Structural Properties of Index Coding Capacity

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***Abstract*—The index coding capacity is investigated through its structural properties. First, the capacity is characterized in three new multiletter expressions involving the clique number, Shannon capacity, and Lovász theta function of the confusion graph, the latter notion introduced by Alon, Hassidim, Lubetzky, Stav, and Weinstein. The main idea is that every confusion graph can be decomposed into a small number of perfect graphs. The clique-number characterization is then utilized to show that the capacity is multiplicative under the lexicographic product of side information graphs, establishing the converse to an earlier result by Blasiak, Kleinberg, and Lubetzky. Second, sufficient and necessary conditions on the criticality of an index coding instance, namely, whether side information can be removed without reducing the capacity, are established based on the notion of unicycle, providing a partial answer to the question first raised by Tahmasbi, Shahrasbi, and Gohari. The necessary condition, along with other existing conditions, can be used to eliminate noncritical instances that do not need to be investigated. As an application of the established multiplicativity and criticality, only 10,634 (0.69%) out of 1,540,944 nonisomorphic six-message index coding instances are identified for further investigation, among which the capacity is still unknown for 119 instances.**

I. Introduction

The index coding problem is a canonical problem in network information theory in which a server has a tuple of n messages $x^n = (x_1, \ldots, x_n)$, $x_j \in \{0,1\}^{t_j}$, and is connected to n receivers via a noiseless broadcast channel. Suppose that receiver $j \in [n] := \{1, 2, \ldots, n\}$ is interested in message x_j and has a set of other messages $x(A_j) := (x_i, i \in A_j), A_j \subseteq [n] \setminus \{j\}$ as side information. Assuming that the server knows side information sets $A_1, \ldots, A_n$, one wishes to characterize the minimum amount of information the server needs to broadcast and to find the optimal coding scheme that achieves this minimum.

More precisely, a $(t_1, \ldots, t_n, r)$ *index code* is defined by

- an encoder $\phi : \prod_{i=1}^n \{0,1\}^{t_i} \to \{0,1\}^r$ that maps n-tuple of messages x^n to an r-bit index and
- n decoders $\psi_j : \{0,1\}^r \times \prod_{k \in A_j} \{0,1\}^{t_k} \to \{0,1\}^{t_j}$ that maps the received index $\phi(x^n)$ and the side information $x(A_j)$ back to x_j for $j \in [n]$.

Thus, for every $x^n \in \prod_{i=1}^n \{0,1\}^{t_i}$,

$$\psi_j(\phi(x^n), x(A_j)) = x_j, \quad j \in [n].$$

A $(t, \ldots, t, r)$ code is written as a (t, r) code. A rate tuple $(R_1, \ldots, R_n)$ is said to be *achievable* for the index coding problem if there exists a $(t_1, \ldots, t_n, r)$ index code such that

$$R_j \leq \frac{t_j}{r}, \quad j \in [n].$$

The *capacity region* $\mathscr{C}$ of the index coding problem is defined as the closure of the set of achievable rate tuples. The *symmetric capacity* (or the *capacity* in short) of the index coding problem is defined as

$$C_{\text{sym}} = \max\{R \colon (R, \ldots, R) \in \mathscr{C}\},$$

and its reciprocal $\beta = 1/C_{\text{sym}}$ is referred to as the *broadcast rate*, which can be equivalently defined as

$$\beta = \inf_t \inf_{(t,r) \text{ codes}} \frac{r}{t} = \lim_{t\to\infty} \inf_{(t,r) \text{ codes}} \frac{r}{t}, \tag{1}$$

where the equality follows by Fekete's lemma [1] and the subadditivity

$$\inf_{(t_1+t_2,r) \text{ codes}} r \leq \inf_{(t_1,r_1) \text{ codes}} r_1 + \inf_{(t_2,r_2) \text{ codes}} r_2.$$

The goal is to characterize the capacity region or the symmetric capacity for the general index coding problem and to determine the coding scheme that can achieve it.

Any instance of the index coding problem is fully determined by the side information sets $A_1, \ldots, A_n$, and is represented compactly as $(j|A_j), j \in [n]$. For example, the 3-message index coding problem with $A_1 = \{2,3\}, A_2 = \{1\}$, and $A_3 = \{1,2\}$ is represented as

$$(1|2,3), (2|1), (3|1,2).$$

The problem can be equivalently specified by a directed graph with n vertices, commonly referred to as the *side information graph*. Each vertex of the side information graph $G = (V, E)$ corresponds to a receiver (and its associated message) and there is a directed edge $i \to j$ if and only if (iff) receiver j knows message i as side information, i.e., $i \in A_j$ (see Fig. 1). Throughout the paper, we identify an instance of the index coding problem with its side information graph G and often write "index coding problem G." We also denote the broadcast rate and the capacity region of problem G with $\beta(G)$ and $\mathscr{C}(G)$ respectively.

The problem of broadcasting to multiple receivers with different side information traces back to the work by Celebiler and Stette [2], Wyner, Wolf, and Willems [3], [4], Yeung [5], and Birk and Kol [6], [7]. The current problem formulation is due to the last. This problem has been shown to be closely related to many other important problems in network information theory such as network coding [8]–[10], locally

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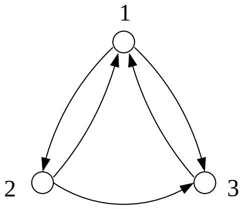


Fig. 1. The graph representation for the index coding problem with $A_1 = \{2, 3\}$, $A_2 = \{1\}$, and $A_3 = \{1, 2\}$.

recoverable distributed storage [11]–[13], guessing games on directed graphs [8], [13], [14], and zero-error capacity of channels [15]. In addition, index coding has its own applications in diverse areas ranging from satellite communication [2]–[7] and multimedia distribution [16] to interference management [17] and coded caching [18], [19]. Due to this significance, the index coding problem has been broadly studied over the past two decades in several disciplines including graph theory, coding theory, and information theory, and various bounds have been established on the capacity region and the broadcast rate. Despite all these efforts, however, the problem is still open in general and the capacity in a computable (single-letter) expression is known only for a handful of special cases (see, for example, [14], [17], [20]–[31]).

Deviating from the common approach of finding the capacity by establishing tight upper and lower bounds, we take a more direct attack at the capacity itself by its structural properties using several graph-theoretic tools. The main contributions are summarized as follows:

- **A new multiletter characterization of the capacity (Theorem 2).** Paralleling the multiletter characterization of the capacity (broadcast rate) via the chromatic number of the confusion graph [32], we establish a multiletter characterization via the *clique number* of the confusion graph. As a corollary, we establish a *nonasymptotic* upper bound on the broadcast rate via the Lovász theta function of the confusion graph that can be computed more efficiently than the existing upper bound using the chromatic number.
- **Multiplicativity of the capacity under the lexicographic product (Theorem 5).** As another corollary of the aforementioned clique-number characterization, we show that if the side information graph is the lexicographic product of two graphs, the capacity is the product of the capacities of the two component graphs, completing an earlier result by Blasiak, Kleinberg, and Lubetzky [33]).
- **Conditions on the criticality of an index coding instance (Theorem 6 and Proposition 10).** Providing a partial answer to the question raised by Tahmasbi, Shahrasbi, and Gohari [34], we establish conditions under which the removal of an edge reduces the capacity. Both sufficient and necessary conditions are based on the notion of unicycle that is closely related to the maximum acyclic induced subgraph bound on the capacity.

The rest of the paper is organized as follows. Sections II and III review graph-theoretic preliminaries and some of the previously known bounds on the capacity, respectively. In Section IV, we introduce the notion of confusion graph associated with a given index coding problem and establish several properties including a tight bound on the chromatic number of a confusion graph in terms of its clique number. In Section V, we characterize the broadcast rate of a general index coding problem via asymptotic expressions involving the clique number, Shannon capacity, and Lovász theta function of the confusion graph. Nonasymptotic upper bounds on the broadcast rate are also established in terms of the Shannon capacity and Lovász theta function of the confusion graph. Based on the clique-number characterization, we prove in Section VI that the broadcast rate is multiplicative under the lexicographic product of side information graphs. In Section VII, we investigate the criticality problem and present sufficient and necessary conditions based on the notion of unicycle. Section VIII concludes the paper with an application of the established structural properties in computing the capacity for index coding problems with six messages.

II. Mathematical Preliminaries

Throughout the paper, a graph $G = (V, E)$ (without a qualifier) means a directed, finite, and simple graph, where $V = V(G)$ is the set of vertices (nodes) and $E = E(G) \subseteq V \times V$ is the set of directed edges. A graph $G = (V, E)$ is said to be *unidirectional* if $(i, j) \in E$ implies $(j, i) \notin E$. Similarly, G is said to be *bidirectional* if $(i, j) \in E$ implies $(j, i) \in E$. Given G, its associated undirected graph $U = U(G)$ is defined by identifying $V(U) = V(G)$ and $E(U) = \{\{i, j\} \colon (i, j) \in E(G) \text{ or } (j, i) \in E(G)\}$. A bidirectional graph G is sometimes identified with its undirected graph. The *complement* $\bar{G}$ of the graph G is defined by $V(\bar{G}) = V(G)$ and $(i, j) \in E(\bar{G})$ iff $(i, j) \notin E(G)$. For any $S \subseteq V(G)$, $G|_S$ denotes the subgraph induced by S, i.e., $V(G|_S) = S$ and $E(G|_S) = \{(i, j) \in E \colon i, j \in S\}$.

An *independent set* I of a graph G is a set of vertices with no edge among them. The *independence number* $\alpha(G)$ is the size of the largest independent set of the graph G. A *clique* K of a graph G is a set of vertices such that there is a (directed) edge from every vertex in K to every other vertex in K. Thus, K is a clique of G iff it is an independent set of $\bar{G}$. The *clique number* $\omega(G)$ is the size of the largest clique of the graph G. It is easy to see that

$$\omega(G) = \alpha(\bar{G}) \tag{2}$$

for any directed or undirected graph G. A *Hamiltonian cycle* of a graph is a cycle that visits each vertex exactly once. A graph possessing a Hamiltonian cycle is said to be *Hamiltonian*.

