

LIE ALGEBRAS AND AROUND: SELECTED QUESTIONS

PASHA ZUSMANOVICH

To my teacher and friend Askar Serkulovich Dzhumadil'daev on his 60th birthday

ABSTRACT. Several open questions are discussed. The topics include cohomology of current and related Lie algebras, algebras represented as the sum of subalgebras, structures and phenomena peculiar to characteristic 2, and variations on themes of Ado, Whitehead, and Banach.

INTRODUCTION

I am presenting here a, perhaps, haphazard collection of questions I am interested in. Being haphazard as it is, this collection features somewhat unexpected and, hopefully, fascinating connections between different topics.

This is a modest contribution in honor of Askar Dzhumadil'daev. I first met him in 1983, when, as an undergraduate student, I started to participate in his seminar on Lie algebras. Since then and throughout many years, I enjoyed his wisdom, unfailing enthusiasm, friendship, and support. Most of what I know in mathematics I owe to him.

1. COHOMOLOGY OF LIE ALGEBRAS COMING FROM KOSZUL DUAL OPERADS

Current Lie algebras – that is, Lie algebras of the form $L \otimes A$, where L is a Lie algebra and A is a commutative associative algebra – as well as algebras close to them, are ubiquitous in mathematics and physics (sufficient is to mention Lie algebras of smooth functions on a manifold prominent in gauge theory, Kac–Moody Lie algebras, modular semisimple Lie algebras, etc.). The Lie bracket is defined in an obvious factor-wise way:

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab,$$

where $x, y \in L$, $a, b \in A$. A vast generalization of this construction comes from the operad theory: if A is an algebra over a binary quadratic operad \mathcal{P} , and B is an algebra over the operad Koszul dual to \mathcal{P} , then the tensor product $A \otimes B$ equipped with the bracket

$$(1) \quad [a \otimes b, a' \otimes b'] = aa' \otimes bb' - a'a \otimes b'b,$$

where $a, a' \in A$, $b, b' \in B$, is a Lie algebra.

Due to a big flexibility of this construction, many interesting Lie algebras can be represented in this form, for a suitable pair of Koszul dual operads and algebras over them. Perhaps the most remarkable examples, beyond current Lie algebras, are various algebras appearing in physics (Schrödinger–Virasoro, Heisenberg–Virasoro, etc.): they are represented as tensor products of algebras over left and right Novikov operads. This remarkable observation was implicit already in the pioneering works of I. Gelfand and S.P. Novikov and their collaborators ([GD] and [BN]); after that, Pei and Bai ([PB] and references therein) noted that Lie algebras in question admit realization as affinizations of certain left Novikov algebras; Dzhumadil'daev

Date: First written April 25, 2016; last revised May 7, 2016.

2010 Mathematics Subject Classification. 17B05; 17B10; 17B50; 17B55; 17B56; 05A99.

Key words and phrases. Lie algebra; Zassenhaus algebra; cohomology; deformations; Koszul dual operads; sum of subalgebras; characteristic 2; Ado Theorem; Whitehead Lemma; ternary maps .

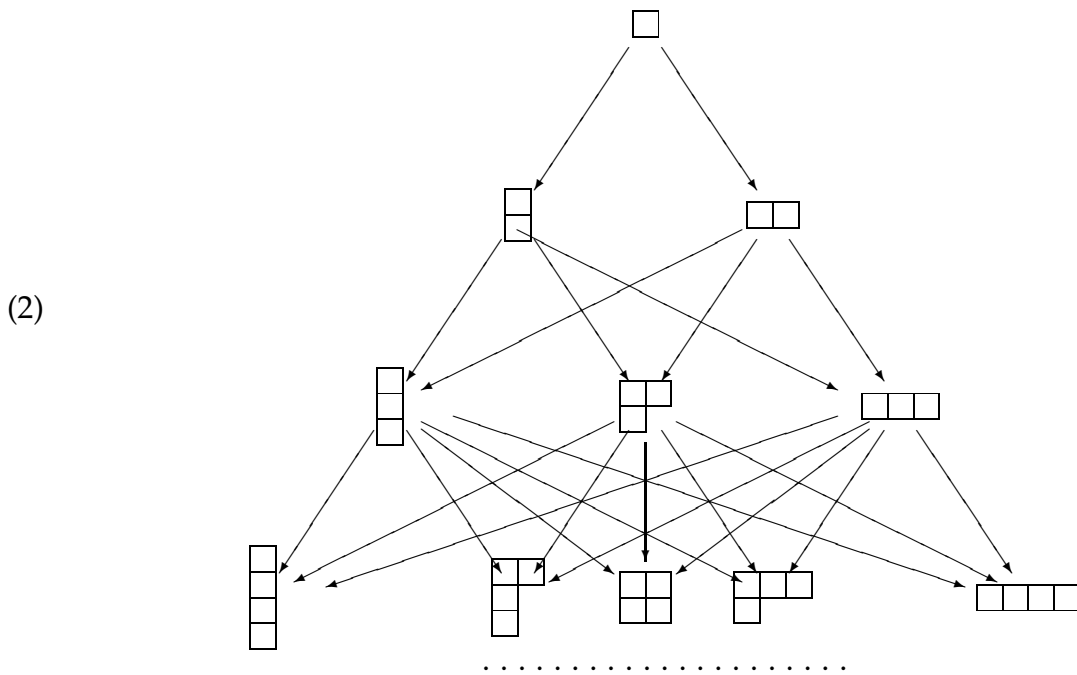
noted in [Dz4] that left and right Novikov operads are Koszul dual to each other; the explicit claim was made in [Z5, §5] by putting all the pieces together.

