m-Modular Wythoff

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Abstract

We discuss a variant of Wythoff's Game, m-Modular Wythoff's Game, and identify the winning and losing positions for this game.

1 Introduction

Nim forms the foundation of the mathematical study of two-player strategy games. In his landmark 1901 paper, *Nim, a game with a complete mathematical theory*, Charles L. Bouton provided a solution to the game of Nim, essentially founding the field of Combinatorial Game Theory [1].

One of the most famous variants of the game of Nim is Wythoff's Game [4], for which the winning strategy is completely understood.

Many variations of these games were studied, but not many of them use moves based on modular congruence. Recently a paper appeared in which a modular extension to Nim was explored [2]. It is called m-Modular Nim, and moves that are predicated upon modular congruence are added to the traditional Nim moves.

In this paper we study a modular extension to Wythoff's game.

We start this paper by describing Wythoff's game and introducing notation in Section 2. We define the m-Modular Wythoff game in Section 3. In addition to Nim moves we allow players to remove tokens from both piles as long as the remainders of the number of tokens removed modulo m are the same. We calculate the P-positions of m-Modular Wythoff for small values of m and show that these positions are a subset of the P-positions of Wythoff's game.

In Section 4 we prove that the number of P-positions of m-Modular Wythoff is finite. Moreover, we completely solve for all of the P-positions of m-Modular Wythoff; we show that there are $2\lfloor m/\phi \rfloor + 1$ of them, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio, and that they form a subset of the P-positions of Wythoff's game.

2 Wythoff's game

First we describe Wythoff's game. In this game we have 2 piles of tokens. Two people take turns making moves. There are two types of moves that are allowed. First, a player can take any positive number of tokens from any one pile. These moves are the same as the moves in a game of Nim. Second, a player can take the same positive number of tokens from both piles. The person who cannot move loses.

A *P-position* is a position from which the *previous* player will win given perfect play. We denote the set of P-positions of Wythoff's game as $\mathcal{P}$. We can observe that all terminal positions are P-positions. An *N-position* is a position from which the *next* player will win given perfect play. We denote the set of N-positions of Wythoff's game as $\mathcal{N}$.

We say that position p_1 *dominates* position p_2 if $p_1 - p_2$ has all nonnegative values. We say that p_1 *strictly dominates* p_2 if p_1 dominates p_2 and $p_1 \neq p_2$.

The set of P-positions of Wythoff's game is well-understood [4].

To make an explicit description we introduce some notation. We construct a set $\mathcal{P}_i$ of P-positions in Wythoff's game recursively as follows. We let $\mathcal{P}_0 = \{(0,0)\}$ and let $\mathcal{P}_i = \mathcal{P}_{i-1} \cup \{(a, a+i), (a+i, a)\}$, where a is the smallest positive integer that is not already part of any of the ordered pairs in P_{i-1}. Thus $\mathcal{P}_1 = \{(0,0), (1,2), (2,1)\}$, $\mathcal{P}_2 = \{(0,0), (1,2), (2,1), (3,5), (5,3)\}$, and so on.

Let $\mathcal{P} = \cup_{i=0}^{\infty} \mathcal{P}_i$. The elements of $\mathcal{P}$ are the P-positions of Wythoff's game [4]. Therefore the new positions that are added at stage i, namely $\mathcal{P}_i \setminus \mathcal{P}_{i-1}$, are simply the positions $(\lfloor i\phi \rfloor, \lfloor i\phi^2 \rfloor) = (\lfloor i\phi \rfloor, \lfloor i\phi \rfloor + i)$ and $(\lfloor i\phi^2 \rfloor, \lfloor i\phi \rfloor) = (\lfloor i\phi \rfloor + i, \lfloor i\phi \rfloor)$. The sequence $\lfloor \phi \rfloor, \lfloor 2\phi \rfloor, \lfloor 3\phi \rfloor, \ldots$ is called *the lower Wythoff sequence* and the sequence $\lfloor \phi^2 \rfloor, \lfloor 2\phi^2 \rfloor, \lfloor 3\phi^2 \rfloor, \ldots$ is called *the upper Wythoff sequence.*

3 m-Modular Wythoff

Now we will describe the *m-Modular Wythoff's game.* We have 2 piles with tokens. Just as with Nim and Wythoff, two players take turns and the person who cannot move loses. The terminal position is when there are no tokens left.

