

The Formulas for the Distribution of the 3-Smooth, 5-Smooth, 7-Smooth and all other Smooth Numbers

Raphael Schumacher
raphschu@ethz.ch

Abstract

In this paper we present and prove rapidly convergent formulas for the distribution of the 3-smooth, 5-smooth, 7-smooth and all other smooth numbers. One of these formulas is another version of a formula due to Hardy and Littlewood for the arithmetic function $N_{a,b}(x)$, which counts the number of positive integers of the form $a^p b^q$ less than or equal to x .

1 Introduction

Let $a \in \mathbb{N}_{\geq 2}$ be a fixed natural number.

Let $N_a(x)$ denote the number of natural numbers of the form a^p which are smaller or equal to x , where $p \in \mathbb{N}_0$.

By definition [1, 2], we have that the 2-smooth numbers are just the powers of 2, namely

$$S_2 := \{1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \dots\}$$

and that the formula for their distribution is

$$N_2(x) = \frac{\log(x)}{\log(2)} + \frac{1}{2} - B_1 \left(\left\{ \frac{\log(x)}{\log(2)} \right\} \right).$$

This follows directly from the more general formula

$$N_a(x) = \frac{\log(x)}{\log(a)} + \frac{1}{2} - B_1 \left(\left\{ \frac{\log(x)}{\log(a)} \right\} \right).$$

Numbers of the form $2^p 3^q$, where $p \in \mathbb{N}_0$ and $q \in \mathbb{N}_0$ are called 3-smooth numbers [1, 2, 3], because these numbers are exactly the numbers which have no prime factors larger than 3. We will denote the sequence of 3-smooth numbers by $S_{2,3}$. Thus, we have that

$$\begin{aligned} S_{2,3} &:= \{2^p 3^q : p \in \mathbb{N}_0, q \in \mathbb{N}_0\} \\ &= \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 32, 36, 48, 54, 64, 72, 81, 96, 108, 128, 144, \dots\}. \end{aligned}$$

More generally, let $a, b \in \mathbb{N}$ be fixed natural numbers such that $a < b$ and $\gcd(a, b) = 1$. Let $N_{a,b}(x)$ denote the number of natural numbers of the form $a^p b^q$ which are smaller or equal to x , where $p, q \in \mathbb{N}_0$. Furthermore, we denote by $\chi_{S_{a,b}}(x)$ the characteristic function of the natural numbers of the form $a^p b^q$.

In his first letter to Hardy [4, 5, 6, 7], Ramanujan gave the formula

$$N_{2,3}(x) \approx \frac{1}{2} \frac{\log(2x) \log(3x)}{\log(2) \log(3)} + \frac{1}{2} \chi_{S_{2,3}}(x),$$

which provides a very close approximation to the number $N_{2,3}(x)$ of 3-smooth numbers less than or equal to x .

In his notebooks [4, 7], Ramanujan later generalized this expression to all $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$, namely

$$N_{a,b}(x) \approx \frac{1}{2} \frac{\log(ax) \log(bx)}{\log(a) \log(b)} + \frac{1}{2} \chi_{S_{a,b}}(x),$$

which is again a very close approximation to $N_{a,b}(x)$.

The analog formula for $N_{a,b,c}(x)$ [8, 9, 10] is

$$N_{a,b,c}(x) \approx \frac{\log\left(x\sqrt{abc}\right)^3}{6 \log(a) \log(b) \log(c)} + \frac{1}{2} \chi_{S_{a,b,c}}(x),$$

which is also a very good approximation to $N_{a,b,c}(x)$.

In the following sections, we present and prove rapidly convergent formulas for the functions $N_{a,b}(x)$ and $N_{2,3}(x)$, having the above Ramanujan approximations as their first term. These two formulas are other versions of a more rapidly convergent formula already found by Hardy and Littlewood around 1920 [11, 12], as it was communicated to us by Emanuele Tron [13]. We also prove very rapidly convergent formulas for the distribution of the 5-smooth, 7-smooth and all other smooth numbers.

At the end of the paper, we give an exact formula for the counting function of the natural numbers of the form $a^{p^2} b^{q^2}$.

We have searched all resulting formulas (which are given in theorems and corollaries) in the literature and on the internet, but we could only find the Hardy-Littlewood formula [11, 12]. Therefore, we believe that all other results are new.

2 The Formulas for $N_{a,b}(x)$ and the 3-Smooth Numbers Counting Function $N_{2,3}(x)$

Let $a, b \in \mathbb{N}$ such that $a < b$ and $\gcd(a, b) = 1$. For $x \in \mathbb{R}_0^+$, we define the function $N_{a,b}(x)$ by

$$N_{a,b}(x) := \sum_{\substack{a^p b^q \leq x \\ p \in \mathbb{N}_0, q \in \mathbb{N}_0}} 1.$$

Moreover, we denote the set of natural numbers of the form $a^p b^q$ by $S_{a,b}$ and its characteristic function by $\chi_{S_{a,b}}(x)$, that is

$$S_{a,b} := \{a^p b^q : p \in \mathbb{N}_0, q \in \mathbb{N}_0\},$$

$$\chi_{S_{a,b}}(x) := \begin{cases} 1 & \text{if } x \in S_{a,b} \\ 0 & \text{if } x \notin S_{a,b} \end{cases}.$$

We have that

$$N_{a,b}(x) = 1 + \sum_{k=0}^{\lfloor \log_a(x) \rfloor} \left\lfloor \log_b \left(\frac{x}{a^k} \right) \right\rfloor + \lfloor \log_a(x) \rfloor.$$

Theorem 1. (*Our Formula for $N_{a,b}(x)$*)

For every real number $x > 1$, we have that

$$\begin{aligned} N_{a,b}(x) = & \frac{1}{2} \frac{\log(ax) \log(bx)}{\log(a) \log(b)} + \frac{\log(a)}{12 \log(b)} + \frac{\log(b)}{12 \log(a)} - \frac{1}{4} - \frac{1}{2} B_1^* \left(\left\{ \frac{\log(x)}{\log(a)} \right\} \right) \\ & - \frac{1}{2} B_1^* \left(\left\{ \frac{\log(x)}{\log(b)} \right\} \right) - \frac{\log(a)}{2 \log(b)} B_2 \left(\left\{ \frac{\log(x)}{\log(a)} \right\} \right) - \frac{\log(b)}{2 \log(a)} B_2 \left(\left\{ \frac{\log(x)}{\log(b)} \right\} \right) \\ & + \frac{\log(a) \log(b)}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\cos \left(\frac{2\pi n \log(x)}{\log(a)} \right) - \cos \left(\frac{2\pi m \log(x)}{\log(b)} \right)}{m^2 \log(a)^2 - n^2 \log(b)^2} + \frac{1}{2} \chi_{S_{a,b}}(x), \end{aligned}$$

where

$$B_1^*({x}) := \begin{cases} {x} - \frac{1}{2} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \end{cases},$$

$$B_2({x}) := {x}^2 - {x} + \frac{1}{6} \quad \forall x \in \mathbb{R}.$$

The above formula converges rapidly.

As usual we denote by $\{x\}$ the fractional part of x .

This is just another version of the following

Theorem 2. (The Hardy-Littlewood formula for $N_{a,b}(x)$)[11, 12]

For every real number $x \geq 1$, we have that

$$N_{a,b}(x) = \frac{1}{2} \frac{\log(ax) \log(bx)}{\log(a) \log(b)} + \frac{\log(a)}{12 \log(b)} + \frac{\log(b)}{12 \log(a)} - \frac{1}{4} - B_1^* \left(\left\{ \frac{\log(x)}{\log(a)} \right\} \right) - B_1^* \left(\left\{ \frac{\log(x)}{\log(b)} \right\} \right) \\ - \frac{1}{2\pi} \sum_{k=1}^{\infty} \left(\frac{\cos \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(b)}{\log(a)} \right)}{k \sin \left(\frac{\pi k \log(b)}{\log(a)} \right)} + \frac{\cos \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(a)}{\log(b)} \right)}{k \sin \left(\frac{\pi k \log(a)}{\log(b)} \right)} \right) + \frac{1}{2} \chi_{S_{a,b}}(x),$$

where the series is to be interpreted as meaning [12]

$$\sum_{k=1}^{\infty} \left(\frac{\cos \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(b)}{\log(a)} \right)}{k \sin \left(\frac{\pi k \log(b)}{\log(a)} \right)} + \frac{\cos \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(a)}{\log(b)} \right)}{k \sin \left(\frac{\pi k \log(a)}{\log(b)} \right)} \right) \\ = \lim_{R \rightarrow \infty} \left(\sum_{k=1}^{\lfloor R \log(a) \rfloor} \frac{\cos \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(b)}{\log(a)} \right)}{k \sin \left(\frac{\pi k \log(b)}{\log(a)} \right)} + \sum_{k=1}^{\lfloor R \log(b) \rfloor} \frac{\cos \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(a)}{\log(b)} \right)}{k \sin \left(\frac{\pi k \log(a)}{\log(b)} \right)} \right),$$

when $R \rightarrow \infty$ in an appropriate manner.