Therefore, it seems to be interesting to develop a general method for computing cohomology and other invariants of Lie algebras given by bracket (1). By this, we mean to express cohomology or other invariants of such Lie algebras in terms of invariants of tensor factors A and B , similarly how it was done earlier for low-degree cohomology of current Lie algebras.

Let us try to pursue further an approach for computation of (co)homology of current Lie algebras described in [Z2, §4]. (For simplicity, we will consider cohomology with coefficients in the trivial module K , assume some finiteness conditions, and zero characteristic of the base field, but similar considerations may be applied in more general settings). Let us decompose the modules $\wedge^n(A \otimes B)$ according to the well-known Cauchy formula:

$$\wedge^n(A \otimes B) \simeq \bigoplus_{\lambda \vdash n} (Y_\lambda(A) \otimes Y_{\lambda^\sim}(B)).$$

Here the direct sum of vector spaces is taken over all Young diagrams of size n , λ^\sim is the Young diagram transposed to λ , and Y_λ is the Young symmetrizer corresponding to λ . Assuming that at least one of the algebras A, B is finite-dimensional, passing to the decomposition of the dual vector spaces, and decomposing the differential $d : (\wedge^n(A \otimes B))^* \rightarrow (\wedge^{n+1}(A \otimes B))^*$ in the Chevalley–Eilenberg complex accordingly, we get the following maps on the Young graph:



Here each Young diagram λ denotes the vector space $Y_\lambda(A)^* \otimes Y_{\lambda^\sim}(B)^*$, and arrows denote the corresponding components of the differential d .

Intuitively it should be clear that the more arrows here vanish, the easier it would be to compute the corresponding cohomology. In the case of the pair of operads (Lie, associative commutative), i.e. for current Lie algebras $L \otimes A$, a miracle happens: approximately half of the arrows vanish (roughly, those going from “left” to “right”), what allows to define a certain spectral sequence on the Chevalley–Eilenberg complex computing the cohomology $H^*(L \otimes A, K)$. In the low cohomology degrees and/or for particular types of algebras, this spectral sequence allows to express $H^*(L \otimes A, K)$ in terms of cohomology and other invariants of the tensor factors L and A . Unfortunately, this miracle fails for the other pairs of Koszul dual operads, even such a classical one as (associative, associative).

Question 1. *What makes the pair (Lie, associative commutative) special in this regard? In which other situations (i.e., for a particular pair of Koszul dual operads, or for a particular kind of algebras over some pair of Koszul dual operads) “many” arrows in the corresponding Young graph (2) will vanish? In particular, for which types of Lie algebras expressed as the tensor products of left Novikov and right Novikov algebras, this will happen?*

