ON THE CONE OF WEIGHTED GRAPHS GENERATED BY TRIANGLES

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Abstract. We examine the facet structure of the cone of weighted graphs generated by triangles. We also explore the application of this cone to the problem of edge-decomposition of graphs into triangles and point out connections with the perimeter inequalities defining the metric polytope.

1. **Introduction**

1.1. **Set-up.** For our purposes, a *weighted graph* on a vertex set V is a function $f : \binom{V}{2} \to \mathbb{R}$ assigning a real number to each edge of the complete graph on V. For $e \in \binom{V}{2}$, we say that $f(e)$ is the *weight* of e. A weighted graph is *nonnegative* if every edge has nonnegative weight. Alternatively, a nonnegative weighted graph on V is a triple (V, E, f), where $G = (V, E)$ is a (simple) graph and $f : E \to \mathbb{R}_+$ is an assignment of positive reals to the edges; here, it is understood that pairs in $\binom{V}{2} \setminus E$ get weight 0. A similar notion may be used in the presence of negative edges.

Here we assume a finite vertex set, typically $V = [n] := \{1, 2, \dots, n\}$. The set of weighted graphs forms a vector space of dimension $\binom{n}{2}$ over the reals. Thus we may identify weighted graphs with vectors in $\mathbb{R}^{\binom{n}{2}}$. As one possible convention, the coordinates can be indexed colexicographically so that, e.g., the vector $(1, 1, 0, 1, 0, 0, -1, 0, 0, 0) \in \mathbb{R}^{10}$ corresponds to a weighted star on $V = \{1, 2, 3, 4, 5\}$.

Recall that a *cone* in $\mathbb{R}^m$ is a set κ which is closed under both addition and scalar multiplication by nonnegative reals. The cone *generated by* $v_1, \dots, v_k$ is $\{\sum_{i=1}^k a_i v_i : a_i \geq 0\}$. For instance, the set of nonnegative weighted graphs forms a cone corresponding to the nonnegative orthant of $\mathbb{R}^{\binom{n}{2}}$, and hence it is generated by the standard basis.

We are interested here in the cone $\tau_n \subset \mathbb{R}^{\binom{n}{2}}$ of weighted graphs on n vertices generated by triangles. Here, a triangle is understood to mean the weighted graph $(V, \{\{x, y\}, \{y, z\}, \{x, z\}\}, 1)$ which takes the value 1 on the edges of a 3-subset $\{x, y, z\} \subseteq V$ and 0 otherwise. When $n = 3$, this cone is simply the ray $\{(t, t, t) : t \geq 0\} \subset \mathbb{R}^3$. In the case $n = 4$, it is easy to see that a weighted graph is spanned by triangles if and only if the sum of weights on disjoint edges is a constant. For $n \geq 5$, the triangles span $\mathbb{R}^{\binom{n}{2}}$ by linear combinations. It follows that τ_n has 'full dimension' $\binom{n}{2}$ in this case.

1.2. **Graph decompositions.** A graph $G = (V, E)$ has an F-*decomposition* if its edge set E can be partitioned by subgraphs, each isomorphic to F. The first interesting case occurs when $F = K_3$, a triangle. A triangle decomposition of K_n is equivalent to a Steiner triple system of order n, which

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exists if and only if $n \equiv 1$ or $3 \pmod 6$. It is NP-complete to decide whether an arbitrary graph G has a triangle decomposition; see [6].

For the existence of a triangle decomposition of G, it is necessary that every vertex of G have even degree and that $|E|$ be divisible by three. These are 'arithmetic' necessary conditions. It is also necessary that $G \in \tau_n$. This is a 'geometric' necessary condition. In fact, we say that a (nonnegative weighted) graph has a *fractional triangle decomposition* if and only if it belongs to τ_n. A few recent papers have explored fractional triangle decompositions. The paper [12] studies the case of planar graphs G. For dense graphs, Dross in [7] obtains a minimum degree threshold sufficient for the existence of a fractional triangle decomposition.

Theorem 1.1 ([7]). *Every graph G on n vertices with $\delta(G) \geq \frac{9}{10}n$ belongs to τ_n.*

The breakthrough paper [1] shows that a minimum degree threshold sufficient for fractional decomposition is also roughly sufficient for the exact decomposition problem. This gives considerable motivation to the question of degree thresholds for fractional decompositions, and of geometric necessary conditions in general. In particular, reducing $\frac{9}{10}$ to $\frac{3}{4}$ in Theorem 1.1 would nearly establish Nash-Williams' conjecture, [13], except for some cases very close to the boundary. This conjecture states that $\delta(G) \geq 3n/4$ together with the arithmetic conditions are sufficient for the existence of a triangle decomposition of G. We note that there exist counterexamples to weakening this minimum degree assumption. For instance, the lexicographic graph product $C_4 \cdot K_{6m+3}$ satisfies the arithmetic conditions for positive integers m, has minimum degree near $3n/4$, but it violates a 'geometric barrier' for K_3-decomposition. In other words, one can find a hyperplane which separates this graph from the cone τ_n. We give more details later.

