GORENSTEIN PROPERTIES AND INTEGER DECOMPOSITION PROPERTIES OF LECTURE HALL POLYTOPES

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Abstract. Though much is known about $\mathbf{s}$-lecture hall polytopes, there are still many unanswered questions. In this paper, we show that $\mathbf{s}$-lecture hall polytopes satisfy the integer decomposition property (IDP) in the case of monotonic $\mathbf{s}$-sequences. Given restrictions on a monotonic $\mathbf{s}$-sequence, we discuss necessary and sufficient conditions for the Fano, reflexive and Gorenstein properties. Additionally, we give a construction for producing Gorenstein/IDP lecture hall polytopes.

1. Introduction

Let $\mathcal{P} \subset \mathbb{R}^d$ be a d-dimensional convex lattice polytope. For $t \in \mathbb{Z}_{>0}$, *lattice point enumerator* $i(\mathcal{P}, t)$ gives the number of lattice points in $t\mathcal{P} = \{t\alpha : \alpha \in \mathcal{P}\}$, the tth dilation of $\mathcal{P}$. That is,

$$i(\mathcal{P}, t) = \#(t\mathcal{P} \cap \mathbb{Z}^d), \quad t \in \mathbb{Z}_{>0}.$$

Provided that $\mathcal{P}$ is a lattice polytope, it is known that this is a polynomial in the variable t of degree d ([6]). The *Ehrhart series* for $\mathcal{P}$, $\mathrm{Ehr}_{\mathcal{P}}(\lambda)$, is the rational generating function

$$\mathrm{Ehr}_{\mathcal{P}}(\lambda) = \sum_{t \geq 0} i(\mathcal{P}, t)\lambda^t = \frac{\delta(\mathcal{P}, \lambda)}{(1-\lambda)^{d+1}}$$

where $\delta(\mathcal{P}, \lambda) = \delta_0 + \delta_1\lambda + \delta_2\lambda^2 + \cdots + \delta_d\lambda^d$ is the *δ-polynomial* of $\mathcal{P}$ and $\delta(\mathcal{P}) = (\delta_0, \delta_1, \delta_2, \ldots, \delta_d)$ the δ-vector of $\mathcal{P}$. The δ-polynomial (δ-vector) is endowed with the following properties:

- $\delta_0 = 1$, $\delta_1 = i(\mathcal{P}, 1) - (d + 1)$, and $\delta_d = \#(\mathcal{P} \setminus \partial\mathcal{P} \cap \mathbb{Z}^d)$;
- $\delta_i \geq 0$ for all $0 \leq i \leq d$ ([15]);
- If $\delta_d \neq 0$, then $\delta_1 \leq \delta_i$ for each $0 \leq i \leq d-1$ ([10]).

The Ehrhart series and δ-polynomials for polytopes have been studied extensively. For a detailed background on these topics, please refer to [4, 6, 8, 16].

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Let $\mathbb{Z}^{d\times d}$ denote the set of $d\times d$ integer matrices. A matrix $A\in\mathbb{Z}^{d\times d}$ is *unimodular* if $\det(A)=\pm 1$. Given lattice polytopes $\mathcal{P}\subset\mathbb{R}^d$ and $\mathcal{Q}\subset\mathbb{R}^d$ of dimension d, we say that $\mathcal{P}$ and $\mathcal{Q}$ are *unimodularly equivalent* if there exists a unimodular matrix $U\in\mathbb{Z}^{d\times d}$ and a vector $\mathbf{w}\in\mathbb{Z}^d$ such that $\mathcal{Q}=f_U(\mathcal{P})+\mathbf{w}$, where f_U is the linear transformation of $\mathbb{R}^d$ defined by U, i.e., $f_U(\mathbf{v})=\mathbf{v}\,U$ for all $\mathbf{v}\in\mathbb{R}^d$.

We say that a lattice polytope $\mathcal{P}$ is *Fano* if $(\mathcal{P}\setminus\partial\mathcal{P})\cap\mathbb{Z}^d=\{\mathbf{0}\}$. We say that $\mathcal{P}$ is *reflexive* if it is Fano and its dual polytope

$$\mathcal{P}^\vee=\left\{y\in\mathbb{R}^d\ :\langle x,y\rangle\le 1 \text{ for all } x\in\mathcal{P}\right\}$$

is a lattice polytope. Moreover, it follows from [9] that the following statements are equivalent:

- $\mathcal{P}$ is unimodularly equivalent to some reflexive polytope;
- $\delta(\mathcal{P},\lambda)$ is of degree d and is symmetric, that is $\delta_i=\delta_{d-i}$ for $0\le i\le\lfloor\frac{d}{2}\rfloor$.

We say that $\mathcal{P}$ is *Gorenstein of index c* where $c\in\mathbb{Z}_{>0}$ if $c\mathcal{P}$ is unimodularly equivalent to a reflexive polytope [5]. Equivalently, $\mathcal{P}$ is Gorenstein if and only if $\delta(\mathcal{P},\lambda)$ is symmetric with $\deg(\delta(\mathcal{P},\lambda))=d-c+1$ ([14]).

We now give the definition of lecture hall polytopes. For a sequence of positive integers $\mathbf{s}=(s_1,s_2,\ldots,s_d)$, the $\mathbf{s}$-*lecture hall polytope* is

$$\mathbf{P}_d^{(\mathbf{s})}:=\left\{\mathbf{x}\in\mathbb{R}^d\ :\ 0\le\frac{x_1}{s_1}\le\frac{x_2}{s_2}\le\cdots\le\frac{x_d}{s_d}\le 1\right\}$$

which alternatively has the vertex representation as the column vectors of the matrix

$$\begin{bmatrix} 0 & s_d & s_d & s_d & \cdots & s_d \\ 0 & 0 & s_{d-1} & s_{d-1} & \cdots & s_{d-1} \\ 0 & 0 & 0 & s_{d-2} & \cdots & s_{d-2} \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & s_1 \end{bmatrix}$$

where x_d is given by the first row and so on with x_1 given by the last row. It should be noted that there is a easy unimodular equivalence $\mathbf{P}_d^{(\mathbf{s})}\cong\mathbf{P}_d^{(s_d,\ldots,s_2,s_1)}$.

For a given $\mathbf{s}=(s_1,s_2,\ldots,s_d)$, we define the $\mathbf{s}$-*inversion sequences* by the set $\mathbf{I}_d^{(\mathbf{s})}:=\{\mathbf{e}\in\mathbb{Z}^d\ :\ 0\le e_i<s_i\}$. Given $\mathbf{e}\in\mathbf{I}_d^{(\mathbf{s})}$, we define the *ascent set* of $\mathbf{e}$ by

$$\operatorname{Asc}\mathbf{e}:=\left\{i\ :0\le i<d \text{ and } \frac{e_i}{s_i}<\frac{e_{i+1}}{s_{i+1}}\right\}$$

with the convention that $e_0=1$ and $s_0=1$. Let $\operatorname{asc}\mathbf{e}:=|\operatorname{Asc}\mathbf{e}|$. The following result of the δ-polynomials of $\mathbf{s}$-lecture hall polytopes for arbitrary $\mathbf{s}$.

