Minimum rank and zero forcing number for butterfly networks

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Abstract

The minimum rank of a simple graph G is the smallest possible rank over all symmetric real matrices A whose nonzero off-diagonal entries correspond to the edges of G. Using the zero forcing number, we prove that the minimum rank of the butterfly network is $\frac{1}{9}\left[(3r+1)2^{r+1}-2(-1)^r\right]$ and that this is equal to the rank of its adjacency matrix.

1 Introduction

Let F be a field, and denote by $S_n(F)$ the set of symmetric $n\times n$ matrices over F. For a simple graph $G(V,E)$ with vertex set $V=\{1,\dots,n\}$, let $S(F,G)$ be the set of matrices in $S_n(F)$ whose non-zero off-diagonal entries correspond to edges of G, i.e.,

$$S(F,G)=\{A\in S_n(F)\ :\ i\neq j\implies (ij\in E(G)\iff a_{ij}\neq 0)\}.$$

The minimum F-rank of a graph G is defined as the minimum rank over all matrices A in $S(F,G)$:

$$\mathrm{mr}^F(G)=\min\left\{\mathrm{rank}(A)\ :\ A\in S(F,G)\right\}.$$

If the index F is omitted then it is understood that $F=\mathbb{R}$. The *minimum rank problem* for a graph G is to determine $\mathrm{mr}(G)$ (and more generally, $\mathrm{mr}^F(G)$), and has been studied intensively for more than ten years, see [9, 10] for surveys of known results and an extensive bibliography.
The concept of *zero-forcing* was introduced by the *AIM Minimum Rank – Special Graphs Work Group* in [1] as a tool to bound the minimum rank of a graph G. For a two-coloring of the vertex set V consider the following color-change rule: a red vertex is converted to blue if it is the only red neighbor of some blue vertex. A vertex set $S\subseteq V$ is called *zero-forcing* if, starting with the vertices in S blue and the vertices in the complement $V\setminus S$ red, all the vertices can be converted to blue by repeatedly applying the color-change rule. The minimum cardinality of a zero-forcing set for the graph G is called the *zero-forcing number* of G, denoted by $Z(G)$. Since its introduction the zero-forcing number has been studied for its own sake as an interesting graph invariant [2, 3, 5, 6, 17]. In [14], the *propagation time* of a graph is introduced as the number of steps it takes for a zero forcing set to turn the entire graph blue. Relations between the metric dimension and the zero forcing number for certain graph classes are established in [7, 8]. Physicists have independently studied the zero forcing parameter, referring to it as the *graph infection number*, in conjunction with the control of quantum systems [18].
The link between the zero forcing number and the minimum rank problem is established by the observation that for a zero-forcing set S and a matrix $A\in S(F,G)$, the rows of A that correspond to the vertices in $V\setminus S$ must be linearly independent, so $\mathrm{rank}(A)\geqslant n-|S|$, and consequently

$$\mathrm{mr}^F(G)\geqslant n-Z(G). \tag{1}$$

Based on this insight, the authors of [1] determined $\mathrm{mr}(G)$ for various graph classes and established equality in (1), independent of the field F, in many cases. In [15], the same is proved for block-clique graphs and unit interval graphs. Recently, the zero forcing number of cartesian products of cycles was established by constructing a matrix in $S(F, G)$ with the required rank [4]. The American Institute for Mathematics maintains the minimum rank graph catalog [13] in order to collect known results about the minimum rank problem for various graph classes.

In this paper, we determine the minimum rank for the butterfly network which is an important and well known interconnection network architecture [16]. Section 2 contains some notation and a precise statement of our main result. In Section 3 we prove an upper bound for the zero-forcing number of the butterfly network by an explicit construction of the corresponding zero forcing set S. By (1) this implies a lower bound for the minimum rank of the butterfly network, and in Section 3 we establish that this bound is tight by showing that the rows of the adjacency matrix corresponding to the vertices in in the complement of the zero-forcing set span the row space of the adjacency matrix of the butterfly network (over any field F).

2 Notation and main result

Let $G = (V, E)$ be a finite simple graph. For a vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u : uv \in E(G)\}$ and the closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. We denote by I_n the $n \times n$ identity matrix, and we use I for I_n when the order n is clear from the context.

For a positive integer r, the butterfly network $\mathrm{BF}(r) = \left(V^{(r)}, E^{(r)}\right)$ has vertex set $V^{(r)} = V_0^{(r)} \cup V_1^{(r)} \cup \cdots \cup V_r^{(r)}$ and edge set $E^{(r)} = E_1^{(r)} \cup E_2^{(r)} \cup \cdots \cup E_r^{(r)}$, where

$$V_i^{(r)} = \{(\boldsymbol{x}, i) \ : \ \boldsymbol{x} \in \{0,1\}^r\} \qquad \text{for } i = 0, 1, \dots, r,$$

$$E_i^{(r)} = \{\{(\boldsymbol{x}, i-1), (\boldsymbol{y}, i)\} \ : \ \boldsymbol{x} \in \{0,1\}^r, \boldsymbol{y} \in \{\boldsymbol{x}, \boldsymbol{x} + \boldsymbol{e}_i\}\} \qquad \text{for } i = 1, 2, \dots, r.$$

Here addition is modulo 2, and $\boldsymbol{e}_i$ is the binary vector of length r with a one in position i and zeros in all other components. For convenience, we identify the binary vector $\boldsymbol{x} = (x_1, \dots, x_r) \in \{0,1\}^r$ with the number $\sum_{i=1}^{r} x_i 2^{i-1}$. Using this identification the butterfly network BF(4) is shown in Figure 1. Our