1.3. **The metric polytope.** The cone τ_n appears 'locally' inside a well-studied polytope. Recall that a *metric* d on a set X is a function $d : X \times X \to \mathbb{R}$ such that, for all $x, y, z \in X$,

(a) $d(x, y) \geq 0$,
(a′) $d(x, y) = 0$ if and only if $x = y$,
(b) $d(x, y) = d(y, x)$,
(c) $d(x, z) \leq d(x, y) + d(y, z)$.

A *semi-metric* is a function that satisfies only conditions (a), (b) and (c). The *metric cone* Met_n consists of all semi-metrics on an n-set. If we bound Met_n by considering only those semi-metrics which satisfy the 'perimeter inequalities'

$$d(x, y) + d(x, z) + d(y, z) \leq 2$$

for all 3-subsets $\{x, y, z\} \subseteq [n]$, then we obtain the *metric polytope* met_n. Each of the triangle inequalities and perimeter inequalities defines a half-space bounding met_n. Taking $X = [n]$ and writing d_{ij} for $d(i, j)$, we can consider the metric d as a vector $(d_{12}, d_{13}, d_{23}, \ldots, d_{n-1,n})$ in $\mathbb{R}^{\binom{n}{2}}$, and thus embed Met_n and met_n in $\mathbb{R}^{\binom{n}{2}}$. It is easy to see that the point $(2/3, \ldots, 2/3)$ is a vertex of met_n. Near this point, only the perimeter inequalities bound met_n. So, upon complementing the inequalities, we see that $(2/3, \ldots, 2/3) - \mathrm{met}_n$ coincides with the dual cone of τ_n near the origin. The following result is an immediate consequence.

Proposition 1.2. *The edges of met_n incident with $(2/3, \ldots, 2/3)$ are in correspondence with the facets of τ_n.*

1.4. **Organization.** The purpose of this note is to initiate a detailed study of the facets of τ_n, especially as they connect with weighted graphs. The next section contains some additional background relevant for our problem, including some example facets to illustrate the connection with triangle decompositions. In Section 3, we identify some simple arithmetic constraints on entries of the normal vectors. Then, in Section 4, we report on a computer-aided classification of facets of τ_n for $n \leq 8$ (this being essentially contained in earlier computations on met_n via Proposition 1.2), and in addition push the computation to a near-classification for $n = 9$. Section 5 contains a 'vertex splitting' operation which generates many infinite families of facets. In spite of this partial inductive structure, there is a surprising level of complexity to τ_n. This appears to be true even when symmetries under the action of of the symmetric group $\mathcal{S}_n$ are considered.

2. **Background**

2.1. **Cones.** First we review some background on cones. For our purposes, all cones are assumed to be 'polyhedral' (finitely generated). Unless otherwise specified, cones are 'pointed' ($u, -u \in \kappa$ implies $u = 0$) and of full dimension in their vector space.

Let κ be a cone. A *face* of κ is a cone $\eta \subseteq \kappa$ such that for all $u \in \eta$, if $u = u_1 + u_2$ with $u_1, u_2 \in \kappa$, then $u_1, u_2 \in \eta$. A face of dimension 1 is called an *extremal ray* of κ, while a face of codimension 1 is called a *facet* of κ.

The discussion from now on focuses on cones in real Euclidean space $\mathbb{R}^m$. The usual inner product $\langle \cdot, \cdot \rangle$ is used. When matrices are involved, we adopt the convention that in $\langle a, b \rangle$, a is a (dual) row vector and b is an (ordinary) column vector. The set $\kappa' = \{y \in \mathbb{R}^m : \langle y, u \rangle \geq 0 \text{ for all } u \in \kappa\}$ is a cone called the *dual* of κ. The dual of a facet of κ is an extremal ray of κ'. For example, the ray defined by $y \in \mathbb{R}^m \setminus \{0\}$ is the dual of the half-space in $\mathbb{R}^m$ having boundary $y^\perp$ and direction y. We say that such a y is a *supporting vector* for any cone contained in this half-space. If y is a supporting vector for κ and $\eta = \kappa \cap y^\perp$ is a face of κ, then y is said to *support κ along η*. A result of fundamental importance is that a cone κ is the intersection of all half-spaces described by supporting vectors of κ. Theorem 2.1 below states this in the concrete setting which shall be used herein.

Given an $m \times n$ matrix A, the set $\mathrm{cone}(A) = \{Ax : x \in \mathbb{R}^n, x \geq 0\}$ is a closed and polyhedral cone in $\mathbb{R}^m$. The dimension of $\mathrm{cone}(A)$ is equal to the rank of A. The following well known result provides necessary and sufficient conditions for a point to belong to $\mathrm{cone}(A)$.