Lemma 1.1 ([13, Theorem 5]). *For any $\mathbf{s}$, the δ-polynomial of $\mathbf{P}_d^{(\mathbf{s})}$ is given by*

$$\delta\left(\mathbf{P}_d^{(\mathbf{s})},\lambda\right)=\sum_{\mathbf{e}\in\mathbf{I}_d^{(\mathbf{s})}}\lambda^{\operatorname{asc}\mathbf{e}}.$$

Moreover, these polynomials are real-rooted and hence unimodal.

The theory of lecture hall polytopes and lecture hall partitions is extensive [12] and many questions have been answered. Some particular motivating work includes the thorough study of Gorenstein properties for $\mathbf{s}$-*lecture hall cones* [1]. These results imply Ehrhart theoretic properties of the *rational* $\mathbf{s}$-*lecture hall polytopes* $\mathbf{R}^d_{(\mathbf{s})}$, but do not imply the same properties for $\mathbf{P}^d_{(\mathbf{s})}$. Additionally, the existence of a unimodular triangulation for the $\mathbf{s}$-lecture hall cone of $\mathbf{s} = (1, 2, \cdots, d)$ was recently shown [2]. However, showing the existence or nonexistence of a unimodular triangulation of $\mathbf{P}^d_{(\mathbf{s})}$ for most $\mathbf{s}$ is still an open question. This motivates the following unanswered questions:

- For what $\mathbf{s}$ is $\mathbf{P}_d^{(\mathbf{s})}$ Fano, reflexive, or Gorenstein?
- For what $\mathbf{s}$ does $\mathbf{P}_d^{(\mathbf{s})}$ satisfies the integer decomposition property?
- If $\mathbf{P}_d^{(\mathbf{s})}$ satisfies the integer decomposition property, for what conditions will it admit a unimodular triangulation?

In this paper, we answer these questions for particular large classes of $\mathbf{s}$ as progress towards a complete characterization. First we consider $\mathbf{P}_d^{(\mathbf{s})}$ when $\mathbf{s}$ is a monotonic sequence. We will show necessary and sufficient conditions for Fano and reflexive in the case when $\mathbf{s}$ is a sequence with $0 \leq s_{i+1} - s_i \leq 1$ for all $0 \leq i \leq d-1$ (or equivalently $0 \leq s_i - s_{i-1} \leq 1$ for all $0 \leq i \leq d-1$), the case when $\mathbf{s}$ is a strictly monotonic sequence, and the case when $\mathbf{s}$ is constant then strictly increasing. In the two latter cases, we can also provided necessary and sufficient conditions for when $\mathbf{P}_d^{(\mathbf{s})}$ is Gorenstein. We continue to show that $\mathbf{P}_d^{(\mathbf{s})}$ satisfies the integer decomposition property for all monotonic $\mathbf{s}$ and show that in some special cases, we can prove that $\mathbf{P}_d^{(\mathbf{s})}$ admits a unimodular triangulation, which is a stronger condition. Furthermore, if we have two lecture hall polytopes $\mathbf{P}_d^{(\mathbf{s})}$ and $\mathbf{P}_e^{(\mathbf{t})}$ which are Gorenstein and/or satisfies the integer decomposition property, we can construct a $(d+e+1)$-dimensional lecture hall polytope with the respective property.

2. Fano, Reflexive, and Gorenstein

Suppose that $\mathbf{s}$ is a monotonic sequence. We give necessary and sufficient conditions for when $\mathbf{P}_d^{(\mathbf{s})}$ is Fano or reflexive in the special cases of $\mathbf{s}$ a strictly increasing sequence and $\mathbf{s}$ a sequence which increases by at most one. In the case of strictly increasing, we can also find necessary and sufficient conditions for when $\mathbf{P}_d^{(\mathbf{s})}$is Gorenstein.

Remark 2.1. *All of the results in this section can be rephrased in the obvious way for when* $\mathbf{s}$ *is decreasing. This follows from the observation* $\mathbf{P}_d^{(s_1,s_2,\ldots,s_d)} \cong \mathbf{P}_d^{(s_d,s_{d-1},\ldots,s_1)}$.

2.1. **Strictly increasing s-sequences.** Suppose that $\mathbf{s} = (s_1, s_2, \ldots, s_d)$ is a sequence of positive integers such that $s_i \lneq s_{i+1}$ for all $i \in \{1, 2, \ldots, d-1\}$. We have the following necessary and sufficient conditions for when $\mathbf{P}_d^{(\mathbf{s})}$ is translation equivalent to a Fano polytope.

Theorem 2.2. *Suppose* $\mathbf{s}$ *is a sequence of strictly increasing positive integers. Then* $\mathbf{P}_d^{(\mathbf{s})}$ *is translation equivalent to a Fano polytope if and only if* $s_1 = 2$ *and* $s_{i+1} \leq 2s_i$ *for all* $1 \leq i \leq d-1$*. Moreover, if* $\mathbf{P}_d^{(\mathbf{s})}$ *is Fano, the unique interior point of* $\mathbf{P}_d^{(\mathbf{s})}$ *is* $(s_d - 1, s_{d-1} - 1, \ldots, s_2 - 1, s_1 - 1)^T$.

Proof. Suppose that $\mathbf{s}$ is a sequence with the property that $s_1 = 2$ and $s_{i+1} \leq 2s_i$. We will show that this implies that $\mathbf{P}_d^{(\mathbf{s})}$ is Fano. It is sufficient to show that $\mathbf{I}_d^{(\mathbf{s})}$ has exactly 1 inversion sequence $\mathbf{e}$ such that asc $\mathbf{e} = d$, as this implies that $\delta_d(\mathbf{P}_d^{(\mathbf{s})}) = 1$ by Lemma 1.1. If we let $\mathbf{e} = (s_1 - 1, s_2 - 1, s_3 - 1, \ldots, s_d - 1)$, we should note that asc $\mathbf{e} = d$ because

$$\frac{s_i - 1}{s_i} < \frac{s_{i+1} - 1}{s_{i+1}}$$

follows from the fact that $-s_{i+1} < -s_i$ which is true by assumption. To claim that this is the only such inversion sequence note that

$$\frac{s_i - 1}{s_i} < \frac{s_{i+1} - 2}{s_{i+1}}$$

is never true for any i because this would imply that $-s_{i+1} < -2s_i$ which is false by assumption. Moreover, in order for $\mathbf{e}$ to have an ascent in position 1, we need $e_1 = 1 = s_1 - 1$, so it follows that there is a single inversion sequence of this type. Hence, Additionally, we should note that because we have

$$0 < \frac{s_1 - 1}{s_1} < \frac{s_2 - 1}{s_2} < \cdots < \frac{s_d - 1}{s_d} < 1$$

it follows that the point $(s_d - 1, s_{d-1} - 1, \ldots, s_2 - 1, s_1 - 1)^T$ does not lie on a supporting hyperplane and is hence the unique interior point of $\mathbf{P}_d^{(\mathbf{s})}$.