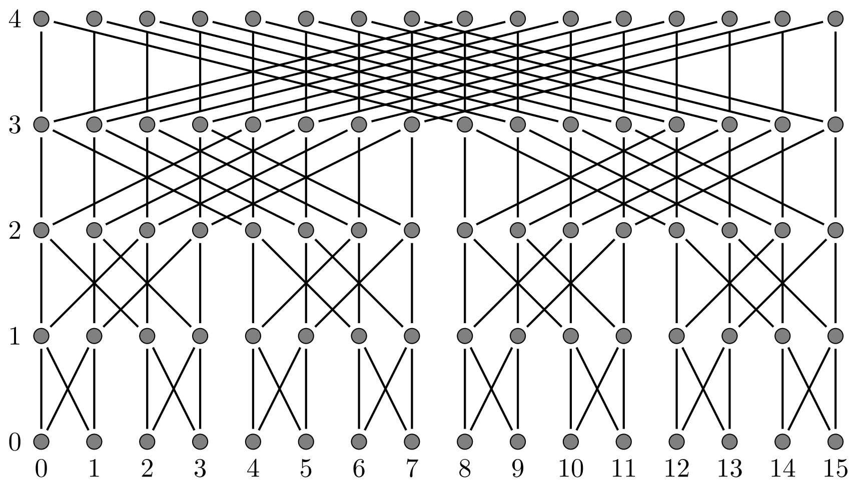

Figure 1: The butterfly network BF(4).

main result is the following theorem.

Theorem 1. *The minimum rank of the butterfly network* $\mathrm{BF}(r)$ *over any field* F *equals*

$$\mathrm{mr}^F(\mathrm{BF}(r)) = \frac{2}{9}\left[(3r+1)2^r - (-1)^r\right],$$

and this is equal to the rank of the adjacency matrix of $\mathrm{BF}(r)$*. Furthermore, for the butterfly network we have equality in* (1)*, i.e.,*

$$Z(\mathrm{BF}(r)) = (r+1)2^r - \mathrm{mr}^F(\mathrm{BF}(r)) = \frac{1}{9}\left[(3r+7)2^r + 2(-1)^r\right].$$

3 The upper bound for $Z(\mathrm{BF}(r))$

Let (J_n) denote the Jacobsthal sequence[1] which is defined by $J_0 = 0$, $J_1 = 1$ and $J_n = J_{n-1} + 2J_{n-2}$ for $n \geqslant 2$. We will need the following relation which follows immediately from the definition:

$$J_{n+2} = 2^n + J_n \quad \text{for every integer } n \geqslant 0. \tag{2}$$

For every r, we define a set $S^{(r)} = S_0^{(r)} \cup S_1^{(r)} \cup \cdots \cup S_r^{(r)}$ by

$$S_i^{(r)} = \left\{(x, i) \;:\; 2^{i+1}\ell \leqslant x \leqslant 2^{i+1}\ell + J_{i+1} - 1 \text{ for some } \ell\right\} \qquad \text{for } i = 0, 1, \dots, r-1$$
$$S_r^{(r)} = \{(x, i) \;:\; 0 \leqslant x \leqslant J_{r+1} - 1\}.$$

For $r = 4$ this is illustrated in Figure 2. Solving the recurrence relation for the numbers J_n, we find a closed form expression for the size of the set $S^{(r)}$.

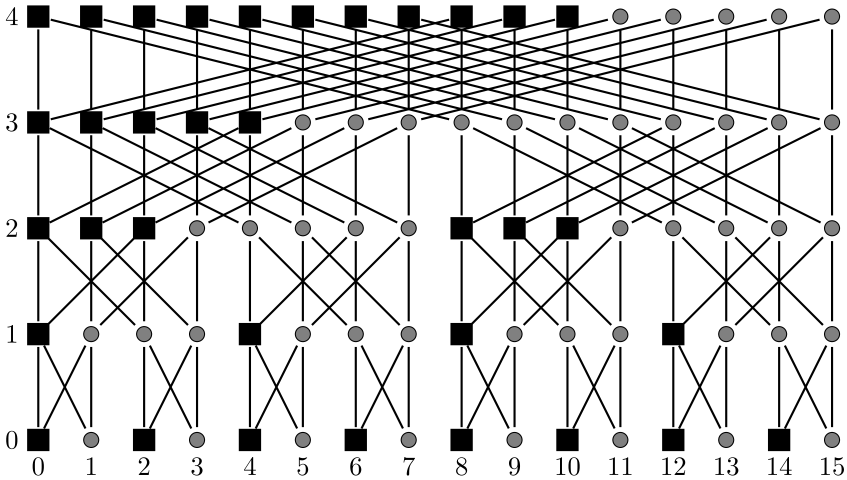

Figure 2: The set $S^{(4)}$ indicated by squares.

Lemma 1. *We have* $\left|S^{(r)}\right| = J_{r+1} + \sum_{i=1}^{r} 2^{r-i} J_i = \frac{1}{9}\left[(3r+7)2^r + 2(-1)^r\right].$ □

Next we want to verify that $S^{(r)}$ is a zero forcing set for $\mathrm{BF}(r)$. For this purpose we set $X_0 = S^{(r)}$ and define a sequence $X_1, X_2, \dots, X_{2r}$ of vertex sets by

$$X_k = X_{k-1} \cup \{(x, r-k) \;:\; \{(x, r-k)\} = N(v) \setminus X_{k-1} \text{ for some } v = (y, r-k+1) \in X_{k-1}\} \tag{3}$$
$$X_{r+k} = X_{r+k-1} \cup \{(x, k) \;:\; \{(x, k)\} = N(v) \setminus X_{r+k-1} \text{ for some } v = (y, k-1) \in X_{r+k-1}\} \tag{4}$$

for $k = 1, \dots, r$. After applying the color-change rule to the coloring with X_{k-1} blue and $V \setminus X_{k-1}$, all the vertices in X_k are blue and therefore it is sufficient to prove that $X_{2r} = V^{(r)}$.