This formula converges very rapidly.

Setting $a = 2$ and $b = 3$, we get immediately two formulas for the distribution of the 3-smooth numbers, namely

Corollary 3. (Our Formula for the 3-Smooth Numbers Counting Function $N_{2,3}(x)$)

For every real number $x > 1$, we have that

$$N_{2,3}(x) = \frac{1}{2} \frac{\log(2x) \log(3x)}{\log(2) \log(3)} + \frac{\log(2)}{12 \log(3)} + \frac{\log(3)}{12 \log(2)} - \frac{1}{4} - \frac{1}{2} B_1^* \left(\left\{ \frac{\log(x)}{\log(2)} \right\} \right) \\ - \frac{1}{2} B_1^* \left(\left\{ \frac{\log(x)}{\log(3)} \right\} \right) - \frac{\log(2)}{2 \log(3)} B_2 \left(\left\{ \frac{\log(x)}{\log(2)} \right\} \right) - \frac{\log(3)}{2 \log(2)} B_2 \left(\left\{ \frac{\log(x)}{\log(3)} \right\} \right) \\ + \frac{\log(2) \log(3)}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\cos \left(\frac{2\pi n \log(x)}{\log(2)} \right) - \cos \left(\frac{2\pi m \log(x)}{\log(3)} \right)}{m^2 \log(2)^2 - n^2 \log(3)^2} + \frac{1}{2} \chi_{S_{2,3}}(x).$$

Corollary 4. (The Hardy-Littlewood formula for $N_{2,3}(x)$)[11, 12]

For every real number $x \geq 1$, we have that

$$N_{2,3}(x) = \frac{1}{2} \frac{\log(2x) \log(3x)}{\log(2) \log(3)} + \frac{\log(2)}{12 \log(3)} + \frac{\log(3)}{12 \log(2)} - \frac{1}{4} - B_1^* \left(\left\{ \frac{\log(x)}{\log(2)} \right\} \right) - B_1^* \left(\left\{ \frac{\log(x)}{\log(3)} \right\} \right) \\ - \frac{1}{2\pi} \sum_{k=1}^{\infty} \left(\frac{\cos \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(3)}{\log(2)} \right)}{k \sin \left(\frac{\pi k \log(3)}{\log(2)} \right)} + \frac{\cos \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(2)}{\log(3)} \right)}{k \sin \left(\frac{\pi k \log(2)}{\log(3)} \right)} \right) + \frac{1}{2} \chi_{S_{2,3}}(x),$$

where the series is to be interpreted as meaning [12]

$$\sum_{k=1}^{\infty} \left(\frac{\cos \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(3)}{\log(2)} \right)}{k \sin \left(\frac{\pi k \log(3)}{\log(2)} \right)} + \frac{\cos \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(2)}{\log(3)} \right)}{k \sin \left(\frac{\pi k \log(2)}{\log(3)} \right)} \right)$$

$$= \lim_{R \rightarrow \infty} \left(\sum_{k=1}^{\lfloor R \log(2) \rfloor} \frac{\cos \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(3)}{\log(2)} \right)}{k \sin \left(\frac{\pi k \log(3)}{\log(2)} \right)} + \sum_{k=1}^{\lfloor R \log(3) \rfloor} \frac{\cos \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(2)}{\log(3)} \right)}{k \sin \left(\frac{\pi k \log(2)}{\log(3)} \right)} \right),$$

when $R \rightarrow \infty$ in an appropriate manner.

Using the computationally more efficient formula

$$N_{2,3}(x) = 1 + \sum_{k=0}^{\lfloor \log_3(x) \rfloor} \left\lfloor \log_2 \left(\frac{x}{3^k} \right) \right\rfloor + \lfloor \log_3(x) \rfloor,$$

we get the following two tables:

x	$N_{2,3}(x)$	Our Formula for $N_{2,3}(x)$	Number of terms (n, m)
1	1	1.0510201857955517	$(n, m) = (4, 4)$ at $x = 1.1$
10	7	7.0071497373839231	$(n, m) = (22, 22)$
10^2	20	20.0045160354084706	$(n, m) = (10, 10)$
10^3	40	40.0039084310672772	$(n, m) = (12, 12)$
10^4	67	67.0408408937206653	$(n, m) = (20, 20)$
10^5	101	101.05072154439969785	$(n, m) = (28, 28)$
10^6	142	142.01315000789587358	$(n, m) = (70, 70)$
10^7	190	190.00707389223323501	$(n, m) = (110, 110)$
10^8	244	244.00659912032029415	$(n, m) = (140, 140)$
10^9	306	306.00585869480145596	$(n, m) = (160, 160)$
10^{10}	376	376.02126583465866742	$(n, m) = (170, 170)$
10^{10^2}	35084	35084.0568926232894816675	$(n, m) = (2000, 2000)$
10^{10^3}	3483931	3483931.035272714689991309386	$(n, m) = (4000, 4000)$

Table 1: Values of $N_{2,3}(x)$

x	$N_{2,3}(x)$	The Hardy-Littlewood Formula for $N_{2,3}(x)$	Number of terms R
1	1	1.00408281281244794423184044310637662236	$R = 1$
10	7	7.01039536792580652845911613960427072715	$R = 6$
10^2	20	20.00554687989157075178992137362449803027	$R = 10$
10^3	40	40.00416733658863125098651349198857667561	$R = 26$
10^4	67	67.04067163854917851848072234444363738009	$R = 32$
10^5	101	101.00383710643693392983460661037688277109	$R = 44$
10^6	142	142.00519665851176957826409909346411626717	$R = 60$
10^7	190	190.00431172466646336030921684292744206625	$R = 100$
10^8	244	244.00043300366963526250817238561664826018	$R = 122$
10^9	306	306.00450681431786167717856515798365069396	$R = 146$
10^{10}	376	376.02231447192801988487484982661961706561	$R = 160$
10^{10^2}	35084	35084.03451234481158685735036751788214481906	$R = 3000$
10^{10^3}	3483931	3483931.03067546896021243171738747049589388966	$R = 3000$
10^{10^4}	348149087	348149087.05625852937187129720297862230958308491	$R = 24000$
10^{10^5}	34812470748	34812470748.06400873722492550333469431071713138958	$R = 200000$

Table 2: Values of $N_{2,3}(x)$

Corollary 5. (Modified version of our formula for $N_{a,b}(x)$)

For every real number $x > 1$, we have

$$\begin{aligned}
N_{a,b}(x) &= \frac{\log(x)^2}{2 \log(a) \log(b)} + \frac{\log(x)}{2 \log(a)} + \frac{\log(x)}{2 \log(b)} + \frac{1}{4} + \frac{\log(a)}{12 \log(b)} + \frac{\log(b)}{12 \log(a)} - \frac{1}{2} B_1^* \left(\left\{ \frac{\log(x)}{\log(a)} \right\} \right) \\
&\quad - \frac{1}{2} B_1^* \left(\left\{ \frac{\log(x)}{\log(b)} \right\} \right) - \frac{\log(a)}{2 \log(b)} B_2 \left(\left\{ \frac{\log(x)}{\log(a)} \right\} \right) - \frac{\log(b)}{2 \log(a)} B_2 \left(\left\{ \frac{\log(x)}{\log(b)} \right\} \right) \\
&\quad + \frac{\log(a) \log(b)}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\cos \left(\frac{2\pi n \log(x)}{\log(a)} \right) - \cos \left(\frac{2\pi m \log(x)}{\log(b)} \right)}{m^2 \log(a)^2 - n^2 \log(b)^2} + \frac{1}{2} \chi_{S_{a,b}}(x).
\end{aligned}$$

Corollary 6. (Modified Hardy-Littlewood formula for $N_{a,b}(x)$)[11, 12]

For every real number $x \geq 1$, we have

$$\begin{aligned}
N_{a,b}(x) &= \frac{\log(x)^2}{2 \log(a) \log(b)} + \frac{\log(x)}{2 \log(a)} + \frac{\log(x)}{2 \log(b)} + \frac{1}{4} + \frac{\log(a)}{12 \log(b)} + \frac{\log(b)}{12 \log(a)} - B_1^* \left(\left\{ \frac{\log(x)}{\log(a)} \right\} \right) \\
&\quad - B_1^* \left(\left\{ \frac{\log(x)}{\log(b)} \right\} \right) - \frac{1}{2\pi} \sum_{k=1}^{\infty} \left(\frac{\cos \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(a)}{\log(b)} \right)}{k \sin \left(\frac{\pi k \log(a)}{\log(b)} \right)} + \frac{\cos \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(b)}{\log(a)} \right)}{k \sin \left(\frac{\pi k \log(b)}{\log(a)} \right)} \right) + \frac{1}{2} \chi_{S_{a,b}}(x),
\end{aligned}$$

where the series is interpreted as mentioned above.