2. ALGEBRAS REPRESENTED AS THE SUM OF SUBALGEBRAS

My mathematical debut, under the guidance of Askar Dzhumadil’daev, was the proof of the Kegel–Kostrikin conjecture about solvability of a modular finite-dimensional Lie algebra L represented as the vector space sum $L = N + M$ of two nilpotent subalgebras N, M ([Z1]; around the same time this was established also by Panyukov, [Pa]). The statement is true over fields of characteristic $p > 2$, but in characteristic 2 there is a counterexample found by Petravchuk, [Pe]. Take the 3-dimensional characteristic 2 analog of \mathfrak{sl}_2 : the simple Lie algebra S , with the basis $\{e_{-1}, e_0, e_1\}$ subject to multiplication

$$[e_{-1}, e_0] = e_{-1}, \quad [e_{-1}, e_1] = e_0, \quad [e_0, e_1] = e_1.$$

Its 2-envelope $S^{[2]}$ is 5-dimensional and admits the decomposition $S^{[2]} = N \oplus M$, where the 2-dimensional abelian subalgebra N is linearly spanned by $e_0 + e_{-1} + e_{-1}^{[2]}$ and $e_0 + e_1 + e_1^{[2]}$, and the 3-dimensional nilpotent subalgebra M is linearly spanned by $e_{-1}^{[2]}$, e_0 and $e_1^{[2]}$ (the vector space sum in this case is direct).

Do such examples exist in higher dimensions? Of course, any (finite- or infinite-dimensional) current Lie algebra $S \otimes A$ admits such a decomposition:

$$S \otimes A = (N \otimes A) \oplus (M \otimes A),$$

so it provides such an example provided it itself is non-solvable (for example, when A contains a unit).

A slightly more interesting example can be obtained as an extension of the corresponding current Lie algebras of the form $S \otimes A + \mathcal{D}$, where \mathcal{D} acts on A by derivations, what includes semisimple Lie algebras. Namely, we have the decomposition

$$(3) \quad S \otimes A + \mathcal{D} = (N \otimes A + \mathcal{D}) \oplus (M \otimes A).$$

An easy induction on n proves that for a Lie algebra $L = S \otimes A + \mathcal{D}$ with A unital, and for any positive integer n , we have

$$L^n = S^n \otimes A + \sum_{\substack{i+j=n \\ i>1, j \geq 1}} S^i \otimes A \mathcal{D}^j(A) + S \otimes \mathcal{D}^{n-1}(A) + \mathcal{D}^n.$$

This implies that if N is nilpotent, and \mathcal{D} is nilpotent as an algebra of derivations of A (and hence is nilpotent as an abstract Lie algebra), then the algebra $N \otimes A + \mathcal{D}$ is nilpotent too. Therefore, (3) provides a decomposition of a nonsolvable Lie algebra into the sum of nilpotent subalgebras.

Yet it would be more interesting to generalize Petravchuk’s decomposition for an arbitrary Zassenhaus algebra $W'_1(n)$ in characteristic 2. Zassenhaus algebras appear prominently in ongoing efforts of classification of simple Lie algebras in characteristic 2 (cf. [St, Vol. I, §7.6], [Sk], [GZ], and references therein). In characteristic $p = 2$, unlike for $p > 2$, the Zassenhaus algebra has dimension $2^n - 1$ and can be defined as the algebra with the basis $\{e_i \mid -1 \leq i \leq 2^n - 3\}$ subject to multiplication

$$[e_i, e_j] = \begin{cases} \binom{i+j+2}{i+1} e_{i+j} & \text{if } -1 \leq i+j \leq 2^n - 2 \\ 0 & \text{otherwise.} \end{cases}$$

The algebra $S = W_1'(2)$ is the first algebra in the series. The 2-envelope $W_1'(n)^{[2]}$ of $W_1'(n)$ coincides with the derivation algebra of $W_1'(n)$, has dimension $2^n + n - 1$, and is spanned, in addition to elements of $W_1'(n)$, by elements $(\text{ad } e_{-1})^{2^k}$, $k = 1, 2, \dots, n - 1$, and $(\text{ad } e_{2^{n-1}-1})^2$.

Question 2. Find a link with combinatorial interpretation of the number $2^n + n - 1$ as the shortest length of a sequence of 0 and 1 containing all subsequences of length n (see [OEIS, A052944]).

Question 3. Is it true that $W_1'(n)^{[2]}$ admits a decomposition into the sum of two nilpotent subalgebras?

Virtually nothing is known about the Kegel–Kostrikin question in the infinite-dimensional case – beyond almost obvious cases when one imposes some sort of finiteness conditions on one or both of the summands; all such cases are reduced quickly to the finite-dimensional situation.