Lemma 2. *For* $k \in \{0, 1, \dots, r\}$*,* $X_k = X_k(0) \cup X_k(1) \cup \cdots \cup X_k(r)$ *with*

$$X_k(i) = S_i^{(r)} \qquad \text{for } i \in \{0, 1, \dots, r-k\} \cup \{r\}$$
$$X_k(i) = \left\{(x, i) \;:\; 2^i k \leqslant x \leqslant 2^i k + J_{i+1} - 1 \text{ for some } \ell\right\} \qquad \text{for } i \in \{r-k+1, \dots, r-1\}.$$

Proof. We proceed by induction on k. For $k = 0$, there is nothing to do since $X_0 = S^{(r)} = S_0^{(r)} \cup \cdots \cup S_r^{(r)}$. Let $k \geqslant 1$ and set $i = r - k$. By (3), we have $X_k(j) = X_{k-1}(j)$ for all $j \neq i$. By induction, this implies

$$X_k(j) = S_j^{(r)} \qquad \text{for } j \in \{0, 1, \dots, r-k-1\} \cup \{r\}$$
$$X_k(j) = \left\{(x, i) \;:\; 2^j k \leqslant x \leqslant 2^j k + J_{i+1} - 1 \text{ for some } \ell\right\} \qquad \text{for } j \in \{r-k+1, \dots, r-1\},$$

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and it remains to be shown that

$$X_k(i)=\left\{(x,i)\ :\ 2^i\ell\leqslant x\leqslant 2^i\ell+J_{i+1}-1\text{ for some }\ell\right\}. \tag{5}$$

Let (x,i) be an arbitrary element of the RHS of (5). If $2^{i+1}\ell\leqslant x\leqslant 2^{i+1}\ell+J_{i+1}-1$ for some integer ℓ then $(x,i)\in X_0\subseteq X_k$. Otherwise

$$2^{i+1}\ell+2^i\leqslant x\leqslant 2^{i+1}\ell+2^i+J_{i+1}-1$$

for some integer ℓ. By induction, the vertex $(y,i+1)$ with $y=x-2^i$ is in X_{k-1} because

$$2^{i+1}\ell\leqslant y\leqslant 2^{i+1}\ell+J_{i+1}-1\leqslant 2^{i+1}\ell+J_{i+2}-1.$$

Let $\ell=2\ell'+\varepsilon$ with $\varepsilon\in\{0,1\}$. The neighbourhood of $(y,i+1)$ is

$$N((y,i+1))=\begin{cases}\{(y,i),(x,i)\} & \text{if } k=1,\\ \{(y,i),(x,i),(y,i+2),(y+(-1)^{\varepsilon}2^{i+1},i+2)\} & \text{if } k>1.\end{cases}$$

Now $(y,i)\in X_0\subseteq X_{k-1}$, and for $k=1$ that's all we need. Using 2, we have

$$2^{i+2}\ell'=2^{i+1}(\ell-\varepsilon)\leqslant 2^{i+1}\ell\leqslant y\leqslant 2^{i+1}\ell+J_{i+1}-1=2^{i+2}\ell'+2^{i+1}\varepsilon+J_{i+1}-1\leqslant 2^{i+2}\ell'+J_{i+3}-1,$$

and therefore $(y,i+2)\in X_{k-1}$. Similarly,

$$2^{i+2}\ell'=2^{i+1}\ell+(-1)^{\varepsilon}2^{i+1}\leqslant y+(-1)^{\varepsilon}2^{i+1}\leqslant 2^{i+2}\ell'+2^{i+1}+J_{i+1}-1\leqslant 2^{i+2}\ell'+J_{i+3}-1,$$

and therefore $(y+(-1)^{\varepsilon}2^{i-1},i+2)\in X_{k-1}$. Consequently, $\{(x,i)\}=N((y,i+1))\setminus X_{k-1}$, and this implies

$$X_k(i)\supseteq\left\{(x,i)\ :\ 2^i\ell\leqslant x\leqslant 2^i\ell+J_{i+1}-1\text{ for some }\ell\right\}.$$

To prove the converse, consider (x,i) with

$$2^i\ell+J_{i+1}\leqslant x\leqslant 2^i(\ell+1)-1.$$

and $\ell=2\ell'+\varepsilon$ as before. We have

$$N((x,i))\cap S_{i+1}^{(r)}=\{(x,i+1),(x+(-1)^{\varepsilon}2^i,i+1)\}.$$

If $(x,i+1)\in X_{k-1}$ then

- $(x+2^{i+1},i+2)\in N((x,i+1))\setminus X_{k-1}$ if $\ell\equiv 0$ or $1 \pmod 4$, and
- $(x,i+2)\in N((x,i+1))\setminus X_{k-1}$ if $\ell\equiv 2$ or $3 \pmod 4$.

Similarly, if $(x+(-1)^{\varepsilon}2^i,i+1)\in X_{k-1}$ then ℓ is odd and

- $(x+2^{i+1},i+2)\in N((x,i+1))\setminus X_{k-1}$ if $\ell\equiv 1 \pmod 4$, and
- $(x,i+2)\in N((x,i+1))\setminus X_{k-1}$ if $\ell\equiv 3 \pmod 4$.

In all cases it follows that $(x,i)\notin X_k$, and this concludes the proof. □

Lemma 3. *For $k\in\{0,1,\dots,r\}$, $X_{r+k}=X_{r+k}(0)\cup X_{r+k}(1)\cup\dots\cup X_{r+k}(r)$ with*

$$X_{r+k}(i)=\begin{cases}V_i^{(r)} & \text{for } i\in\{0,\dots,k\},\\ X_r(i) & \text{for } i\in\{k+1,k+2,\dots,r\}.\end{cases}$$