Now, suppose that $\mathbf{s}$ is not of the prescribed form. We will show that $\mathbf{P}_d^{(\mathbf{s})}$ is not Fano. There are three possible cases:

(i) $s_1 = 1$;
(ii) $s_1 \geq 3$;
(iii) $s_1 = 2$ and $s_{i+1} > 2s_i$ for some $1 \leq i \leq d-1$.

Each of these cases preclude $\mathbf{P}_d^{(\mathbf{s})}$ from being Fano.

For (i), if $s_1 = 1$, it is clear from the vertex description of the polytope that $\mathbf{P}_d^{(\mathbf{s})} \cong \mathrm{Pyr}(\mathbf{P}_{d-1}^{(s_2, s_3, \ldots, s_d)})$ and hence $\delta_d(\mathbf{P}_d^{(\mathbf{s})}) = 0$.

For (ii), if $s_1 \geq 3$, it is easy to see that $\mathbf{P}_d^{(3,4,\ldots,d+2)} \subseteq \mathbf{P}_d^{(\mathbf{s})}$. We can see that $\delta_d(\mathbf{P}_d^{(3,4,\ldots,d+2)}) \geq 2$ because both the inversion sequences $\mathbf{e} = (1, 2, \ldots, d)$ and $\mathbf{e}' = (2, 3, \ldots, d+1)$ have the property asc $\mathbf{e} =$ asc $\mathbf{e}' = d$. So, $\mathbf{P}_d^{(3,4,\ldots,d+2)}$ has at least 2 interior points, which must also be interior points of $\mathbf{P}_d^{(\mathbf{s})}$, meaning it is not Fano.

For (iii), if we have $s_1 = 2$ but that there exists at least one $1 \leq i \leq d-1$ such that $s_{i+1} > 2s_i$. If there exist multiple such i, choose the smallest. We can see that $\mathbf{P}_d^{(\mathbf{t})} \subseteq \mathbf{P}_d^{(\mathbf{s})}$, where $\mathbf{t} = (s_1, \ldots, s_i, 2s_i + 1, 2s_i + 2, \ldots, 2s_i + (d-i+1))$. If we consider this smaller polytope, we can again ascertain that $\delta_d(\mathbf{P}_d^{(\mathbf{t})}) \geq 2$. Note that $\mathbf{e} = (s_1 - 1, \ldots s_i - 1, 2s_i, 2s_i + 1, \ldots, 2s_i + (d-i))$ has asc $\mathbf{e} = d$ as

$$\frac{s_i - 1}{s_i} < \frac{2s_i}{2s_i + 1}$$

follows from $-s_i - 1 < 0$ and the other inequalities follow from previous arguments. However, $\mathbf{e}' = (s_1 - 1, \ldots s_i - 1, 2s_i - 1, 2s_i, \ldots, 2s_i + (d-i-1))$ also has the property asc $\mathbf{e}' = d$ because

$$\frac{s_i - 1}{s_i} < \frac{2s_i - 1}{2s_i + 1}$$

is follows from $-1 < 0$ and

$$\frac{2s_i + k}{2s_i + k + 2} < \frac{2s_i + k + 1}{2s_i + k + 3}$$

follows from $0 < 4s_i + 2k + 6$. Hence, $\mathbf{P}_d^{(\mathbf{t})}$, and therefore $\mathbf{P}_d^{(\mathbf{s})}$, has at least two interior points, and is not Fano. □

We can go further to provide necessary and sufficient conditions for when $\mathbf{P}_d^{(\mathbf{s})}$ is translation equivalent to a reflexive polytope.

Theorem 2.3. *Suppose that* $\mathbf{s}$ *is a sequence of strictly increasing positive integers such that* $\mathbf{P}_d^{(\mathbf{s})}$ *is Fano. Then* $\mathbf{P}_d^{(\mathbf{s})}$ *is reflexive (up to translation) if and only if for each* $0 \leq i \leq d-1$, $k_i = s_{i+1} - s_i$ *has the property* $k_i | s_i$ *and* $k_i | s_{i+1}$.

Proof. If $\mathbf{P}_d^{(\mathbf{s})}$ is Fano, by Theorem 2.2 we know that the interior point is $(s_d - 1, s_{d-1} - 1, \ldots, s_2 - 1, s_1 - 1)^T$. If we translate $\mathbf{P}_d^{(\mathbf{s})}$ such that the interior point is the origin, the resulting polytope has vertices given by the columns of

$$\begin{bmatrix} 1 - s_d & 1 & 1 & 1 & \cdots & 1 \\ 1 - s_{d-1} & 1 - s_{d-1} & 1 & 1 & \cdots & 1 \\ 1 - s_{d-2} & 1 - s_{d-2} & 1 - s_{d-2} & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ -1 & -1 & -1 & \cdots & -1 & 1 \end{bmatrix}$$

This polytope has $\mathcal{H}$-representation

- $x_d \leq 1$
- $s_{i+1}x_i - s_i x_{i+1} \leq s_{i+1} - s_i$ for all $1 \leq i \leq d-1$
- $-x_1 \leq 1$

using the convention of x_d given by the first row and so on with x_1 given by the last row, as it is clear that each vertex satisfies d equations with equality and 1 with strict inequality.

It follows then that $(\mathbf{P}_d^{(\mathbf{s})})^\vee$ is a lattice polytope if and only if $k_i | s_i$ and $k_i | s_{i+1}$ where $k_i = s_{i+1} - s_i$. $\square$

We have the following corollary.

Corollary 2.4. *Suppose* $\mathbf{s}$ *is a sequence of strictly increasing positive integers. Then* $\mathbf{P}_d^{(\mathbf{s})}$ *is Gorenstein of index 2 if and only* $\mathbf{s} = (\frac{t_1}{2}, \frac{t_2}{2}, \ldots, \frac{t_d}{2})$ *where* $\mathbf{t} = (t_1, \ldots, t_d)$ *is a sequence such that* $\mathbf{P}_d^{(\mathbf{t})}$ *is reflexive. Moreover, there is no sequence* $\mathbf{s}$ *of strictly increasing positive integers such that* $\mathbf{P}_d^{(\mathbf{s})}$ *is Gorenstein of index* ≥ 3.