As a first step, one may wish to prove that such an algebra satisfies a nontrivial (Lie) identity. According to [Z4, Corollary 2.5], a Lie algebra L does not satisfy a nontrivial identity if and only if a free Lie algebra is embedded into an ultraproduct of L . As the ultraproduct construction obviously commutes with the decomposition into the sum of subalgebras, the question whether a Lie algebra $L = N + M$ does not satisfy a nontrivial identity is equivalent to the question whether its ultraproduct $L^{\mathcal{U}} = N^{\mathcal{U}} + M^{\mathcal{U}}$ does not contain a free Lie subalgebra. As being nilpotent (of a fixed nilpotency index) is the first-order property, by the Łoś theorem the Lie algebras $N^{\mathcal{U}}$ and $M^{\mathcal{U}}$ are also nilpotent. Thus the question whether the sum of two nilpotent Lie algebras satisfies a nontrivial identity, is equivalent to an a priori more special

Question 4. Is it true that an infinite-dimensional Lie algebra represented as the sum of two nilpotent subalgebras, cannot contain a free Lie algebra as a subalgebra?

In the theory of (associative) PI algebras, there is a similar long-standing open question: whether the sum of two PI algebras is PI? (See [KLM] and (numerous) references therein). By the same reasonings as in the Lie case, this question is equivalent to

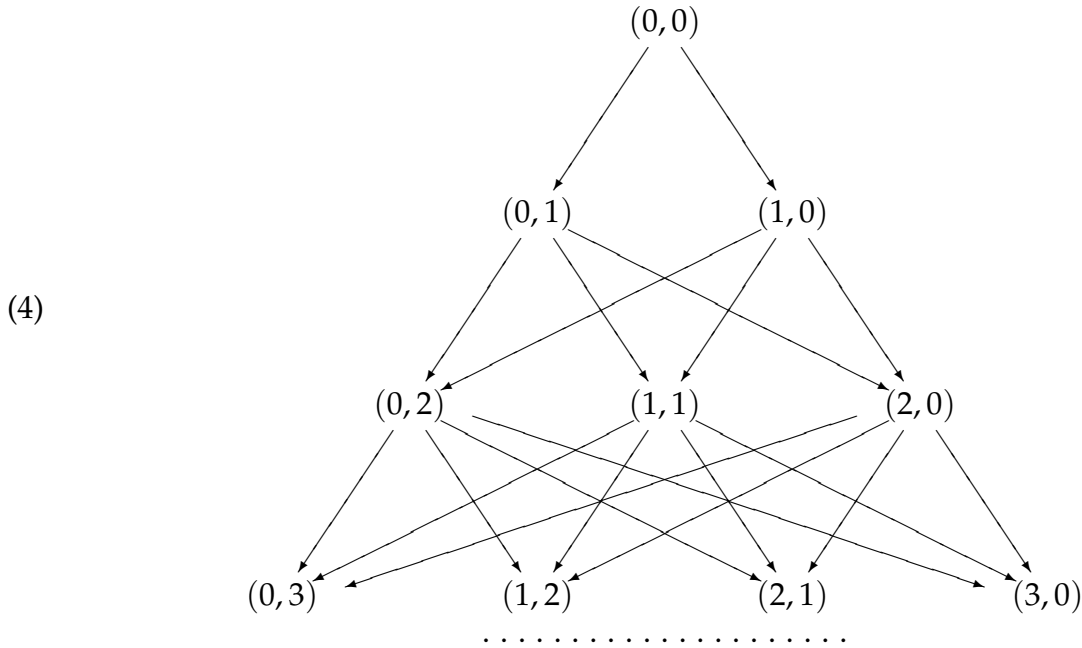
Question 5. Is it true that an associative algebra represented as the sum of two PI subalgebras, cannot contain a free associative algebra as a subalgebra?

Another interesting topic is to develop a machinery to express the (co)homology (Chevalley–Eilenberg, Hochschild, etc.) of such algebras in terms of factors and their action on each other. In the case when the sum of subalgebras is direct, one may attempt to mimic the approach of §1, albeit in a more simple situation, as we have direct sums instead of tensor products. If say, a Lie algebra $L = M \oplus N$ is represented as the vector space direct sum of subalgebras M and N , then, instead of the Cauchy formula we have a more simple isomorphism of vector spaces:

$$\bigwedge^n (N \oplus M) \simeq \bigoplus_{\substack{i+j=n \\ i,j \geq 0}} \bigwedge^i (N) \otimes \bigwedge^j (M).$$

Passing to the dual vector spaces, and decomposing the differentials in the Chevalley–Eilenberg complex, as in §1, we get the following picture (now instead of the Young graph

we have a more simpler triangle):



Here each pair (i, j) denotes the vector space $\wedge^i(N)^* \otimes \wedge^j(M)^*$.