Proof. We proceed by induction on k. For $k = 0$, there is nothing to do since $X_r = X_r(1) \cup X_r(2) \cup \ldots \cup X_r(r) \cup V_0(r)$, which is true by Lemma 2. Let $k \geqslant 1$ and set $i = k$. By (4), we have $X_{r+k}(j) = X_{r+k-1}(j)$ for all $j \neq i$. By induction, this implies

$$\begin{aligned} X_{r+k}(j) &= V_j^{(r)} && \text{for } j \in \{0, 1, \ldots, k-1\} \\ X_{r+k}(j) &= X_r(j) && \text{for } j \in \{k+1, \ldots, r-1\}, \end{aligned}$$

and it remains to be shown that

$$X_{r+k}(k) = V_k^{(r)}.$$

Let (x, i) be an arbitrary element of $V_k^{(r)}$. If $2^{i+1}\ell \leqslant x \leqslant 2^{i+1}\ell + J_{i+1} - 1$ and $2^{i+1}\ell + 2^i \leqslant x \leqslant 2^{i+1}\ell + 2^i + J_{i+1} - 1$ for some integer ℓ, then $(x, i) \in X_{r-k} \subseteq X_{r+k}$. Otherwise

$$2^{i+1}\ell + 2^i + J_{i+1} - 1 \leqslant x \leqslant 2^{i+1}\ell + 2^i$$

for some integer ℓ. By induction, the vertex $(y, i-1)$ with $y = x - 2^{i-1}$ is in X_{r+k-1} because $y \in V_{k-1}^{(r)}$. More precisely,

$$2^{i+1}\ell + 2^i + J_{i+1} - 1 - 2^{i-1} \leqslant y \leqslant 2^{i+1}\ell + 2^i - 2^{i-1}.$$

Let $\ell' = 2\ell - \varepsilon$ with $\varepsilon \in \{0, 1\}$. The neighbourhood of $(y, i-1)$ is

$$N((y, i-1)) = \begin{cases} \{(y, i), (x, i)\} & \text{if } k = 1, \\ \{(y, i), (x, i), (y, i-2), (y + (-1)^\varepsilon 2^{i-2}, i-2)\} & \text{if } k > 1. \end{cases}$$

Now $(y, i-2), (y + (-1)^\varepsilon 2^{i-2}, i-2) \in X_{r+k-1}$. Using (2), we have

$$2^i\ell' + 2^i + J_{i+1} - 1 - 2^{i-1} \leqslant y \leqslant 2^i\ell' + 2^i - 2^{i-1}.$$

and therefore $(y, i) \in X_{r+k-1}$. Consequently, $\{(x, i)\} = N((y, i-1)) \setminus X_{r+k-1}$, and this implies

$$X_{r+k}(i) = V_k^{(r)}.$$

□

Combining Lemmas 2 and 3, we have proved that $S^{(r)}$ is indeed a zero forcing set for $\mathrm{BF}(r)$.

Lemma 4. *For every $r \geqslant 1$, $S^{(r)}$ is a zero forcing set for the butterfly network $\mathrm{BF}(r)$.* □

From Lemmas 1 and 4 we obtain an upper bound for the zero forcing number of the butterfly network.

Proposition 1. *For every $r \geqslant 1$, $Z(\mathrm{BF}(r)) \leqslant \frac{1}{9}\left[(3r + 7)2^r + 2(-1)^r\right]$.* □

4 The lower bound for $Z(\mathrm{BF}(r))$

By (1), the corank of the adjacency matrix of a graph G provides a lower bound for the zero forcing number of G, and consequently we can conclude the proof of Theorem 1 by establishing the following result.

Proposition 2. *Let F be a field, and let A_r denote the adjacency matrix of $\mathrm{BF}(r)$ over F. Then*

$$\operatorname{rank}(A_r) \leqslant (r+1)2^r - \frac{1}{9}\left[(3r+7)2^r + 2(-1)^r\right] = \frac{2}{9}\left[(3r+1)2^r - (-1)^r\right].$$

We will prove this by verifying that the rows corresponding to vertices in $S^{(r)}$ are linear combinations of the rows corresponding to vertices in the complement of $S^{(r)}$. For this purpose it turns out to be convenient to number the vertices recursively as indicated in Figure 3. Formally this vertex numbering is given by a bijection $f : \{0, 1, 2, \ldots\}^2 \to \{1, 2, 3, \ldots\}$ defined as follows. For a positive integer x, let $\rho(x)$ be the unique integer such that $2^{\rho x - 1} \leqslant i < 2^{\rho(x)}$. In addition, let $\rho(0) = -1$. Then

$$f(x, i) = \begin{cases} i2^i + i + 1 & \text{if } i \geqslant \rho(x), \\ \rho(x)2^{\rho(x)-1} + f\left(x - 2^{\rho(x)-1},\, i\right) & \text{if } i < \rho(x). \end{cases} \tag{6}$$

With respect to the vertex numbering given by (6) the adjacency matrices for BF(1) and BF(2) are

$$A_1 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \qquad A_2 = \left(\begin{array}{cccc|cccc|cccc} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right),$$

and in general, A_r has the structure illustrated in Figure 4 where I is the identity matrix of size $2^{r-1} \times 2^{r-1}$.

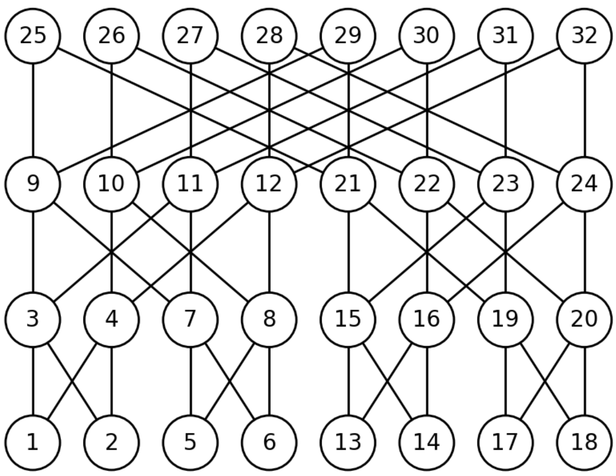

Figure 3: The butterfly network BF(3).

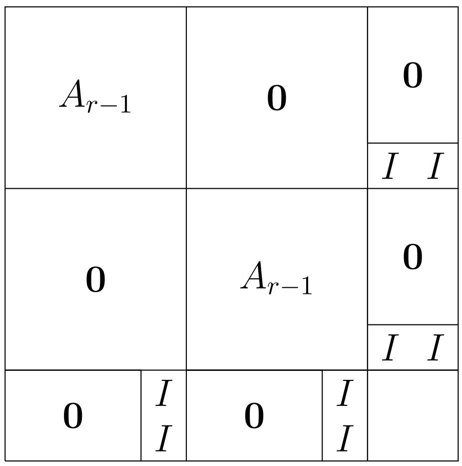

Figure 4: The structure of the adjacency matrix of the butterfly graph.