Proof. This follows immediately from the observation that $r\mathbf{P}_d^{(\mathbf{s})} = \mathbf{P}_d^{(rs_1, rs_2, \ldots, rs_d)}$ and the condition that $s_1 = 2$ when $\mathbf{P}_d^{(\mathbf{s})}$ is reflexive. $\square$

2.2. **Constant then strictly increasing s-sequences.** Suppose that we have a sequence of positive integers $\mathbf{s} = (s_1, s_2, \ldots, s_i, s_{i+1}, \ldots, s_d)$ such that $s_1 = s_2 = \cdots = s_i$ and $s_j < s_{j+1}$ for all $j \geq i$. We will given necessary and sufficient conditions for when $\mathbf{P}_d^{(\mathbf{s})}$ is translation equivalent to a Fano polytope for such sequences.

Theorem 2.5. *Suppose that* $\mathbf{s}$ *is a sequence such that* $s_1 = \cdots = s_i$ *for some* $1 \leq i \leq d$ *,and* $s_j < s_{j+1}$ *for all* $i \leq j \leq d-1$. *The polytope* $\mathbf{P}_d^{(\mathbf{s})}$ *is translation equivalent to a Fano polytope if and only if* $s_1 = \cdots = s_i = i+1$ *and for all* $j \geq i$, $s_{j+1} \leq 2s_j$. *Moreover, the unique interior point is* $(s_d - 1, \ldots, s_{i+1} - 1, i, i-1, \ldots, 2, 1)^T$.

Proof. Suppose that $\mathbf{s}$ is a sequence of this form such that $s_1 = \cdots = s_i = i+1$ and $s_{j+1} \leq 2s_j$ for all $j \geq i$. We will show that $\delta_d(\mathbf{P}_d^{(\mathbf{s})}) = 1$ by showing that there is a unique inversion sequence $\mathbf{e}$ such that asc $\mathbf{e} = d$. Let $\mathbf{e} = (1, 2, \ldots, i, s_{i+1} - 1, s_{i+2} - 1, \ldots, s_d - 1)$. It is clear that this sequence has d ascents, as $\frac{c}{i+1} < \frac{c+1}{i+1}$ for all $1 \leq c \leq i$,

$$\frac{s_j - 1}{s_j} < \frac{s_{j+1} - 1}{s_{j+1}}$$

for all $j > i$ because $s_j < s_{j+1}$, and

$$\frac{i}{i+1} < \frac{s_{i+1} - 1}{s_{i+1}} = 1 - \frac{1}{s_{i+1}}$$

because $s_{i+1} > i+1$. To claim that this is the unique such inversion sequence, note that the only way to obtain an ascent each of the first i positions is have the sequence begin $1, 2, \ldots, i$. From previous work, we know that

$$\frac{s_j - 1}{s_j} < \frac{s_{j+1} - 2}{s_{j+1}}$$

cannot hold by the assumption $s_{j+1} \leq 2s_j$ for all $j \geq i$. This ensures that no other such inversion sequence with d ascents exists. Thus, we have $\delta_d(\mathbf{P}_d^{(\mathbf{s})}) = 1$ so the polytope is Fano. Additionally, because we have

$$0 < \frac{1}{i+1} < \cdots < \frac{i}{i+1} < \frac{s_{i+1} - 1}{s_{i+1}} < \cdots < \frac{s_d - 1}{s_d} < 1$$

the point $(s_d - 1, \ldots, s_{i+1} - 1, i, i-1, \ldots, 2, 1)^T$ is in $\mathbf{P}_d^{(\mathbf{s})}$ and cannot lie on any supporting hyperplane and is hence the unique interior point.

Now, suppose that $\mathbf{s}$ does not have the desired properties. We will show that $\mathbf{P}_d^{(\mathbf{s})}$ is not Fano. There are 3 possibilities:

(i) $s_1 = \cdots = s_i \leq i$;
(ii) $s_1 = \cdots = s_i \geq i+2$;
(iii) $s_1 = \cdots = s_i = i+1$, but there exists some $j \geq i$ such that $2s_j < s_{j+1}$

Each of these cases preclude $\mathbf{P}_d^{(\mathbf{s})}$ from being Fano.

For (i), note that it is impossible for there to be an ascent in each of the first i positions. Hence, we have $\delta_d(\mathbf{P}_d^{(\mathbf{s})}) = 0$.

For (ii), notice that $\mathbf{P}_d^{(i+2,\ldots,i+2,i+3,i+4,\ldots,d+2)} \subset \mathbf{P}_d^{(\mathbf{s})}$. If we consider inversion sequences in $\mathbf{I}_d^{(i+2,\ldots,i+2,i+3,i+4,\ldots,d+2)}$, we have that both $\mathbf{e} = (1, 2, \ldots, i, i+1, i+2, \ldots, d)$ $\mathbf{e}' = (2, 3, \ldots, i+1, i+2, i+3, \ldots, d+1)$ have the property asc $\mathbf{e} =$ asc $\mathbf{e}' = d$ and hence $\delta_d(\mathbf{P}_d^{(i+2,\ldots,i+2,i+3,i+4,\ldots,d+2)}) \geq 2$, which implies it has at least two interior points, which are also interior points of $\mathbf{P}_d^{(\mathbf{s})}$.

For (iii), note that $\mathbf{P}_d^{(2,3,\ldots,i+1,s_{i+1},\ldots,s_d)} \subset \mathbf{P}_d^{(\mathbf{s})}$. By the proof of Theorem 2.2, we know that $\delta_d(\mathbf{P}_d^{(2,3,\ldots,i+1,s_{i+1},\ldots,s_d)}) \geq 2$, which implies that $\delta_d(\mathbf{P}_d^{(\mathbf{s})}) \geq 2$. $\square$

Now that we have a complete characterization of when $\mathbf{P}_d^{(\mathbf{s})}$ is Fano for $\mathbf{s}$ of this type, we can now give necessary and sufficient conditions for when they are reflexive.

Theorem 2.6. *Suppose that* $\mathbf{s}$ *is a sequence such that* $s_1 = \cdots = s_i$ *for some* $1 \leq i \leq d$ *,and* $s_j < s_{j+1}$ *for all* $i \leq j \leq d-1$ *and suppose that* $\mathbf{P}_d^{(\mathbf{s})}$ *is Fano. Then* $\mathbf{P}_d^{(\mathbf{s})}$ *is reflexive if and only if for all* $i \leq j \leq d-1$ *we have* $k_j | s_j$ *and* $k_j | s_{j+1}$ *where* $k_j = s_{j+1} - s_j$.