3 The Formula for the Distribution of the 5-Smooth Numbers

Let $a, b, c \in \mathbb{N}$ such that $a < b < c$ and $\gcd(a, b, c) = 1$.
For $x \in \mathbb{R}_0^+$, we define the function $N_{a,b,c}(x)$ by

$$N_{a,b,c}(x) := \sum_{\substack{a^p b^q c^l \leq x \\ p \in \mathbb{N}_0, q \in \mathbb{N}_0, l \in \mathbb{N}_0}} 1.$$

We define also

$$S_{a,b,c} := \{a^p b^q c^l : p \in \mathbb{N}_0, q \in \mathbb{N}_0, l \in \mathbb{N}_0\},$$

$$\chi_{S_{a,b,c}}(x) := \begin{cases} 1 & \text{if } x \in S_{a,b,c} \\ 0 & \text{if } x \notin S_{a,b,c} \end{cases}.$$

Thus, we have that

$$N_{a,b,c}(x) = \sum_{k=0}^{\lfloor \log_a(x) \rfloor} \sum_{l=0}^{\lfloor \log_b\left(\frac{x}{a^k}\right) \rfloor} \left(\left\lfloor \log_c\left(\frac{x}{a^k b^l}\right) \right\rfloor + 1 \right).$$

We have the following

Theorem 7. (Formula for $N_{a,b,c}(x)$)
For every real number $x \geq 1$, we have that

$$\begin{aligned} N_{a,b,c}(x) = & \frac{\log(x)^3}{6 \log(a) \log(b) \log(c)} + \frac{\log(x)^2}{4 \log(a) \log(b)} + \frac{\log(x)^2}{4 \log(a) \log(c)} + \frac{\log(x)^2}{4 \log(b) \log(c)} + \frac{\log(x)}{4 \log(a)} \\ & + \frac{\log(x)}{4 \log(b)} + \frac{\log(x)}{4 \log(c)} + \frac{\log(a) \log(x)}{12 \log(b) \log(c)} + \frac{\log(b) \log(x)}{12 \log(a) \log(c)} + \frac{\log(c) \log(x)}{12 \log(a) \log(b)} \\ & + \frac{\log(a)}{24 \log(b)} + \frac{\log(a)}{24 \log(c)} + \frac{\log(b)}{24 \log(a)} + \frac{\log(b)}{24 \log(c)} + \frac{\log(c)}{24 \log(a)} + \frac{\log(c)}{24 \log(b)} + \frac{1}{8} \\ & - B_1^* \left(\left\{ \frac{\log(x)}{\log(a)} \right\} \right) - B_1^* \left(\left\{ \frac{\log(x)}{\log(b)} \right\} \right) - B_1^* \left(\left\{ \frac{\log(x)}{\log(c)} \right\} \right) \\ & - \frac{1}{4\pi} \sum_{k=1}^{\infty} \left(\frac{\cos\left(2\pi k \frac{\log(x) - \frac{1}{2} \log(b)}{\log(a)}\right)}{k \sin\left(\frac{\pi k \log(b)}{\log(a)}\right)} + \frac{\cos\left(2\pi k \frac{\log(x) - \frac{1}{2} \log(a)}{\log(b)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(b)}\right)} \right) \\ & - \frac{1}{4\pi} \sum_{k=1}^{\infty} \left(\frac{\cos\left(2\pi k \frac{\log(x) - \frac{1}{2} \log(c)}{\log(b)}\right)}{k \sin\left(\frac{\pi k \log(c)}{\log(b)}\right)} + \frac{\cos\left(2\pi k \frac{\log(x) - \frac{1}{2} \log(b)}{\log(c)}\right)}{k \sin\left(\frac{\pi k \log(b)}{\log(c)}\right)} \right) \\ & - \frac{1}{4\pi} \sum_{k=1}^{\infty} \left(\frac{\cos\left(2\pi k \frac{\log(x) - \frac{1}{2} \log(a)}{\log(c)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(c)}\right)} + \frac{\cos\left(2\pi k \frac{\log(x) - \frac{1}{2} \log(c)}{\log(a)}\right)}{k \sin\left(\frac{\pi k \log(c)}{\log(a)}\right)} \right) \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{8\pi} \sum_{k=1}^{\infty} \left(\frac{\sin\left(2\pi k \frac{\log(x) + \frac{1}{2}\log(b) - \frac{1}{2}\log(c)}{\log(a)}\right)}{k \sin\left(\frac{\pi k \log(b)}{\log(a)}\right) \sin\left(\frac{\pi k \log(c)}{\log(a)}\right)} + \frac{\sin\left(2\pi k \frac{\log(x) - \frac{1}{2}\log(b) + \frac{1}{2}\log(c)}{\log(a)}\right)}{k \sin\left(\frac{\pi k \log(b)}{\log(a)}\right) \sin\left(\frac{\pi k \log(c)}{\log(a)}\right)} \right) \\
& -\frac{1}{8\pi} \sum_{k=1}^{\infty} \left(\frac{\sin\left(2\pi k \frac{\log(x) + \frac{1}{2}\log(a) - \frac{1}{2}\log(c)}{\log(b)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(b)}\right) \sin\left(\frac{\pi k \log(c)}{\log(b)}\right)} + \frac{\sin\left(2\pi k \frac{\log(x) - \frac{1}{2}\log(a) + \frac{1}{2}\log(c)}{\log(b)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(b)}\right) \sin\left(\frac{\pi k \log(c)}{\log(b)}\right)} \right) \\
& -\frac{1}{8\pi} \sum_{k=1}^{\infty} \left(\frac{\sin\left(2\pi k \frac{\log(x) + \frac{1}{2}\log(a) - \frac{1}{2}\log(b)}{\log(c)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(c)}\right) \sin\left(\frac{\pi k \log(b)}{\log(c)}\right)} + \frac{\sin\left(2\pi k \frac{\log(x) - \frac{1}{2}\log(a) + \frac{1}{2}\log(b)}{\log(c)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(c)}\right) \sin\left(\frac{\pi k \log(b)}{\log(c)}\right)} \right) + \frac{1}{2} \chi_{S_{a,b,c}}(x).
\end{aligned}$$

This formula converges very rapidly.

In the above formula, the series are to be interpreted as meaning

$$\begin{aligned}
& \sum_{k=1}^{\infty} \left(\frac{\cos\left(2\pi k \frac{\log(x) - \frac{1}{2}\log(b)}{\log(a)}\right)}{k \sin\left(\frac{\pi k \log(b)}{\log(a)}\right)} + \frac{\cos\left(2\pi k \frac{\log(x) - \frac{1}{2}\log(a)}{\log(b)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(b)}\right)} \right) \\
& = \lim_{R \rightarrow \infty} \left(\sum_{k=1}^{\lfloor R \log(a) \rfloor} \frac{\cos\left(2\pi k \frac{\log(x) - \frac{1}{2}\log(b)}{\log(a)}\right)}{k \sin\left(\frac{\pi k \log(b)}{\log(a)}\right)} + \sum_{k=1}^{\lfloor R \log(b) \rfloor} \frac{\cos\left(2\pi k \frac{\log(x) - \frac{1}{2}\log(a)}{\log(b)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(b)}\right)} \right)
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{k=1}^{\infty} \left(\frac{\sin\left(2\pi k \frac{\log(x) + \frac{1}{2}\log(b) - \frac{1}{2}\log(c)}{\log(a)}\right)}{k \sin\left(\frac{\pi k \log(b)}{\log(a)}\right) \sin\left(\frac{\pi k \log(c)}{\log(a)}\right)} + \frac{\sin\left(2\pi k \frac{\log(x) - \frac{1}{2}\log(b) + \frac{1}{2}\log(c)}{\log(a)}\right)}{k \sin\left(\frac{\pi k \log(b)}{\log(a)}\right) \sin\left(\frac{\pi k \log(c)}{\log(a)}\right)} \right) \\
& = \lim_{R \rightarrow \infty} \left(\sum_{k=1}^{\lfloor R \log(a) \rfloor} \left(\frac{\sin\left(2\pi k \frac{\log(x) + \frac{1}{2}\log(b) - \frac{1}{2}\log(c)}{\log(a)}\right)}{k \sin\left(\frac{\pi k \log(b)}{\log(a)}\right) \sin\left(\frac{\pi k \log(c)}{\log(a)}\right)} + \frac{\sin\left(2\pi k \frac{\log(x) - \frac{1}{2}\log(b) + \frac{1}{2}\log(c)}{\log(a)}\right)}{k \sin\left(\frac{\pi k \log(b)}{\log(a)}\right) \sin\left(\frac{\pi k \log(c)}{\log(a)}\right)} \right) \right),
\end{aligned}$$

when $R \rightarrow \infty$ in an appropriate manner.