Using the vertex numbering given by (6) the upper bound construction for a zero forcing set $S^{(r)} \in V(\mathrm{BF}(r))$ can be written recursively as $S^{(1)} = \{1, 3\}$ and

$$S^{(r)} = S^{(r-1)} \cup \left\{ i + r2^{r-1} \; : \; i \in S^{(r-1)},\, i \leqslant (r-1)2^{r-1} \right\} \cup \{r2^r + 1, \ldots, r2^r + J_{r+1}\} \tag{7}$$

for $r \geqslant 2$. In order to prove that $S^{(r)}$ is a minimum zero forcing set for BF(r) it is sufficient to show that every row $i \in S^{(r)}$ of A_r can be written as a linear combination of the rows in $\overline{S}^{(r)} = \{1, \ldots, (r+1)2^r\} \setminus S^{(r)}$. We proceed by induction on r. Let $A_r(i)$ denote the i-th row of A_r. The induction base is provided by checking the cases $r = 1$ and $r = 2$. For $S^{(1)} = \{1, 3\}$, we have

$$A_1(1) = A_1(2), \tag{8}$$
$$A_1(3) = A_1(4), \tag{9}$$

and for $S^{(2)} = \{1,5,3,9,10,11\}$ we have

$$A_2(1) = A_2(2), \tag{10}$$
$$A_2(5) = A_2(6), \tag{11}$$
$$A_2(3) = A_2(4) + A_2(7) - A_2(8), \tag{12}$$
$$A_2(9) = A_2(2) + A_2(6) - A_2(12), \tag{13}$$
$$A_2(10) = A_2(12), \tag{14}$$
$$A_2(11) = A_2(2) + A_2(6) - A_2(12).. \tag{15}$$

The next two lemmas follow directly from the recursive structure illustrated in Figure 4.

Lemma 5. *If $i \leqslant (r-1)2^{r-1}$ and*

$$A_{r-1}(i) = \sum_{j\in K^+} A_{r-1}(j) - \sum_{j\in K^-} A_{r-1}(j) \text{ for some } K^+, K^- \subseteq \overline{S}^{(r-1)},$$

then

$$A_r(i) = \sum_{j\in K^+} A_r(j) - \sum_{j\in K^-} A_r(j) \quad \text{and}$$
$$A_r\left(i + r2^{r-1}\right) = \sum_{j\in K'^+} A_r(j) - \sum_{j\in K'^-} A_r(j)$$

where $K^+, K^- \subseteq \overline{S}^{(r)}$ and $K'^{\varepsilon} = \{j + r2^{r-1} \ : \ j \in K^{\varepsilon}\} \subseteq \overline{S}^{(r)}$ for $\varepsilon \in \{+,-\}$. □

Lemma 6. *If $(r-1)2^{r-1} + 1 \leqslant i \leqslant (r-1)2^{r-1} + J_r$ and*

$$A_{r-1}(i) = \sum_{j\in K^+} A_{r-1}(j) - \sum_{j\in K^-} A_{r-1}(j) \text{ for some } K^+, K^- \subseteq \overline{S}^{(r-1)},$$

then

$$A_r(i) = \sum_{j\in K'^+} A_r(j) - \sum_{j\in K'^-} A_r(j)$$

where

$$K'^+ = K^+ \cup \left\{j + r2^{r-1} \ : \ j \in K^-\right\} \cup \left\{i + r2^{r-1}\right\} \subseteq \overline{S}^{(r)},$$
$$K'^- = K^- \cup \left\{j + r2^{r-1} \ : \ j \in K^+\right\} \subseteq \overline{S}^{(r)}.$$

Lemmas 5 and 6 take care of the first two components in the recursion for $S^{(r)}$ in (7). It remains to check the rows $r2^r + i$ for $i \in \{1,\dots,J_{r+1}\}$. For $J_{r-1} + 1 \leqslant i \leqslant 2^{r-1}$ the required linear dependence is $A_r\left(r2^r + i\right) = A_r\left(r2^r + i + 2^{r-1}\right)$, because $i + 2^{r-1} > J_{r+1}$ and therefore $r2^r + i + 2^{r-1} \in \overline{S}^{(r)}$. For $i > 2^{r-1}$ we have $i \leqslant 2^{r-1} + J_{r-1}$ and $A_r(r2^r + i) = A_r(r2^r + i - 2^{r-1})$, and consequently it is sufficient to consider $i \in \{1,\dots,J_{r-1}\}$. The induction step for these cases will be from BF$(r-2)$ to BF(r), so we have to take the recursion for the adjacency matrix one step further which is illustrated in Figure 5. The basic idea is as follows. Let $i \in \{1,\dots,J_{r-1}\}$. Then $(r-2)2^{r-2} + i \in S^{(r-2)}$, and by induction there are sets K^+, $K^-\overline{S}^{(r-2)}$ such that

$$A_{r-2}\left((r-2)2^{r-2} + i\right) = \sum_{j\in K^+} A_{r-2}(j) - \sum_{j\in K^-} A_{r-2}(j), \tag{16}$$