Question 6. *Are there any patterns (vanishing or otherwise) in the graph (4) in the general situation? In some special cases?*

Ideally, a positive answer to this question should allow to develop a cohomological machinery which would unify and generalize various situations: some particular instances of the Hochschild–Serre spectral sequence, a stuff related to “Tate Lie algebras”, “Japanese cocycles” (see, e.g., [BD, §2.7]), etc.

A somewhat similar machinery is contained in an interesting and seemingly entirely forgotten paper [Du] (there, the author presents an alternative derivation of the Lyndon–Hochschild–Serre spectral sequence for the semidirect product of groups, but similar considerations seem to be applicable as well to the group-theoretic analog of our situation, i.e. for a group $G = AB$ decomposed into the product of its subgroups A and B ; the promised sequels to [Du] treating the Lie-algebraic and associative cases have never appeared).

3. DEFORMATIONS AND “COMMUTATIVE” COHOMOLOGY IN CHARACTERISTIC 2

Classification of finite-dimensional simple Lie algebras over algebraically closed fields of characteristic zero is a classical piece of mathematics, crystallized at the end of XIX–beginning of XX centuries. It took the mankind another some 100 years to achieve the same classification over fields of characteristic $p > 3$ (see [St]). The cases $p = 2$ and 3 remain widely open. In [GZ], an attempt was made to advance the case $p = 2$ basing on earlier results of Skryabin [Sk]. The general line of attack is more or less the same as in “big” characteristics: one first classifies algebras of small toral rank, and then, taking advantage of appropriate root space decompositions, glue the results together; also, many questions are reduced to computation of deformations of certain classes of algebras. In particular, in [Sk] simple Lie algebras having a Cartan subalgebra of toral rank 1 are characterized as certain filtered deformations of semisimple Lie algebras L such that

(5)
$$S \otimes \mathcal{O}_1(n) \subset L \subseteq \text{Der}(S) \otimes \mathcal{O}_1(n) + K\partial,$$

where $\mathcal{O}_1(n)$ is the divided powers algebra, ∂ its standard derivation, and either $n = 2$ and $S \simeq W_1'(n)$, or $n = 1$ and S is isomorphic to a two-variable Hamiltonian algebra.

In [GZ], these deformations were computed in the simplest case $S \simeq S$, what allowed to classify simple Lie algebras of absolute toral rank 2 and having a Cartan subalgebras of toral rank 1 – a small but necessary step in the classification program. To further advance along this road, one need to compute these deformations in all the cases.

Question 7. *Compute deformations of semisimple Lie algebras in characteristic 2 of the form (5).*

In the process of these computations, it became apparent that a new cohomology theory peculiar to characteristic 2 plays a role. This cohomology is defined via the standard formula for the coboundary map in the Chevalley–Eilenberg complex, with the alternating cochains being replaced by symmetric ones. Note that we can (profitably) consider commutative 2-cocycles of Lie algebras in arbitrary characteristic ([Dz3] and [DZ]), albeit they do not lead to any cohomology; while in characteristic 2 we have a bona fide cohomology theory. Unlike the usual cohomology, this complex is a priori not restricted by the dimension of the algebra, so new interesting phenomena, similar to those appearing in cohomology of Lie superalgebras (in any characteristic), occur. Generally, this “commutative” cohomology is different from the Chevalley–Eilenberg cohomology. For example, while the second cohomology of the Zassenhaus algebra $W_1'(n)$ with coefficients in the trivial module is zero (note that this is in striking difference with the cases of “big” characteristics; if $p > 3$, the corresponding cohomology is 1-dimensional, leading to the modular analog of the famous Virasoro algebra, cf. [Dz1]), the analogous “commutative” cohomology has dimension n and is generated by “commutative” 2-cocycles

$$e_i \vee e_j \mapsto \begin{cases} 1 & \text{if } i = j = 2^k - 2, \text{ or } \{i, j\} = \{-1, 2^{k+1} - 3\} \\ 0 & \text{otherwise.} \end{cases}$$

for $k = 0, \dots, n - 1$.

(This can be established by considering subalgebras of $W_1'(n)$ spanned by a “small” number of root vectors – what corresponds to the cases $n = 2$ and 3 – similarly how it was done in computation of commutative 2-cocycles on simple Lie algebras of classical type in [Dz3]).

Besides a few isolated computations like just presented, virtually nothing is known about this kind of cohomology, so any result about it would be of interest. For example, to compute deformations in Question 7, one need to compute low-degree “commutative” cohomology with various coefficients of simple Lie algebras involved – the Zassenhaus and Hamiltonian algebras.