A. Chromatic Number

A (vertex) coloring of an undirected (finite simple) graph U is a mapping that assigns a color to each vertex such that no two adjacent vertices share the same color. The *chromatic number* $\chi(U)$ is the minimum number of colors such that a coloring of the graph exists. More generally, a b-fold coloring assigns a set of b colors to each vertex such that no two adjacent vertices share the same color. The b-fold chromatic number $\chi^{(b)}(U)$ is the minimum number of colors such that a b-fold coloring exists. The *fractional chromatic number* of the graph is defined as

$$\chi_f(U) = \lim_{b\to\infty} \frac{\chi^{(b)}(U)}{b} = \inf_b \frac{\chi^{(b)}(U)}{b},$$

where the limit exists since $\chi^{(b)}(U)$ is subadditive. Consequently,

$$\chi_f(U) \le \chi(U). \tag{3}$$

Let $\mathcal{I}$ be the collection of all independent sets in U. The chromatic number and the fractional chromatic number are also characterized via the following optimization problem

$$\begin{aligned} \text{minimize} \ & \sum_{S\in\mathcal{I}} \rho_S \\ \text{subject to} \ & \sum_{S\in\mathcal{I}:\, j\in S} \rho_S \ge 1, \quad j \in V. \end{aligned}$$

When the optimization variables ρ_S, $S \in \mathcal{I}$, take integer values in $\{0,1\}$, then the (integral) solution is the chromatic number. If this constraint is relaxed and $\rho_S \in [0,1]$, then the (rational) solution is the fractional chromatic number [35]. The (fractional) chromatic number can be related to the independence and clique numbers.

Lemma 1 (Scheinerman and Ullman [35])**.** *For any undirected graph U with n vertices,*

$$\frac{n}{\alpha(U)} \le \chi_f(U).$$

Lemma 2. *For any graph U we have*

$$\omega(U) \le \chi_f(U) \le \chi(U).$$

An undirected graph $U = (V,E)$ is said to be *perfect* if for every induced subgraph $U|_S$, $S \subseteq V$, the clique number equals the chromatic number, i.e., $\omega(U|_S) = \chi(U|_S)$. Perfect graphs can be characterized as follows.

Proposition 1 (Chudnovsky, Robertson, Seymour, and Thomas [36])**.** *A graph U is perfect iff no induced subgraph of U is an odd cycle of length at least five (odd hole) or the complement of one (odd antihole).*

Let $U = (V,E)$ be an undirected graph with $V = [n]$. For each clique K of U, the *incidence vector* is an n-dimensional vector whose jth component is equal to 1 if $j \in K$ and 0 otherwise. The *clique polytope* of U is defined as

$$\begin{aligned} P_{\mathrm{K}}(U) = \{x \in \mathbb{R}^n_{\ge 0} \colon & x \text{ is a convex combination of} \\ & \text{incidence vectors of cliques of } U\}. \end{aligned} \tag{4}$$

Another (convex) polytope associated with U is defined as

$$P(U) = \{x \in \mathbb{R}^n_{\ge 0} \colon \sum_{i\in I} x_i \le 1 \text{ for all independent sets } I\}. \tag{5}$$

Since every incidence vector x of a clique satisfies $\sum_{i\in I} x_i \le 1$ for an independent set I, $P_{\mathrm{K}}(U) \subseteq P(U)$ for every U. Lovász's perfect graph theorem states that equality holds iff U is perfect.

Lemma 3 (Lovász [37])**.** *For any graph U the following statements are equivalent:*

- *U is perfect.*
- *$P_{\mathrm{K}}(U) = P(U)$.*
- *$\bar{U}$ is perfect.*

We now state a result on chromatic numbers that will be useful later. The chromatic number of a graph can be upper bounded by decomposing it into smaller graphs. The following decomposition result will be proved in Appendix A.

Lemma 4. *Let $U_1 = (V, E_1)$ and $U_2 = (V, E_2)$ be two undirected graphs on the set of vertices V. Consider the graph $U = (V, E_1 \cup E_2)$ defined on the same vertex set V in which each edge either belongs to E_1 or E_2. Then*

$$\chi(U) \le \chi(U_1) + \chi(U_2).$$

B. Graph Products

Generally speaking, a graph product is a binary operation on two graphs that produces a graph on the Cartesian product of the original vertex sets with the edge set constructed from the original edge sets according to certain rules. In the following, $v_1 \sim v_2$ denotes that there exists an edge between v_1 and v_2.

Given two undirected graphs U_1 and U_2, the *disjunctive product* $U = U_1 \vee U_2$ [35], [38] is defined as $V(U) = V(U_1) \times V(U_2)$ and $(u_1,u_2) \sim (v_1,v_2)$ iff

$$u_1 \sim v_1 \quad \text{or} \quad u_2 \sim v_2.$$

Throughout the paper, $U^{\vee k}$ denotes the disjunctive product of k copies of U.

Given two undirected graphs U_1 and U_2, the *strong product* $U = U_1 \boxtimes U_2$ [39] is defined as $V(U) = V(U_1) \times V(U_2)$ and $(u_1,u_2) \sim (v_1,v_2)$ iff

$$\begin{aligned} & (u_1 = v_1 \text{ and } u_2 \sim v_2), \\ \text{or} \quad & (u_1 \sim v_1 \text{ and } u_2 = v_2), \\ \text{or} \quad & (u_1 \sim v_1 \text{ and } u_2 \sim v_2). \end{aligned}$$

Throughout the paper, $U^{\boxtimes k}$ denotes the strong product of k copies of U. The following lemma elucidates the relation between the disjunctive product and the strong product.

Lemma 5. *For any two undirected graphs U_1 and U_2*

$$\overline{U_1 \vee U_2} = \overline{U}_1 \boxtimes \overline{U}_2.$$

Given two graphs G_1 and G_2, the *lexicographic product* $G = G_1 \circ G_2$ [39] is defined as $V(G) = V(G_1) \times V(G_2)$ and $((u_1,u_2),(v_1,v_2)) \in E(G)$ iff

$$(u_1, v_1) \in E(G_1) \quad \text{or} \quad (u_1 = v_1 \text{ and } (u_2, v_2) \in E(G_2)).$$

The lexicographic product $G_1 \circ G_2$ can be thought of as replacing each vertex $i \in V(G_1)$ with a copy of G_2. Therefore, the edges among the vertices of each copy of G_2 remain the same as in G_2 and there exists a directed edge from every vertex in copy i of G_2 to every vertex in copy j of G_2 iff $(i,j) \in E(G_1)$ (see Fig. 2 for an example).

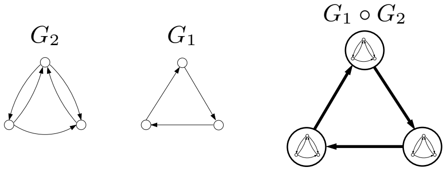


Fig. 2. Graphs G_1 and G_2 and their lexicographic product $G_1 \circ G_2$. The bold arrows indicate that there is an edge from every vertex in the circle attached to the tail of the arrow to every vertex in the circle attached to the head of the arrow.

The lexicographic product can be generalized as follows. Let G_0 be a graph with m vertices and $G_1, \dots, G_m$ be m graphs with $n_1, \dots, n_m$ vertices respectively. The *generalized lexicographic product* $G_0 \circ (G_1, \dots G_m)$ is defined to be a graph on $n_1 + \cdots + n_m$ vertices in which vertex $i \in V(G_0)$ is replaced with G_i, i.e., the edges among the vertices of G_i remain the same as before and there is a directed edge from every vertex of G_i to every vertex of G_j iff $(i, j) \in E(G_0)$.

C. Shannon Capacity of a Graph and Lovász Function

Consider a graph U whose vertices represent input symbols of a noisy channel and two vertices are connected iff the corresponding channel inputs are confusable as they may result in the same channel output. The goal is to find the zero-error capacity of the channel represented by the graph U. If we are limited to use the channel only once, then we can send up to $\lfloor \log(\alpha(U)) \rfloor$ bits without an error. However, if we are allowed to use the channel t times, then we can construct the following graph to capture the confusabilities. Assign each t-tuple of the input symbols to a vertex and the vertices for two tuples x^t and z^t connect iff for every i, $x_i = z_i$ or $x_i \sim z_i$ in U. We can easily check that the resulting graph is the strong product $U^{\boxtimes t}$. Thus, by using the channel t times, we can send $\lfloor \log(\alpha(U^{\boxtimes t})) \rfloor$ bits without an error. Based on this observation [40], the *Shannon capacity* of a graph U is defined as

$$\Theta(U) = \sup_t \sqrt[t]{\alpha(U^{\boxtimes t})} = \lim_{t \to \infty} \sqrt[t]{\alpha(U^{\boxtimes t})}. \tag{6}$$

In other words, $\log(\Theta)$ indicates the number of bits per input symbol that can be sent through the channel without error. By definition,

$$\alpha(U) \leq \Theta(U). \tag{7}$$

Shannon [40] showed that for perfect graphs $\alpha(U) = \Theta(U)$. The equality does not hold in general, however. In fact, computing the Shannon capacity of a general graph is a very hard problem. Lovász [41] derived an upper bound on the Shannon capacity referred to as the Lovász theta function, which is easily computable and results in determining the Shannon capacity of some graphs. Before defining the Lovász theta function, we need the following definition. An *orthonormal representation* of an undirected graph U with n vertices is a set of unit vectors $(v_1, \dots, v_n)$ such that if i and j are nonadjacent vertices of U, then v_i and v_j are orthogonal, i.e., $v_i^T v_j = 0$. For example, a set of n pairwise orthogonal unit vectors is an orthonormal representation of any undirected n-node graph. The *value* of an orthonormal representation is defined as

$$\min_{c: \|c\|=1} \max_{i \in [n]} \frac{1}{\left(c^T v_i\right)^2}.$$

The unit vector c attaining the minimum is referred to as the *handle* of the representation. The *Lovász theta function* of U, denoted as $\vartheta(U)$, is defined to be the minimum value over all orthonormal representations of U. A representation is said to be optimal if it attains this minimum.

Lemma 6 (Lovász [41])**.** *For any undirected graph U,*

$$\Theta(U) \leq \vartheta(U).$$

By (2), (7), Lemma 6, and Theorem 10 in [41], the Lovász theta function is sandwiched by other graph-theoretic quantities that are NP-hard to compute.

Lemma 7. *For any undirected graph U,*

$$\omega(U) \leq \vartheta(\bar{U}) \leq \chi(U).$$

However, the Lovász theta function $\vartheta(U)$ is polynomially computable in $|V(U)|$ [42].

III. Bounds on the Capacity

The simplest approach to index coding is a coding scheme by Birk and Kol [6] that partitions the side information graph G by cliques and transmits the binary sums (parities) of all the messages in each clique.

Proposition 2 (Clique covering bound)**.** *Let $\beta_{\mathrm{CC}}(G)$ be the minimum number of cliques that partition G, or equivalently, the chromatic number of $U(\bar{G})$, which is the solution to the integer program*

$$\begin{aligned} \textit{minimize} \quad & \sum_{S \in \mathcal{K}} \rho_S \\ \textit{subject to} \quad & \sum_{S \in \mathcal{K}: j \in S} \rho_S \geq 1, \quad j \in V(G), \\ & \rho_S \in \{0, 1\}, \quad S \in \mathcal{K}, \end{aligned} \tag{8}$$

where $\mathcal{K}$ is the collection of all cliques in G. Then for any index coding problem G, $\beta(G) \leq \beta_{\mathrm{CC}}(G)$.