or equivalently

$$\sum_{j\in K^+} A_{r-2}(j) - \sum_{j\in K'^-} A_{r-2}(j), \tag{17}$$

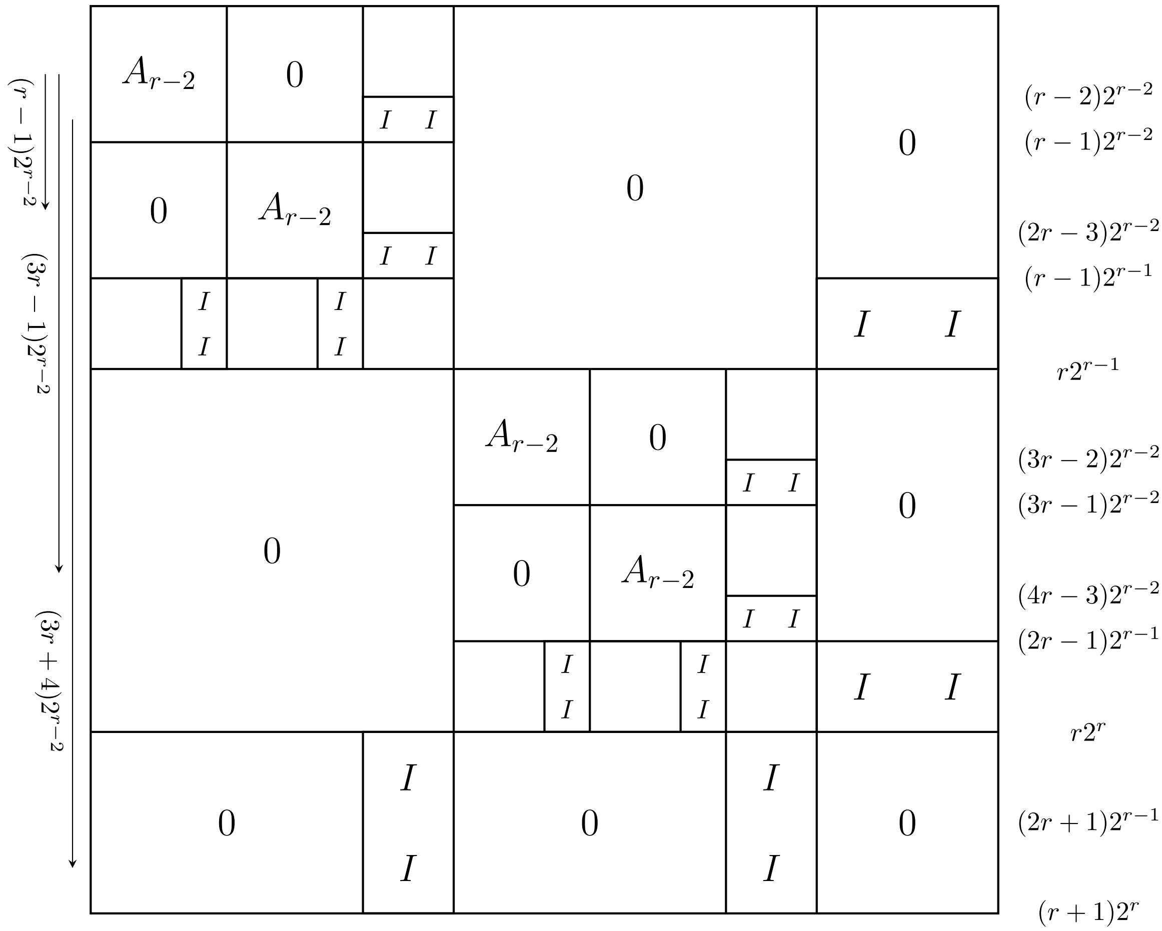

Figure 5: The second level of the recursion for A_r.

where $K'^- = K^- \cup \{(r-2)2^{r-2}+i\}$. This is a linear dependence of the rows of A_{r-2} with coefficients in $\{1,-1\}$ and involving exactly one of the rows $(r-2)2^{r-2}+1,\dots,(r-2)2^{r-2}+J_{r-1}$, namely $(r-2)2^{r-2}+i$. Putting $K = K^+ \cup K'^-$ we have

$$K \cap \{(r-2)2^{r-2}+1,\dots,(r-2)2^{r-2}+J_{r-1}\} = \{(r-2)2^{r-2}+i\}\,. \tag{18}$$

We now translate the $|K|$ rows in this linear dependence by $(r-1)2^{r-2}$ and $(3r-1)2^{r-2}$ as indicated in Figure 5. The combination of the $2|K|$ translated rows is a $\{0,1,-1\}$-vector $\boldsymbol{x}$ which has all its nonzero entries in columns with indices in $\{(r-1)2^{r-1}+1,\dots,r2^{r-1}\} \cup \{(2r-1)2^{r-1}+1,\dots,r2^r\}$, and has $x_k = 1$ for $k \in \{(r-1)2^{r-1}+i,\,(2r-1)2^{r-1}+i\}$ which are the one-entries of the row $A_r(r2^r+i)$. Finally we use some of the rows $r2^r+J_{r+1}+1,\dots,(r+1)2^r$ with the appropriate sign to eliminate the other nonzero entries of $\boldsymbol{x}$.
More precisely, we define $\tilde{K} = \tilde{K}^+ \cup \tilde{K}^- \subseteq \{1,\dots,(r+1)2^r\}$ with $\tilde{K}^+ = \tilde{K}_1^+ \cup \tilde{K}_2^+$ and $\tilde{K}^- = \tilde{K}_1^- \cup \tilde{K}_2^-$ where

$$\tilde{K}_1^+ = \{j+(r-1)2^{r-2} \;:\; j \in K'^-\} \cup \{j+(3r-1)2^{r-2} \;:\; j \in K'^-\} \tag{19}$$

$$\tilde{K}_1^- = \{j+(r-1)2^{r-2} \;:\; j \in K^+\} \cup \{j+(3r-1)2^{r-2} \;:\; j \in K^+\} \tag{20}$$

$$\tilde{K}_2^+ = \{j+(3r+4)2^{r-2} \;:\; j \in K^+ \text{ with } j > (r-2)2^{r-2}\} \cup \{j+(3r+5)2^{r-2} \;:\; j \in K^+ \text{ with } j > (r-2)2^{r-2}\}\,. \tag{21}$$

$$\tilde{K}_2^- = \{j+(3r+4)2^{r-2} \;:\; j \in K^- \text{ with } j > (r-2)2^{r-2}\} \cup \{j+(3r+5)2^{r-2} \;:\; j \in K'^- \text{ with } j > (r-2)2^{r-2}\}\,. \tag{22}$$

The construction of $\tilde{K}$ is illustrated for $r=4$ and $i=1$ in Figure 6. The next two lemmas state that $\tilde{K}$