Proof. By Theorem 2.5, we know that the interior point is $(s_d - 1, \ldots, s_{i+1} - 1, i, i-1, \ldots, 2, 1)^T$. If we translate $\mathbf{P}_d^{(\mathbf{s})}$ so the interior point is the origin, the resulting polytope has vertices given as the columns of

$$\begin{bmatrix} 1-s_d & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1-s_{d-1} & 1-s_{d-1} & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1-s_{d-2} & 1-s_{d-2} & 1-s_{d-2} & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1-s_{i+1} & 1-s_{i+1} & 1-s_{i+1} & \cdots & 1-s_{i+1} & 1 & 1 & \cdots & 1 & 1 \\ -i & -i & -i & \cdots & -i & -i & 1 & \cdots & 1 & 1 \\ 1-i & 1-i & 1-i & \cdots & 1-i & 1-i & 1-i & \cdots & 2 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \ddots & & \vdots \\ -1 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 & i-1 \end{bmatrix}.$$

This polytope has $\mathcal{H}$-representation

- $-x_1 \leq 1$;

- $x_d \leq 1$;
- $x_{j-1} - x_j \leq 1$ for all $2 \leq j \leq i$;
- $s_{j+1}x_j - s_jx_{j+1} \leq s_{j+1} - s_j$ for all $i \leq j \leq d-1$.

using the convention that x_d is given by the first row and so on with x_1 given by the last row. It is easy to see that each column of the matrix satisfies precisely d equations with equality and 1 with strict inequality validating the $\mathcal{H}$-representation. It follows then that the dual polytope $(\mathbf{P}_d^{(\mathbf{s})})^\vee$ is a lattice polytope exactly when $k_j | s_j$ and $k_j | s_{j+1}$ where $k_j = s_{j+1} - s_j$ for $i \leq j \leq d-1$. $\square$

We can additionally give a description of Gorenstein lecture hall polytopes where $\mathbf{s}$ is of this form.

Corollary 2.7. *Suppose that $\mathbf{s}$ is a sequence such that $s_1 = \cdots = s_i$ for some $1 \leq i \leq d$,and $s_j < s_{j+1}$ for all $i \leq j \leq d-1$. Then $\mathbf{P}_d^{(\mathbf{s})}$ is Gorenstein of index $k \in \mathbb{Z}_{>0}$ if and only if there exists a sequence $\mathbf{t} = (t_1, \ldots, t_d)$ such that $t_j = ks_j$ for all j (which implies that $t_1 = \cdots = t_i$ and $t_j < t_{j+1}$ for $j \geq i$) and $\mathbf{P}_d^{(\mathbf{t})}$ is reflexive.*

Proof. This is immediate with the observation that $k\mathbf{P}_d^{(\mathbf{s})} = \mathbf{P}_d^{(\mathbf{t})}$ and applying the conditions given in Theorem 2.6. $\square$

2.3. **s-sequences increasing by at most 1.** We now consider an additional subclass of $\mathbf{s}$-sequences. Suppose the $\mathbf{s} = (s_1, s_2, \ldots, s_d)$ is a sequence of positive integers such that $s_i \leq s_{i+1}$ and $0 \leq s_{i+1} - s_i \leq 1$ for all $1 \leq i \leq d-1$. We have the following characterizations for when $\mathbf{P}_d^{(\mathbf{s})}$ is Fano and reflexive.

Theorem 2.8. *Suppose that $\mathbf{s} = (s_1, s_2, \ldots, s_d)$ is a sequence of positive integers such that $s_i \leq s_{i+1}$ and $0 \leq s_{i+1} - s_i \leq 1$ for all $1 \leq i \leq d-1$. Then $\mathbf{P}_d^{(\mathbf{s})}$ is translation equivalent to a Fano polytope if and only if $s_d = d+1$. Moreover, the unique interior point is $(d, d-1, \ldots, 2, 1)^T$.*

Proof. Suppose that $s_d = d+1$. We will show that there is a unique $\mathbf{e} \in \mathbf{I}_d^{(\mathbf{s})}$ such that $\operatorname{asc} \mathbf{e} = d$. It is clear that the sequence $\mathbf{e} = (1, 2, \ldots, d)$ satisfies this property, as both $\frac{i}{k} < \frac{i+1}{k}$ and $\frac{i}{k} < \frac{i+1}{k+1}$ are true which implies $\frac{i}{s_i} < \frac{i+1}{s_{i+1}}$. Moreover, to have maximum ascents, we must have $e_i < e_{i+1}$, which means that if $e_d \leq d-1$, $e_1 = 0$ implying that there is no ascent in the first position. Thus, the sequence $\mathbf{e} = (1, 2, \ldots, d)$ is the only inversion sequence with d ascents, giving $\delta_d(\mathbf{P}_d^{(\mathbf{s})}) = 1$. It also follows that the unique interior point of $\mathbf{P}_d^{(\mathbf{s})}$ is $(d, d-1, \ldots, 2, 1)^T$, as

$$0 < \frac{1}{s_1} < \frac{2}{s_2} < \cdots < \frac{d}{s_d} < 1$$

implies that the point is in $\mathbf{P}_d^{(\mathbf{s})}$ and not on any supporting hyperplane.

Now, note that if $s_d \geq d+2$, both the inversion sequences $(1, 2, 3, \ldots, d)$ and $(2, 3, \ldots, d+1)$ has d ascents. Thus, $\delta_d(\mathbf{P}_d^{(\mathbf{s})}) \geq 2$ in this case.

If we have that $s_d \leq d$, it follows that $\mathbf{P}_d^{(\mathbf{s})} \subseteq \mathbf{P}_d^{(\mathbf{t})}$ where $\mathbf{t} = (d, d, \ldots, d)$. Since it is clear that for $\mathbf{e} \in \mathbf{I}_d^{(\mathbf{t})}$ we have $i \in \operatorname{Asc} \mathbf{e}$ if and only if $e_{i-1} < e_i$ and since $e_i \in \{0, 1, ..., d-1\}$ there is no sequence with $\operatorname{asc} \mathbf{e} = d$. Thus, we have $\delta_d(\mathbf{P}_d^{(\mathbf{s})}) = \delta_d(\mathbf{P}_d^{(\mathbf{t})}) = 0$. □

Theorem 2.9. *Suppose that $\mathbf{s} = (s_1, s_2, \ldots, s_d)$ is a sequence of positive integers such that $s_i \leq s_{i+1}$ and $0 \leq s_{i+1} - s_i \leq 1$ for all $1 \leq i \leq d-1$. and suppose that $\mathbf{P}_d^{(\mathbf{s})}$ is Fano. Then $\mathbf{P}_d^{(\mathbf{s})}$ is reflexive if and only if $k_i | s_i$ and $k_i | s_{i+1}$ where $k_i = (i+1)s_i - i s_{i+1}$.*