This bound, which is achieved by *time division* over a clique partition, has been extended in several directions. First, Birk and Kol [6] showed that one can use an MDS code over a finite field and perform time division over arbitrary subgraphs (partial cliques) instead of cliques. The number of parity symbols needed for a subgraph H is characterized by the difference $\kappa(H)$ between the number of vertices in H and the minimum indegree within H.

Proposition 3 (Partial clique covering bound)**.** *If $G_1, \dots, G_m$ partition G, then the optimal broadcast rate is upper bounded by*

$$\beta_{\mathrm{PC}}(G_1, \dots, G_m) = \sum_{i=1}^{m} \kappa(G_i) \tag{9}$$

and thus by

$$\beta_{\mathrm{PC}} = \min_{G_1, \dots, G_m} \beta_{\mathrm{PC}}(G_1, \dots, G_m),$$

where the minimum is over all partitions.

Remark 1. If the graph G with n vertices is Hamiltonian, then the minimum indegree is at least one and thus $\beta_{\mathrm{PC}} \le n-1$, or equivalently, the symmetric rate $(\frac{1}{n-1}, \ldots, \frac{1}{n-1})$ is achievable for problem G.

By the standard time-sharing argument, Blasiak, Kleinberg, and Lubetzky [22] extended the clique covering bound to the *fractional clique covering bound*, which is equivalent to the fractional chromatic number of $U(\bar{G})$, namely, the solution to the linear program obtained by relaxing the integer constraint $\rho_S \in \{0,1\}$ in (8) to $\rho_S \in [0,1]$.

Remark 2. The integral, partial, and fractional clique covering bounds can be readily extended to the corresponding *inner* bounds on the capacity region. For example, by fractional clique covering, a rate tuple $(R_1, \ldots, R_n)$ is achievable for the index coding problem $(j \,|\, A_j), j \in [n]$, if there exists $(\rho_S \in [0,1], S \in \mathcal{K})$ such that

$$\begin{aligned} \sum_{S \in \mathcal{K}} \rho_S &\le 1, \\ \sum_{S \in \mathcal{K}: j \in S} \rho_S &\ge R_j, \quad j \in V(G). \end{aligned} \tag{10}$$

Tighter bounds can be found in [25], [26], [28], [43]. In this paper, we only need the simpler integral, partial, and fractional clique covering bounds.

As for bounding the broadcast rate from below, Bar-Yossef, Birk, Jayram, and Kol [20] proposed the following.

Proposition 4 (Maximum acyclic induced subgraph (MAIS) bound)**.** *For any index coding problem G*

$$\beta_{\mathrm{MAIS}}(G) := \max_{S \subseteq V(G) : G|_S \text{ is acyclic}} |S| \le \beta(G).$$

Remark 3. Since every independent set is acyclic, Proposition 4 implies that for any G, $\alpha(G) \le \beta(G)$.

Remark 4. When G is bidirectional (undirected) and perfect we have $\omega(\bar{G}) = \alpha(G) = \chi(\bar{G})$. Hence, the upper bound of Proposition 2 matches the lower bound of Remark 3 and the broadcast rate is known [20].

Remark 5. The MAIS bound can be generalized to an outer bound $\mathscr{R}_{\mathrm{MAIS}}$ on the capacity region [25] as follows. If a rate tuple $(R_1, \ldots, R_n)$ is achievable for index coding problem G, then

$$\sum_{j \in S} R_j \le 1 \tag{11}$$

for all S such that $G|_S$ is acyclic. This bound is a special case of the polymatroidal outer bound [33], [44], [45].

Remark 6. When G is bidirectional (undirected), the polytope associated with G in (5) is equivalent to the MAIS outer bound in (11). It is also easy to see that the rate tuple given by each incidence vector of cliques in G is achievable by clique covering and thus the polytope associated with G in (4) is achievable by fractional clique covering. Therefore, by Lemma 3, if G is bidirectional and perfect, then the capacity region is equal to the MAIS outer bound in (11), which is achieved by fractional clique covering [14].

IV. Confusion Graphs

The notion of confusion graph for the index coding problem was originally introduced by Alon, Hassidim, Lubetzky, Stav, and Weinstein [32]. In the context of guessing games, an equivalent notion was introduced independently by Gadouleau and Riis [46]. Consider a directed graph $G = (V, E)$ with $V = [n]$. Let $A_j = \{i \in V \colon (i,j) \in E\}$, $j \in [n]$, and let $\mathbf{t} = (t_1, \ldots, t_n)$ be a length-n integer tuple. Two q-ary n-tuples $x^n, z^n \in \prod_{i=1}^n \{0, \ldots, q-1\}^{t_i}$ are said to be *confusable at position $l \in [t_j]$ of node $j \in [n]$* if $x_{jl} \ne z_{jl}$ and $x_i = z_i$ for all $i \in A_j$.

Given a directed graph G and a length-n integer tuple $\mathbf{t} = (t_1, \ldots, t_n)$, the *confusion graph $\Gamma_{\mathbf{t}}^{(jl)}(G)$ at position l of node j* is an undirected graph with $\prod_{i=1}^n q^{t_i}$ vertices such that every vertex corresponds to a q-ary tuple x^n and two vertices are connected iff the corresponding q-ary tuples are confusable at position l of receiver j.

Aggregating over all positions, we say that $x^n, z^n \in \prod_{i=1}^n \{0, \ldots, q-1\}^{t_i}$ are *confusable* if they are confusable at some position l of some node j. The *confusion graph $\Gamma_{\mathbf{t}}(G)$* is defined as before based on confusion between each pair of vertices, or equivalently,

$$E\left(\Gamma_{\mathbf{t}}(G)\right) = \bigcup_{j=1}^{n} \bigcup_{l=1}^{t_j} E(\Gamma_{\mathbf{t}}^{(jl)}(G)). \tag{12}$$

If $\mathbf{t} = (t, \ldots, t)$, then $\Gamma_{\mathbf{t}}(G)$ is simply denoted by $\Gamma_t(G)$. Fig. 3 shows $\Gamma_{\mathbf{t}}^{(11)}(G)$, $\Gamma_{\mathbf{t}}^{(21)}(G)$, and $\Gamma_{\mathbf{t}}^{(31)}(G)$ as well as $\Gamma_{\mathbf{t}}(G)$ corresponding to $\mathbf{t} = (1,1,1)$ for G in Fig. 1.

By Lemma 4 and (12), the chromatic number of $\Gamma_t(G)$ can be upper bounded by those of its components.

Proposition 5. $\chi\left(\Gamma_{\mathbf{t}}(G)\right) \le \sum_{j=1}^{n} \sum_{l=1}^{t_j} \chi\left(\Gamma_{\mathbf{t}}^{(jl)}(G)\right)$.

Each component confusion graph $\Gamma_{\mathbf{t}}^{(jl)}(G)$ has the following properties.

Lemma 8. *$\Gamma_{\mathbf{t}}^{(jl)}(G)$ does not have any chordless cycle of length greater than four.*

Lemma 9. *The complement of $\Gamma_{\mathbf{t}}^{(jl)}(G)$ does not have any chordless cycle of length greater than four.*

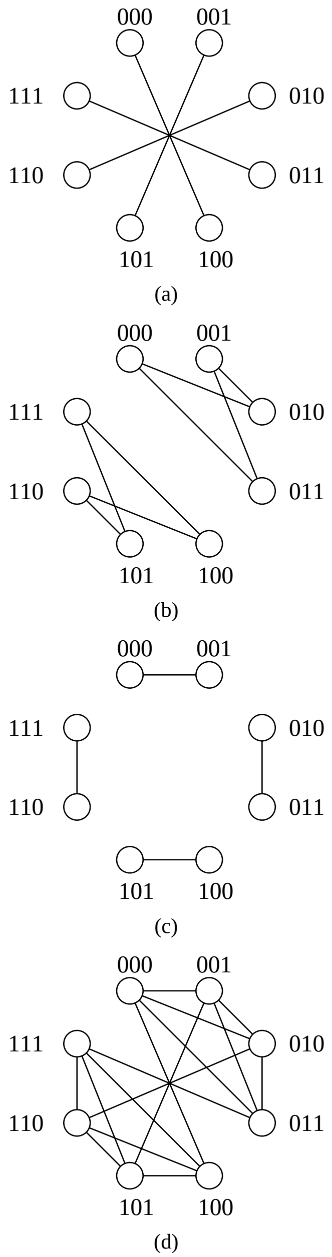
The proofs of the lemmas are given in Appendices B and C. By Proposition 1, Lemma 8, and Lemma 9, the following is immediate.

Proposition 6. *$\Gamma_{\mathbf{t}}^{(jl)}(G)$ is perfect.*

As the main contribution of this section, we now establish an upper bound on the chromatic number of a confusion graph in terms of its clique number.

Theorem 1. *Given a directed graph G, a length-n integer tuple $\mathbf{t} = (t_1, \ldots, t_n)$, and a positive integer q, the confusion graph $\Gamma_{\mathbf{t}}(G)$ satisfies*

$$\chi\left(\Gamma_{\mathbf{t}}(G)\right) \le \left(\sum_{j=1}^{n} t_j\right) \omega\left(\Gamma_{\mathbf{t}}(G)\right). \tag{13}$$



(a) (b) (c) (d)

Fig. 3. Confusion graphs for the directed graph G shown in Fig. 1 corresponding to the integer tuple $\mathbf{t} = (t_1, t_2, t_3) = (1, 1, 1)$. (a) $\Gamma_{\mathbf{t}}^{(11)}(G)$. (b) $\Gamma_{\mathbf{t}}^{(21)}(G)$. (c) $\Gamma_{\mathbf{t}}^{(31)}(G)$. (d) $\Gamma_{\mathbf{t}}(G)$.