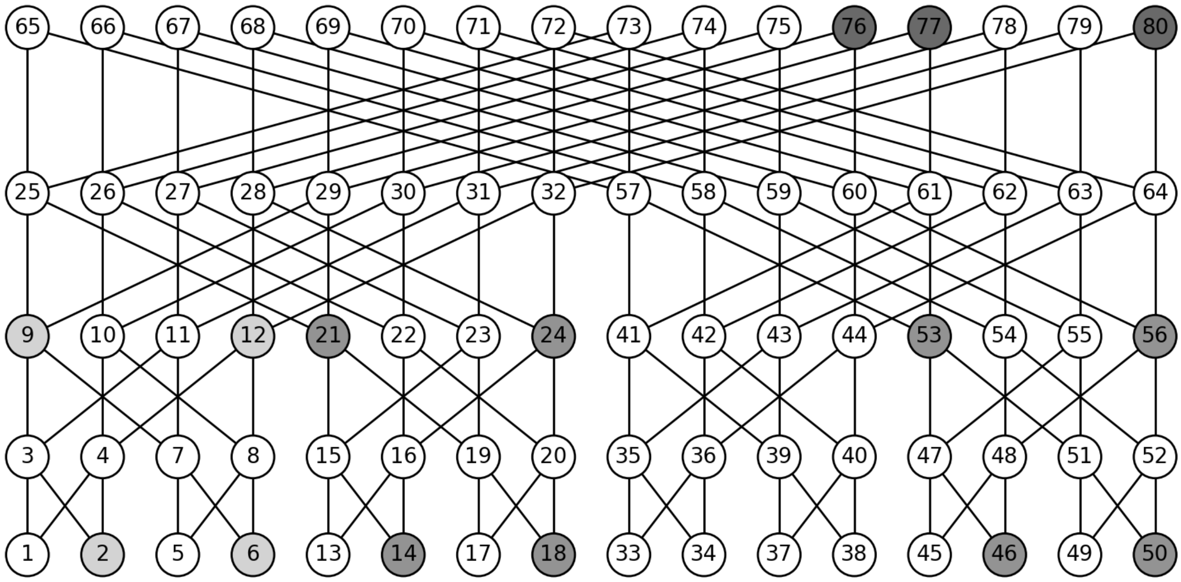

Figure 6: The construction of $\tilde{K}$ for $r = 4$ and $i = 1$. Here $K^+ = \{2, 6\}$, $K'^- = \{9, 12\}$, $\tilde{K}_1^- = \{14, 18, 46, 50\}$, $\tilde{K}_1^+ = \{21, 24, 53, 56\}$, $\tilde{K}_2^- = \{76, 77, 80\}$, $K_2''^+ = \emptyset$.

has the required properties.

Lemma 7. *Let $r \geqslant 3$, $i \in \{1, \ldots, J_{r-1}\}$, suppose $K \subseteq \overline{S}^{(r-2)}$ satisfies* (16)*, and define $\tilde{K}$ by* (19) *to* (22)*. Then $\tilde{K} \subseteq \overline{S}^{(r)}$.*

Proof. Note that by construction

$$\tilde{K}_1 = \tilde{K}_1^+ \cup \tilde{K}_1^- \subseteq \left[(r-1)2^{r-2} + 1, (r-1)2^{r-1}\right] \cup \left[(3r-1)2^{r-2} + 1, (2r-1)2^{r-1}\right]. \tag{23}$$

Suppose there is an element $j \in K$ such that $k = j + (r-1)2^{r-1} \in \tilde{K}_1 \cap S^{(r)}$. Using (7), we obtain

$$\begin{aligned} & k \in S^{(r)} = S^{(r-1)} \cup \left\{p + r2^{r-1} \ : \ p \in S^{(r-1)},\, p \leqslant (r-1)2^{r-1}\right\} \cup \{r2^r + 1, \ldots, r2^r + J_{r+1}\} \\ \implies & k \in S^{(r-1)} = S^{(r-2)} \cup \left\{p + (r-1)2^{r-2} \ : \ p \in S^{(r-2)},\, p \leqslant (r-2)2^{r-2}\right\} \\ & \qquad \cup \left\{(r-1)2^{r-2} + 1, \ldots, (r-1)2^{r-1} + J_r\right\} \\ \implies & k = p + (r-1)2^{r-2} \text{ for some } p \in S^{(r-2)} \text{ with } p \leqslant (r-2)2^{r-2}, \end{aligned}$$

which contradicts the assumption that $j \in K \subseteq \overline{S}^{(r-2)} \cup \{(r-2)2^{r-2} + i\}$. Similarly, for $k = j + (3r-1)2^{r-1} \in \tilde{K}_1 \cap S^{(r)}$ we obtain

$$\begin{aligned} & k \in S^{(r)} = S^{(r-1)} \cup \left\{p + r2^{r-1} \ : \ p \in S^{(r-1)},\, p \leqslant (r-1)2^{r-1}\right\} \cup \{r2^r + 1, \ldots, r2^r + J_{r+1}\} \\ \implies & k = p + r2^{r-1} \text{ for some } p \in S^{(r-1)} \text{ with } p \leqslant (r-1)2^{r-1} \\ \implies & k = q + (r-1)2^{r-2} \text{ for some } q \in S^{(r-2)} \text{ with } q \leqslant (r-2)2^{r-2}, \end{aligned}$$

where we use $k > (3r-1)2^{r-2}$ for the last implication. Again we obtain a contradiction to the assumption that $j \in K \subseteq \overline{S}^{(r-2)} \cup \{(r-2)2^{r-2} + i\}$. Finally, the elements of $\tilde{K}_2 = \tilde{K}_2^+ \cup \tilde{K}_2^-$ are in $\overline{S}^{(r)}$ since for $j \in K^+ \cup K^-$ we have

$$j > (r-2)2^{r-2} \implies j > (r-2)2^{r-2} + J_{r-1} \implies j + (3r+4)2^{r-2} > r2^r + 2^{r-1} + J_{r-1} = 2^r + J_{r+1},$$

and for $j \in K'$,

$$j > (r-2)2^{r-2} \implies j + (3r+5)2^{r-2} > r2^r + 2^{r-1} + 2^{r-2} > r2^r + J_{r+1},$$

and this concludes the proof of the lemma. □

Lemma 8. *Let $r \geqslant 3$, $i \in \{1, \dots, J_{r-1}\}$, suppose $K^+, K^- \subseteq \overline{S}^{(r-2)}$ satisfy (16), and define $\tilde{K}^+$ and $\tilde{K}^-$ by (19) to (22). Then*

$$A_r\left(r2^r + i\right) = \sum_{j \in \tilde{K}^+} A_r(j) - \sum_{j \in \tilde{K}^-} A_r(j). \tag{24}$$