Proof. By Theorem 2.8, the interior point of $\mathbf{P}_d^{(\mathbf{s})}$ is $(d, d-1, \ldots, 2, 1)^T$. So, if we translate the polytope such that the origin is the interior point, we have the polytope with vertices

$$\begin{bmatrix} -d & 1 & 1 & 1 & \cdots & 1 \\ 1-d & 1-d & s_{d-1} - d + 1 & s_{d-1} - d + 1 & \cdots & s_{d-1} - d + 1 \\ 2-d & 2-d & 2-d & s_{d-2} - d + 2 & \cdots & s_{d-2} - d + 2 \\ \vdots & \vdots & \vdots & & \ddots & \cdots \\ -1 & -1 & -1 & \cdots & -1 & s_1 - 1 \end{bmatrix}$$

which, using the convention of x_d given by the first row and so on with x_1 given by the last row, has the $\mathcal{H}$-representation

- $x_d \leq 1$
- $s_{i+1} x_i - s_i x_{i+1} \leq (i+1)s_i - i s_{i+1}$ for all $1 \leq i \leq d-1$
- $-x_1 \leq 1$

as it is not hard to see that each vertex satisfies d equations with equality and 1 equation with strict inequality. It is now clear that $(\mathbf{P}_d^{(\mathbf{s})})^\vee$ is a lattice polytope if and only if $k_i | s_i$ and $k_i | s_{i+1}$ for $k_i = (i+1)s_i - i s_{i+1}$. □

3. Integral decomposition property and triangulations

We say $\mathcal{P}$ satisfies the *integral decomposition property (IDP)* if for all $\mathbf{z} \in k\mathcal{P} \cap \mathbb{Z}^d$ there exists $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in \mathcal{P} \cap \mathbb{Z}^d$ such that

$$\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_k = \mathbf{z}.$$

If $\mathcal{P}$ satisfies then integer decomposition property, we say that $\mathcal{P}$ is IDP. For $\mathbf{s}$-lecture hall polytopes where $\mathbf{s}$ is monotonic sequence, we have the following theorem.

Theorem 3.1. *Let $\mathbf{s} = (s_1, s_2, \ldots, s_d)$ be a monotone sequence of positive integers. Then the polytope $\mathbf{P}_d^{(\mathbf{s})}$ is IDP.*

Proof. Without loss of generality, suppose that $\mathbf{s}$ is increasing. We will show that given $k \geq 2$, for any $\mathbf{x} \in k\mathbf{P}_d^{(\mathbf{s})} \cap \mathbb{Z}^d$, there exists some $\mathbf{y} \in \mathbf{P}_d^{(\mathbf{s})} \cap \mathbb{Z}^d$ such that $(\mathbf{x} - \mathbf{y}) \in (k-1)\mathbf{P}_d^{(\mathbf{s})} \cap \mathbb{Z}^d$. Note that this is sufficient, because this result allows integral closure to follow from induction on k.

First note that $k\mathbf{P}_d^{(\mathbf{s})} = \mathbf{P}_d^{(ks_1,ks_2,\ldots,ks_d)}$, which is clear by definition. Let $\mathbf{x} = (x_d, x_{d-1}, \ldots, x_1)^T \in k\mathbf{P}_d^{(\mathbf{s})} \cap \mathbb{Z}^d$, so we have that $\mathbf{x}$ satisfies

$$0 \leq \frac{x_1}{ks_1} \leq \frac{x_2}{ks_2} \leq \cdots \leq \frac{x_d}{ks_d} \leq 1.$$

Note that since $\mathbf{s}$ is increasing, given any $C \in \mathbb{Z}_{>0}$ by the above we must have that $x_i \leq Cs_i$ implies that $x_{i-1} \leq Cs_{i-1}$ and likewise $x_i > Cs_i$ implies $x_{i+1} > Cs_{i+1}$. So, let $1 \leq j \leq d$ be the minimum index such that $x_j > (k-1)s_j$. Then we let

$$\mathbf{y} = (x_d - (k-1)s_d, \ldots, x_j - (k-1)s_j, 0, \ldots, 0)^T$$

with $\mathbf{y} = \mathbf{0}$ if there is no such j.

We know that the lattice point is in $\mathbf{P}_d^{(\mathbf{s})}$ because for any $j \leq i < d$ we have

$$\frac{x_i - (k-1)s_i}{s_i} \leq \frac{x_{i+1} - (k-1)s_{i+1}}{s_{i+1}}$$

is equivalent to $\dfrac{x_i}{ks_i} \leq \dfrac{x_{i+1}}{ks_{i+1}}$ and $0 < x_i - (k-1)s_i \leq s_i$ by construction.

It is left to verify that $(\mathbf{x} - \mathbf{y}) = ((k-1)s_d, \ldots, (k-1)s_j, x_{j-1}, \ldots, x_1)^T \in \mathbf{P}_d^{((k-1)s_1,\ldots,(k-1)s_d)} \cap \mathbb{Z}^d$. However, this is immediate, because $\dfrac{x_i}{(k-1)s_i} \leq \dfrac{x_{i+1}}{(k-1)s_{i+1}}$ is equivalent to $\dfrac{x_i}{ks_i} \leq \dfrac{x_{i+1}}{ks_{i+1}}$ and it is clear that since $x_{j-1} \leq (k-1)s_{j-1}$ by assumption that

$$\frac{x_{j-1}}{(k-1)s_{j-1}} \leq \frac{(k-1)s_j}{(k-1)s_j} = 1.$$

Thus, we have the $\mathbf{P}_d^{(\mathbf{s})}$ is IDP. $\square$

Recall that a *triangulation* of a lattice polytope $\mathcal{P}$ is a subdivison of $\mathcal{P}$ into d-dimensional simplices. We say that a triangulation is *unimodular* if each simplex Δ of the triangulation is unimodularly equivalent to the standard d-simplex or equivalently, each simplex has smallest possible normalized volume $\mathrm{Vol}(\Delta) = 1$. One should note that a polytope $\mathcal{P}$ possessing a unimodular triangulation means that $\mathcal{P}$ can be covered by IDP polytopes which implies that $\mathcal{P}$ is IDP. We will show the existence for a unimodular triangulation of $\mathbf{P}_d^{(\mathbf{s})}$ provided that for all $1 \leq i \leq d-1$, $s_{i+1} = n_i s_i$ where $n_i \in \mathbb{Z}_{>0}$.

First, we define chimney polytopes. Given a polytope $\mathcal{P} \subset \mathbb{R}^d$ and two integral linear functionals ℓ and u such that $\ell \leq u$, then the *chimney polytope* associated to $\mathcal{P}$, ℓ, and u is

$$\mathrm{Chim}(\mathcal{P}, \ell, u) := \{(\mathbf{x}, y) \in \mathbb{R}^d \times \mathbb{R} \,:\, \mathbf{x} \in \mathcal{P},\ \ell(\mathbf{x}) \leq y \leq u(\mathbf{x})\}.$$

For chimney polytopes we have the following theorem regarding triangulations.

Lemma 3.2 ([7, Theorem 2.8]). *If $\mathcal{P}$ admits a unimodular triangulation, then so does* $\mathrm{Chim}(\mathcal{P}, \ell, u)$.