The players are allowed to take any positive number of tokens from one of the piles. These are called *Type I* moves. They are the same moves as the moves in Nim [1]. We also allow *Type II* moves, in which a positive number of tokens is removed from both piles given that the number of tokens taken from each pile both have the same remainder modulo m. The moves in the regular Wythoff's game are a subset of the moves in our game.

3.1 Examples

Let us start by analyzing the 2-modular Wythoff's game. As usual in combinatorial game theory we can find P-positions by starting from the terminal positions at the end of the game, so we note that $(0,0)$ is a P-position. The positions that are one move away from the terminal position are N-positions. These are the positions in which exactly one of the coordinates is zero, as well as the positions that have positive coordinates of the same parity. After that we can determine that $(1,2)$ and $(2,1)$ are P-positions. Now we see that all positions other than $(1,2)$ and $(2,1)$ in which one of the coordinates is 1 is an

N-position, and any position with coordinates greater than 1 that are of different parity also must be N-positions. Therefore, all other positions are N-positions and we have found our finite set of P-positions.

The P-positions for m-Modular Wythoff's game for small values of m are represented in Table 1. As one can observe, the P-positions form a subset of the P-positions of Wythoff's game.

m	P-positions
2	(0, 0), (1, 2), (2, 1)
3	(0, 0), (1, 2), (2, 1)
4	(0, 0), (1, 2), (2, 1), (3, 5), (5, 3)
5	(0, 0), (1, 2), (2, 1), (3, 5), (5, 3), (4, 7), (7, 4)

Table 1: P-positions.

With our notation we can describe the P-positions for small m as in Table 2.

m	P-positions
2	$\mathcal{P}_1$
3	$\mathcal{P}_1$
4	$\mathcal{P}_2$
5	$\mathcal{P}_3$

Table 2: P-positions for small values of m.

We will show later that P-positions in the m-Modular Wythoff's game are P-positions in Wythoff's game such that the smaller number in the position does not exceed m.

4 P-positions

First, we make some observations of m-Modular Wythoff.

Lemma 1. *If position* (q_1, q_2) *strictly dominates* (s_1, s_2) *and* $q_1 - q_2 \equiv s_1 - s_2 \pmod{m}$, *then there exists a Type II move from* (q_1, q_2) *to* (s_1, s_2).

Proof. The move from (q_1, q_2) to (s_1, s_2) is $(q_1 - s_1, q_2 - s_2)$. It is a Type II move, because $q_1 - s_1 \equiv q_2 - s_2 \pmod{m}$. □

Lemma 2. *Any two Type II moves performed consecutively are equivalent to a single Type II move.*

Proof. Consider two Type II moves, one in which we remove k_1 tokens from the first pile and k_2 tokens from the second pile where $k_1 \equiv k_2 \pmod{m}$, and another in which we remove k_3 tokens from the first pile and k_4 tokens from the second pile where $k_3 \equiv k_4 \pmod{m}$. Then $k_1 + k_3 \equiv k_2 + k_4 \pmod{m}$, which means that removing $k_1 + k_3$ tokens from the first pile and $k_2 + k_4$ tokens from the second pile is a Type II move as well. □

Before describing P-positions in our game, we introduce more notation. For all positive integers m, let a_m be the number of positive lower Wythoff numbers strictly less than m. Equivalently, we define a_m as the unique integer for which $\lfloor a_m\phi \rfloor < m \leq \lfloor (a_m + 1)\phi \rfloor$. In other words, $a_m = \lfloor m/\phi \rfloor$. Then the sequence $a_1, a_2, a_3, \ldots$ is the sequence A005206 in OEIS [3] (shifted by one).

For visualization, we list here the initial terms of the sequence of lower Wythoff numbers as well as the sequence a_m, both starting at index 1:

- Lower Wythoff: 1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, ...
- a_m: 0, 1, 1, 2, 3, 3, 4, 4, 5, 6, 6, 7, 8, 8, 9, 9, 10, 11, 11, 12, ...

We claim that the set of P-positions of m-Modular Wythoff form the set $\mathcal{P}_{a_m}$, which by definition contains $2a_m + 1 = 2\lfloor m/\phi \rfloor + 1$ elements. Before proving our claim, we make a few simple observations regarding the set of positions $\mathcal{P}_{a_m}$.

