A note on the order of iterated line digraphs

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Abstract

Given a digraph G, we propose a new method to find the recurrence equation for the number of vertices n_k of the k-iterated line digraph $L^k(G)$, for $k \geq 0$, where $L^0(G) = G$. We obtain this result by using the minimal polynomial of a quotient digraph $\pi(G)$ of G. We show some examples of this method applied to the so-called cyclic Kautz, the unicyclic, and the acyclic digraphs. In the first case, our method gives the enumeration of the ternary length-2 squarefree words of any length.

Mathematics Subject Classifications: 05C20, 05C50.
Keywords: Line digraph, adjacency matrix, minimal polynomial, regular partition, quotient digraph, recurrence

1 Preliminaries

In this section we recall some basic notation and results concerning digraphs and their spectra. A digraph $G = (V, E)$ consists of a (finite) set $V = V(G)$ of vertices and a set $E = E(G)$ of arcs (directed edges) between vertices of G. As the initial and final vertices of an arc are not necessarily different, the digraphs may have *loops* (arcs from a vertex to itself), and *multiple arcs*, that is, there can be more than one arc from each vertex to any other. If $a = (u, v)$ is an arc from u to v, then vertex u (and arc a) is *adjacent to* vertex v, and vertex v (and arc a) is *adjacent from* u. Let $G^+(v)$ and $G^-(v)$ denote the set of arcs adjacent from and to vertex v, respectively. A digraph G is *d-regular* if $|G^+(v)| = |G^-(v)| = d$ for all $v \in V$.

In the line digraph $L(G)$ of a digraph G, each vertex of $L(G)$ represents an arc of G, that is, $V(L(G)) = \{uv|(u, v) \in E(G)\}$; and vertices uv and wz of

$L(G)$ are adjacent if and only if $v = w$, namely, when arc (u, v) is adjacent to arc (w, z) in G. For $k \geq 0$, we consider the sequence of line digraph iterations $L^0(G) = G, L(G), L^2(G), \ldots, L^k(G) = L(L^{k-1}(G)), \ldots$ It can be easily seen that every vertex of $L^k(G)$ corresponds to a walk $v_0, v_1, \ldots, v_k$ of length k in G, where $(v_{i-1,}, v_i) \in E$ for $i = 1, \ldots, k$. Then, if there is one arc between pairs of vertices and $\boldsymbol{A}$ is the adjacency matrix of G, the uv-entry of the power $\boldsymbol{A}^k$, denoted by $a_{uv}^{(k)}$, is the number of k-walks from vertex u to vertex v, and the order n_k of $L^k(G)$ turns out to be

$$n_k = \boldsymbol{j}\boldsymbol{A}^k\boldsymbol{j}^\top, \tag{1.1}$$

where $\boldsymbol{j}$ stands for the all-1 vector. If there are multiple arcs between pairs of vertices, then the corresponding entry in the matrix is not 1, but the number of these arcs. If G is a d-regular digraph with n vertices then its line digraph $L^k(G)$ is d-regular with $n_k = d^k n$ vertices.

Recall also that a digraph G is *strongly connected* if there is a (directed) walk between every pair of its vertices. If G is strongly connected, different from a directed cycle, and it has diameter D, then its line digraph $L^k(G)$ has diameter $D + k$. See Fiol, Yebra, and Alegre [4] for more details. The interest of the line digraph technique is that it allows us to obtain digraphs with small diameter and large connectivity. For a comparison between the line digraph technique and other techniques to obtain digraphs with minimum diameter see Miller, Slamin, Ryan and Baskoro [7]. Since these techniques are related to the degree/diameter problem, we refer also to the comprehensive survey on this problem by Miller and Širáň [6].

For the concepts and/or results not presented here, we refer the reader to some of the basic textbooks and papers on the subject; about digraphs see, for instance, Chartrand and Lesniak [2] or Diestel [3], and Godsil [5] about the quotient graphs.

This paper is organized as follows. In Section 2, we recall the definition of regular partitions and we give some lemmas about them. In Section 3 we prove our main result. In Section 4, we give examples in which the sequence on the number of vertices of iterated line digraphs is increasing, tending to a positive constant, or tending to zero.