With this in mind, we can now state and prove a theorem for $\mathbf{P}_d^{(\mathbf{s})}$ where $\mathbf{s}$ is increasing of a particular form.

Theorem 3.3. *Let* $\mathbf{s}$ *be an increasing sequence of positive integers such that* $s_{i+1} = k_i s_i$ *for some* $k_i \in \mathbb{Z}_{>0}$ *for all* $1 \le i \le d-1$. *Then* $\mathbf{P}_d^{(\mathbf{s})}$ *admits a unimodular triangulation.*

Proof. Note that if $\mathbf{s}$ has the property $s_d = k_{d-1}s_{d-1}$ for some $k_{d-1} \in \mathbb{Z}_{>0}$, we can express $\mathbf{P}_d^{(\mathbf{s})}$ as a chimney polytope, namely

$$\mathbf{P}_d^{(\mathbf{s})} \cong \mathrm{Chim}(\mathbf{P}_{d-1}^{(s_1,\ldots,s_{d-1})}, k_{d-1}x_{d-1}, s_d)$$

where $\mathbf{s_d}$ is constant function of value s_d. It is easy to see this isomorphism as all of the supporting hyperplanes for $\mathrm{Chim}(\mathbf{P}_{d-1}^{(s_1,\ldots,s_{d-1})}, k_{d-1}x_{d-1}, s_d)$ are those of $\mathbf{P}_{d-1}^{(s_1,\ldots,s_{d-1})}$ with the addition of $x_d \le s_d$ and $k_{d-1}x_{d-1} \le x_d$. However, these hyperplanes are precisely the supporting hyperplanes of $\mathbf{P}_d^{(\mathbf{s})}$.

Now, note that any 1 dimensional lecture hall polytope trivially has a unimodular triangulation. So, if $\mathbf{s}$ has the property that $s_{i+1} = k_i s_i$ for a positive integer k_i for each i, then applying Theorem 3.2 to this inductive chimney polytope construction of $\mathbf{P}_d^{(\mathbf{s})}$ yields the existence of a unimodular triangulation. $\square$

Remark 3.4. *We should note that Theorem 3.3 implies that* $\mathbf{P}_d^{(\mathbf{s})}$ *where* $\mathbf{s}$ *has the property* $s_{i+1} = \frac{s_i}{k_i}$ *for some positive integer* k_i *for all* i *also admits a unimodular triangulation.*

4. Constructing new examples

In this section, we construct new Gorenstein and IDP lecture hall polytopes. We will do this by identifying an $\mathbf{s}$-lecture hall polytope as the free sum of two smaller lecture hall polytopes which are Gorenstein and/or IDP.

Recall that given two lattice polytopes $\mathcal{P} \subset \mathbb{R}^{d_\mathcal{P}}$ and $\mathcal{Q} \subset \mathbb{R}^{d_\mathcal{Q}}$ such that $0_{d_\mathcal{P}} \in \mathcal{P}$ and $0_{d_\mathcal{Q}} \in \mathcal{Q}$, the *free sum* of $\mathcal{P}$ and $\mathcal{Q}$ is the $(d_\mathcal{P} + d_\mathcal{Q})$-dimensional polytope given by $\mathcal{P} \oplus \mathcal{Q} = \mathrm{conv}\{(0_\mathcal{P} \times \mathcal{Q}) \cup (\mathcal{P} \times 0_\mathcal{Q})\}$. We can view lecture hall polytopes as free sum of smaller lecture hall polytopes.

Proposition 4.1. *For integer sequences* $\mathbf{s} = (s_1, \ldots, s_d)$ *and* $\mathbf{t} = (t_1, \ldots, t_e)$, *we have* $\mathbf{P}_{d+e}^{(\mathbf{s},\mathbf{t})} \cong \mathbf{P}_d^{(\mathbf{s})} \oplus \mathbf{P}_e^{(\tilde{\mathbf{t}})}$, *where* $(\mathbf{s}, \mathbf{t}) = (s_1, \ldots, s_d, t_1, \ldots, t_e)$ *and* $\tilde{\mathbf{t}} = (t_d, t_{d-1}, \ldots, t_1)$.

Proof. Translate by the vector $(t_e, \ldots, t_2, t_1, 0, 0, \ldots, 0)^T$. $\square$

The following generalization of Braun's formula gives us conditions on the δ-polynomial of a free sum of two polytopes.

Lemma 4.2 ([3, Theorem 1.4]). *Let* $\mathcal{P} \subset \mathbb{R}^d$ *and* $\mathcal{Q} \subset \mathbb{R}^e$ *be integral convex polytopes each containing its respective origin. Then* $\delta(\mathcal{P} \oplus \mathcal{Q}, \lambda) = \delta(\mathcal{P}, \lambda)\delta(\mathcal{Q}, \lambda)$ *holds if* *and only if either $\mathcal{P}$ or $\mathcal{Q}$ satisfies that the equation of each facet is of the form $\sum_{i=1}^{f} a_i x_i = b$ where a_i is an integer, $b \in \{0,1\}$, and $f \in \{d,e\}$.*

We can now give a construction for larger lecture hall polytopes which must be Gorenstein.

Theorem 4.3. *Suppose that $\mathbf{s} = (s_1, s_2, \ldots, s_d)$ and $\mathbf{t} = (t_1, t_2, \ldots, t_e)$ are integer sequences such that $\mathbf{P}_d^{(\mathbf{s})}$ is Gorenstein of index k and $\mathbf{P}_e^{(\mathbf{t})}$ is Gorenstein of index ℓ. Then $\mathbf{P}_{d+e+1}^{(\mathbf{s},1,\mathbf{t})}$ is Gorenstein of index $k+\ell$.*

Proof. Note that by Proposition 4.1, we have that $\mathbf{P}_{d+e+1}^{(\mathbf{s},1,\mathbf{t})} \cong \mathbf{P}_{d+1}^{(\mathbf{s},1)} \oplus \mathbf{P}_e^{(\tilde{\mathbf{t}})}$. By the $\mathcal{H}$-representation, we know that $\mathbf{P}_{d+1}^{(\mathbf{s},1)}$ satisfies that the equation of each facet is of the form $\sum_{i=1}^{d+1} a_i x_i = b$ where a_i is an integer, $b \in \{0,1\}$. Moreover, from the $\mathcal{V}$-represntation it is clear that $\mathbf{P}_{d+1}^{(\mathbf{s},1)} \cong \mathrm{Pyr}(\mathbf{P}_d^{(\mathbf{s})})$, so it has the same δ-vector and is thus Gorenstein. By Lemma 4.2, we then know that $\delta(\mathbf{P}_{d+e+1}^{(\mathbf{s},1,\mathbf{t})}, \lambda) = \delta(\mathbf{P}_d^{(\mathbf{s})}, \lambda)\delta(\mathbf{P}_e^{(\mathbf{t})}, \lambda)$ because $\mathbf{P}_e^{(\tilde{t})} \cong \mathbf{P}_e^{(t)}$. Therefore, $\delta(\mathbf{P}_{d+e+1}^{(\mathbf{s},1,\mathbf{t})}, \lambda)$ is symmetric polynomial of degree $(d+e+1) - (k+\ell) + 1$ and we have the desired. □

Additionally, necessary and sufficient conditions for the integral closure of a free sum of two polytopes are known. These are given in the following theorem.

