

ON SUMS OF PRIME FACTORS

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ABSTRACT. We study the arithmetic function $\text{sopfr}(n)$ (OEIS A001414) which gives the sum of prime factors (with repetition) of a number n . In particular we obtain the asymptotic formula

$$\sum_{n \leq x} \text{sopfr}(n) \sim \frac{\pi^2}{12} \frac{x^2}{\log x},$$

which holds as well for the function $\text{sopf}(n)$ (OEIS A008472) that just gives the sum of distinct prime factors of n . This asymptotic formula was already stated by R. Jakimcyuk [3] which was brought to our attention after the completion of the first version of this manuscript.

1. INTRODUCTION

The number and distribution of primes p less than a given number $x \in \mathbb{R}$ is a classical problem in number theory. Let $\pi(x) = \sum_{p \leq x} 1$, $x \in \mathbb{R}$, be the prime counting function and write $F(x) \sim G(x)$ if $\lim_{x \rightarrow \infty} F(x)/G(x) = 1$. By the prime number theorem we have that

$$\pi(x) \sim \frac{x}{\log x}.$$

Using the prime number theorem it can easily be shown that

$$(1) \quad P(x) = \sum_{p \leq x} p \sim \frac{1}{2} \frac{x^2}{\log x};$$

see f.e. [1, Section 2.7]. The aim of this note is to extend the sum on the left hand side of (1) to all integers in the following way. For $n \in \mathbb{N}$, $n > 1$, we write

$$n = \prod_{i=1}^r p_i^{\alpha_i} = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \dots \cdot p_r^{\alpha_r},$$

for the unique prime factorization of n , in which the p_i are the different prime factors of n and the α_i are the corresponding multiplicities. We define the arithmetic function

$$\text{sopfr}(n) = \begin{cases} \sum_{i=1}^r \alpha_i p_i = \alpha_1 p_1 + \alpha_2 p_2 + \dots + \alpha_r p_r & \text{for } n > 1, \\ 0 & \text{for } n = 1. \end{cases}$$

Clearly, n is prime if and only if $\text{sopfr}(n) = n$. Moreover, $\text{sopfr}(n)$ is completely additive due to the uniqueness of the prime factorization of every integer n . The function $\text{sopfr}(n)$ fluctuates wildly attaining sharp local maxima whenever n is prime and small values whenever n has many small prime factors. As an example, assume that n is a Mersenne prime, i.e., $n = 2^p - 1$ for a prime p , then $\text{sopfr}(n) = 2^p - 1$ and $\text{sopfr}(n+1) = \text{sopfr}(2^p) = 2p$. As a second example, consider the numbers $10^9 + 7$, $10^9 + 8$, $10^9 + 9$. The first and the last of these numbers are primes and we obtain:

$$\begin{aligned} \text{sopfr}(10^9 + 7) &= 10^9 + 7 \\ \text{sopfr}(10^9 + 8) &= 3 \cdot 2 + 2 \cdot 3 + 1 \cdot 7 + 2 \cdot 109 + 1 \cdot 167 = 404 \\ \text{sopfr}(10^9 + 9) &= 10^9 + 9 \end{aligned}$$

It is thus interesting to study

$$B(x) = \sum_{n \leq x} \text{sopfr}(n)$$

and compare it with the growth of $P(x)$. The value of $B(x)$ is obviously larger than $P(x)$ for all $x \geq 4$. The main result of this note is the following asymptotic formula for $B(x)$.

Theorem 1.1.

$$B(x) \sim \frac{\pi^2}{12} \frac{x^2}{\log x}$$

We can relate this formula in a straightforward way to the function $P(x)$.

Corollary 1.2.

$$B(x) \sim \frac{\pi^2}{6} P(x)$$

The interesting observation is that $\pi^2/6 = 1.664\dots$ is considerably larger than 1, however, the order of the growth of the two sums is the same.

2. PROOF OF THEOREM 1.1

For a real number x we write $[x]$ for the integer part of x . We start from the definition $B(x) = \sum_{n \leq x} \text{sopfr}(n)$ and rewrite $B(x)$ as a sum over all primes $p \leq x$. We count the contribution of each p to $B(x)$ in a systematic way as follows. First, we note that exactly $[x/p]$ integers $n \leq x$ have p as a prime factor, from these integers exactly $[x/p^2]$ are also divisible by p^2 , and from these exactly $[x/p^3]$ are divisible by p^3 as well. Continuing in this manner we see that the contribution of each p to $B(x)$ is $p([x/p] + [x/p^2] + [x/p^3] + \dots)$. Therefore,

$$(2) \quad B(x) = \sum_{p \leq x} p \left(\left[\frac{x}{p} \right] + \left[\frac{x}{p^2} \right] + \left[\frac{x}{p^3} \right] + \dots \right).$$