2 Regular partitions

Let G be a digraph with adjacency matrix $\boldsymbol{A}$. A partition $\pi = (V_1, \ldots, V_m)$ of its vertex set V is called *regular* (or *equitable*) whenever, for any $i, j = 1, \ldots, m$, the *intersection numbers* $b_{ij}(u) = |G^+(u) \cap V_j|$, where $u \in V_i$, do not depend on the vertex u but only on the subsets (usually called *classes* or *cells*) V_i and V_j. In this case, such numbers are simply written as b_{ij}, and the $m \times m$ matrix $\boldsymbol{B} = (b_{ij})$ is referred to as the *quotient matrix* of $\boldsymbol{A}$ with respect to π. This is also represented by the *quotient (weighted) digraph* $\pi(G)$ (associated to the partition π), with vertices representing the cells, and an arc with weight b_{ij}

from vertex V_i to vertex V_j if and only if $b_{ij} \neq 0$. Of course, if $b_{ii} > 0$ for some $i = 1, \dots, m$, the quotient digraph $\pi(G)$ has loops.

The *characteristic matrix* of (any) partition π is the $n \times m$ matrix $\boldsymbol{S} = (s_{ui})$ whose i-th column is the characteristic vector of V_i, that is, $s_{ui} = 1$ if $u \in V_i$, and $s_{ui} = 0$ otherwise. In terms of such a matrix, we have the following characterization of regular partitions.

Lemma 2.1. *Let $G = (V, E)$ be a digraph with adjacency matrix $\boldsymbol{A}$, and vertex partition π with characteristic matrix $\boldsymbol{S}$. Then π is regular if and only if there exists an $m \times m$ matrix $\boldsymbol{C}$ such that $\boldsymbol{SC} = \boldsymbol{AS}$. Moreover, $\boldsymbol{C} = \boldsymbol{B}$, the quotient matrix of $\boldsymbol{A}$ with respect to π.*

Proof. Let $\boldsymbol{C} = (c_{ij})$ be an $m \times m$ matrix. For any fixed $u \in V_i$ and $j = 1, \dots, m$, we have

$$(\boldsymbol{SC})_{uj} = \sum_{k=1}^{m} s_{uk} c_{kj} = c_{ij}, \quad (\boldsymbol{AS})_{uj} = \sum_{v \in V} a_{uv} s_{vj} = |G^{+}(u) \cap V_j| = b_{ij}(u),$$

and the result follows. □

Most of the results about regular partitions in graphs can be generalized for regular partitions in digraphs. For instance, using the above lemma it can be proved that all the eigenvalues of the quotient matrix $\boldsymbol{B}$ are also eigenvalues of $\boldsymbol{A}$. Moreover, we have the following result.

Lemma 2.2. *Let G be a digraph with adjacency matrix $\boldsymbol{A}$. Let $\pi = (V_1, \dots, V_m)$ be a regular partition of G, with quotient matrix $\boldsymbol{B}$. Then, the number of k-walks from each vertex $u \in V_i$ to all vertices of V_j is the ij-entry of $\boldsymbol{B}^k$.*

Proof. We use induction. The result is clearly true for $k = 0$, since $\boldsymbol{B}^0 = \boldsymbol{I}$, and for $k = 1$ because of the definition of $\boldsymbol{B}$. Suppose that the result holds for some $k > 1$. Then the set of walks of length $k + 1$ from $u \in V_i$ to the vertices of V_j is in bijective correspondence with the set of k-walks from u to vertices $v \in V_h$ adjacent to some vertex of V_j. Then, the number of such walks is $\sum_{h=1}^{m} (\boldsymbol{B}^k)_{ih} b_{hj} = (\boldsymbol{B}^{k+1})_{ij}$, as claimed. □

As a consequence of this lemma, the number of vertices of $L^k(G)$ is

$$n_k = \sum_{i=1}^{m} |V_i| \sum_{j=1}^{m} (\boldsymbol{B}^k)_{ij} = \boldsymbol{s}\boldsymbol{B}^k \boldsymbol{j}^{\top}, \tag{2.1}$$

where $\boldsymbol{s} = (|V_1|, \dots, |V_m|)$ and $\boldsymbol{j} = (1, \dots, 1)$.