Lemma 3. *If $(a, b) \in \mathcal{P}_{a_m}$, where $a < b$, then $a < m$, $b < m\phi$, and $b - a < m/\phi = m(\phi - 1)$.*

For any nonnegative integer $r < m$, there is a position in $\mathcal{P}_{a_m}$ in which r is either the first or second coordinate.

For any positive integer $s < m/\phi$, there are two positions, (q_1, q_2) and (q_2, q_1), in $\mathcal{P}_{a_m}$ such that $q_2 - q_1 = s$. In particular, $q_1 = \lfloor s\phi \rfloor$ and $q_2 = \lfloor s\phi \rfloor + s$.

Proof. Let (a, b) be a position in $\mathcal{P}_{a_m}$ where $a < b$. Then by our definition of $\mathcal{P}_{a_m}$ we know that a is a lower Wythoff number equal to at most $\lfloor a_m\phi \rfloor < m$, so $a < m$.

Likewise, b is an upper Wythoff number equal to at most $\lfloor a_m\phi \rfloor + a_m < m + a_m = m + \lfloor m/\phi \rfloor < m + m/\phi = m\phi$, so $b < m\phi$.

Furthermore, by our definition of $\mathcal{P}_{a_m}$ it is evident that $b - a$ is at most $a_m = \lfloor m/\phi \rfloor$, so $b - a < m/\phi = m(\phi - 1)$.

Let r be a nonnegative integer less than m. Then assume for the sake of contradiction that there does not exist a position in $\mathcal{P}_{a_m}$ in which r is either the first or second coordinate. By definition, the lower Wythoff number within the pair of positions $\mathcal{P}_{a_m+1} \setminus \mathcal{P}_{a_m}$ is equal to $\lfloor (a_m + 1)\phi \rfloor$ and is also the smallest positive integer such that there does not exist a position in $\mathcal{P}_{a_m}$ in which it is either the first or second coordinate. Thus $\lfloor (a_m + 1)\phi \rfloor \leq r$, but also $\lfloor (a_m + 1)\phi \rfloor \geq m$, a contradiction, as $r < m$. Thus there does exist a position in $\mathcal{P}_{a_m}$ in which r is either the first or second coordinate.

Now let s be a positive integer such that $s < m/\phi$. Then $s \leq \lfloor m/\phi \rfloor = a_m$, so by the definition of P_{a_m} the positions $(\lfloor s\phi \rfloor, \lfloor s\phi \rfloor + s)$ and $(\lfloor s\phi \rfloor + s, \lfloor s\phi \rfloor)$ are in P_{a_m}. □

Using the facts from this lemma, we are ready to describe P-positions in m-Modular Wythoff.

Theorem 4. *The P-positions of m-Modular Wythoff form the set $\mathcal{P}_{a_m}$.*

Proof. **First, we show that there are no valid moves between any two elements of $\mathcal{P}_{a_m}$.**

Because these are P-positions in Wythoff's game, there are no valid Wythoff moves between them. In other words, there can be no Type I moves between them, and there can be no Type II moves between them in which the same number of tokens is removed from both piles. We will now consider if other Type II moves are possible between them.

Let (a, b) be a non-terminal position in $\mathcal{P}_{a_m}$ for which $a < b$ from which we remove (k_1, k_2) tokens such that $k_1 \equiv k_2 \pmod{m}$ but $k_1 \neq k_2$. We know that $a < m$, so $k_1 \leq a < m$. As $k_2 \leq b < m\phi < 2m$, it must be the case that $k_2 = k_1 + m$.

The resulting position is then $(a - k_1, b - k_1 - m)$. Assume for the sake of contradiction that $(a-k_1, b-k_1-m) \in \mathcal{P}_{a_m}$. The difference between the number of tokens in the two piles is $m - (b - a)$, which is nonnegative since $m > m/\phi > b-a$. Thus $a-k_1$ is the larger pile. This means that $\lfloor (m-(b-a))\phi^2 \rfloor = a-k_1 < a$ and thus $(m - (b - a))\phi^2 < a$, so $m - (b - a) < a/\phi^2 = a(2 - \phi) < m(2 - \phi)$. We also know that $b - a < m(\phi - 1)$. Summing up these inequalities, we get $m < m$, a contradiction, so $(a - k_1, b - k_1 - m) \notin \mathcal{P}_{a_m}$ as desired.