Proof: Consider

$$\begin{aligned}\chi(\Gamma_{\mathbf{t}}(G)) &\le \sum_{j=1}^{n}\sum_{l=1}^{t_j} \chi(\Gamma_{\mathbf{t}}^{(jl)}(G)) &(14)\\ &= \sum_{j=1}^{n}\sum_{l=1}^{t_j} \omega(\Gamma_{\mathbf{t}}^{(jl)}(G)) &(15)\\ &\le \sum_{j=1}^{n}\sum_{l=1}^{t_j} \omega(\Gamma_{\mathbf{t}}(G)) &(16)\\ &= \sum_{j=1}^{n} t_j \; \omega(\Gamma_{\mathbf{t}}(G)),\end{aligned}$$

where (14) follows by Proposition 5, (15) follows by Proposition 6, and (16) follows by (12). □

V. Multiletter Characterizations of the Capacity

Consider an index coding problem G. Using the notion of confusion graph introduced in Section IV, Alon, Hassidim, Lubetzky, Stav, and Weinstein [32] showed that

$$\frac{\beta_t(G)}{t} := \inf_{(t,r) \text{ codes}} \frac{r}{t} = \frac{1}{t}\lceil \log(\chi(\Gamma_t(G))) \rceil. \qquad (17)$$

To prove this, consider a coloring of the vertices of the confusion graph $\Gamma = \Gamma_t(G)$ with $\chi(\Gamma)$ colors. This partitions the vertices of Γ into $\chi(\Gamma)$ independent sets. By the definition of the confusion graph, no two message tuples in each independent set are confusable and therefore assigning a unique index to each independent set yields a valid index code. The total number of codewords of this index code is $\chi(\Gamma)$, which requires $r = \lceil \log(\chi(\Gamma)) \rceil$ bits to be broadcast. Hence, $\beta_t(G) \le \lceil \log(\chi(\Gamma_t(G))) \rceil$. Conversely, consider any (t, r) index code that assigns (at most) 2^r distinct indices to message tuples. By definition, all the message tuples mapped to an index form an independent set of the confusion graph $\Gamma = \Gamma_t(G)$. Moreover, every message tuple is mapped to some index so that these independent sets partition $V(\Gamma)$. Thus, $\chi(\Gamma) \le 2^r$, or equivalently, $r \ge \lceil \log(\chi(\Gamma)) \rceil$, and hence $\beta_t(G) \ge \lceil \log(\chi(\Gamma_t(G))) \rceil$.

Based on (17), Alon, Hassidim, Lubetzky, Stav, and Weinstein [32] established the following upper bound on the broadcast rate

$$\beta(G) \le \frac{1}{t}\lceil \log\left(\chi\left(\Gamma_t(G)\right)\right) \rceil, \qquad (18)$$

for every positive integer t, and established a multiletter characterization of the broadcast rate as

$$\beta(G) = \lim_{t\to\infty} \frac{1}{t} \log(\chi(\Gamma_t(G))). \qquad (19)$$

In our earlier work [47], this characterization was strengthened using the fractional chromatic number as

$$\beta(G) = \lim_{t\to\infty} \frac{1}{t} \log(\chi_f(\Gamma_t(G))). \qquad (20)$$

We now further strengthen this result and characterize the broadcast rate in terms of the clique number of the confusion graph.

Theorem 2. *For any side information graph G,*

$$\beta(G) = \lim_{t\to\infty} \frac{1}{t} \log(\omega(\Gamma_t(G))). \qquad (21)$$

Proof: By setting $\mathbf{t} = (t, \ldots, t)$ in Theorem 1 and recalling Lemma 2, we have

$$\omega(\Gamma_t(G)) \le \chi(\Gamma_t(G)) \le nt \cdot \omega(\Gamma_t(G)). \qquad (22)$$

Hence,

$$\lim_{t\to\infty} \frac{1}{t} \log(\chi(\Gamma_t(G))) = \lim_{t\to\infty} \frac{1}{t} \log(\omega(\Gamma_t(G))), \qquad (23)$$

which, combined with (19), completes the proof. □

Note that since $\omega(\Gamma) \le \chi_f(\Gamma) \le \chi(\Gamma)$ for any graph Γ, Equation (20) can be derived as a corollary of Theorem 2.

Combining (7), Lemma 6, and Lemma 7, we have for any positive integer t

$$\omega(\Gamma_t) = \alpha(\overline{\Gamma}_t) \leq \Theta(\overline{\Gamma}_t) \leq \vartheta(\overline{\Gamma}_t) \leq \chi(\Gamma_t). \tag{24}$$

Thus, we can characterize the broadcast rate in terms of the Shannon capacity and the Lovász theta function of the complement of the confusion graph.

Corollary 1.

$$\begin{aligned}\beta(G) &= \lim_{t\to\infty} \frac{1}{t} \log\left(\Theta\left(\overline{\Gamma_t(G)}\right)\right) \\ &= \lim_{t\to\infty} \frac{1}{t} \log\left(\vartheta\left(\overline{\Gamma_t(G)}\right)\right).\end{aligned}$$

In summary, the broadcast rate can be characterized as the first order in the exponent of six well-known graph theoretic quantities associated with $\Gamma_t(G)$ and its complement, namely, $\omega(\Gamma_t)$, $\alpha(\overline{\Gamma}_t)$, $\Theta(\overline{\Gamma}_t)$, $\vartheta(\overline{\Gamma}_t)$, $\chi(\Gamma_t)$, and $\chi_f(\Gamma_t)$.

In the following, we present nonasymptotic upper bounds on the broadcast rate $\beta(G)$ in terms of the Shannon capacity and the Lovász theta function that hold for every positive integer t and, due to (24), are tighter than the upper bound in (18).

Theorem 3. *For any side information graph G and any positive integer t,*

$$\beta(G) \leq \frac{1}{t} \log\left(\Theta\left(\overline{\Gamma_t(G)}\right)\right). \tag{25}$$

Proof: Consider

$$\omega(\Gamma_{tk}) \leq \omega(\Gamma_t^{\vee k}) = \alpha(\overline{\Gamma_t^{\vee k}}) = \alpha(\overline{\Gamma}_t^{\boxtimes k}), \tag{26}$$

where the inequality holds since the set of edges of $\Gamma_t^{\vee k}$ contains the set of edges of Γ_{tk}, and the last equality follows by Lemma 5. Now for any t,

$$\begin{aligned}\beta(G) &= \lim_{k\to\infty} \frac{\log\left(\omega\left(\Gamma_k\right)\right)}{k} && (27)\\ &= \lim_{k\to\infty} \frac{\log\left(\omega\left(\Gamma_{tk}\right)\right)}{tk} && (28)\\ &\leq \lim_{k\to\infty} \frac{\log\left(\alpha\left(\overline{\Gamma}_t^{\boxtimes k}\right)\right)}{tk} && (29)\\ &= \lim_{k\to\infty} \frac{\log\left(\sqrt[k]{\alpha\left(\overline{\Gamma}_t^{\boxtimes k}\right)}\right)}{t} \\ &= \frac{1}{t}\log\left(\lim_{k\to\infty} \sqrt[k]{\alpha\left(\overline{\Gamma}_t^{\boxtimes k}\right)}\right) \\ &= \frac{1}{t}\log\left(\Theta\left(\overline{\Gamma}_t\right)\right), && (30)\end{aligned}$$

where (27) follows by Theorem 2, (28) holds since the limit of a subsequence is equal to the limit of the sequence, (29) follows by (26), and (30) follows by the definition of the Shannon capacity in (6). □

Corollary 2. For any side information graph G and any positive integer t,

$$\beta(G) \leq \frac{1}{t} \log\left(\vartheta\left(\overline{\Gamma_t(G)}\right)\right). \tag{31}$$

Remark 7. Unlike the upper bounds in (18) and (25) in terms of the chromatic number and the Shannon capacity, the upper bound in (31) can be computed in polynomial time in the number of vertices of the confusion graph (see [42]).

Remark 8. Equation (20) can be generalized to characterize the capacity region $\mathscr{C}$ of the index coding problem G as the closure of all rate tuples $(R_1, \ldots, R_n)$ such that

$$R_j \leq \frac{t_j}{\log(\chi_f(\Gamma_{\mathbf{t}}(G)))}, \quad j \in [n], \tag{32}$$

for some $\mathbf{t} = (t_1, \ldots, t_n)$ [47]. By a sandwich argument similar to (22), $\mathscr{C}$ can be also characterized in terms of $\omega(\Gamma_{\mathbf{t}}(G))$ asymptotically as $\mathbf{t} \to \infty$.

Remark 9. Similar to the index coding problem, the optimal rate region of the locally recoverable distributed storage problem with recovery graph G [11], [12] is characterized as the closure of all rate tuples $(R'_1, \ldots, R'_n)$ such that

$$R'_j \geq \frac{t_j}{\log(\alpha(\Gamma_{\mathbf{t}}(G)))}, \quad j \in [n], \tag{33}$$

for some $\mathbf{t} = (t_1, \ldots, t_n)$ [11], [13]. Based on the vertex transitivity of $\Gamma_{\mathbf{t}}$ which, inter alia, implies that $\log(\alpha(\Gamma_{\mathbf{t}})) = \sum_{i=1}^n t_i - \log(\chi_f(\Gamma_{\mathbf{t}}))$, the relationship between the index coding capacity region in (32) and the distributed storage optimal rate region in (33) can be made precise. See [13] for the details.

VI. Lexicographic Product of Side Information

We first establish an upper bound on the broadcast rate of the index coding problem whose side information graph is a general lexicographic product (recall the definition in Section II).

Theorem 4. *Let $G = ([m], E)$ be a directed graph with m vertices and $F_1, \ldots, F_m$ be m directed graphs with $\beta(F_1) \leq \cdots \leq \beta(F_m)$. Then*

$$\begin{aligned}&\beta(G \circ (F_1, \ldots, F_m)) \\ &\quad \leq \beta(F_1)\beta(G) + \sum_{i=1}^{m-1} (\beta(F_{i+1}) - \beta(F_i))\beta(G|_{[m]\setminus[i]}).\end{aligned} \tag{34}$$

The proof of the theorem is given in Appendix D.

Remark 10. For the special case in which G has two vertices, the upper bound in Theorem 4 is tight [34], [46], [47]. In particular, if G has either no edges or one edge (see Fig. 4(a) and 4(b)), then $\beta(G \circ (F_1, F_2)) = \beta(F_1) + \beta(F_2)$, and if G is a complete graph on two vertices (see Fig. 4(c)), then $\beta(G \circ (F_1, F_2)) = \beta(F_2) = \max\{\beta(F_1), \beta(F_2)\}$.

The following states another special case for which the bound in Theorem 4 is tight.

Theorem 5. *For any two directed graphs G and F,*

$$\beta(G \circ F) = \beta(G)\beta(F).$$

In words, the broadcast rate is multiplicative under the lexicographic product of index coding side information graphs. Achievability was shown by Blasiak, Kleinberg, and Lubetzky [33]. It also follows from Theorem 4 by setting $F_1 = \cdots =$

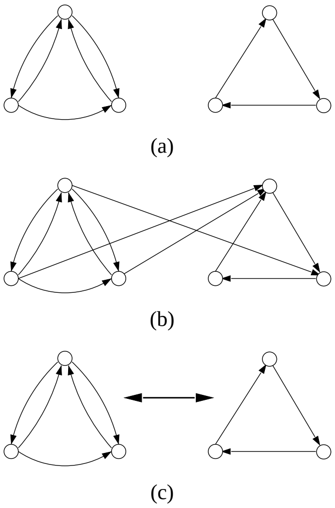


Fig. 4. Graph examples with (a) no interaction, (b) one-way interaction, and (c) complete interaction among its two parts (↔ indicates that there is a bidirectional edge between every vertex on the left and every vertex on the right).