3 Main result

In the following result, we obtain a recurrence equation on the number of vertices n_k of the k-iterated line digraph of a digraph G.

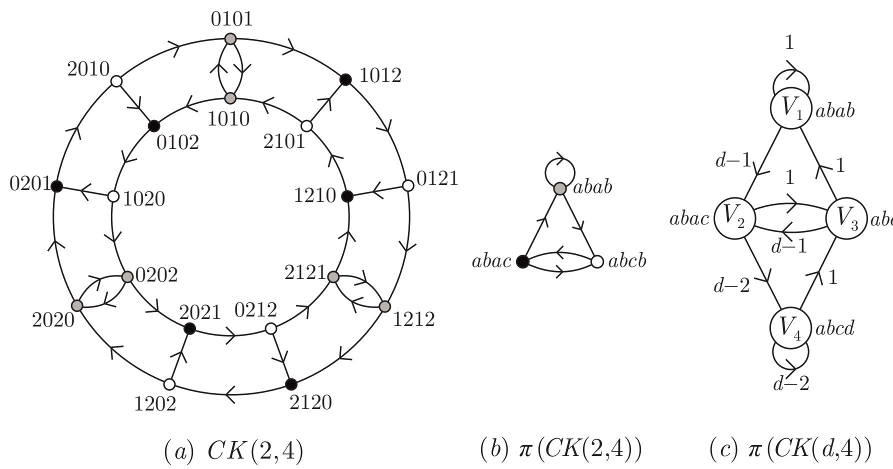

Figure 1: The cyclic Kautz digraph $CK(2,4)$, its quotient $\pi(CK(2,4))$, and the quotient digraph of $CK(d,4)$.

Theorem 3.1. *Let $G = (V,E)$ be a digraph on n vertices, and consider a regular partition $\pi = (V_1,\dots,V_m)$ with quotient matrix $\boldsymbol{B}$. Let $m(x) = x^r - \alpha_{r-1}x^{r-1} - \cdots - \alpha_0$ be the minimal polynomial of $\boldsymbol{B}$. Then, the number of vertices n_k of the k-iterated line digraph $L^k(G)$ satisfies the recurrence*

$$n_k = \alpha_{r-1}n_{k-1} + \cdots + \alpha_0 n_{k-r}, \qquad k = r, r+1, \dots \tag{3.1}$$

initialized with the values n_k, for $k = 0,1,\dots,r-1$, given by (2.1)*.*

Proof. Since the polynomial $x^{k-r}m(x)$ annihilates $\boldsymbol{B}$ for any $k \geq 0$, we have

$$\boldsymbol{B}^k = \alpha_{r-1}\boldsymbol{B}^{k-1} + \cdots + \alpha_0\boldsymbol{B}^{k-r}.$$

Then, by (2.1), we get the recurrence

$$\begin{aligned} n_k = \boldsymbol{s}\boldsymbol{B}^k\boldsymbol{j}^\top &= \alpha_{r-1}\boldsymbol{s}\boldsymbol{B}^{k-1}\boldsymbol{j}^\top + \cdots + \alpha_0\boldsymbol{s}\boldsymbol{B}^{k-r}\boldsymbol{j}^\top \\ &= \alpha_{r-1}n_{k-1} + \cdots + \alpha_0 n_{k-r}, \end{aligned}$$

with the first values n_k, for $k = 0,\dots,r-1$, given as claimed. □

4 Examples

In what follows, we give examples of the three possible behaviours of the sequence $n_0, n_1, n_2, \dots$ Namely, when it is increasing, tending to a positive constant, or tending to zero.