Lemma 4.4 ([11, Theorem 0.1])**.** *Let $\mathcal{P} \subset \mathbb{R}^d$ and $\mathcal{Q} \subset \mathbb{R}^e$ be integral convex polytopes each containing its respective origin. Suppose that $\mathcal{P}$ and $\mathcal{Q}$ satisfy $\mathbb{Z}(\mathcal{P} \cap \mathbb{Z}^d) = \mathbb{Z}^d$, $\mathbb{Z}(\mathcal{Q} \cap \mathbb{Z}^e) = \mathbb{Z}^e$, and*

$$(\mathcal{P} \oplus \mathcal{Q}) \cap \mathbb{Z}^{d+e} = \mu(\mathcal{P} \cap \mathbb{Z}^d) \cup \nu(\mathcal{Q} \cap \mathbb{Z}^e)$$

where μ and ν are the canonical injections defined $\mu : \mathbb{R}^d \to \mathbb{R}^{d+e}$ by $\alpha \mapsto (\alpha, 0_e)$ and $\nu : \mathbb{R}^e \to \mathbb{R}^{d+e}$ by $\beta \mapsto (0_d, \beta)$. Then the free sum $\mathcal{P} \oplus \mathcal{Q}$ is IDP if and only if the following two conditions hold:

- *each of $\mathcal{P}$ and $\mathcal{Q}$ is IDP;*
- *either $\mathcal{P}$ or $\mathcal{Q}$ has the property that the equation of each facet is of the form $\sum_{i=1}^{f} a_i x_i = b$ where a_i is an integer, $b \in \{0,1\}$, and $f \in \{d,e\}$.*

We can now give a construction for larger IDP lecture hall polytopes.

Theorem 4.5. *Suppose that $\mathbf{s} = (s_1, s_2, \ldots, s_d)$ and $\mathbf{t} = (t_1, t_2, \ldots, t_e)$ are integer sequences such that $\mathbf{P}_d^{(\mathbf{s})}$ and $\mathbf{P}_e^{(\mathbf{t})}$ are IDP. Then $\mathbf{P}_{d+e+1}^{(\mathbf{s},1,\mathbf{t})}$ is IDP.*

Proof. Note that for any 2 lecture hall polytopes $\mathbf{P}_d^{(\mathbf{s})}$ and $\mathbf{P}_e^{(\mathbf{t})}$, we have $\mathbb{Z}(\mathbf{P}_d^{(\mathbf{s})} \cap \mathbb{Z}^d) = \mathbb{Z}^d$ and $\mathbb{Z}(\mathbf{P}_e^{(\mathbf{t})} \cap \mathbb{Z}^e) = \mathbb{Z}^e$ follow immediately.

Now, by Proposition 4.1, we have that $\mathbf{P}_{d+e+1}^{(\mathbf{s},1,\mathbf{t})} \cong \mathbf{P}_{d+1}^{(\mathbf{s},1)} \oplus \mathbf{P}_e^{(\tilde{\mathbf{t}})}$. By the $\mathcal{H}$-representation, we know that $\mathbf{P}_{d+1}^{(\mathbf{s},1)}$ satisfies that the equation of each facet is of the form $\sum_{i=1}^{d+1} a_i x_i = b$ where a_i is an integer, $b \in \{0,1\}$. To see that

$$(\mathbf{P}_{d+1}^{(\mathbf{s},1)} \oplus \mathbf{P}_e^{(\tilde{\mathbf{t}})}) \cap \mathbb{Z}^{d+1+e} = \mu(\mathbf{P}_{d+1}^{(\mathbf{s},1)} \cap \mathbb{Z}^{d+1}) \cup \nu(\mathbf{P}_e^{(\tilde{\mathbf{t}})} \cap \mathbb{Z}^e)$$

holds, note that the right side is clearly contained in the left side. If we consider an element $\mathbf{x}$ such that

$$\mathbf{x} \in (\mathbf{P}_{d+1}^{(\mathbf{s},1)} \oplus \mathbf{P}_{e}^{(\tilde{\mathbf{t}})}) \cap \mathbb{Z}^{d+1+e} \setminus \left(\mu(\mathbf{P}_{d+1}^{(\mathbf{s},1)} \cap \mathbb{Z}^{d+1}) \cup \nu(\mathbf{P}_{e}^{(\tilde{\mathbf{t}})} \cap \mathbb{Z}^{e})\right),$$

we have that $\sum_{i=1}^{d+e+1} c_i v_i = 1$ where c_i is constant and v_i is the ith vertex. However, we also must have that $x_{d+1} = 1$, which implies that $\sum_{i=1}^{d+1} c_i = 1$ from the definition of the free sum. So this implies that $\mathbf{x} \in \mu(\mathbf{P}_{d+1}^{(\mathbf{s},1)} \cap \mathbb{Z}^{d+1})$ which is a contradiction. The result now follows from Lemma 4.4. □

5. Concluding Remarks

While we have been able to ascertain many previously unknown properties of lecture hall polytopes, full characterizations of all of these properties remain elusive. We conclude with two conjectures.

Conjecture 5.1. *For any* $\mathbf{s} = (s_1, \cdots, s_d)$*,* $\mathbf{P}_d^{(\mathbf{s})}$ *is IDP.*

For many randomly generated $\mathbf{s}$, we have found $\mathbf{P}_d^{(\mathbf{s})}$ to be IDP and we have been unable to find an example of a non IDP lecture hall polytope. Additionally, the convenient description of dilates of lecture hall polytopes, namely $c\mathbf{P}_d^{(\mathbf{s})} = \mathbf{P}_d^{(cs_1, cs_2, \cdots, cs_d)}$, suggests that one may be able to generalize our arguments for monotone sequences to arbitrary $\mathbf{s}$.

Conjecture 5.2. *For any* $\mathbf{s} = (s_1, \cdots, s_d)$*,* $\mathbf{P}_d^{(\mathbf{s})}$ *admits a unimodular triangulation.*

We have come across no examples of lecture hall polytopes which do not admit a unimodular triangulation. However, using Gröbner bases has not proved fruitful given that though a variable ordering and monomial ordering which yield a quadratic squarefree Gröbner basis seem to always exist, it is not always the same ordering. A positive answer to this conjecture would resolve Conjecture 5.1 as well. Moreover, a counterexample, or a positive partial result such as the monotone case would be of great interest.

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