$F_m = F$. The proof of the converse is based on the clique-number characterization of the broadcast rate in Theorem 2 and the following.

Lemma 10. *For any (t, r) index code for problem $G \circ F$,*

$$\lfloor \log(\omega(\Gamma_t(F))) \rfloor \beta(G) \le r.$$

The proof of the lemma is relegated to Appendix E.

Proof of the converse for Theorem 5: Consider

$$\begin{aligned}\beta(G \circ F) &= \lim_{t\to\infty} \inf_{(t,r) \text{ codes for } G\circ F} \frac{r}{t} \\ &\ge \lim_{t\to\infty} \frac{1}{t} \lfloor \log(\omega(\Gamma_t(F))) \rfloor \beta(G) \qquad (35)\\ &= \lim_{t\to\infty} \frac{1}{t} \log(\omega(\Gamma_{\mathbf{t}}(F)))\beta(G) \\ &= \beta(F)\beta(G), \qquad (36)\end{aligned}$$

where (35) follows by Lemma 10, and (36) follows by Theorem 2. □

Example 1. The graph shown in Fig. 5(a) can be considered as the lexicographic product $G \circ F$ of two smaller graphs G and F shown in Fig. 5(b) and 5(c) respectively with $\beta(G) = 2$ and $\beta(F) = 2$. By Theorem 5, instead of directly computing the broadcast rate for this six-message problem, we can use the known broadcast rates of smaller problems and get $\beta(G \circ F) = 4$. Note that although this six-message problem has a certain symmetric structure, it does not fall into the class of cyclically symmetric index coding problems studied by Maleki, Cadambe, and Jafar [29].

The bound in Theorem 4 is not tight in general, as illustrated by the following.

Example 2. Consider the following 7-message index coding problem G

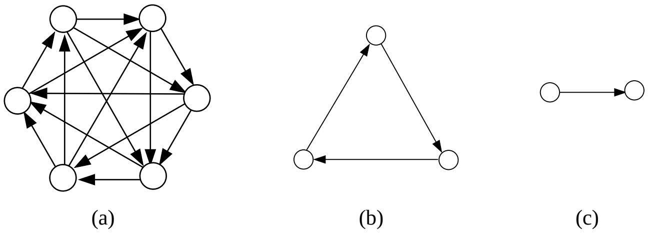


Fig. 5. (a) A 6-node graph that is the lexicographic product $G \circ F$ of two smaller graphs G and F. (b) The 3-node graph G. (c) The 2-node graph F.

$$(1|2,3,4,6,7),\ (2|1,3,6,7),\ (3|1,4,5,7),\ (4|1,2,5,6),$$
$$(5|3,4,6,7),\ (6|2,4,5,7),\ (7|2,3,5,6),$$

for which $\beta(G) = 2.5$ [17]. Let $F_1, \ldots, F_6$ be 1-message problems and F_7 be the 2-message problem

$$(1|2),\ (2|-).$$

Then Theorem 4 yields

$$\beta\left(G \circ (F_1, \ldots, F_7)\right) \le 1 \times 2.5 + (2-1) \times 1 = 3.5.$$

This bound is not tight since the composite coding scheme [25], [26] achieves the tighter upper bound of 10/3 on $\beta(G \circ (F_1, \ldots, F_7))$.

Remark 11. The upper bound on the broadcast rate in Theorem 4 can be generalized to an inner bound on the capacity region as follows. Denoting the capacity regions of the index coding problems $G \circ (F_1, \ldots, F_m)$, G, and $F_1, \ldots, F_m$ by $\mathscr{C}$, $\mathscr{C}_0$, and $\mathscr{C}_1, \ldots, \mathscr{C}_m$ respectively, we have

$$\bigcup_{\boldsymbol{\alpha}\in\mathscr{C}_0} \big\{(\alpha_1\mathbf{R}_1, \ldots, \alpha_m\mathbf{R}_m) \colon \mathbf{R}_i \in \mathscr{C}_i, i \in [m]\big\} \subseteq \mathscr{C}. \qquad (37)$$

For the special case in which G has two vertices, the inner bound in (37) is tight [34], [47], generalizing the results in Remark 10. If G has either no edge or only one edge, then

$$\mathscr{C} = \bigcup_{\alpha\in[0,1]} \big\{(\alpha\mathbf{R}_1, (1-\alpha)\mathbf{R}_2) \colon \mathbf{R}_1 \in \mathscr{C}_1, \mathbf{R}_2 \in \mathscr{C}_2\big\}.$$

In other words, in this case, the capacity region of $G \circ (F_1, F_2)$ is achieved by time division between the optimal coding schemes for two subproblems F_1 and F_2. If G is a complete graph on two vertices, then

$$\mathscr{C} = \big\{(\mathbf{R}_1, \mathbf{R}_2) \colon \mathbf{R}_1 \in \mathscr{C}_1, \mathbf{R}_2 \in \mathscr{C}_2\big\}.$$

In other words, the capacity region of $G \circ (F_1, F_2)$ is achieved by simultaneously using the optimal coding schemes for F_1 and F_2.

VII. Critical Index Coding Instances

As Remark 11 suggests, if an edge e of the side information graph G belongs to a directed cut, removing e does not reduce the capacity region. The Farkas lemma [48, Th. 2.2] states that each edge in a directed graph either lies on a directed cycle or belongs to a directed cut but not both. Hence, if edge e does not lie on any directed cycle, it can be removed from G

without affecting the capacity region. This was first observed by Tahmasbi, Shahrasbi, and Gohari [34], who then asked for general conditions under which an edge of the side information graph can be removed without reducing the capacity region.

Let e be an edge of side information graph $G = (V, E)$. We denote the graph resulting from removing e from G by G_e, i.e.,

$$V(G_e) = V(G) \text{ and } E(G_e) = E(G) \setminus \{e\}.$$

Given the index coding problem G, the edge $e \in E$ is said to be *critical* if $\mathscr{C}(G_e) \neq \mathscr{C}(G)$, or in other words, if the removal of e from G strictly reduces the capacity region. The index coding problem G itself is said to be *critical* if every $e \in E(G)$ is critical. Thus, each critical graph (= index coding problem) cannot be made "simpler" into another one of the same capacity region.

Remark 11 can be paraphrased into the following necessary condition for criticality.

Proposition 7 (Union-of-cycles condition [34])**.** *If G is critical, then every edge belongs to a directed cycle.*

This simple condition, however, is not sufficient. For the index coding problem shown in Fig. 1, although the edge 2 → 3 lies on a directed cycle, it can be shown that the capacity region is characterized by

$$\begin{aligned} R_1 &\leq 1, \\ R_2 + R_3 &\leq 1, \end{aligned}$$

with or without this edge.

To observe another simple necessary condition for criticality, consider an index coding problem G with side information sets $A_1, \ldots, A_n$. These sets are said to be *degraded* if there exist $i, j \in V(G)$ such that $i \in A_j$ and $A_i \subseteq A_j$. In this case, the edge $i \to j$ can be removed since x_i can be recovered at node j. This observation leads to the following necessary condition.

Proposition 8 (Nondegradedness condition)**.** *If G is critical, then side information sets must be nondegraded.*

Satisfying the above two necessary conditions at the same time is still not sufficient for criticality. As an example, it can be checked that the side information graph shown in Fig. 6 satisfies both union-of-cycles and nondegradedness conditions. However, it is not a critical graph as the capacity region is characterized by

$$\begin{aligned} R_1 + R_3 &\leq 1, \\ R_1 + R_4 &\leq 1, \\ R_2 + R_4 &\leq 1, \\ R_2 + R_5 &\leq 1, \\ R_3 + R_5 &\leq 1, \\ R_1 + R_2 + R_3 + R_4 + R_5 &\leq 2, \end{aligned}$$

with or without the edge $4 \to 1$.

In order to find a tighter necessary condition, we now focus on a sufficient condition. Given a graph $G = (V, E)$, the vertex induced subgraph $G|_S$ is referred to as a *unicycle* if its set of edges is a (chordless) Hamiltonian cycle over S. Note that if the subgraph $G|_S$ is a unicycle, then $G|_{S'}$ cannot be a unicycle for any S' that is a proper subset or superset of S. As an example, in Fig. 7(a), $G|_{\{1,2,3\}}$ is a unicycle, but G itself is not a unicycle. As another example, for the graph in Fig. 7(b), $G|_{\{1,2,3\}}$ and $G|_{\{1,3,4\}}$ are both unicycles.

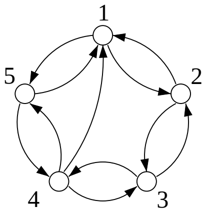


Fig. 6. A 5-message index coding problem. The edge $4 \to 1$ lies on a directed cycle and $A_4 \not\subseteq A_1$. However, removing this edge does not affect the capacity region. The capacity region is achieved by the composite coding scheme [25] with or without this edge.

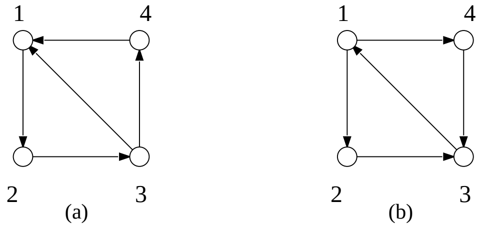


Fig. 7. (a) $G|_{\{1,2,3\}}$ is a unicycle, but G is not a unicycle. (b) $G|_{\{1,2,3\}}$ and $G|_{\{1,3,4\}}$ are both unicycles.

The following states a sufficient condition for the criticality of a problem.