Theorem 2.1 (Farkas Lemma)**.** *Let A be an $m \times n$ matrix, and $b \in \mathbb{R}^m$. The equation $Ax = b$ has an entrywise nonnegative solution x if and only if $\langle y, b \rangle \geq 0$ for all $y \in \mathbb{R}^m$ such that $y^\top A \geq 0$.*

Remarks. One direction of this result is immediate. Suppose $Ax = b$ has a nonnegative solution $x \in \mathbb{R}^n$, and let y be such that $y^\top A \geq 0$. Then $\langle y, b \rangle = \langle y, Ax \rangle = \langle yA, x \rangle \geq 0$. The converse is deeper, asserting the existence of a 'separating hyperplane' between $\mathrm{cone}(A)$ and a given point not in this cone.

It is enough to check the condition in Theorem 2.1 for y corresponding to facets of $\mathrm{cone}(A)$. Adapting the simplex algorithm or Fourier-Motzkin elimination gives a procedure to enumerate the facets of $\mathrm{cone}(A)$; hence, it is a finite problem to determine whether $Ax = b$ has a nonnegative solution x. However this is seldom easy in practice.

2.2. **Graphs and special vectors.** We return to the setting of (edge-weighted) graphs. For a simple graph G on vertex set $[n]$, let $\mathbb{1}_G$ denote the characteristic vector of $E(G)$ in $\mathbb{R}^{\binom{n}{2}}$, where again coordinates are indexed by $\binom{[n]}{2}$. That is,

$$\mathbb{1}_G(e) = \begin{cases} 1 & \text{if } e \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Let us define W as the matrix whose columns are the characteristic vectors of all triangles in K_n. We have, for $e \in \binom{[n]}{2}$ and $K \in \binom{[n]}{3}$,

$$W(e, K) = \begin{cases} 1 & \text{if } e \subseteq K, \\ 0 & \text{otherwise.} \end{cases}.$$

Up to the isomorphism from the vector space of edge-weighted graphs to $\mathbb{R}^{\binom{n}{2}}$, we see that $\tau_n = \text{cone}(W)$. In particular, the existence of a fractional triangle decomposition of G is equivalent to the existence of a nonnegative solution x to $Wx = \mathbb{1}_G$. In light of Theorem 2.1 and the ensuing discussion, we are motivated to understand the facets of τ_n. A vector $y \in \mathbb{R}^{\binom{n}{2}}$, or alternatively an edge-weighted graph on n vertices, is a *facet normal* of τ_n if: (1) $\langle y, \mathbb{1}_K \rangle \geq 0$ for all triangles K, and (2) the span of triangles K for which $\langle y, \mathbb{1}_K \rangle = 0$ has codimension 1. For instance, the graph consisting of a single nonzero edge (of weight 1) is a facet normal of τ_n, which we call *trivial*.

Example 2.2. Let $n \geq 6$. The edge-weighted graph y on vertex set $[n]$ with

$$y(e) = \begin{cases} -1 & \text{if } e = \{1, n\}, \\ 1 & \text{if } e = \{i, n\} \text{ for } i \in \{2, \dots, n-1\}, \\ 0 & \text{otherwise} \end{cases}$$

is a facet normal of τ_n. This is easy to verify directly. Any triangle avoiding vertex n has all of its edges of weight zero, and any triangle containing edge $\{1, n\}$ also has weight $1 - 1 = 0$. If we include among this family of zero-sum triangles any triangle of positive weight, say $\{2, 3, n\}$, it is possible to span any other such triangle. For more details, including why $n \geq 6$ is needed, see [9].

It is natural to call the facet y described by Example 2.2 a *star* facet of τ_n. It has a combinatorial interpretation for triangle decomposition, albeit of very mild significance: the inequality $\langle y, \mathbb{1}_K \rangle \geq 0$ is asserting that, should G have a fractional triangle decomposition, there cannot exist any vertex of degree 1 in G. The star y centered at such a vertex would have $\langle y, \mathbb{1}_G \rangle = -1$.

Example 2.3. Let $n \geq 5$ and suppose (A, B) is a partition of $[n]$ with $|A|, |B| \geq 2$. The vector y defined by

$$y(e) = \begin{cases} 2 & \text{if } e \subseteq \binom{A}{2} \cup \binom{B}{2}, \\ -1 & \text{otherwise} \end{cases}$$

is a facet normal of τ_n. This follows from Propositions 4.1 and 5.1 to follow. Let us call this vector an $(|A|, |B|)$-*cut*.

Suppose $4 \mid n$ and recall the graph $G = C_4 \cdot K_{n/4}$ mentioned in Secton 1. There exists an equipartition of the vertices (A, B) so that the number of edges of G within A or B equals $4\binom{n/4}{2} = n^2/8 - n/2$, while the number of edges crossing the partition equals $n^2/4$. Taking y to be the $(n/2, n/2)$-cut facet defined by (A, B), we see that $\langle y, \mathbb{1}_G \rangle < 0$. In fact, this same cut witnesses many other graphs with minimum degree near $3n/4$ also failing to have a (fractional) triangle decomposition.