4.1 Cyclic Kautz digraphs

The *cyclic Kautz digraph $CK(d,\ell)$*, introduced by Böhmová, Dalfó, and Huemer in [1], has vertices labeled by all possible sequences $a_1 \dots a_\ell$ with $a_i \in$

$\{0,1,\ldots,d\}$, $a_i \neq a_{i+1}$ for $i = 1,\ldots,\ell-1$, and $a_1 \neq a_\ell$. Moreover, there is an arc from vertex $a_1a_2\ldots a_\ell$ to vertex $a_2\ldots a_\ell a_{\ell+1}$, whenever $a_1 \neq a_\ell$ and $a_2 \neq a_{\ell+1}$. By this definition, we observe that the cyclic Kautz digraph $CK(d,\ell)$ is a subdigraph of the well-known Kautz digraph $K(d,\ell)$, defined in the same way, but without the requirement $a_1 \neq a_\ell$.

For example, Figure 1(a) shows the cyclic Kautz digraph $CK(2,4)$. Notice that, in general, such digraphs are not d-regular and, hence, the number of vertices of their iterated line digraphs are not obtained by repeatedly multiplying by d. Instead, we can apply our method, as shown next with $CK(2,4)$. This digraph has a regular partition π of its vertex set into three classes (each one with 6 vertices): *abcb* (the second and the last digits are equal), *abab* (the first and the third digits are equal, and also the second and the last), and *abac* (the first and the third digits are equal). Then, the quotient matrix of π (which in this case coincides with the adjacency matrix of $\pi(CK(2,4))$) is

$$\boldsymbol{B} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

and it has minimal polynomial $m(x) = x^3 - x^2 - x$. Consequently, by Theorem 3.1, the number of vertices of $L^k(CK(2,4))$ satisfies the recurrence $n_k = n_{k-1} + n_{k-2}$ for $k \geq 3$. In fact, in this case, $\boldsymbol{s}(\boldsymbol{B}^2 - \boldsymbol{B} - \boldsymbol{I})\boldsymbol{j}^\top = 0$, and the above recurrence applies from $k = 2$. This, together with the initial values $n_0 = 18$ and $n_1 = \boldsymbol{sBj}^\top = 30$, yields the Fibonacci sequence, $n_2 = 48, n_3 = 78, n_4 = 126\ldots$, as Böhmová, Dalfó, and Huemer [1] proved by using a combinatorial approach. Moreover, n_k is also the number of ternary length-2 squarefree words of length $k+4$ (that is, words on a three-letter alphabet that do not contain an adjacent repetition of any subword of length ≤ 2); see the sequence A022089 in the On-Line Encyclopedia of Integer Sequences [8].

In fact our method allows us to generalize this result and, for instance, derive a formula for the order of $L^k(CK(d,4))$ for any value of the degree $d \geq 2$. To this end, it is easy to see that a quotient digraph of $CK(d,4)$ for $d > 2$ is as shown in Figure 1(c), where now we have to distinguish four classes of vertices. Then, the corresponding quotient matrix is

$$\boldsymbol{B} = \begin{pmatrix} 1 & d-1 & 0 & 0 \\ 0 & 0 & 1 & d-2 \\ 1 & d-1 & 0 & 0 \\ 0 & 0 & 1 & d-2 \end{pmatrix},$$

and it has minimal polynomial is $m(x) = x^3 - (d-1)x^2 - x$. In turn, this leads to the recurrence formula $n_k = (d-1)n_{k-1} + n_{k-2}$, with initial values $n_0 = d^4 + d$ and $n_1 = d^5 - d^4 + d^3 + 2d^2 - d$, which are computed by using (2.1) with the vector

$$\begin{aligned} \boldsymbol{s} &= (|V_1|, |V_2|, |V_3|, |V_4|) \\ &= ((d+1)d, (d+1)d(d-1), (d+1)d(d-1), (d+1)d(d-1)(d-2)). \end{aligned}$$

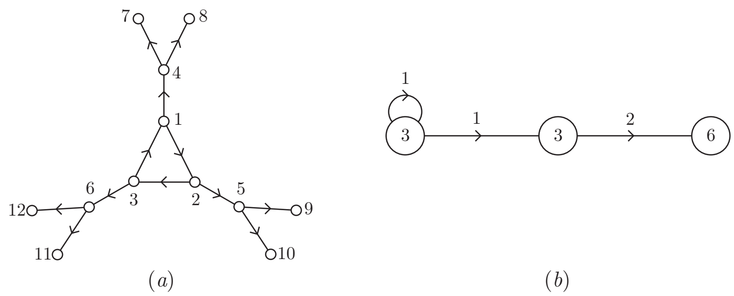

Figure 2: The unicyclic digraph $G_{3,2}$ and its quotient digraph.