Proof. Setting

$$\boldsymbol{x} = \sum_{j \in \tilde{K}_1^+} A_r(j) - \sum_{j \in \tilde{K}_1^-} A_r(j), \qquad \boldsymbol{y} = A_r(r2^r + i) - \sum_{j \in \tilde{K}_2^+} A_r(j) + \sum_{j \in \tilde{K}_2^-} A_r(j)$$

equation (24) is equivalent to $\boldsymbol{x} = \boldsymbol{y}$. From (23) and (16) it follows that

$$\operatorname{supp}(\boldsymbol{x}) \subseteq \left[(r-1)2^{r-1}+1,\, r2^{r-1}\right] \cup \left[(2r-1)2^{r-1}+1,\, r2^r\right],$$

and by construction, for every $j \in \{1, \dots, 2^{r-2}\}$,

$$x\left((r-1)2^{r-1}+j\right) = x\left((r-1)2^{r-1}+2^{r-2}+j\right) = x\left((2r-1)2^{r-1}+j\right) = x\left((2r-1)2^{r-1}+2^{r-2}+j\right).$$

Denoting this value by $\tilde{x}(j)$, we have

$$\tilde{x}(j) = \begin{cases} 1 & \text{if } (r-2)2^{r-2}+j \in K'^-, \\ -1 & \text{if } (r-2)2^{r-2}+j \in K^+, \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

From (21) and (22) it follows that

$$\tilde{K}_2^+ \cup \tilde{K}_2^- \cup \left\{(2r+1)2^{r-1}+i\right\} \subseteq \left[(2r+1)2^{r+1}+1,\, (r+1)2^r\right],$$

and therefore

$$\operatorname{supp}(\boldsymbol{y}) \subseteq \left[(r-1)2^{r-1}+1,\, r2^{r-1}\right] \cup \left[(2r-1)2^{r-1}+1,\, r2^r\right],$$

After replacing $A_r\left(r2^r+i\right)$ by $A_r\left(r2^r+2^{r-1}+i\right)$ (which we can do since the two rows are equal), the rows contributing to $\boldsymbol{y}$ come in pairs $\left(j, j+2^{r-2}\right)$ where both rows in each pair have the same sign in $\boldsymbol{y}$. Therefore

$$y\left((r-1)2^{r-1}+j\right) = y\left((r-1)2^{r-1}+2^{r-2}+j\right) = y\left((2r-1)2^{r-1}+j\right) = y\left((2r-1)2^{r-1}+2^{r-2}+j\right).$$

Finally, for $j \in \{1, \dots, 2^{r-2}\}$ we have $y\left((r-1)2^{r-1}+j\right) = 1$ if and only if $(2r+1)2^{r-1}+j = j' + (3r+4)2^{r-1}$ for some $j' \in K'^-$, or equivalently $j' = (r-2)2^{r-2}+j \in K'^-$. Similarly, we have $y\left((r-1)2^{r-1}+j\right) = -1$ if and only if $j' = (r-2)2^{r-2}+j \in K^+$, and comparing this with (25) we conclude $\boldsymbol{x} = \boldsymbol{y}$, as required. ☐

Proof of Proposition 1. The statement follows by induction with base (8)–(15), using Lemmas 5, 6, 7 and 8 for the induction step. ☐

Finally, Theorem 1 is a consequence of Propositions 1 and 2.

5 Additional comments

For a graph $G = (V, E)$ and a zero-forcing set $S \subseteq V$, the *propagation time* $\operatorname{pt}(S)$ has been defined in [14] as the length m of the increasing sequence $S = S_0 \subsetneq S_1 \subseteq \cdots \subsetneq S_m = V$, where

$$S_i = S_{i-1} \cup \{w \,:\, \{w\} = N(v) \cap S_{i-1} \text{ for some } v \in S_{i-1}\} \quad \text{for } i = 1, 2 \dots.$$

The propagation time of $\operatorname{pt}(G)$ of the graph G is the minimum of the propagation times $\operatorname{pt}(S)$ over all minimum zero-forcing sets S. The construction in Section 3 gives the upper bound $\operatorname{pt}(\mathrm{BF}(r)) \leqslant 2r$, and we leave it as an open problem to determine the propagation time of $\mathrm{BF}(r)$. A concept closely related to zero-forcing is *power domination* which was introduced in [12]. A vertex set $S \subseteq V$ is called power dominating if the closed neighbourhood $N[S] = S \cup \{w \,:\, vw \in E \text{ for some } v \in S\}$ is a zero forcing set.

It was shown in [4] that $Z(G)/\Delta$ provides a lower bound for the size of a power dominating set in G where Δ is the maximum degree of G. This implies that the power domination number of the butterfly network $\mathrm{BF}(r)$, i.e., the minimum size of a power pominating set, is at least

$$\left\lceil \frac{1}{36}\left[(3r+7)2^r + 2(-1)^r\right] \right\rceil .$$

This bound does not appear to be tight and we leave for future work the problems of finding the power domination number of the butterfly network as well as its *power propagation time* which is defined in [11] as

$$\mathrm{ppt}(G) = 1 + \min\{\mathrm{pt}\left(N[S]\right) \,:\, S \text{ is a minimum power dominating set in } G\}.$$

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