Question 8. *Compute the “commutative” cohomology for various Lie algebras in characteristic 2.*

Question 9. *Is it possible to represent the “commutative” cohomology as a derived functor?*

4. VARIATIONS ON A THEME OF ADO

The Ado Theorem, one of the cornerstones of the modern theory of Lie algebras, says that each finite-dimensional Lie algebra has a finite-dimensional faithful representation. Somewhat surprisingly, its proof is not that straightforward as one may expect for such a basic result: it involves universal enveloping algebras – infinite dimensional objects, and is strikingly different for the cases of zero and positive characteristics. There exists a substantial body of literature with variants of the proof of the Ado theorem, but all of them follow, more or less, the same pattern. In [Z6] a different proof was given, not appealing to the notion of universal enveloping algebra and intrinsic to the category of finite-dimensional Lie algebras. Unfortunately, the proof is valid for nilpotent Lie algebras and in characteristic zero only.

Question 10. *Give a characteristic-free, “short” and “natural” (i.e., not utilizing the notion of universal enveloping algebra or any other infinite-dimensional objects) proof of the full Ado Theorem.*

In the standard proofs of the Ado Theorem, the case of nilpotent Lie algebras is the most laborious part. Then, the general case is derived from the nilpotent one via the passage to the algebraic envelope, and employing a certain structure of a faithful module built, again, with the help of universal enveloping algebra (and the PBW theorem).

To get a partial answer to Question 10, we may try to move along the same route, but employing ideas of [Z6]. As any finite-dimensional Lie algebra is embedded into its algebraic envelope (first constructed by Malcev [Mal], and independently by Goto [G] and Matsushima [Mat]), it is enough to prove the Theorem for algebraic Lie algebras. In characteristic zero, the Levi–Malcev decomposition of any algebraic Lie algebra is of the form $L = S + T + N$ (direct sum of vector spaces), where S is the semisimple part, T is a torus acting on the nilradical N , and $[S, T] = 0$. As in the proof of the nilpotent case of the Theorem in [Z6], it is enough to establish the existence of a faithful representation of L not vanishing on any nonzero central element of L , and the latter lie in N . If, say, N is \mathbb{N} -graded, then arguing like in [Z6, Lemma 2.5], we may construct a representation of N in $L \otimes tK[t]/(t^n)$, for a suitable n , with required properties. Since S and T act on N by derivations, we may extend this representation to the whole L . In this way we get a proof of the Theorem for Lie algebras whose algebraic envelope has an \mathbb{N} -graded nilpotent radical.

As for characteristic-free requirement, it is easy to see that all the reasonings of [Z6] remain valid over a field of characteristic p , if the index of nilpotency of the Lie algebra in question is $< p$. But to give a full-blown characteristic-free proof will require, apparently, new ideas.

5. VARIATIONS ON A THEME OF WHITEHEAD

The Second Whitehead Lemma is another classical result saying that the second cohomology of a finite-dimensional semisimple Lie algebra of characteristic zero, with coefficients in arbitrary finite-dimensional module, vanishes.

In [Z3] a curious “almost converse” was proved: a finite-dimensional Lie algebra of characteristic zero such that the second cohomology in any its finite-dimensional module vanishes, is either semisimple, or one-dimensional, or is the direct sum of a semisimple and one-dimensional algebra. According to [Mak], over an algebraically closed field this is exactly the list of finite-dimensional Lie algebras having the tame representation type.

Question 11. *What is the reason that these two classes of Lie algebras coincide?*

Note that in the modular case the situation is entirely different, due to the result of Dzhu-madil’daev [Dz2]: for any finite-dimensional Lie algebra over a field of positive characteristic, and any degree not exceeding the dimension of the algebra, there is a finite-dimensional module with non-vanishing cohomology in that degree. In particular, the one-dimensional Lie algebra is the only finite-dimensional Lie algebra with vanishing second cohomology in any finite-dimensional module.

Question 12. *Is it true that over an (algebraically closed) field of positive characteristic, the only finite-dimensional Lie algebra of tame representation type is one-dimensional?*

If this question has an affirmative answer, then one may ask for a characteristic-free variant of Question 11; a satisfactory answer should establish a bijection between these two classes of Lie algebras, without addressing peculiarities related to characteristic of the base field.

It is known that analogs of the Second Whitehead Lemma hold for other classes of algebraic structures: Jordan algebras, alternative algebras, Lie triple systems, etc.