Theorem 6 (Union-of-unicycles condition)**.** *If every edge of G belongs to a vertex induced subgraph that is a unicycle, then G is critical.*

Proof: It suffices to show that removing each edge of $G = (V, E)$ that belongs to a unicycle strictly reduces the capacity region. Let e be an edge of $G|_S$, where $S \subseteq V$ and $G|_S$ is a unicycle. The rate tuple $(R_1, \ldots, R_n)$ such that

$$R_i = \begin{cases} 0, & i \notin S, \\ \frac{1}{|S|-1}, & i \in S, \end{cases} \tag{38}$$

is achievable for index coding problem G by partial clique covering (see Proposition 3 and Remark 1). The vertex-induced subgraph $G_e|_S$, however, is acyclic (since the Hamiltonian cycle of $G|_S$ is broken and by definition there is no other cycle). Therefore, by the MAIS outer bound, any rate tuple $(R'_1, \ldots, R'_n) \in \mathscr{C}(G_e)$ must satisfy

$$\sum_{i \in S} R'_i \leq 1. \tag{39}$$

The rate tuple in (38), however, does not satisfy (39) and thus is not in $\mathscr{C}(G_e)$. This implies that removing edge e from G strictly reduces the capacity region ($\mathscr{C}(G_e) \neq \mathscr{C}(G)$) and hence e is critical. □

Remark 12. If a graph satisfies the union-of-unicycles condition, it trivially satisfies the union-of-cycles condition. We now argue that, as expected, satisfying the union-of-unicycles condition also implies the nondegradedness condition. Assume that G has degraded side information sets. Then, there exists an edge $i \to j$ such that $A_i \subseteq A_j$. We show that this edge cannot belong to a unicycle. If the edge $i \to j$ does not belong to any cycle, then trivially it does not belong to any unicycle. Otherwise, it suffices to show that none of the cycles that contain this edge is a unicycle. Assume that $i \to j$ lies on a cycle $C = (i, j, \ldots, v)$, which by degradedness must have at least three vertices. Then, by definition, $v \in A_i$ and, by the assumption, $v \in A_j$. Therefore, $(j, \ldots, v)$ is also a cycle and C is not a unicycle.

The converse to Theorem 6, however, does not hold in general.

Example 3. The capacity region of the index coding problem with side information graph shown in Fig. 8 is characterized by

$$\begin{aligned}
R_1 + R_2 &\le 1,\\
R_1 + R_3 &\le 1,\\
R_1 + R_4 &\le 1,\\
R_2 + R_4 &\le 1,\\
R_2 + R_5 &\le 1,\\
R_3 + R_5 &\le 1,
\end{aligned}$$

which is achievable by the composite coding scheme [25]. Although the edge $2 \to 5$ does not belong to any unicycle, removing it from the side information graph reduces the capacity region to

$$\begin{aligned}
R_1 + R_2 &\le 1,\\
R_1 + R_3 &\le 1,\\
R_1 + R_4 &\le 1,\\
R_2 + R_4 &\le 1,\\
R_2 + R_5 &\le 1,\\
R_3 + R_5 &\le 1,\\
R_1 + R_2 + R_3 + R_4 + R_5 &\le 2,
\end{aligned} \tag{40}$$

which is also achievable by the composite coding scheme [25].

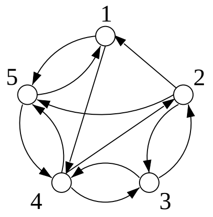


Fig. 8. A critical 5-message index coding problem. Although the edge $2 \to 5$ does not belong to any unicycle, it is critical. The capacity region is achieved by composite coding [25] with or without the edge $2 \to 5$.

The above example illustrates that the union-of-unicycles condition does not capture "criticality" with respect to the capacity region. In the following, we argue that this condition in fact is sufficient and necessary for the criticality with respect to the MAIS outer bound. The proof is relegated to Appendix F.

Proposition 9. *Edge e belongs to a unicycle iff the MAIS bound $\mathscr{R}_{\text{MAIS}}(G_e)$ on $\mathscr{C}(G_e)$ is a proper subset of the MAIS bound $\mathscr{R}_{\text{MAIS}}(G)$ on $\mathscr{C}(G)$.*

Proposition 9 implies the following partial converse to Theorem 6.

Proposition 10. *If $G = (V, E)$ is critical, then*

1) *every edge $e \in E$ belongs to a unicycle, or*
2) *the MAIS bound is not tight for G_e, i.e., $\mathscr{R}_{\text{MAIS}}(G_e) \neq \mathscr{C}(G_e)$, for every $e \in E$ that does not belong to any unicycle.*

In other words, e is not critical if it does not belong to any unicycle and the MAIS bound is tight for G_e.

Proof: It suffices to show that if G is critical and there exists an edge e that does not belong to any unicycle, then the MAIS bound is not tight for G_e. Since G is critical, we have $\mathscr{C}(G_e) \subsetneq \mathscr{C}(G)$. Assume by contradiction that the MAIS bound is tight for G_e. Then

$$\mathscr{R}_{\text{MAIS}}(G_e) = \mathscr{C}(G_e) \subsetneq \mathscr{C}(G) \subseteq \mathscr{R}_{\text{MAIS}}(G),$$

which contradicts Proposition 9. □

Recall that the edge $2 \to 5$ in Fig. 8 is critical and does not belong to any unicycle. As is suggested by Proposition 10 and verified by (40), the MAIS bound is not tight for the side information graph resulting from removing this edge.

Remark 13. The three necessary conditions in Propositions 7, 8, and 10 can be rewritten as follows. For an index coding problem with the edge e from i to j, if e does not belong to a directed cycle, or $A_i \subset A_j$, or e does not belong to a unicycle and the MAIS bound is tight for G_e, then e can be removed without reducing the capacity region.

The next three examples demonstrate that these necessary conditions are mutually independent.

Example 4. The six-message problem

$$(1|5, 6),\ (2|6),\ (3|6),\ (4|6),\ (5|1),\ (6|2, 3, 4, 5)$$

satisfies the union-of-cycles and nondegradedness conditions. However, it does not satisfy the necessary condition in Proposition 10, as the edge $5 \to 6$ does not belong to any unicycle and the MAIS bound is tight (and is achieved by the composite coding scheme) after removing this edge.

Example 5. The six-message problem

$$(1|4, 5),\ (2|5, 6),\ (3|5),\ (4|1, 6),\ (5|1, 2),\ (6|2, 3, 4, 5)$$

satisfies the union-of-cycles condition and the necessary condition in Proposition 10. However, $A_3 \subset A_6$ and thus it does not satisfy the nondegradedness condition.

Example 6. The six-message problem

$$(1|4, 6),\ (2|5, 6),\ (3|5),\ (4|1, 6),\ (5|1, 2),\ (6|2, 4, 5)$$

satisfies the nondegradedness condition and the necessary condition in Proposition 10. However, the edge $5 \to 3$ does not belong to any cycle and thus the problem does not satisfy the union-of-cycles condition.

For the rest of this section, we present a few results that relate the capacity of index coding problem G and its MAIS bound to those of simpler problems. Consider the graph $G = (V, E)$ and let G' be the graph resulting from removing all edges of G that do not belong to any unicycle, i.e.,

$$\begin{aligned} V(G') &= V(G), \\ E(G') &= \{e \in E(G) \colon e \text{ in a unicycle of } G\}. \end{aligned} \tag{41}$$

Proposition 11. $\mathscr{R}_{\text{MAIS}}(G') = \mathscr{R}_{\text{MAIS}}(G)$.

In words, the set of edges of G that do not belong to any unicycle, is the (maximum) set of edges that can be removed from G without changing the MAIS bound. The proof of the proposition, which is implied by Proposition 9, is presented in Appendix G.

This observation leads to a condition under which the capacity of index coding problem G is equal to the capacity of the simpler problem G'.

Proposition 12. *If the MAIS bound is tight for G', then*

$$\mathscr{R}_{\text{MAIS}}(G') = \mathscr{C}(G') = \mathscr{C}(G) = \mathscr{R}_{\text{MAIS}}(G).$$

Consequently, if the MAIS bound is tight for G', then G is not critical and all the edges that do not belong to any unicycle can be removed without reducing the capacity.

Proof of Proposition 12: Since

$$\mathscr{R}_{\text{MAIS}}(G') = \mathscr{C}(G') \subseteq \mathscr{C}(G) \subseteq \mathscr{R}_{\text{MAIS}}(G),$$

the proof follows by Proposition 11. □

Remark 14. It can be similarly shown that the result of Proposition 12 also holds for the broadcast rate. If $\beta_{\text{MAIS}}(G') = \beta(G')$, then $\beta_{\text{MAIS}}(G) = \beta(G) = \beta(G') = \beta_{\text{MAIS}}(G')$.

Example 7. Consider the side information graph G shown in Fig. 9, where edges $5 \to 3$, $3 \to 1$, and $6 \to 5$ do not belong to any unicycle. It can be shown that the capacity region for problem G' is achieved by composite coding [25] and is characterized by

$$\begin{aligned} R_1 + R_3 + R_4 &\leq 1, \\ R_1 + R_3 + R_5 &\leq 1, \\ R_2 + R_3 + R_4 + R_6 &\leq 1, \\ R_2 + R_3 + R_5 + R_6 &\leq 1, \end{aligned} \tag{42}$$

which is equal to its MAIS bound. Thus, by Proposition 12, G is not critical and its capacity is also characterized by (42).

Proposition 12, together with Remark 6, implies the following.

Proposition 13. *If G' is bidirectional and $U(G')$ is perfect, then $\mathscr{C}(G) = \mathscr{R}_{\text{MAIS}}(G)$ which is achieved by the fractional clique covering scheme.*

This result can be recast to an earlier result by Yi, Sun, Jafar, and Gesbert [14], using the following two lemmas that are proved in Appendices H and I.

Lemma 11. *Consider $G = (V, E)$ and let G' be the graph as defined in* (41)*. The following statements are equivalent.*

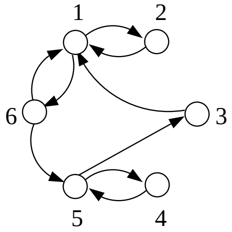


Fig. 9. A noncritical 6-message index coding problem with nondegraded side information sets. The edges $5 \to 3$, $3 \to 1$, and $6 \to 5$ lie on a directed cycle, but do not belong to any unicycle. The capacity region is equal to the MAIS outer bound and is achieved by composite coding [25] with or without these edges.

(1) For each clique K in $U(\bar{G})$, $G|_K$ is acyclic.
(2) For each $S \subseteq V(G)$, if $G|_S$ contains a cycle, then there exists a bidirectional edge in $G|_S$, i.e., $\exists\ i, j \in S$ such that $(i, j) \in E(G)$ and $(j, i) \in E(G)$.
(3) No unidirectional edge of G belongs to a unicycle.
(4) G' is bidirectional.

Lemma 12. *If G' is bidirectional, then $\overline{U(G')} = U(\bar{G})$.*

By Lemma 3 (U is perfect iff $\bar{U}$ is perfect), Lemma 11, and Lemma 12, we can now restate Proposition 13 as follows.

Proposition 14 (Yi, Sun, Jafar, and Gesbert [14])**.** *If $U(\bar{G})$ is perfect and for each clique K in $U(\bar{G})$, $G|_K$ is acyclic, then $\mathscr{C}(G) = \mathscr{R}_{\text{MAIS}}(G)$ which is achieved by the fractional clique covering scheme.*

Note that this proposition includes Remark 6 as a special case.