Now,

$$\begin{aligned} \sum_{p \leq x} p \left[\frac{x}{p^2} \right] &\leq \sum_{p \leq x} p \frac{x}{p^2} = x \sum_{p \leq x} \frac{1}{p} \\ &= x(\log \log x + \mathcal{O}(1)) = x \log \log x + \mathcal{O}(x), \end{aligned}$$

by a classical result of Mertens [4] and

$$\begin{aligned} \sum_{p \leq x} p \left(\left[\frac{x}{p^3} \right] + \dots \right) &\leq \sum_{p \leq x} p \left(\frac{x}{p^3} + \dots \right) = x \sum_{p \leq x} \left(\frac{1}{p^2} + \dots \right) \\ &= x \sum_{p \leq x} \frac{1}{p^2} \frac{1}{1 - \frac{1}{p}} = x \sum_{p \leq x} \frac{1}{p(p-1)} = \mathcal{O}(x) \end{aligned}$$

since $\sum_{p \leq x} \frac{1}{p(p-1)}$ converges. Therefore, we are left with the estimation of

$$(3) \quad \sum_{p \leq x} p \left[\frac{x}{p} \right].$$

Remark 2.1. Note that the trivial estimate $\sum_{p \leq x} p \left[\frac{x}{p} \right] \leq x\pi(x)$ satisfies $x\pi(x) \sim x^2/\log x$.

Remark 2.2. From our argument it is clear that (3) can be used to rewrite $\sum_{n \leq x} \text{sopfr}(n)$, in which $\text{sopfr}(n)$ gives the sum of distinct prime factors of n . Thus, the order of growth is the same as for $B(x)$.

Recalling that $P(x) = \sum_{p \leq x} p$, we see that the contribution of the primes contained in the interval $(\frac{x}{2}, x]$ to the sum (3) is

$$\sum_{\frac{x}{2} < p \leq x} p = P(x) - P\left(\frac{x}{2}\right).$$

The contribution of the primes in $(\frac{x}{3}, \frac{x}{2}]$ is

$$2 \sum_{\frac{x}{3} < p \leq \frac{x}{2}} p = 2 \left(P\left(\frac{x}{2}\right) - P\left(\frac{x}{3}\right) \right),$$

and the contribution of the primes in $(\frac{x}{4}, \frac{x}{3}]$ is

$$3 \sum_{\frac{x}{4} < p \leq \frac{x}{3}} p = 3 \left(P\left(\frac{x}{3}\right) - P\left(\frac{x}{4}\right) \right).$$

Thus,

$$\begin{aligned} \sum_{p \leq x} p \left[\frac{x}{p} \right] &= P(x) - P\left(\frac{x}{2}\right) + 2 \left(P\left(\frac{x}{2}\right) - P\left(\frac{x}{3}\right) \right) + 3 \left(P\left(\frac{x}{3}\right) - P\left(\frac{x}{4}\right) \right) + \dots \\ &= P(x) + P\left(\frac{x}{2}\right) + P\left(\frac{x}{3}\right) + \dots = \sum_{1 \leq n \leq \frac{x}{2}} P\left(\frac{x}{n}\right). \end{aligned}$$

We would like to use this formula to obtain an asymptotic formula for (3). Therefore, we need an estimate for the error term in (1). Starting from the prime number theorem in the form

$$(4) \quad \pi(x) = \frac{x}{\log x} + \mathcal{O}\left(\frac{x}{\log^2 x}\right),$$

we use partial summation to obtain

$$(5) \quad P(x) = \frac{x^2}{2 \log x} + \mathcal{O}\left(\frac{x^2}{\log^2 x}\right).$$

Next, we use this formula to rewrite our expression as follows.

$$(6) \quad \begin{aligned} \sum_{1 \leq n \leq \frac{x}{2}} P\left(\frac{x}{n}\right) &= \sum_{1 \leq n \leq \frac{x}{2}} \left(\frac{1}{2} \left(\frac{x}{n}\right)^2 \frac{1}{\log \frac{x}{n}} + \mathcal{O}\left(\left(\frac{x}{n}\right)^2 \frac{1}{\log^2 \frac{x}{n}}\right) \right) \\ &= \frac{x^2}{2} \sum_{1 \leq n \leq \frac{x}{2}} \frac{1}{n^2 \log \frac{x}{n}} + x^2 \mathcal{O}\left(\sum_{1 \leq n \leq \frac{x}{2}} \frac{1}{n^2 \log^2 \frac{x}{n}}\right). \end{aligned}$$