Setting $a = 2$, $b = 3$ and $c = 5$ and interpreting the series, like before, as meaning (for example)

$$\begin{aligned}
& \sum_{k=1}^{\infty} \left(\frac{\cos\left(2\pi k \frac{\log(x) - \frac{1}{2}\log(3)}{\log(2)}\right)}{k \sin\left(\frac{\pi k \log(3)}{\log(2)}\right)} + \frac{\cos\left(2\pi k \frac{\log(x) - \frac{1}{2}\log(2)}{\log(3)}\right)}{k \sin\left(\frac{\pi k \log(2)}{\log(3)}\right)} \right) \\
& = \lim_{R \rightarrow \infty} \left(\sum_{k=1}^{\lfloor R \log(2) \rfloor} \frac{\cos\left(2\pi k \frac{\log(x) - \frac{1}{2}\log(3)}{\log(2)}\right)}{k \sin\left(\frac{\pi k \log(3)}{\log(2)}\right)} + \sum_{k=1}^{\lfloor R \log(3) \rfloor} \frac{\cos\left(2\pi k \frac{\log(x) - \frac{1}{2}\log(2)}{\log(3)}\right)}{k \sin\left(\frac{\pi k \log(2)}{\log(3)}\right)} \right)
\end{aligned}$$

and

$$\begin{aligned} & \sum_{k=1}^{\infty} \left(\frac{\sin \left(2\pi k \frac{\log(x) + \frac{1}{2} \log(3) - \frac{1}{2} \log(5)}{\log(2)} \right)}{k \sin \left(\frac{\pi k \log(3)}{\log(2)} \right) \sin \left(\frac{\pi k \log(5)}{\log(2)} \right)} + \frac{\sin \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(3) + \frac{1}{2} \log(5)}{\log(2)} \right)}{k \sin \left(\frac{\pi k \log(3)}{\log(2)} \right) \sin \left(\frac{\pi k \log(5)}{\log(2)} \right)} \right) \\ &= \lim_{R \rightarrow \infty} \left(\sum_{k=1}^{\lfloor R \log(2) \rfloor} \left(\frac{\sin \left(2\pi k \frac{\log(x) + \frac{1}{2} \log(3) - \frac{1}{2} \log(5)}{\log(2)} \right)}{k \sin \left(\frac{\pi k \log(3)}{\log(2)} \right) \sin \left(\frac{\pi k \log(5)}{\log(2)} \right)} + \frac{\sin \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(3) + \frac{1}{2} \log(5)}{\log(2)} \right)}{k \sin \left(\frac{\pi k \log(3)}{\log(2)} \right) \sin \left(\frac{\pi k \log(5)}{\log(2)} \right)} \right) \right), \end{aligned}$$

when $R \rightarrow \infty$ in an appropriate manner, we get for the sequence

$$\begin{aligned} S_{2,3,5} &:= \{2^p 3^q 5^l : p \in \mathbb{N}_0, q \in \mathbb{N}_0, l \in \mathbb{N}_0\} \\ &= \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 27, 30, 32, 36, 40, 45, 48, \dots\}, \end{aligned}$$

of 5-smooth numbers (regular numbers or Hamming numbers) [9, 10], the following formula

Corollary 8. (Formula for the 5-Smooth Numbers Counting Function $N_{2,3,5}(x)$)

For every real number $x \geq 1$, we have that

$$\begin{aligned} N_{2,3,5}(x) &= \frac{\log(x)^3}{6 \log(2) \log(3) \log(5)} + \frac{\log(x)^2}{4 \log(2) \log(3)} + \frac{\log(x)^2}{4 \log(2) \log(5)} + \frac{\log(x)^2}{4 \log(3) \log(5)} + \frac{\log(x)}{4 \log(2)} \\ &+ \frac{\log(x)}{4 \log(3)} + \frac{\log(x)}{4 \log(5)} + \frac{\log(2) \log(x)}{12 \log(3) \log(5)} + \frac{\log(3) \log(x)}{12 \log(2) \log(5)} + \frac{\log(5) \log(x)}{12 \log(2) \log(3)} \\ &+ \frac{\log(2)}{24 \log(3)} + \frac{\log(2)}{24 \log(5)} + \frac{\log(3)}{24 \log(2)} + \frac{\log(3)}{24 \log(5)} + \frac{\log(5)}{24 \log(2)} + \frac{\log(5)}{24 \log(3)} + \frac{1}{8} \\ &- B_1^* \left(\left\{ \frac{\log(x)}{\log(2)} \right\} \right) - B_1^* \left(\left\{ \frac{\log(x)}{\log(3)} \right\} \right) - B_1^* \left(\left\{ \frac{\log(x)}{\log(5)} \right\} \right) \\ &- \frac{1}{4\pi} \sum_{k=1}^{\infty} \left(\frac{\cos \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(3)}{\log(2)} \right)}{k \sin \left(\frac{\pi k \log(3)}{\log(2)} \right)} + \frac{\cos \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(2)}{\log(3)} \right)}{k \sin \left(\frac{\pi k \log(2)}{\log(3)} \right)} \right) \\ &- \frac{1}{4\pi} \sum_{k=1}^{\infty} \left(\frac{\cos \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(5)}{\log(3)} \right)}{k \sin \left(\frac{\pi k \log(5)}{\log(3)} \right)} + \frac{\cos \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(3)}{\log(5)} \right)}{k \sin \left(\frac{\pi k \log(3)}{\log(5)} \right)} \right) \\ &- \frac{1}{4\pi} \sum_{k=1}^{\infty} \left(\frac{\cos \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(2)}{\log(5)} \right)}{k \sin \left(\frac{\pi k \log(2)}{\log(5)} \right)} + \frac{\cos \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(5)}{\log(2)} \right)}{k \sin \left(\frac{\pi k \log(5)}{\log(2)} \right)} \right) \\ &- \frac{1}{8\pi} \sum_{k=1}^{\infty} \left(\frac{\sin \left(2\pi k \frac{\log(x) + \frac{1}{2} \log(3) - \frac{1}{2} \log(5)}{\log(2)} \right)}{k \sin \left(\frac{\pi k \log(3)}{\log(2)} \right) \sin \left(\frac{\pi k \log(5)}{\log(2)} \right)} + \frac{\sin \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(3) + \frac{1}{2} \log(5)}{\log(2)} \right)}{k \sin \left(\frac{\pi k \log(3)}{\log(2)} \right) \sin \left(\frac{\pi k \log(5)}{\log(2)} \right)} \right) \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{8\pi} \sum_{k=1}^{\infty} \left(\frac{\sin\left(2\pi k \frac{\log(x) + \frac{1}{2}\log(2) - \frac{1}{2}\log(5)}{\log(3)}\right)}{k \sin\left(\frac{\pi k \log(2)}{\log(3)}\right) \sin\left(\frac{\pi k \log(5)}{\log(3)}\right)} + \frac{\sin\left(2\pi k \frac{\log(x) - \frac{1}{2}\log(2) + \frac{1}{2}\log(5)}{\log(3)}\right)}{k \sin\left(\frac{\pi k \log(2)}{\log(3)}\right) \sin\left(\frac{\pi k \log(5)}{\log(3)}\right)} \right) \\
& -\frac{1}{8\pi} \sum_{k=1}^{\infty} \left(\frac{\sin\left(2\pi k \frac{\log(x) + \frac{1}{2}\log(2) - \frac{1}{2}\log(3)}{\log(5)}\right)}{k \sin\left(\frac{\pi k \log(2)}{\log(5)}\right) \sin\left(\frac{\pi k \log(3)}{\log(5)}\right)} + \frac{\sin\left(2\pi k \frac{\log(x) - \frac{1}{2}\log(2) + \frac{1}{2}\log(3)}{\log(5)}\right)}{k \sin\left(\frac{\pi k \log(2)}{\log(5)}\right) \sin\left(\frac{\pi k \log(3)}{\log(5)}\right)} \right) + \frac{1}{2} \chi_{S_{2,3,5}}(x).
\end{aligned}$$

This formula converges very rapidly.