As a concrete application of Proposition 12, consider a side information graph G satisfying

$$A_j \subseteq \{j-1, j+1\}, \quad j \in [n].$$

If $A_j = \{j-1, j+1\}$ or $A_j = \{j-1\}$ for all $j \in [n]$, then every edge belongs to a unicycle. (For these cases, the capacity is known [29] and is achieved by the fractional local clique covering scheme [23].) Otherwise, G' is a bidirectional (undirected) perfect graph (by Proposition 1). Therefore, by Remark 6, the MAIS bound is tight for index coding problem G' and Proposition 12 implies the following.

Corollary 3. For the class of index coding problems satisfying

$$A_j \subseteq \{j-1, j+1\}, \quad j \in [n],$$

any edge that does not belong to a unicycle can be removed without reducing the capacity region. Thus, for this class of index coding problems, the union-of-unicycles sufficient condition is also necessary for the problem to be critical.

Example 8. In the side information graph shown in Fig. 10 (a), edges $5 \to 4$, $4 \to 3$, and $2 \to 1$ do not belong to any unicycle. Hence, the two side information graphs shown in Fig. 10 have the same capacity region.

VIII. Application: Index Coding With Six Messages

The number of instances of the index coding problem with n messages, which is equal to the number of nonisomorphic

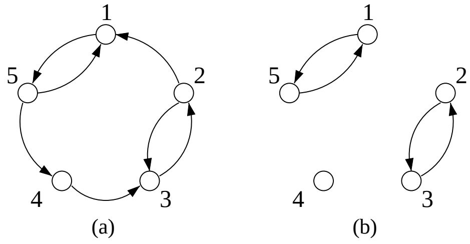


Fig. 10. Two 5-node index coding problems with the same capacity region.

directed graphs with n vertices [49, Seq. A000273], blows up quickly with n. Even when n is as small as six, there are 1,540,944 nonisomorphic instances. In this section, we apply the structural properties discussed earlier to identify the 6-message index coding instances for which the capacity can be characterized via the capacities of "simpler" problems. By Theorem 5 and Remark 11, if G can be decomposed into smaller graphs, then the capacity of G can be expressed as a simple function of the capacities of smaller problems with five or fewer messages, for which the capacity is known [25]. At the same time, by Propositions 7, 8, and 10 (see also Remark 13), if the graph G does not satisfy the three necessary conditions, then a violating edge e can be removed to form a new graph G_e of the same capacity (which may or may not be known as G_e still has 6 vertices).

Among the above conditions for simplification, we focus on the following four properties on G. If any of them is satisfied, then G can be simplified.

P_1: G is not strongly connected.
P_2: The complement of G is disconnected.
P_3: G is not a union-of-unicycles ($G \neq G'$) and the MAIS bound is tight for G'.
P_4: G has degraded side information subsets.

Note that if the complement of G is disconnected, then G is strongly connected. Hence, P_1 and P_2 are mutually exclusive. The properties P_1 and P_2 allow decomposition into smaller problems, while P_1, P_3, and P_4 allow removal of some edge. Finally, P_1, P_2, and P_3 (for the case of $n = 6$) lead to simpler problems with known capacity, while P_4 may result in a simpler problem with still unknown capacity.

Table I shows the number of 6-message instances that satisfy each of the mentioned properties.

TABLE I
THE NUMBER OF 6-MESSAGE INDEX CODING INSTANCES THAT SATISFY PROPERTIES P_1-P_4.

Structural Property	Number of six-message instances
P_1	493,936
P_2	10,101
P_3	$\geq$ 1,513,890
P_4	1,336,566
$\neg(P_1 \vee P_2 \vee P_3 \vee P_4)$	$\leq$ 10,634

It can be easily checked that the side information graphs corresponding to the six-message instances in Examples 4 to 6 have connected complement and thus do not satisfy property P_2. This proves that there are instances satisfying $(P_1 \wedge \neg P_2 \wedge \neg P_3 \wedge \neg P_4)$ or $(P_3 \wedge \neg P_1 \wedge \neg P_2 \wedge \neg P_4)$, or $(P_4 \wedge \neg P_1 \wedge \neg P_2 \wedge \neg P_3)$. Moreover, the six-message problem

$$(1|6),\ (2|6),\ (3|6),\ (4|6),\ (5|6),\ (6|1,2,3,4,5)$$

satisfies P_2 but not P_1, P_3, or P_4. Therefore, checking all of these four properties is useful in removing instances that do not need further investigation.

Among the remaining 10,634 instances that are not simplified, the polymatroidal bound [25] is achieved by a simplified form of composite coding [25], [26] for 10,515 instances. This leaves at most 119 instances that require further investigation via a tighter bound using non-Shannon inequalities [30] and more general coding schemes. In addition, there are at most 853 *noncritical* instances that simplify to one of the 119 critical instances of unknown capacity. In summary, the capacity is now fully characterized for at least 1,540,944 − 119 − 853 = 1,539,972 index coding problems (99.9%) with 6 messages.

IX. Acknowledgments

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Appendix A
Proof of Lemma 4

Let V' be the set of vertices incident to the edges in $E_2 \setminus E_1$ and let $U' = (V', E_2 \setminus E_1)$. In order to color the vertices of U, we first color the vertices in $V \setminus V'$ with $\chi(U_1)$ colors using the optimal coloring for U_1. Next, we color U' with $\chi(U_2)$ additional colors using the optimal coloring for U_2, which is valid since $V' \subseteq V$ and $E_2 \setminus E_1 \subseteq E_2$. This guarantees that any pair of adjacent vertices are assigned different colors, whether both of them belong to V' or to $V \setminus V'$ or one to each. Therefore, there exists a proper coloring of U with at most $\chi(U_1) + \chi(U_2)$ colors and thus $\chi(U) \leq \chi(U_1) + \chi(U_2)$.

Appendix B
Proof of Lemma 8

It suffices to show that every cycle of length greater than four has a chord. Let $v_1^n, v_2^n, \ldots, v_k^n$ be the vertices (each associated with an n-message tuple) of a length-k cycle of $\Gamma_{\mathbf{t}}^{(jl)}(G)$ for $k \geq 5$. Then $v_1^n \sim v_2^n$, $v_2^n \sim v_3^n$, …, $v_{k-1}^n \sim v_k^n$. Therefore, $v_{1j}(l) \neq v_{2j}(l)$, $v_{2j}(l) \neq v_{3j}(l)$, …, $v_{(k-1)j}(l) \neq v_{kj}(l)$, and $v_{1,A_j} = v_{2,A_j} = \cdots = v_{k,A_j}$. If $v_{1j}(l) \neq v_{3j}(l)$, then since $v_{1,A_j} = v_{3,A_j}$, we have $v_1^n \sim v_3^n$ and the length-k cycle has a chord. Otherwise, since $v_{1j}(l) = v_{3j}(l) \neq v_{4j}(l)$ and $v_{1,A_j} = v_{4,A_j}$, we have $v_1^n \sim v_4^n$ and again the cycle has a chord.

Appendix C
Proof of Lemma 9

It suffices to show that every cycle of length greater than four has a chord. Let $v_1^n, v_2^n, \ldots, v_k^n$ be the vertices of a length-k cycle of $\bar{\Gamma} = \Gamma_{\mathbf{t}}^{(jl)}(G)$ for $k \geq 5$. Then $v_1^n \sim v_2^n$, $v_2^n \sim v_3^n$, …, $v_{k-1}^n \sim v_k^n$ in $\bar{\Gamma}$. If $v_{1j}(l) = \cdots = v_{kj}(l)$, then $v_1^n, v_2^n, \dots, v_k^n$ form a clique in $\bar{\Gamma}$ and thus the cycle is not chordless. Hence, assume without loss of generality that $v_{1j}(l) \neq v_{2j}(l)$, which implies $v_{1,A_j} \neq v_{2,A_j}$. We now consider two cases.

Case 1 ($v_{2j}(l) = v_{3j}(l)$): In this case, if $v_{1,A_j} \neq v_{3,A_j}$, then $v_1^n \sim v_3^n$ in $\bar{\Gamma}$ and the length-k cycle has a chord. Suppose $v_{1,A_j} = v_{3,A_j}$ and consider $v_{4j}(l)$. If $v_{4j}(l) = v_{2j}(l)$, then $v_2^n \sim v_4^n$ in $\bar{\Gamma}$ which is a chord for the length-k cycle. Suppose $v_{4j}(l) \neq v_{2j}(l)$. Then, since $v_3^n \sim v_4^n$ in $\bar{\Gamma}$ we have $v_{3,A_j} \neq v_{4,A_j}$ and hence $v_{1,A_j} \neq v_{4,A_j}$. Therefore, $v_1^n \sim v_4^n$ in $\bar{\Gamma}$ and the length-k cycle has a chord.

Case 2 ($v_{2j}(l) \neq v_{3j}(l)$): In this case, if $v_{1j}(l) = v_{3j}(l)$, then $v_1^n \sim v_3^n$ in $\bar{\Gamma}$ which is a chord. Suppose $v_{1j}(l) \neq v_{3j}(l)$. If $v_{1,A_j} \neq v_{3,A_j}$, then $v_1^n \sim v_3^n$ in $\bar{\Gamma}$ which is a chord. Suppose $v_{1,A_j} = v_{3,A_j}$. If $v_{3j}(l) = v_{4j}(l)$, then the situation will be the same as case 1. Otherwise, we have $v_{3,A_j} \neq v_{4,A_j}$ which implies $v_{1,A_j} \neq v_{4,A_j}$ and thus $v_1^n \sim v_4^n$ in $\bar{\Gamma}$ which is a chord.

Appendix D
Proof of Theorem 4

Fix $\epsilon > 0$. By the definition of the broadcast rate in (1) and (17), for sufficiently large t, there exists a (t, r_i) index code for problem F_i satisfying

$$\beta(F_i) \leq \frac{r_i}{t} \leq \beta(F_i) + \epsilon, \quad i \in [m]. \tag{43}$$

Let $I := \{i \in [m-1] \colon r_{i+1} - r_i > 0\}$, and k be a sufficiently large integer such that there exist a (kr_1, s_0) index code for problem G satisfying

$$\frac{s_0}{kr_1} \leq \beta(G) + \epsilon, \tag{44}$$

and a $(k(r_{i+1} - r_i), s_i)$ index code for problem $G|_{[m]\setminus[i]}$ satisfying

$$\frac{s_i}{k(r_{i+1} - r_i)} \leq \beta(G|_{[m]\setminus[i]}) + \epsilon, \quad i \in I. \tag{45}$$

Consider the following coding scheme that consists of m inner codes and at most m outer codes. First, for each $i \in V(G)$, the (t, r_i) index code for problem F_i is applied to the messages indexed by $\{i\} \times V(F_i)$. This inner code is deployed k times to generate kr_i bits. As the second step, the outer codes are used to send these $k \sum_{i\in[m]} r_i$ bits. The (kr_1, s_0) index code for problem G is used to send the first kr_1 bits generated from copies of problems $F_1, \dots, F_m$ (If $r_i < r_1$ for some i, zero-pad to get sufficient number of bits). Next, for each $i \in I$, the $(k(r_{i+1} - r_i), s_i)$ index code for problem $G|_{[m]\setminus[i]}$ is used to send $k(r_{i+1} - r_i)$ bits generated from copies $F_{i+1}, \dots, F_m$ (If required, zero-pad to get sufficient number of bits). Using the above scheme, all the bits generated from messages indexed by $\{i\} \times V(F_i)$, $i \in [m]$, are sent.