Lemma 2.4. *Let y be a nontrivial facet normal of τ_n. Then every edge in $\binom{[n]}{2}$ is contained in a triangle K such that $\langle y, \mathbb{1}_K \rangle = 0$.*

PROOF. If e is an edge in no such triangle, then it is possible to decrease the weight of e in y such that the resulting vector still supports τ_n. Therefore, the standard form of y can only be $\mathbb{1}_e$. $\square$

2.3. **Stabilizers.** The symmetric group $\mathcal{S}_n$ acts on $\mathbb{R}^{\binom{[n]}{2}}$ in a natural way. For an edge-weighted graph y and permutation $\alpha \in \mathcal{S}_n$, we have $y^\alpha(e) = y(\alpha^{-1}e)$; that is, the action is induced on edges (edge weights) by permutations of the vertices.

Let us define the *stabilizer* of y to be $\text{stab}(y) = \{\alpha \in \mathcal{S}_n : y^\alpha = y\}$. By the orbit-stabilizer theorem, the number of distinct edge-weighted graphs isomorphic to y on n vertices is $n!/|\text{stab}(y)|$.

Suppose G has automorphism group Γ. Testing whether $G \in \tau_n$ amounts to checking $\langle \overline{y}, \mathbb{1}_G \rangle \geq 0$ on all $\overline{y}$ of the form

$$\overline{y} = \frac{1}{|\Gamma|} \sum_{\alpha \in \Gamma} y^\alpha$$

for some facet normal y of τ_n. The set of weighted graphs invariant under Γ is a subspace of $\mathbb{R}^{\binom{[n]}{2}}$. Therefore, its intersection with τ_n is a sub-cone. For example, if $\Gamma = \mathcal{S}_a \times \mathcal{S}_b$ for $n = a + b$, then the (a, b)-cut and nonnegative orthant give a complete description; see [8]. It would be interesting to study invariant sub-cones for other specific groups $\Gamma \leq \mathcal{S}_n$.

2.4. **A matrix formulation.** As an alternative to placing edge-weighted graphs in correspondence with $\mathbb{R}^{\binom{n}{2}}$, we can use the $n \times n$ symmetric matrices with zero diagonal. Under this slight change in notation, τ_n is equivalent to the cone generated by the $\binom{n}{3}$ matrices

$$P^\top \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} P,$$

where P is comprised of three rows of an $n \times n$ permutation matrix. This alternate presentation has some advantages. For example, the characteristic polynomial $\chi_y(t)$ of the matrix corresponding to an edge-weighted graph y is preserved under vertex permutation, making it a useful invariant. Moreover, the factorization of $\chi_y(t)$ in $\mathbb{Q}[t]$ appears to be related to the stabilizer of y.

3. Arithmetic and combinatorial structure

As we shall see, classifying the facets of τ_n is very challenging. In this section, though, we sample some of the structure that is forced upon facets. We begin with an easy observation on the relative sizes of extreme entries in facet normals.

Proposition 3.1. *Let y be a nontrivial facet normal of τ_n. Let a and b denote, respectively, the maximum of the positive entries and minimum of the negative entries in y. Then we have $-b/2 \leq a \leq -2b$.*

PROOF. First, suppose $a \geq -b$. Using Lemma 2.4, choose in y a zero-sum triangle K containing an edge of weight a. The other two edges of K must have negative weight, since a is largest among the entries in magnitude. It follows that one of these other edges in K has weight at most $-a/2$. This establishes $a \leq -2b$. The case $a < -b$ is similar. $\square$

Remarks. It is an easy exercise to see that $a = 1$, $b = -2$ is not possible for facets (although it is possible for supporting vectors). Indeed, it may be the case that the first inequality can be strengthened to $a > -b/2$ or even $a \geq -b$.

Presumably, Proposition 3.1 only scratches the surface of constraints on the signs and relative magnitudes in facet normals. We do not explore this further here, although it seems reasonable to guess that entries have some central tendency and are roughly symmetric about their mean.

Proposition 3.2. *Let y be a facet normal of τ_n and let G be the simple graph carrying the nonpositive entries of y. If G is bipartite, then it is complete bipartite.*

Proof. Suppose (A, B) is a bipartition of the vertices of G such that all edges in $\binom{A}{2}$ and $\binom{B}{2}$ are positive. Let z be the cut facet corresponding to (A, B); see Example 2.3. Since all entries of y on edges within A and B are positive, it follows that for some $\epsilon > 0$, $y - \epsilon z$ also supports τ_n. This is a contradiction to the indecomposability of y. $\square$

Next, we offer a purely arithmetic constraint. Let us say that a facet normal is in *standard form* if its entries are integers with greatest common divisor equal to 1.

Proposition 3.3. *Let $n \geq 5$ and suppose y is a facet normal of τ_n in standard form. The following are equivalent:*

(a) $\langle y, \mathbb{1}_K \rangle \equiv 0 \pmod 3$ *for all triangles K;*
(b) *no entry of y is* $0 \pmod 3$*; and*
(c) *all entries of y are* $1 \pmod 3$ *or all entries of y are* $2 \pmod 3$.