Using this formula for the 5-Smooth Numbers Counting Function $N_{2,3,5}(x)$, we get the following table:

x	$N_{2,3,5}(x)$	Formula for $N_{2,3,5}(x)$	Number of terms R
1	1	1.0191146914343678209209456	$R = 3$
10	9	9.0066388420020729763649195	$R = 11$
10^2	34	34.01798108016701636663657078	$R = 32$
10^3	86	86.01831146911104727455077198	$R = 40$
10^4	175	175.01259815271196528318821070	$R = 52$
10^5	313	313.01116052291470126065468770	$R = 100$
10^6	507	507.04384962202822061525989835	$R = 104$
10^7	768	768.05762686767314864195183397	$R = 110$
10^8	1105	1105.00435666776355760375109758	$R = 260$
10^9	1530	1530.00198789289107971841182114	$R = 300$
10^{10}	2053	2053.01709151724653660944693303	$R = 306$
10^{10^2}	1697191	1697191.10060827971167051326275935	$R = 20000$

Table 3: Values of $N_{2,3,5}(x)$

4 The Formula for the Distribution of the 7-Smooth Numbers

Let $a, b, c, d \in \mathbb{N}$ such that $a < b < c < d$ and $\gcd(a, b, c, d) = 1$.

For $x \in \mathbb{R}_0^+$, we define the function $N_{a,b,c,d}(x)$ by

$$N_{a,b,c,d}(x) := \sum_{\substack{a^p b^q c^l d^f \leq x \\ p \in \mathbb{N}_0, q \in \mathbb{N}_0, l \in \mathbb{N}_0, f \in \mathbb{N}_0}} 1.$$

We define also

$$S_{a,b,c,d} := \{a^p b^q c^l d^f : p \in \mathbb{N}_0, q \in \mathbb{N}_0, l \in \mathbb{N}_0, f \in \mathbb{N}_0\},$$

$$\chi_{S_{a,b,c,d}}(x) := \begin{cases} 1 & \text{if } x \in S_{a,b,c,d} \\ 0 & \text{if } x \notin S_{a,b,c,d} \end{cases}.$$

Thus, we have that

$$N_{a,b,c,d}(x) = \sum_{k=0}^{\lfloor \log_a(x) \rfloor} \sum_{l=0}^{\lfloor \log_b\left(\frac{x}{a^k}\right) \rfloor} \sum_{m=0}^{\lfloor \log_c\left(\frac{x}{a^k b^l}\right) \rfloor} \left(\left\lfloor \log_d\left(\frac{x}{a^k b^l c^m}\right) \right\rfloor + 1 \right).$$

We have the following

Theorem 9. (Formula for $N_{a,b,c,d}(x)$)
For every real number $x \geq 1$, we have that

$$\begin{aligned} N_{a,b,c,d}(x) = & \frac{\log(x)^4}{24 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(x)^3}{12 \log(a) \log(b) \log(c)} + \frac{\log(x)^3}{12 \log(a) \log(b) \log(d)} \\ & + \frac{\log(x)^3}{12 \log(a) \log(c) \log(d)} + \frac{\log(x)^3}{12 \log(b) \log(c) \log(d)} + \frac{\log(a) \log(x)^2}{24 \log(b) \log(c) \log(d)} \\ & + \frac{\log(b) \log(x)^2}{24 \log(a) \log(c) \log(d)} + \frac{\log(c) \log(x)^2}{24 \log(a) \log(b) \log(d)} + \frac{\log(d) \log(x)^2}{24 \log(a) \log(b) \log(c)} \\ & + \frac{\log(x)^2}{8 \log(a) \log(b)} + \frac{\log(x)^2}{8 \log(a) \log(c)} + \frac{\log(x)^2}{8 \log(a) \log(d)} + \frac{\log(x)^2}{8 \log(b) \log(c)} + \frac{\log(x)^2}{8 \log(b) \log(d)} \\ & + \frac{\log(x)^2}{8 \log(c) \log(d)} + \frac{\log(x)}{8 \log(a)} + \frac{\log(x)}{8 \log(b)} + \frac{\log(x)}{8 \log(c)} + \frac{\log(x)}{8 \log(d)} + \frac{\log(a) \log(x)}{24 \log(b) \log(c)} \\ & + \frac{\log(a) \log(x)}{24 \log(b) \log(d)} + \frac{\log(a) \log(x)}{24 \log(c) \log(d)} + \frac{\log(b) \log(x)}{24 \log(a) \log(c)} + \frac{\log(b) \log(x)}{24 \log(a) \log(d)} + \frac{\log(b) \log(x)}{24 \log(c) \log(d)} \\ & + \frac{\log(c) \log(x)}{24 \log(a) \log(b)} + \frac{\log(c) \log(x)}{24 \log(a) \log(d)} + \frac{\log(c) \log(x)}{24 \log(b) \log(d)} + \frac{\log(d) \log(x)}{24 \log(a) \log(b)} + \frac{\log(d) \log(x)}{24 \log(a) \log(c)} \\ & + \frac{\log(d) \log(x)}{24 \log(b) \log(c)} + \frac{1}{16} + \frac{\log(a)}{48 \log(b)} + \frac{\log(a)}{48 \log(c)} + \frac{\log(a)}{48 \log(d)} + \frac{\log(b)}{48 \log(a)} + \frac{\log(b)}{48 \log(c)} \\ & + \frac{\log(b)}{48 \log(d)} + \frac{\log(c)}{48 \log(a)} + \frac{\log(c)}{48 \log(b)} + \frac{\log(c)}{48 \log(d)} + \frac{\log(d)}{48 \log(a)} + \frac{\log(d)}{48 \log(b)} + \frac{\log(d)}{48 \log(c)} \\ & + \frac{\log(a) \log(b)}{144 \log(c) \log(d)} + \frac{\log(a) \log(c)}{144 \log(b) \log(d)} + \frac{\log(a) \log(d)}{144 \log(b) \log(c)} + \frac{\log(b) \log(c)}{144 \log(a) \log(d)} \\ & + \frac{\log(b) \log(d)}{144 \log(a) \log(c)} + \frac{\log(c) \log(d)}{144 \log(a) \log(b)} - \frac{\log(a)^3}{720 \log(b) \log(c) \log(d)} - \frac{\log(b)^3}{720 \log(a) \log(c) \log(d)} \\ & - \frac{\log(c)^3}{720 \log(a) \log(b) \log(d)} - \frac{\log(d)^3}{720 \log(a) \log(b) \log(c)} - \frac{7}{8} B_1^* \left(\left\{ \frac{\log(x)}{\log(a)} \right\} \right) \end{aligned}$$

Setting $a = 2$, $b = 3$, $c = 5$ and $d = 7$, we get for the sequence

$$\begin{aligned} S_{2,3,5,7} &:= \{2^p 3^q 5^l 7^f : p \in \mathbb{N}_0, q \in \mathbb{N}_0, l \in \mathbb{N}_0, f \in \mathbb{N}_0\} \\ &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 24, 25, 27, 28, 30, 32, 35, 36, 40, 42, 45, 48, \dots\}, \end{aligned}$$

of 7-smooth numbers (Humble numbers or "highly composite numbers") [1, 2, 14], immediately the following