As for the decoding, first the decoders of the outer codes are utilized to recover the kr_i bits corresponding to messages indexed by $\{i\} \times V(F_i)$. Next, each of the decoders of the inner codes is used k times to recover the kt bits of each message.

This coding scheme uses $s_0 + \sum_{i\in I} s_i$ bits to send kt bits for each message of the problem $G \circ (F_1, \dots, F_m)$. Now consider

$$\begin{aligned}
s_0 &+ \sum_{i\in I} s_i \\
&\leq kr_1(\beta(G) + \epsilon) \\
&\quad + \sum_{i\in I} k(r_{i+1} - r_i)(\beta(G|_{[m]\setminus[i]}) + \epsilon) \qquad (46)\\
&\leq kt(\beta(F_1) + \epsilon)(\beta(G) + \epsilon) \\
&\quad + \sum_{i\in I} kt(\beta(F_{i+1}) - \beta(F_i) + \epsilon)(\beta(G|_{[m]\setminus[i]}) + \epsilon), \qquad (47)\\
&\leq kt(\beta(F_1) + \epsilon)(\beta(G) + \epsilon) \\
&\quad + \sum_{i\in[m-1]} kt(\beta(F_{i+1}) - \beta(F_i) + \epsilon)(\beta(G|_{[m]\setminus[i]}) + \epsilon),
\end{aligned} \tag{48}$$

where (46) follows by (44) and (45), and (47) follows by (43), and (48) follows by the assumption of the theorem. Letting $\epsilon \to 0$ completes the proof.

Appendix E
Proof of Lemma 10

Let $K = \{y_1, y_2, \dots, y_{|K|}\}$ be a maximum clique in $\Gamma_t(F)$ and let $k = \lfloor \log(|K|) \rfloor = \lfloor \log(\omega(\Gamma_t(F))) \rfloor$. By the definition of the broadcast rate in (1), $\beta(G) \leq r_G/k$ for any (k, r_G) index code for problem G. Hence, it suffices to show that given any (t, r) index code for problem $G \circ F$, a (k, r_G) index code for problem G can be constructed such that $r_G \leq r$.

Let $m = |V(G)|$ and $n = |V(F)|$. We denote a tuple of mn messages of problem $G \circ F$ by $x = (x_1, \dots, x_m)$, where $x_i = (x_{i1}, \dots, x_{in})$ and $x_{ij} \in \{0,1\}^t$ for $i \in [m]$ and $j \in [n]$. Consider the one-to-one mapping

$$f \colon \{0,1\}^k \to \{y_1, y_2, \dots, y_{2^k}\}$$

that maps the k-bit binary representation of $i-1$ to y_i, $i \in [2^k]$.

Let $\phi_{G\circ F}$ be the encoder associated with the (t, r) index code for problem $G \circ F$. For any message tuple $v = (v_1, \dots, v_m)$, $v_i \in \{0,1\}^k$, of problem G define

$$\phi_G(v_1, \dots, v_m) = \phi_{G\circ F}(f(v_1), \dots, f(v_m)). \tag{49}$$

The function ϕ_G in (49) is the encoder of an index code for problem G iff any two message tuples to which the same codeword is assigned are nonconfusable. Hence, it suffices to show that if $\phi_G(v_1, \dots, v_m) = \phi_G(v_1', \dots, v_m')$, then $(v_1, \dots, v_m)$ and $(v_1', \dots, v_m')$ are nonconfusable for problem G.

Suppose $\phi_G(v_1, \dots, v_m) = \phi_G(v_1', \dots, v_m')$. Then $\phi_{G\circ F}(f(v_1), \dots, f(v_m)) = \phi_{G\circ F}(f(v_1'), \dots, f(v_m'))$. By the definition of the mapping f, for every $i \in [m]$, either $f(v_i) = f(v_i')$ or $f(v_i) \sim f(v_i')$ in $\Gamma_t(F)$. As $\phi_{G\circ F}$ is the encoder of an index code for problem $G \circ F$, $(f(v_1), \dots, f(v_m))$ and $(f(v_1'), \dots, f(v_m'))$ are nonconfusable for problem $G \circ F$ and thus, if $f(v_i) \sim f(v_i')$ in $\Gamma_t(F)$, then $f(v_j) \neq f(v_j')$ for some $j \in A_i(G)$. Hence, since f is one-to-one, for every $i \in [m]$, either $v_i = v_i'$ or $v_j \neq v_j'$ for some $j \in A_i(G)$. Therefore, $(v_1, \dots, v_m)$ and $(v_1', \dots, v_m')$ are nonconfusable for problem G and (49) defines the encoder of a (k, r_G) index code for problem G such that the set of codewords is a subset of the set of codewords of the (t, r) index code for problem $G \circ F$, which implies $r_G \leq r$.

Appendix F
Proof of Proposition 9

Sufficiency. If the MAIS bound on $\mathscr{C}(G_e)$ is a proper subset of the MAIS bound on $\mathscr{C}(G)$, there exists a subset $S \subseteq V$ such that $G|_S$ contains a cycle and $G_e|_S$ is acyclic. Let S_{min} be a minimal such subset. Then, $G|_{S_{\text{min}}}$ is a unicycle that contains e.

Necessity. Let $G|_S$, $S \subseteq V$, be a unicycle that contains e. By the definition of unicycle, $G_e|_S$ is acyclic. Therefore, by the MAIS outer bound, any rate tuple $(R_1, \ldots, R_n) \in \mathscr{C}(G_e)$ must satisfy

$$\sum_{j \in S} R_j \leq 1. \tag{50}$$

However, since $G|_S$ is not acyclic, (50) is not implied by the MAIS outer bound on $\mathscr{C}(G)$.

Appendix G
Proof of Proposition 11

Proposition 9, together with the following, implies Proposition 11.

Lemma 13. *If e_1 and e_2 do not belong to any unicycle of G, then e_2 does not belong to any unicycle of G_{e_1}.*

Proof: If e_2 does not belong to any cycle of G, then it trivially does not belong to any unicycle of G_{e_1}. Suppose e_2 belongs to some cycle in G. It suffices to show that for every cycle C of G that contains e_2, $C \setminus e_1$ is not a unicycle of G_{e_1}. Let $e_1 = (u_1, u_2)$, $e_2 = (v_l, v_1)$, and $C = (v_1, \ldots, v_l)$ be a cycle of G that contains e_2. By the assumption, C is not a unicycle and thus $l \geq 3$. If $|\{u_1, u_2\} \cap \{v_1, \ldots, v_l\}| < 2$, then removing e_1 does not affect C and hence $C \setminus e_1$ is not a unicycle of G_{e_1}. Suppose $|\{u_1, u_2\} \cap \{v_1, \ldots, v_l\}| = 2$ and consider three cases.

Case 1: $e_1 = (v_i, v_{i+1})$ for some $i \in [l-1]$. In this case, removing e_1 breaks the cycle C and hence $C \setminus e_1$ is not a unicycle of G_{e_1}.

Case 2: $e_1 = (v_i, v_j)$ for some $1 \leq i < j \leq l$, $(i, j) \neq (1, l)$. In this case, $(v_1, \ldots, v_i, v_j, \ldots, v_l)$ is a cycle of G that contains both e_1 and e_2 and thus, by the assumption, is not a unicycle and has a chord, which is also a chord of $C \setminus e_1$. Thus, $C \setminus e_1$ is not a unicycle of G_{e_1}.

Case 3: $e_1 = (v_j, v_i)$ for some $1 \leq i < j \leq l$, $(i, j) \neq (1, l)$. In this case, $(v_i, \ldots, v_j)$ is a cycle of G that contains e_1 and thus, by the assumption, is not a unicycle and has a chord, which is also a chord of $C \setminus e_1$. Thus, $C \setminus e_1$ is not a unicycle of G_{e_1}.

□

Appendix H
Proof of Lemma 11

$(1) \Rightarrow (2)$*:* Assume that (2) does not hold. Then there exists a subset S such that $G|_S$ contains a cycle but does not have any bidirectional edge. By the definition of $U(\bar{G})$, S is a clique of $U(\bar{G})$, which contradicts (1).

$(2) \Rightarrow (1)$*:* Assume that (1) does not hold. Then there exists a clique K in $U(\bar{G})$ such that $G|_K$ has a cycle. By the definition of $U(\bar{G})$, $G|_K$ has no bidirectional edge, which contradicts (2).

$(2) \Rightarrow (3)$*:* Assume that there exists a unidirectional edge e and $S \subseteq V$, $|S| \geq 3$, such that $G|_S$ is a unicycle and $e \in E(G|_S)$. By the definition of unicycle, all of the edges of $G|_S$ are unidirectional, which contradicts (2).

$(3) \Rightarrow (2)$*:* Assume that (2) does not hold. Then there exists a subset S, $|S| \geq 3$ such that $G|_S$ has a cycle but does not have any bidirectional edge. A minimal such S forms a unicycle and hence all of its unidirectional edges belong to a unicycle, which contradicts (3).

$(3) \Rightarrow (4)$*:* To form G', every edge of G that do not belong to a unicycle is removed. Hence, if (3) holds, then all unidirectional edges of G are removed to form bidirectional G'.

$(4) \Rightarrow (3)$*:* G' is formed by removing edges of G that do not belong to any unicycle. Hence, G' is bidirectional implies that no unidirectional edge of G belongs to a unicycle.

Appendix I
Proof of Lemma 12

Since G' is bidirectional and every bidirectional edge belongs to a unicycle, we have

$$\{i, j\} \in E(U(G')) \iff (i, j) \in E(G) \text{ and } (j, i) \in E(G).$$

By definition,

$$\{i, j\} \notin E(U(\bar{G})) \iff (i, j) \in E(G) \text{ and } (j, i) \in E(G).$$

Thus, $\overline{U(G')} = U(\bar{G})$.

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