Question 13. *Do some sort of converses hold for all these analogs of the Second Whitehead Lemma?*

6. VARIATIONS ON A THEME OF BANACH

In [Z7] an attempt was made to trace the possible origins of a (vaguely formulated) question by Stefan Banach about ternary maps which are superpositions of binary maps.

This is possibly the most isolated topic in the present collection, though a nice relation with Lie theory exists: an answer to possible interpretations of the Banach question may be obtained using an idea borrowed from a pioneering paper by Jacobson about Lie triple systems.

In the process the following question arose. Let us count the number of ternary maps $f : X \times X \times X \rightarrow X$ on a finite set X of n elements, which can be represented as a superposition of binary maps $* : X \times X \rightarrow X$. We get the following table:

n	$T_L(n)$	$T_{LR}(n)$	$T_{comm}(n)$
1	1	1	1
2	14	21	5
3	19292	38472	48

Here $T_L(n)$ denotes the number of ternary maps represented in the form

$$(6) \quad f(x, y, z) = (x * y) * z$$

for some binary map $*$, $T_{LR}(n)$ denotes the number of ternary maps represented either in the form (6), or in the form

$$(7) \quad f(x, y, z) = x * (y * z),$$

and $T_{comm}(n)$ denotes the number of ternary symmetric maps (i.e., invariant under any permutation in S_3) represented in the form (6). In the given range, the latter number coincides with the number of ternary symmetric maps represented in the form (6) for some *commutative* $*$.

As of time of this writing, the 3-term sequences for $T_L(n)$ and $T_{LR}(n)$ were absent in The Online Encyclopedia of Integer Sequences, and among a dozen or so sequences containing the 3-term sequence for $T_{comm}(n)$, nothing seems to be relevant.

Question 14. *Continue this table. Give formulas (closed form, or recurrent) for the numbers $T_L(n)$, $T_{LR}(n)$, $T_{comm}(n)$.*

Question 15. *Is it true that for every n , any symmetric ternary map represented in the form (6) for some $*$, can be represented in the form (6) for some commutative $*$?*

ACKNOWLEDGEMENTS

The financial support of the Regional Authority of the Moravian-Silesian Region (grant MSK 44/3316), and of the Ministry of Education and Science of the Republic of Kazakhstan (grant 0828/GF4) is gratefully acknowledged.

REFERENCES

- [BN] A.A. Balinskii and S.P. Novikov, *Poisson brackets of hydrodynamic type, Frobenius algebras and Lie algebras*, Doklady Akad. Nauk SSSR **283** (1985), 1036–1039 (in Russian); Soviet Math. Dokl. **32** (1985), 228–231 (English translation).
- [BD] A. Beilinson and V. Drinfeld, *Chiral Algebras*, AMS, 2004.
- [Du] A.M. DuPré, *Combinatorial extension cohomology I. Groups*, Adv. Math. **106** (1994), 96–117.
- [Dz1] A.S. Dzhumadil'daev, *Central extensions of the Zassenhaus algebra and their irreducible representations*, Mat. Sbornik **126** (1985), 473–489 (in Russian); Math. USSR Sbornik **54** (1986), 457–474 (English translation).