Corollary 10. (Formula for the 7-Smooth Numbers Counting Function $N_{2,3,5,7}(x)$)
For every real number $x \geq 1$, we have that

$$\begin{aligned} N_{2,3,5,7}(x) &= \frac{\log(x)^4}{24 \log(2) \log(3) \log(5) \log(7)} + \frac{\log(x)^3}{12 \log(2) \log(3) \log(5)} + \frac{\log(x)^3}{12 \log(2) \log(3) \log(7)} \\ &+ \frac{\log(x)^3}{12 \log(2) \log(5) \log(7)} + \frac{\log(x)^3}{12 \log(3) \log(5) \log(7)} + \frac{\log(2) \log(x)^2}{24 \log(3) \log(5) \log(7)} \\ &+ \frac{\log(3) \log(x)^2}{24 \log(2) \log(5) \log(7)} + \frac{\log(5) \log(x)^2}{24 \log(2) \log(3) \log(7)} + \frac{\log(7) \log(x)^2}{24 \log(2) \log(3) \log(5)} \\ &+ \frac{\log(x)^2}{8 \log(2) \log(3)} + \frac{\log(x)^2}{8 \log(2) \log(5)} + \frac{\log(x)^2}{8 \log(2) \log(7)} + \frac{\log(x)^2}{8 \log(3) \log(5)} + \frac{\log(x)^2}{8 \log(2) \log(7)} \\ &+ \frac{\log(x)^2}{8 \log(5) \log(7)} + \frac{\log(x)}{8 \log(2)} + \frac{\log(x)}{8 \log(3)} + \frac{\log(x)}{8 \log(5)} + \frac{\log(x)}{8 \log(7)} + \frac{\log(2) \log(x)}{24 \log(3) \log(5)} \\ &+ \frac{\log(2) \log(x)}{24 \log(3) \log(7)} + \frac{\log(2) \log(x)}{24 \log(5) \log(7)} + \frac{\log(3) \log(x)}{24 \log(2) \log(5)} + \frac{\log(3) \log(x)}{24 \log(2) \log(7)} + \frac{\log(3) \log(x)}{24 \log(5) \log(7)} \\ &+ \frac{\log(5) \log(x)}{24 \log(2) \log(3)} + \frac{\log(5) \log(x)}{24 \log(2) \log(7)} + \frac{\log(5) \log(x)}{24 \log(3) \log(7)} + \frac{\log(7) \log(x)}{24 \log(2) \log(3)} + \frac{\log(7) \log(x)}{24 \log(2) \log(5)} \\ &+ \frac{\log(7) \log(x)}{24 \log(3) \log(5)} + \frac{1}{16} + \frac{\log(2)}{48 \log(3)} + \frac{\log(2)}{48 \log(5)} + \frac{\log(2)}{48 \log(7)} + \frac{\log(3)}{48 \log(2)} + \frac{\log(3)}{48 \log(5)} \\ &+ \frac{\log(3)}{48 \log(7)} + \frac{\log(5)}{48 \log(2)} + \frac{\log(5)}{48 \log(3)} + \frac{\log(5)}{48 \log(7)} + \frac{\log(7)}{48 \log(2)} + \frac{\log(7)}{48 \log(3)} + \frac{\log(7)}{48 \log(5)} \\ &+ \frac{\log(2) \log(3)}{144 \log(5) \log(7)} + \frac{\log(2) \log(5)}{144 \log(3) \log(7)} + \frac{\log(2) \log(7)}{144 \log(3) \log(5)} + \frac{\log(3) \log(5)}{144 \log(2) \log(7)} \\ &+ \frac{\log(3) \log(7)}{144 \log(2) \log(5)} + \frac{\log(5) \log(7)}{144 \log(2) \log(3)} - \frac{\log(2)^3}{720 \log(3) \log(5) \log(7)} - \frac{\log(3)^3}{720 \log(2) \log(5) \log(7)} \\ &- \frac{\log(5)^3}{720 \log(2) \log(3) \log(7)} - \frac{\log(7)^3}{720 \log(2) \log(3) \log(5)} - \frac{7}{8} B_1^* \left(\left\{ \frac{\log(x)}{\log(2)} \right\} \right) \\ &- \frac{7}{8} B_1^* \left(\left\{ \frac{\log(x)}{\log(3)} \right\} \right) - \frac{7}{8} B_1^* \left(\left\{ \frac{\log(x)}{\log(5)} \right\} \right) - \frac{7}{8} B_1^* \left(\left\{ \frac{\log(x)}{\log(7)} \right\} \right) \\ &- \frac{1}{8\pi} \sum_{k=1}^{\infty} \left(\frac{\cos \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(3)}{\log(2)} \right)}{k \sin \left(\frac{\pi k \log(3)}{\log(2)} \right)} + \frac{\cos \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(2)}{\log(3)} \right)}{k \sin \left(\frac{\pi k \log(2)}{\log(3)} \right)} \right) \end{aligned}$$

$$\begin{aligned}
& - \frac{1}{16\pi} \sum_{k=1}^{\infty} \left(\frac{\sin \left(2\pi k \frac{\log(x) + \frac{1}{2} \log(3) - \frac{1}{2} \log(7)}{\log(5)} \right)}{k \sin \left(\frac{\pi k \log(3)}{\log(5)} \right) \sin \left(\frac{\pi k \log(7)}{\log(5)} \right)} + \frac{\sin \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(3) + \frac{1}{2} \log(7)}{\log(5)} \right)}{k \sin \left(\frac{\pi k \log(3)}{\log(5)} \right) \sin \left(\frac{\pi k \log(7)}{\log(5)} \right)} \right) \\
& - \frac{1}{16\pi} \sum_{k=1}^{\infty} \left(\frac{\sin \left(2\pi k \frac{\log(x) + \frac{1}{2} \log(2) - \frac{1}{2} \log(7)}{\log(5)} \right)}{k \sin \left(\frac{\pi k \log(2)}{\log(5)} \right) \sin \left(\frac{\pi k \log(7)}{\log(5)} \right)} + \frac{\sin \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(2) + \frac{1}{2} \log(7)}{\log(5)} \right)}{k \sin \left(\frac{\pi k \log(2)}{\log(5)} \right) \sin \left(\frac{\pi k \log(7)}{\log(5)} \right)} \right) \\
& - \frac{1}{16\pi} \sum_{k=1}^{\infty} \left(\frac{\sin \left(2\pi k \frac{\log(x) + \frac{1}{2} \log(2) - \frac{1}{2} \log(3)}{\log(7)} \right)}{k \sin \left(\frac{\pi k \log(2)}{\log(7)} \right) \sin \left(\frac{\pi k \log(3)}{\log(7)} \right)} + \frac{\sin \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(2) + \frac{1}{2} \log(3)}{\log(7)} \right)}{k \sin \left(\frac{\pi k \log(2)}{\log(7)} \right) \sin \left(\frac{\pi k \log(3)}{\log(7)} \right)} \right) \\
& - \frac{1}{16\pi} \sum_{k=1}^{\infty} \left(\frac{\sin \left(2\pi k \frac{\log(x) + \frac{1}{2} \log(3) - \frac{1}{2} \log(5)}{\log(7)} \right)}{k \sin \left(\frac{\pi k \log(3)}{\log(7)} \right) \sin \left(\frac{\pi k \log(5)}{\log(7)} \right)} + \frac{\sin \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(3) + \frac{1}{2} \log(5)}{\log(7)} \right)}{k \sin \left(\frac{\pi k \log(3)}{\log(7)} \right) \sin \left(\frac{\pi k \log(5)}{\log(7)} \right)} \right) \\
& - \frac{1}{16\pi} \sum_{k=1}^{\infty} \left(\frac{\sin \left(2\pi k \frac{\log(x) + \frac{1}{2} \log(2) - \frac{1}{2} \log(5)}{\log(7)} \right)}{k \sin \left(\frac{\pi k \log(2)}{\log(7)} \right) \sin \left(\frac{\pi k \log(5)}{\log(7)} \right)} + \frac{\sin \left(2\pi k \frac{\log(x) - \frac{1}{2} \log(2) + \frac{1}{2} \log(5)}{\log(7)} \right)}{k \sin \left(\frac{\pi k \log(2)}{\log(7)} \right) \sin \left(\frac{\pi k \log(5)}{\log(7)} \right)} \right) \\
& + \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos \left(\frac{\pi k \log(3)}{\log(2)} \right) \cos \left(\frac{\pi k \log(5)}{\log(2)} \right) \cos \left(\frac{\pi k \log(7)}{\log(2)} \right) \cos \left(2\pi k \frac{\log(x)}{\log(2)} \right)}{k \sin \left(\frac{\pi k \log(3)}{\log(2)} \right) \sin \left(\frac{\pi k \log(5)}{\log(2)} \right) \sin \left(\frac{\pi k \log(7)}{\log(2)} \right)} \\
& + \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos \left(\frac{\pi k \log(2)}{\log(3)} \right) \cos \left(\frac{\pi k \log(5)}{\log(3)} \right) \cos \left(\frac{\pi k \log(7)}{\log(3)} \right) \cos \left(2\pi k \frac{\log(x)}{\log(3)} \right)}{k \sin \left(\frac{\pi k \log(2)}{\log(3)} \right) \sin \left(\frac{\pi k \log(5)}{\log(3)} \right) \sin \left(\frac{\pi k \log(7)}{\log(3)} \right)} \\
& + \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos \left(\frac{\pi k \log(2)}{\log(5)} \right) \cos \left(\frac{\pi k \log(3)}{\log(5)} \right) \cos \left(\frac{\pi k \log(7)}{\log(5)} \right) \cos \left(2\pi k \frac{\log(x)}{\log(5)} \right)}{k \sin \left(\frac{\pi k \log(2)}{\log(5)} \right) \sin \left(\frac{\pi k \log(3)}{\log(5)} \right) \sin \left(\frac{\pi k \log(7)}{\log(5)} \right)} \\
& + \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos \left(\frac{\pi k \log(2)}{\log(7)} \right) \cos \left(\frac{\pi k \log(3)}{\log(7)} \right) \cos \left(\frac{\pi k \log(5)}{\log(7)} \right) \cos \left(2\pi k \frac{\log(x)}{\log(7)} \right)}{k \sin \left(\frac{\pi k \log(2)}{\log(7)} \right) \sin \left(\frac{\pi k \log(3)}{\log(7)} \right) \sin \left(\frac{\pi k \log(5)}{\log(7)} \right)} + \frac{1}{2} \chi_{S_{2,3,5,7}}(x).
\end{aligned}$$