Proof. It is enough to prove (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (c). We let $\mathbb{F}_3$ denote the ternary field.

Suppose y is a ternary vector satisfying (a). Then it is easy to check that y agrees on the opposite edges of any 4-cycle in K_n. Consider any two adjacent edges, e and f, say. Since $n \geq 5$, there exist 4-cycles for which e and f are both opposite the same other edge. Therefore y is constant. Being in standard form, it is also nonzero modulo 3, and we obtain (c).

Now, suppose no entry of y is $0 \pmod 3$. Let $\mathcal{K}_0$ be the set of triangles K such that $\langle y, 1_K \rangle = 0$. Since y is a facet normal, there exists an edge f such that $\{1_K : K \in \mathcal{K}_0\} \cup \{1_f\}$ spans $\mathbb{Q}^{\binom{n}{2}}$ over the rationals. Let e be any edge. Choose rationals a_K and b such that $1_e = \sum_K a_K 1_K + b 1_f$. Since $y_e \neq 0$, we have $0 \neq y_e = 0 + b\langle y, 1_f \rangle$. Therefore, $b \neq 0$; say $b = p/q$ in lowest terms. Then

$$q 1_e = \sum_K q a_K 1_K + p 1_f. \tag{3.1}$$

Taking inner products of both sides with 1 gives $q = p + 3q \sum_K a_K \equiv p \pmod 3$. Since $\gcd(p, q) = 1$, we may assume $p \equiv q \equiv 1 \pmod 3$. Returning then to (3.1), we have

$$1_e - 1_f \equiv \sum_K t_K 1_K \tag{3.2}$$

in $\mathbb{F}_3^{\binom{n}{2}}$ for some $t_K \in \mathbb{F}_3$. Taking inner products of both sides with y gives $y_e - y_f \equiv 0$ in $\mathbb{F}_3$. Since e was arbitrary, we obtain (c). $\square$

As a result of Proposition 3.3, we can classify all facets of τ_n according to their normal vectors in standard form. We have those which contain an entry $0 \pmod 3$, and those whose entries are all 1 (or all 2) modulo 3. It is natural to label these categories 0, 1, 2, respectively.

For example, the family of star facets belongs to category 0, and the family of cut facets belongs to category 2. The first example of a facet in category 1 appears when $n = 7$. We discuss facets of τ_n for small n in the next section, and explore some interesting examples that arise.

4. **Classification for small** n

$n = 5$

The case $n = 5$ is the first in which τ_n has full dimension. We study it via the inclusion matrix W.

Proposition 4.1. *Let $n = 5$. The inverse of the 10×10 inclusion matrix W, with rows indexed by triangles and columns indexed by pairs, is the matrix whose (K, e)-entry equals $1/3$ if $|K \cap e| \in \{0, 2\}$, and equals $-1/6$ otherwise.*

Proof. Let U be the given matrix. The inner product of row K of U and column K' of W is computed in cases. If $K' = K$, the product is $3 \cdot 1/3$. If $|K' \cap K| = 1$ or 2, exactly one edge e of K' satisfies $|K \cap e| \in \{0, 2\}$ (this being $K^c = K' \setminus K$ or $K \cap K'$, respectively). In either of these cases, the inner product is $1/3 - 2 \cdot 1/6 = 0$. So $UW = I$. □

Corollary 4.2. *Up to isomorphism, the only facet of τ_5 is the $(2, 3)$-cut, giving a total of 10 facets.*

Two extreme rays in a cone are *adjacent* if they span a face of dimension 2. Likewise, facets are adjacent if they intersect in a face of codimension 2. It is clear that the facet structure of τ_5 is 'simplicial' (any two are adjacent) since the ten facet normals are linearly independent in $\mathbb{R}^{10}$.

$n = 6$

Using the built-in function '`Cone()`' in Sage [14], we obtain a classification of facets of τ_6. This is shown in Table 1. Each row gives a vector, with coordinates corresponding to the colex order on $\binom{[6]}{2}$ for an isomorphism class. The right column indicates the number of distinct copies induced under the action of $\mathcal{S}_6$. The three nontrivial isomorphism types are displayed as weighted graphs in Figure 1. A {green,red}-edge-coloring illustrates the positive and negative weights, with magnitudes as labeled.

representative	#	deg
(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)	15	32
(1, 1, 0, 1, 0, 0, 1, 0, 0, 0, -1, 0, 0, 0, 0)	30	14
(2, 2, 2, -1, -1, -1, -1, -1, -1, 2, -1, -1, -1, 2, 2)	10	57
(2, 2, 2, 2, 2, 2, -1, -1, -1, -1, -1, -1, -1, -1, 2)	15	32
total	70	

Table 1. Isomorphism classes of facet normals of τ_6, in standard form

We conclude by summarizing the structure of τ_6.

