of $\mathbb{S}((n+1)^2,w)$ for any $n\geq 0$ and arbitrary two-row weight w:

Theorem 3.10. *Let $\vec{s}=\{s_i\}$ and $\vec{t}=\{t_i\}$ be sequences of positive integers such that $t_i<s_i$ for all i. If $\lambda=(n+1)^2$ and w is the two-row weight w shown below (where x,y are arbitrary positive integers), then $|\mathbb{S}(\lambda,w)|$ equals the solution to the $(\vec{s},\vec{t})$-tennis ball problem after n turns.*

x	s_1-t_1	$\ldots$	$s_{n-1}-t_{n-1}$	s_n-t_n
t_1	t_2	$\ldots$	t_n	y

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