- [Dz2] ———, *Cohomology of truncated coinduced representations of Lie algebras of positive characteristic*, Mat. Sbornik **180** (1989), 456–468 (in Russian); Math. USSR Sbornik **66** (1990), 461–473 (English translation).
- [Dz3] ———, *Algebras with skew-symmetric identity of degree 3*, Sovremennaya Matematika i ee Prilozheniya **60** (2008) (E.B. Vinberg Festschrift), 13–31 (in Russian); J. Math. Sci. **161** (2009), 11–30 (English translation).
- [Dz4] ———, *Codimension growth and non-Koszulity of Novikov operad*, Comm. Algebra **39** (2011), 2943–2952; loosely corresponds to arXiv:0902.3187 + arXiv:0902.3771.
- [DZ] ——— and P. Zusmanovich, *Commutative 2-cocycles on Lie algebras*, J. Algebra **324** (2010), 732–748; arXiv:0907.4780.
- [GD] I.M. Gelfand and I.Ya. Dorfman, *Hamiltonian operators and algebraic structures related to them*, Funkts. Anal. Prilozh. **13** (1979), *N*4, 13–30 (in Russian); Funct. Anal. Appl. **13** (1979), 248–262 (English translation); reprinted in I.M. Gelfand, *Collected Papers*, Vol. I, Springer, 1987.
- [G] M. Goto, *On algebraic Lie algebras*, J. Math. Soc. Japan **1** (1948), 29–45.
- [GZ] A. Grishkov and P. Zusmanovich, *Deformations of current Lie algebras. I. Small algebras in characteristic 2*; arXiv:1410.3645.
- [KLM] I. Kaygorodov, A. Lopatin, and Yu. Popov, *Identities of sum of two PI-algebras in the case of positive characteristic*, Intern. J. Algebra Comp. **25** (2015), 1265–1273; arXiv:1411.7395.
- [Mak] E.A. Makedonskii, *On wild and tame finite-dimensional Lie algebras*, Funkts. Anal. Prilozh. **47** (2013), *N*4, 30–44 (in Russian); Funct. Anal. Appl. **47** (2013), 271–283 (English translation); arXiv:1202.1401.
- [Mal] A.I. Malcev, *Solvable Lie algebras*, Izvestiya Akad. Nauk SSSR **9** (1945), 329–356; reprinted in *Izbrannyye Trudy. Tom I*, Nauka, Moscow, 1976, 155–176 (in Russian); Amer. Math. Soc. Transl. 1950, *N*27, 36 pp.; reprinted in *Lie Groups*, Amer. Math. Soc. Transl. **9** (1962), 228–262 (English translation).
- [Mat] Y. Matsushima, *On the faithful representations of Lie groups*, J. Math. Soc. Japan **1** (1949), 254–261; reprinted in *Collected Papers of Yozô Matsushima*, World Scientific, 1992, 51–58.
- [OEIS] The On-Line Encyclopedia of Integer Sequences; <http://oeis.org/>.
- [Pa] V.V. Panyukov, *On the solubility of Lie algebras of positive characteristic*, Uspekhi Mat. Nauk **45** (1990), *N*4, 163–164 (in Russian); Russ. Math. Surv. **45** (1990), *N*4, 181–182 (English translation).
- [PB] Y. Pei and C. Bai, *Novikov algebras and Schrödinger–Virasoro Lie algebras*, J. Phys. A **44** (2011), 045201.
- [Pe] A.P. Petravchuk, *Lie algebras which can be decomposed into the sum of an abelian subalgebra and a nilpotent subalgebra*, Ukrain. Mat. Zh. **40** (1988), 385–388 (in Russian); Ukrain. Math. J. **40** (1988), 331–334 (English translation).
- [Sk] S. Skryabin, *Toral rank one simple Lie algebras of low characteristics*, J. Algebra **200** (1998), 650–700.
- [St] H. Strade, *Simple Lie Algebras over Fields of Positive Characteristic. Vol. I. Structure Theory. Vol. II. Classifying the Absolute Toral Rank Two Case. Vol. III. Completion of the Classification*, De Gruyter, 2004, 2009, 2013.
- [Z1] P. Zusmanovich, *A Lie algebra that can be written as a sum of two nilpotent subalgebras, is solvable*, Mat. Zametki **50** (1991), *N*3, 38–43 (in Russian); Math. Notes **50** (1991), 909–912; arXiv:0911.5418.
- [Z2] ———, *Low-dimensional cohomology of current Lie algebras and analogs of the Riemann tensor for loop manifolds*, Lin. Algebra Appl. **407** (2005), 71–104; arXiv:math/0302334.
- [Z3] ———, *A converse to the Second Whitehead Lemma*, J. Lie Theory **18** (2008), 295–299; Erratum: **24** (2014), 1207–1208; arXiv:0704.3864.
- [Z4] ———, *On the utility of Robinson–Amitsur ultrafilters*, J. Algebra **388** (2013), 268–286; arXiv:0911.5414.
- [Z5] ———, *A compendium of Lie structures on tensor products*, Zapiski Nauchnykh Seminarov POMI **414** (2013) (N.A. Vavilov Festschrift), 40–81; reprinted in J. Math. Sci. **199** (2014), 266–288; arXiv:1303.3231.
- [Z6] ———, *Yet another proof of the Ado theorem*, J. Lie Theory **26** (2016), 673–681; arXiv:1507.02233.
- [Z7] ———, *On the last question of Stefan Banach*, Expos. Math., to appear; arXiv:1408.2982.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OSTRAVA, 30. DUBNA 22, 70103 OSTRAVA, CZECH REPUBLIC
E-mail address: pasha.zusmanovich@osu.cz