Proposition 4.3 (See also [2]). *The facets of τ_6 fall into four isomorphism classes: trivial, star, $(3, 3)$-cut, and $(4, 2)$-cut.*

Our computation shows that the four types of facets have respective degrees $32, 14, 57, 32$. It is noteworthy that each type is adjacent to a mixture of the other types. In general, adjacent facets of

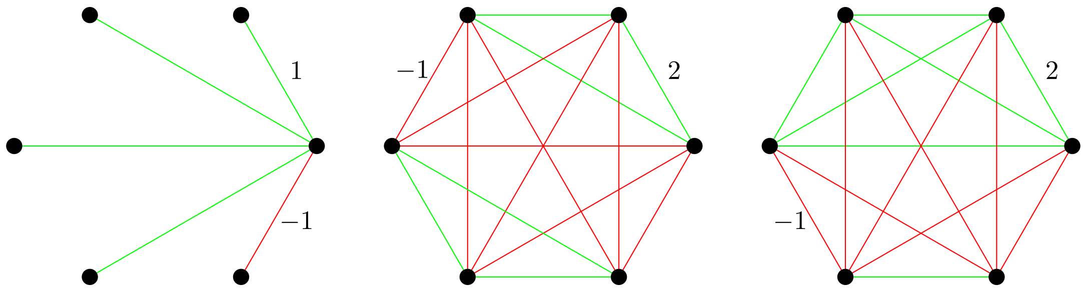

FIGURE 1. Weighted graphs for the nontrivial facet normals of τ_6

τ_n appear to possess interesting combinatorial properties, although we do not presently have a test for adjacency which is 'algebra-free'.

$$n = 7$$

For $n = 7$, the computation of cone facets is still quite fast. Our classification appears in Table 2 with headings as before. See also [2]. Weighted graphs for the nontrivial isomorphism types are given in Figure 2. As edge-colored graphs, shades of green represent positive edges of different weights.

representative	#	deg
(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)	21	340
(1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 1, 1, -1, -1, 0, 0, 0, 0, 1)	420	20
(1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0)	42	75
(2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, -1, 1, 1, 1, 1)	105	75
(2, 2, 2, 2, 2, 2, -1, -1, -1, -1, -1, -1, -1, -1, 2, -1, -1, -1, -1, 2, 2)	35	340
(2, 2, 2, 2, 2, 2, 2, 2, 2, 2, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, 2)	21	75
(4, 4, -2, 1, 1, 1, 1, 1, 1, -2, -2, 4, -2, 1, 1, -2, -2, 4, 1, 1, 4)	252	20
total	896	

TABLE 2. Isomorphism classes of facet normals of τ_7, in standard form

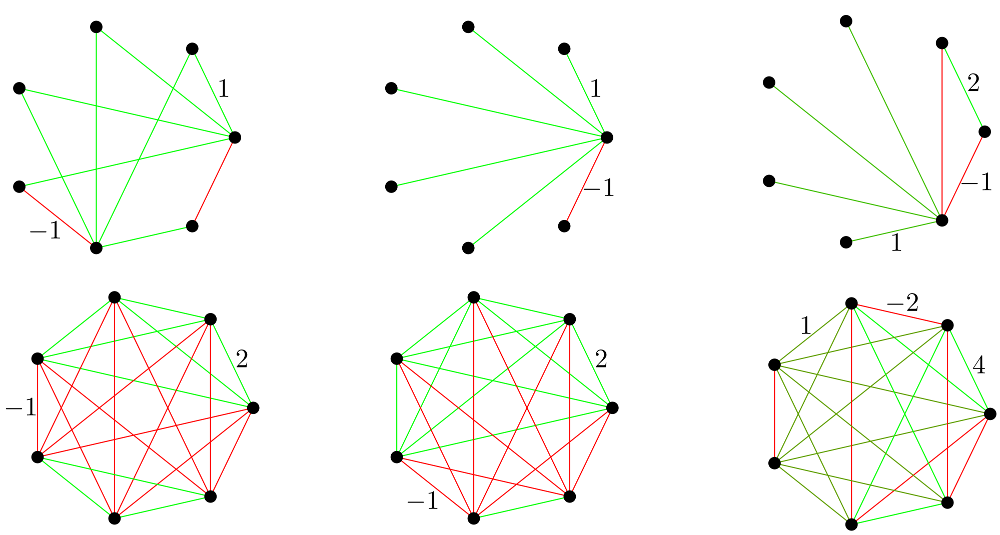

FIGURE 2. Weighted graphs for the nontrivial facet normals of τ_7

$n = 8$

The classification of facets of τ_8 is computed similarly as for τ_7 and is displayed in Table 3.