This formula converges very rapidly.

Every series is interpreted as mentioned above.

Using this formula for the 7-Smooth Numbers Counting Function $N_{2,3,5,7}(x)$, we get the following table:

x	$N_{2,3,5,7}(x)$	Formula for $N_{2,3,5,7}(x)$	Number of terms R
1	1	1.030388812940249824617233653730019551	$R = 3$
10	10	10.01263249440259984789405319823431872556	$R = 3$
10^2	46	46.03668521491726375130238293886497852216	$R = 20$
10^3	141	141.01285390547424275647701138240776403195	$R = 80$
10^4	338	338.0186997720522261698185344005048234745	$R = 80$
10^5	694	694.00540895426731024839939099335158382934	$R = 100$
10^6	1273	1273.02115574787663113791230619711970129327	$R = 1500$
10^7	2155	2155.01133325568473975698180880511876853632	$R = 1500$
10^8	3427	3427.01611847162744035197962908126411814549	$R = 1500$
10^9	5194	5194.03771424320772544603355297308020543638	$R = 1600$
10^{10}	7575	7575.01767118495435682818874877606239707862	$R = 9000$

Table 4: Values of $N_{2,3,5,7}(x)$

5 The Formula for the Distribution of all Smooth Numbers

Let $a_1, a_2, a_3, \dots, a_n \in \mathbb{N}$ such that $a_1 < a_2 < a_3 < \dots < a_n$ and $\gcd(a_1, a_2, a_3, \dots, a_n) = 1$. For $x \in \mathbb{R}_0^+$, we define the function $N_{a_1, a_2, a_3, \dots, a_n}(x)$ by

$$N_{a_1, a_2, a_3, \dots, a_n}(x) := \sum_{\substack{a_1^{q_1} a_2^{q_2} a_3^{q_3} \dots a_n^{q_n} \leq x \\ q_1 \in \mathbb{N}_0, q_2 \in \mathbb{N}_0, q_3 \in \mathbb{N}_0, \dots, q_n \in \mathbb{N}_0}} 1.$$

We define also

$$S_{a_1, a_2, a_3, \dots, a_n} := \{a_1^{q_1} a_2^{q_2} a_3^{q_3} \dots a_n^{q_n} : q_1 \in \mathbb{N}_0, q_2 \in \mathbb{N}_0, q_3 \in \mathbb{N}_0, \dots, q_n \in \mathbb{N}_0\},$$

$$\chi_{S_{a_1, a_2, a_3, \dots, a_n}}(x) := \begin{cases} 1 & \text{if } x \in S_{a_1, a_2, a_3, \dots, a_n} \\ 0 & \text{if } x \notin S_{a_1, a_2, a_3, \dots, a_n} \end{cases}.$$

Thus, we have that

$$N_{a_1, a_2, a_3, \dots, a_n}(x) = \sum_{k_1=0}^{\lfloor \log_{a_1}(x) \rfloor} \sum_{k_2=0}^{\left\lfloor \log_{a_2} \left(\frac{x}{a_1^{k_1}} \right) \right\rfloor} \dots \sum_{k_{n-1}=0}^{\left\lfloor \log_{a_{n-1}} \left(\frac{x}{a_1^{k_1} a_2^{k_2} \dots a_{n-2}^{k_{n-2}}} \right) \right\rfloor} \left(\left\lfloor \log_{a_n} \left(\frac{x}{a_1^{k_1} a_2^{k_2} a_3^{k_3} \dots a_{n-1}^{k_{n-1}}} \right) \right\rfloor + 1 \right).$$

Expressions of this form for $N_{a_1, a_2, a_3, \dots, a_n}(x)$ are called "Klauder-Ness Expressions" [15, 16]. We have the following

Theorem 11. (Formula for $N_{a_1, a_2, a_3, \dots, a_n}(x)$)
For every real number $x \geq 1$, we have that

$$\begin{aligned} N_{a_1, a_2, a_3, \dots, a_n}(x) &= \text{Res}_{s=0} \left(\frac{x^s}{s \prod_{k=1}^n \left(1 - \frac{1}{a_k^s}\right)} \right) - \frac{1}{2^{n-1}} \sum_{k=1}^n B_1^* \left(\left\{ \frac{\log(x)}{\log(a_k)} \right\} \right) \\ &+ \frac{1}{2^{n-1} \pi} \sum_{m=1}^n \sum_{r=1}^{n-1} \sum_{\substack{i_1 < i_2 < i_3 < \dots < i_r \\ \{i_1, i_2, i_3, \dots, i_r\} \\ \subset \{a_1, a_2, a_3, \dots, a_m, \dots, a_n\}}} \sum_{k=1}^{\infty} \frac{\sin \left(2\pi k \frac{\log(x)}{\log(a_m)} - \frac{\pi r}{2} \right)}{k} \prod_{l=1}^r \cot \left(\frac{\pi k \log(i_l)}{\log(a_m)} \right) \\ &+ \frac{1}{2} \chi_{S_{a_1, a_2, a_3, \dots, a_n}}(x), \end{aligned}$$

where the series are to be interpreted as meaning

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{\sin \left(2\pi k \frac{\log(x)}{\log(a_m)} - \frac{\pi r}{2} \right)}{k} \prod_{l=1}^r \cot \left(\frac{\pi k \log(i_l)}{\log(a_m)} \right) \\ &= \lim_{R \rightarrow \infty} \left(\sum_{k=1}^{\lfloor R \log(a_m) \rfloor} \frac{\sin \left(2\pi k \frac{\log(x)}{\log(a_m)} - \frac{\pi r}{2} \right)}{k} \prod_{l=1}^r \cot \left(\frac{\pi k \log(i_l)}{\log(a_m)} \right) \right), \end{aligned}$$

when $R \rightarrow \infty$ in an appropriate manner.

This formula converges again very rapidly.

Proof. We have that

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\chi_{S_{a_1, a_2, a_3, \dots, a_n}}(k)}{k^s} &= \prod_{k=1}^n \left(\sum_{m=0}^{\infty} \frac{1}{a_k^{ms}} \right) \\ &= \prod_{k=1}^n \frac{1}{1 - e^{-\log(a_k)s}}. \end{aligned}$$

Therefore, by Perron's formula, we get that

$$\begin{aligned}
N_{a_1, a_2, a_3, \dots, a_n}(x) &= \frac{1}{2\pi i} \int_{\gamma} \left(\prod_{k=1}^n \frac{1}{1 - e^{-\log(a_k)s}} \right) \frac{x^s}{s} ds \\
&= \text{Res}_{s=0} \left(\frac{x^s}{s \prod_{k=1}^n \left(1 - \frac{1}{a_k^s}\right)} \right) \\
&\quad + \sum_{m=1}^n \sum_{k=1}^{\infty} \left(\text{Res}_{s=\frac{2\pi i k}{\log(a_m)}} \left(\frac{x^s}{s \prod_{k=1}^n \left(1 - \frac{1}{a_k^s}\right)} \right) + \text{Res}_{s=-\frac{2\pi i k}{\log(a_m)}} \left(\frac{x^s}{s \prod_{k=1}^n \left(1 - \frac{1}{a_k^s}\right)} \right) \right) \\
&\quad + \frac{1}{2} \chi_{S_{a_1, a_2, a_3, \dots, a_n}}(x),
\end{aligned}$$

where $\gamma =$ line from $1 - i\infty$ to $1 + i\infty$.