Solving the recurrence, we get the closed formula

$$n_k = \frac{2^k d}{\sqrt{\Delta}} \left(\frac{(d^2+d)\sqrt{\Delta} - d^3 - d - 2}{(1-d-\sqrt{\Delta})^{k+1}} + \frac{(d^2+d)\sqrt{\Delta} + d^3 + d + 2}{(1-d+\sqrt{\Delta})^{k+1}} \right),$$

where $\Delta = d^2 - 2d + 5$ and, hence, n_k is an increasing sequence.

4.2 Unicyclic digraphs

A unicyclic digraph is a digraph with exactly one (directed) cycle. As usual, we denote a cycle on n vertices by C_n. For example, consider the digraph $G_{n,d}$, obtained by joining to every vertex of C_n one 'out-tree' with d leaves (or 'sinks'), as shown in Figure 2(a) for the case $G_{3,2}$. This digraph has the regular partition $\pi = (V_1, V_2, V_3)$, where V_1 is the set of vertices of the cycle, V_2 the central vertices of the trees, and V_3 the set of leaves. (In the figure $V_1 = \{1,2,3\}$, $V_2 = \{4,5,6\}$, and $V_3 = \{7,8,9,10,11,12\}$). This partition gives the quotient digraph $\pi(G)$ of Figure 2(b), and the quotient matrix

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix},$$

with minimal polynomial $m(x) = x^3 - x^2$. Then, by Theorem 3.1, the order of $L^k(G)$ satisfies the recurrence $n_k = n_{k-1}$ for $k \geq 0$, since $\boldsymbol{s}(\boldsymbol{B}^k - \boldsymbol{B}^{k-1})\boldsymbol{j}^\top = 0$ for $k = 1, 2$, where $\boldsymbol{s} = (n, n, nd)$. Thus, we conclude that all the iterated line digraphs $L^k(G)$ have constant order $n_k = n_0 = n(d+2)$, that is, n_k tends to a positive constant. (In fact, this is because in this case $L(G)$—and, hence, $L^k(G)$—is isomorphic to G.)

4.3 Acyclic digraphs

Finally, let us consider an example of an acyclic digraph, that is, a digraph without directed cycles, such as the digraph G of Figure 3(a). Its quotient

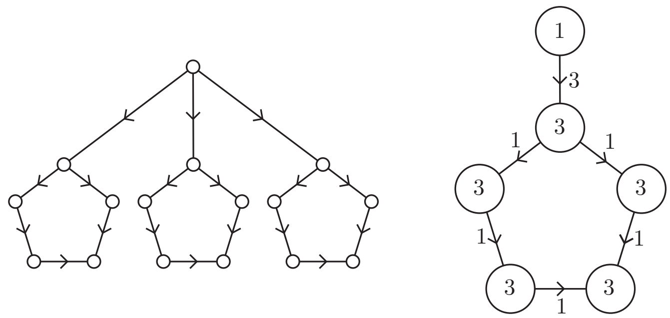

Figure 3: An acyclic digraph and its quotient digraph.

digraph is depicted in Figure 3(b), with quotient matrix

$$\boldsymbol{B} = \begin{pmatrix} 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and minimal polynomial $m(x) = x^5$. This indicates that $n_k = 0$ for every $k \geq 5$ (as expected, because G has not walks of length larger than or equal to 5). Moreover, from (2.1), the first values are $n_0 = 16$, $n_1 = 18$, $n_2 = 15$, $n_3 = 9$, and $n_4 = 3$.

Acknowledgments. This research is supported by the *Ministerio de Economía y Competitividad* and the *European Regional Development Fund* under project MTM2011-28800-C02-01, and by the Catalan Government under project 2014SGR1147.

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