representative	#	deg
(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)	28	18848
(1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 1, 1, 1)	560	82
(1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 1, 1, -1, 0, 1, 1, -1, 1, 0, -1, 0, 0, 0, 0, 1, 1)	3360	52
(1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0)	56	82
(1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1, -1, -1, 0, 0, 0, 0, 0, 1)	840	902
(2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, -1, 1, 1, 1, 1, 1)	168	1580
(2, 1, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, -1, 0, 0, 0, -1, -1, 0, 1, 1, 1, 1)	3360	125
(2, 1, 1, 1, 1, 0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 1, 1, -1, -1, -1, 0, 0, 0, 0, 1)	3360	245
(2, 1, 1, 1, 1, 0, 1, 1, 0, 0, 1, 1, 0, 0, 0, -1, -1, 0, 0, 0, 0, -1, -1, 0, 0, 0, 0, 2)	420	27
(2, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1, 1, 1, 1, 1)	280	347
(2, 2, 2, 2, 2, 2, -1, -1, -1, -1, -1, -1, -1, -1, 2, -1, -1, -1, -1, 2, 2, -1, -1, -1, -1, 2, 2, 2)	35	11878
(2, 2, 2, 2, 2, 2, 2, 2, 2, 2, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, 2, -1, -1, -1, -1, -1, 2, 2)	56	4641
(2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, 2)	28	245
(3, 2, 1, 2, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, -1, 0, 0, 0, 0, -2, -1, 0, 0, 2, 2, 2)	10080	27
(4, 4, -2, 1, 1, 1, 1, 1, 1, 4, 1, 1, 1, -2, -2, -2, 4, -2, 1, 1, 1, -2, -2, 4, 1, 1, 1, 4)	2016	95
(4, 4, 4, 4, 4, -2, 1, 1, 1, 1, 1, 1, 1, 1, -2, -2, -2, 4, -2, 1, 1, -2, -2, -2, 4, 1, 1, 4)	5040	60
(5, 5, 2, 2, -1, -1, 2, -1, -1, 2, -1, 2, 2, -1, -1, -1, 2, 2, -1, -1, 2, -4, -1, -1, 2, 2, 5, 5)	2520	109
(7, 7, 4, 4, 1, 1, 4, 1, 1, -2, 1, 4, -2, 1, 1, 1, -2, 4, 1, 1, -2, -5, -2, -2, 1, 1, 4, 4)	10080	27
(8, 5, -1, 5, -1, 2, 2, 2, 5, 5, 2, 2, -1, -1, -4, -4, 2, -1, -1, 2, 2, -4, -4, 5, 5, 2, 2, 8)	10080	27
total	52367	

TABLE 3. Isomorphism classes of facet normals of τ_8, in standard form

The total count of 52367 facets also appears in [2] as the degree of anti-cuts in the metric polytope met_8.

Here, for the first time, we encounter facets in category $2 \pmod 3$ other than cuts. The significance of these new facets is still poorly understood. We also notice that the trivial facet and cut facets have the richest neighborhood structure; this is likely connected with the relatively large number of triangles on which the corresponding facet normals vanish.

The number of facets of τ_n for $n = 5, 6, 7, 8$ is now sequence A246427 in the OEIS database; see [15].

$n = 9$

It is presently out of reach to compute and classify all facets of τ_n for $n \geq 9$. However, for $n = 9$, we sampled a large number of 'random' facets using standard elimination steps. We found 143 isomorphism classes of facets of τ_9, accounting for nearly 12 million distinct facets. This possibly represents a complete classification, since all but five types have had their neighborhoods exhaustively checked (and are adjacent to no new types). The trivial and $(5, 4)$-cut facets have by far the largest degrees and may be particularly challenging to fully check. See [3] for more detail on the 'adjacency decomposition' method for symmetric cones and polytopes. A list of the known facets of τ_9 can be found at the first author's webpage: `http://www.math.uvic.ca/~dukes/facets-tri9.txt`

A new feature that emerges at $n = 9$ is the existence of automorphism-free facets.

Example 4.4. With coordinates given in colex order, the facet of τ_9 which is normal to

$$y = (4, 2, 2, 2, 0, 0, 1, 1, -1, 1, 1, -1, -1, 1, 2, 0, 0, 2, 0, -1, 1, -1, 1, 1, -1, 0, 0, 1, -2, -2, 0, 2, 1, 3, 2, 3)$$

has no automorphisms, and hence generates 9! distinct facets of τ_9 under the action of $\mathcal{S}_9$.

We find it interesting that the number of facets of τ_9 is already so large. As n grows, if the number of facets of τ_n exceeds $2^{\binom{n}{2}}$, then it would follow that certain inequalities are only useful to exclude (non-simple) multigraphs from the cone. A starting estimate on the number of facets of τ_n via the metric polytope (see Section 6) can be obtained from [10].

5. Lifting facets

Our next result concerns lifting facets of τ_n to facets of τ_{n+1} via a 'vertex splitting' operation.

Proposition 5.1. *Let $n \geq 5$ and suppose y is a facet normal of τ_n. Suppose there exists a triangle $K \subset [n-1]$ with $\langle y, \mathbb{1}_K \rangle > 0$. Define the vector y^{spl} on $\binom{[n+1]}{2}$ by*

$$y^{spl}(e) = \begin{cases} y(e) & \text{if } e \subset [n], \\ y(\{i,n\}) & \text{if } e = \{i, n+1\} \text{ for } i \in [n-1], \\ -2\min\{y(\{i,n\}) : i \in [n-1]\} & \text{if } e = \{n, n+1\}. \end{cases}$$

Then y^{spl} is a facet normal of τ_{n+1}.