Second, we show that there exists a move from any position not in $\mathcal{P}_{a_m}$ to a position in $\mathcal{P}_{a_m}$.

Let (q_1, q_2) be a position not in $\mathcal{P}_{a_m}$ where $q_1 \leq q_2$.

Suppose $q_1 < m$. Then there exists a position in $\mathcal{P}_{a_m}$ for which q_1 is one of the pile sizes. If q_1 is an upper Wythoff number $\lfloor i\phi \rfloor + i$ for some integer i, then there exists a Type I move from (q_1, q_2) to $(\lfloor i\phi \rfloor + i, \lfloor i\phi \rfloor) \in \mathcal{P}_{a_m}$ since $q_2 \geq q_1 \geq \lfloor i\phi \rfloor$.

Otherwise, q_1 is a lower Wythoff number, so $q_1 = \lfloor i\phi \rfloor$ for some integer $i \leq a_m$. First, consider the case in which $q_2 > q_1\phi = \lfloor i\phi \rfloor\phi$. Then because $i\phi - (i - 1)\phi = \phi > 1$, it follows that $\lfloor i\phi \rfloor > (i - 1)\phi$. Then $\lfloor i\phi \rfloor/\phi > i - 1$, so $\lfloor i\phi \rfloor(\phi - 1) > i - 1$. Thus we have that $\lfloor i\phi \rfloor\phi > \lfloor i\phi \rfloor + i - 1 = \lfloor i\phi^2 \rfloor - 1$. Therefore $q_2 > \lfloor i\phi \rfloor\phi > \lfloor i\phi^2 \rfloor - 1$, so $q_2 \geq \lfloor i\phi^2 \rfloor$. Thus there exists a Type I move from (q_1, q_2) to $(\lfloor i\phi \rfloor, \lfloor i\phi^2 \rfloor) \in \mathcal{P}_{a_m}$.

Now consider the case in which $q_2 \leq q_1\phi$. Then $q_2 - q_1 \leq q_1(\phi - 1) = q_1/\phi = \lfloor i\phi \rfloor/\phi < i \leq a_m$. Thus $(\lfloor (q_2 - q_1)\phi \rfloor, \lfloor (q_2 - q_1)\phi \rfloor + (q_2 - q_1)) \in \mathcal{P}_{a_m}$, and there exists a Type II move from (q_1, q_2) to that position, since the two positions have the same difference between their two piles and $\lfloor (q_2 - q_1)\phi \rfloor < \lfloor i\phi \rfloor = q_1$.

Suppose now $q_1 \geq m$. Then we can write $q_2 = q_1 + mx + r$, where $0 \leq r < m$ is the remainder when $q_2 - q_1$ is divided by m.

If $r < m/\phi$, then there exists $(s_1, s_2) \in \mathcal{P}_{a_m}$ such that $s_2 - s_1 = r$ and (q_1, q_2) dominates it, as $q_1 \geq m > s_1$. That means there exists a Type II move from (q_1, q_2) to (s_1, s_2).

Otherwise $r > m/\phi = m(\phi - 1)$. The remainder when $q_1 - q_2$ is divided by m is $m - r < m(2 - \phi) < m(\phi - 1)$. Thus there exists a position $(s_1, s_2) \in \mathcal{P}_{a_m}$ for which $s_1 - s_2 = m - r$, so $q_1 - q_2 \equiv s_1 - s_2 \pmod{m}$. Moreover, $s_2 = \lfloor (m-r)\phi \rfloor < m(2 - \phi)\phi < m \leq q_1$, and $s_1 = \lfloor (m - r)\phi^2 \rfloor < m(2 - \phi)\phi^2 = m \leq q_1 \leq q_2$.

Thus (q_1, q_2) dominates (s_1, s_2), and there exists a Type II move from (q_1, q_2) to (s_1, s_2). □

Corollary 5. *The number of P-positions of m-Modular Wythoff is finite and equal to* $2\lfloor m/\phi \rfloor + 1$.

Proof. By definition $\mathcal{P}_{a_m}$ contains $2a_m + 1 = 2\lfloor m/\phi \rfloor + 1$ elements. □

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