To estimate the first sum, we use again partial summation. We set

$$u(n) = \frac{1}{n^2} \quad \text{and} \quad f(n) = \left(\log \frac{x}{n}\right)^{-1}.$$

Then,

$$\begin{aligned} U(x) &= \sum_{n \leq x} u(n) = \sum_{n \leq x} \frac{1}{n^2} = \frac{\pi^2}{6} + \mathcal{O}\left(\frac{1}{x}\right) \\ f'(n) &= (-1) \left(\log \frac{x}{n}\right)^{-2} \frac{n}{x} \left(-\frac{x}{n^2}\right) = \frac{1}{n \log^2 \frac{x}{n}}. \end{aligned}$$

Consequently,

$$\begin{aligned} \sum_{1 \leq n \leq \frac{x}{2}} \frac{1}{n^2 \log \frac{x}{n}} &= \sum_{n \leq \frac{x}{2}} u(n) f(n) = U\left(\frac{x}{2}\right) f\left(\frac{x}{2}\right) - \int_1^{\frac{x}{2}} U(t) f'(t) dt \\ &= \left(\frac{\pi^2}{6} + \mathcal{O}\left(\frac{1}{x}\right)\right) \frac{1}{\log 2} - \int_1^{\frac{x}{2}} \left(\frac{\pi^2}{6} + \mathcal{O}\left(\frac{1}{t}\right)\right) \frac{1}{t \log^2 \frac{x}{t}} dt \\ &= \frac{\pi^2}{6 \log 2} + \mathcal{O}\left(\frac{1}{x}\right) - \frac{\pi^2}{6} \int_1^{\frac{x}{2}} \frac{dt}{t \log^2 \frac{x}{t}} + \mathcal{O}\left(\int_1^{\frac{x}{2}} \frac{dt}{t^2 \log^2 \frac{x}{t}}\right) \\ &= \frac{\pi^2}{6} \frac{1}{\log x} + \mathcal{O}\left(\frac{1}{\log^2 x}\right), \end{aligned}$$

since

$$\begin{aligned} \int_1^{\frac{x}{2}} \frac{dt}{t \log^2 \frac{x}{t}} &\stackrel{u=\frac{x}{t}}{=} \int_x^2 \frac{u}{x \log^2 u} \left(-\frac{x}{u^2}\right) du \\ &= \int_2^x \frac{du}{u \log^2 u} = -\frac{1}{\log u} \Big|_2^x = \frac{1}{\log 2} - \frac{1}{\log x}, \end{aligned}$$

and similarly

$$\int_1^{\frac{x}{2}} \frac{dt}{t^2 \log^2 \frac{x}{t}} = \mathcal{O}\left(\frac{1}{\log^2 x}\right).$$

Setting $f(t) = (\log \frac{x}{t})^{-2}$ we can handle the sum in the \mathcal{O} term of (6) in a similar fashion and obtain

$$(7) \quad \sum_{1 \leq n \leq \frac{x}{2}} \frac{1}{n^2 \log^2 \frac{x}{n}} = \frac{\pi^2}{6} \frac{1}{\log^2 x} + \mathcal{O}\left(\frac{1}{\log^3 x}\right) = \mathcal{O}\left(\frac{1}{\log^2 x}\right).$$

Putting the pieces together and using (6) we get

$$\sum_{1 \leq n \leq \frac{x}{2}} P\left(\frac{x}{n}\right) = \frac{\pi^2}{12} \frac{x^2}{\log x} + \mathcal{O}\left(\frac{x^2}{\log^2 x}\right).$$

Consequently, we obtain the right asymptotic order of $B(x)$ via (2)

$$(8) \quad \begin{aligned} B(x) &= \frac{\pi^2}{12} \frac{x^2}{\log x} + \mathcal{O}\left(\frac{x^2}{\log^2 x}\right) + \mathcal{O}(x \log \log x) + \mathcal{O}(x) \\ &= \frac{\pi^2}{12} \frac{x^2}{\log x} + \mathcal{O}\left(\frac{x^2}{\log^2 x}\right). \end{aligned}$$

Remark 2.3. Note that even under the assumption of the Riemann Hypothesis (RH) the error term coming from the prime number theorem will be dominant in (8). Assuming the RH we can replace the error term in (4) by the slightly sharper term; see f.e. [2] for details.

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