Proof. It is straightforward to check that y^{spl} is nonnegative on all triangles. Let $\mathcal{K} \subseteq \binom{[n+1]}{3}$ contain K, along with all zero-sum triangles with respect to y^{spl}. Since y is a facet normal of τ_n, every edge in $\binom{[n+1]}{2}$, except possibly $\{n, n+1\}$, is a linear combination of triangles in $\mathcal{K}$. By the choice of weight on $\{n, n+1\}$, there exists j such that $\{j, n, n+1\} \in \mathcal{K}$. Then, since $\{j, n\}$ and $\{j, n+1\}$ are spanned by $\mathcal{K}$, so is $\{n, n+1\}$. □

The second author's dissertation presents a similar construction which allows copies of a facet of τ_n to be glued together along a common positive triangle to produce facets of τ_m for $m > n \geq 5$. This allows for somewhat more general lifts of facets. See [9, Proposition 3.8] for details.

Example 5.2. The weighted graph shown in Figure 3 corresponds to a facet of τ_8. Repeatedly applying Proposition 5.1 to leaf vertices gives an infinite family of facets of τ_n, $n \geq 8$, as follows. Given any partition (A, B) of $\{3, \dots, n\}$ with $|A|, |B| \geq 3$, a facet normal arises from the vector y defined by

$$y(e) = \begin{cases} -1 & \text{if } e = \{1,2\}, \\ 1 & \text{if } e = \{1,a\} \text{ for } a \in A, \text{ or } e = \{2,b\} \text{ for } b \in B, \\ 0 & \text{otherwise.} \end{cases}$$

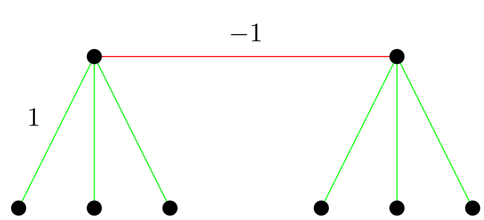

Figure 3. The weighted graph corresponding to the 'binary star' facet of τ_8

The terminology 'binary star' was used for such facets in [9]. These have some significance for triangle decompositions. Consider the question of whether G has a fractional triangle decomposition under the assumption that it has at least $\frac{3}{4}\binom{n}{2}$ edges and has minimum degree $\delta(G) \geq cn$. The binary star y reveals that we cannot take $c < \frac{1}{2}$. For instance, the graph G built from a complete graph K_{n-2} on $A \cup B$ with A joined to 2, B joined to 1, and the edge $\{1,2\}$ satisfies $\langle y, \mathbb{1}_G \rangle = -1$.

6. **Conclusion**

We have seen many properties of the facet structure of τ_n and connected some of these with the triangle decomposition problem for graphs. There are still many more questions than answers. As next steps, we believe some of these questions deserve consideration.

(1) Can τ_n be 'approximated' by a cone with simpler facet structure?
(2) Do there exist tighter bounds on the entries of a facet normal, improving on Proposition 3.1?
(3) Are there other ways to lift facets from τ_n to τ_{n+1}, perhaps generalizing Proposition 5.1?
(4) Can it be argued that no facet is adjacent only to trivial facets, bypassing the need to search the largest neighborhood for $n = 9$?
(5) Is there a combinatorial description of adjacency of facets?

As mentioned in Section 1.3, our problem is essentially (properly) contained in the study of the vertices of metric polytope met_n. However, for general n it does not seem that the triangle inequalities can shed much light on τ_n. That is, analysis of the neighborhood of $(2/3, \ldots, 2/3)$ in met_n probably requires separate treatment, which we hope this paper initiates to some extent.

It is worth briefly mentioning the 'cut cone' in $\mathbb{R}^{\binom{n}{2}}$, which is generated by all vertex cuts in the complete graph K_n. This cone is superficially similar to τ_n and its structure has been quite thoroughly investigated; see for instance [4, 5]. While we can find no obvious explicit connection between τ_n and the cut cone, we are hopeful that the results and conjectures for the latter can guide future work on τ_n.

It should be mentioned that the cone τ_n even long ago attracted some interest in quantum physics for its connection with the 'N-representability problem'; see for instance [11].

Finally, our cone τ_n is a special case of the family of cones introduced in [8]. This more general setting considers the cone generated by the inclusion matrix of t-subsets $\binom{[n]}{t}$ versus k-subsets $\binom{[n]}{k}$, where t, k, n are positive integers satisfying $k \geq t$ and $n \geq k + t$. Alternatively, this is the cone of weighted t-uniform hypergraphs on n vertices generated by k-vertex cliques $K_k^{(t)}$. This class of cones was not studied in much detail in [8]. However, some interesting supporting vectors connected with association schemes were shown to imply various inequalities for t-designs.

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