Using the relation

$$\lim_{s \rightarrow \pm \frac{2\pi i k}{\log(a_m)}} \left(\frac{s \mp \frac{2\pi i k}{\log(a_m)}}{1 - e^{-\log(a_m)s}} \right) = \frac{1}{\log(a_m)} \quad \forall k \in \mathbb{N},$$

we can compute the " a_m -Residues" to

$$\begin{aligned}
\text{Res}_{s=\frac{2\pi i k}{\log(a_m)}} \left(\frac{x^s}{s \prod_{k=1}^n \left(1 - \frac{1}{a_k^s}\right)} \right) &= - \frac{i \prod_{l=1, l \neq m}^n a_l^{\frac{2\pi i k}{\log(a_m)}} x^{\frac{2\pi i k}{\log(a_m)}}}{2\pi k \prod_{l=1, l \neq m}^n \left(a_l^{\frac{2\pi i k}{\log(a_m)}} - 1 \right)} \quad \text{for all } k \in \mathbb{N} \\
\text{Res}_{s=-\frac{2\pi i k}{\log(a_m)}} \left(\frac{x^s}{s \prod_{k=1}^n \left(1 - \frac{1}{a_k^s}\right)} \right) &= - \frac{(-1)^n i x^{-\frac{2\pi i k}{\log(a_m)}}}{2\pi k \prod_{l=1, l \neq m}^n \left(a_l^{\frac{2\pi i k}{\log(a_m)}} - 1 \right)} \quad \text{for all } k \in \mathbb{N}.
\end{aligned}$$

Using the relations

$$\begin{aligned}
\sin \left(2\pi k \frac{\log(x)}{\log(a_m)} \right) &= \frac{1}{2} i x^{-\frac{2\pi i k}{\log(a_m)}} - \frac{1}{2} i x^{\frac{2\pi i k}{\log(a_m)}} \\
\cos \left(2\pi k \frac{\log(x)}{\log(a_m)} \right) &= \frac{1}{2} x^{-\frac{2\pi i k}{\log(a_m)}} + \frac{1}{2} x^{\frac{2\pi i k}{\log(a_m)}} \\
\cot \left(\frac{\pi k \log(i_l)}{\log(a_m)} \right) &= i \frac{i_l^{\frac{2\pi i k}{\log(a_m)}} + 1}{i_l^{\frac{2\pi i k}{\log(a_m)}} - 1}
\end{aligned}$$

$$\begin{aligned} \sin\left(2\pi k \frac{\log(x)}{\log(a_m)} - \frac{\pi r}{2}\right) &= \begin{cases} (-1)^{\frac{r}{2}} \sin\left(2\pi k \frac{\log(x)}{\log(a_m)}\right), & \text{if } r = \text{even} \\ (-1)^{\frac{r+1}{2}} \cos\left(2\pi k \frac{\log(x)}{\log(a_m)}\right), & \text{if } r = \text{odd} \end{cases} \\ &= \begin{cases} (-1)^{\frac{r}{2}} \left(\frac{1}{2}ix^{-\frac{2\pi ik}{\log(a_m)}} - \frac{1}{2}ix^{\frac{2\pi ik}{\log(a_m)}}\right), & \text{if } r = \text{even} \\ (-1)^{\frac{r+1}{2}} \left(\frac{1}{2}x^{-\frac{2\pi ik}{\log(a_m)}} + \frac{1}{2}x^{\frac{2\pi ik}{\log(a_m)}}\right), & \text{if } r = \text{odd} \end{cases} \end{aligned}$$

and

$$\sin\left(2\pi k \frac{\log(x)}{\log(a_m)} - \frac{\pi r}{2}\right) = (-1)^{\frac{1}{2}(r \bmod 4 + r \bmod 2)} \left(\frac{1}{2}i^{(r+1) \bmod 2} x^{-\frac{2\pi ik}{\log(a_m)}} + (-1)^{r+1} \frac{1}{2}i^{(r+1) \bmod 2} x^{\frac{2\pi ik}{\log(a_m)}}\right),$$

we establish (by expanding everything out) that

$$\begin{aligned} &\sum_{r=1}^{n-1} \sum_{\substack{i_1 < i_2 < i_3 < \dots < i_r \\ \{i_1, i_2, i_3, \dots, i_r\} \\ \subset \{a_1, a_2, a_3, \dots, \widehat{a_m}, \dots, a_n\}}} \sin\left(2\pi k \frac{\log(x)}{\log(a_m)} - \frac{\pi r}{2}\right) \prod_{l=1}^r \cot\left(\frac{\pi k \log(i_l)}{\log(a_m)}\right) + \sin\left(2\pi k \frac{\log(x)}{\log(a_m)}\right) \\ &= \sum_{r=1}^{n-1} \sum_{\substack{i_1 < i_2 < i_3 < \dots < i_r \\ \{i_1, i_2, i_3, \dots, i_r\} \\ \subset \{a_1, a_2, a_3, \dots, \widehat{a_m}, \dots, a_n\}}} (-1)^{\frac{1}{2}(r \bmod 4 + r \bmod 2)} \left(\frac{1}{2}i^{(r+1) \bmod 2} x^{-\frac{2\pi ik}{\log(a_m)}} + (-1)^{r+1} \frac{1}{2}i^{(r+1) \bmod 2} x^{\frac{2\pi ik}{\log(a_m)}}\right) \\ &\quad \cdot \prod_{l=1}^r \left(i^{\frac{\frac{2\pi ik}{\log(a_m)}}{\log(i_l)} + 1} + 1\right) + \left(\frac{1}{2}ix^{-\frac{2\pi ik}{\log(a_m)}} - \frac{1}{2}ix^{\frac{2\pi ik}{\log(a_m)}}\right) \\ &= \sum_{r=1}^{n-1} \sum_{\substack{i_1 < i_2 < i_3 < \dots < i_r \\ \{i_1, i_2, i_3, \dots, i_r\} \\ \subset \{a_1, a_2, a_3, \dots, \widehat{a_m}, \dots, a_n\}}} (-1)^{\frac{1}{2}(r \bmod 4 + r \bmod 2)} \frac{1}{2}i^{r+(r+1) \bmod 2} \prod_{l=1}^r \left(\frac{i_l^{\frac{2\pi ik}{\log(a_m)} + 1}}{i_l^{\frac{2\pi ik}{\log(a_m)} - 1}}\right) x^{-\frac{2\pi ik}{\log(a_m)}} + \frac{1}{2}ix^{-\frac{2\pi ik}{\log(a_m)}} \\ &\quad + \sum_{r=1}^{n-1} \sum_{\substack{i_1 < i_2 < i_3 < \dots < i_r \\ \{i_1, i_2, i_3, \dots, i_r\} \\ \subset \{a_1, a_2, a_3, \dots, \widehat{a_m}, \dots, a_n\}}} (-1)^{r+1 + \frac{1}{2}(r \bmod 4 + r \bmod 2)} \frac{1}{2}i^{r+(r+1) \bmod 2} \prod_{l=1}^r \left(\frac{i_l^{\frac{2\pi ik}{\log(a_m)} + 1}}{i_l^{\frac{2\pi ik}{\log(a_m)} - 1}}\right) x^{\frac{2\pi ik}{\log(a_m)}} - \frac{1}{2}ix^{\frac{2\pi ik}{\log(a_m)}} \\ &= (-1)^{n-1} \frac{1}{2}ix^{-\frac{2\pi ik}{\log(a_m)}} \left(\frac{\sum_{\substack{\{\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_{n-1}\} \\ \subset \{\pm 1, \pm 1, \pm 1, \dots, \pm 1\}}} \prod_{l \neq m}^n \epsilon_k \left(a_l^{\frac{2\pi ik}{\log(a_m)} + \epsilon_k}\right)}{\prod_{l \neq m}^n \left(a_l^{\frac{2\pi ik}{\log(a_m)} - 1}\right)}\right) \\ &\quad - \frac{1}{2}ix^{\frac{2\pi ik}{\log(a_m)}} \left(\frac{\sum_{\substack{\{\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_{n-1}\} \\ \subset \{\pm 1, \pm 1, \pm 1, \dots, \pm 1\}}} \prod_{l \neq m}^n \left(a_l^{\frac{2\pi ik}{\log(a_m)} + \epsilon_k}\right)}{\prod_{l \neq m}^n \left(a_l^{\frac{2\pi ik}{\log(a_m)} - 1}\right)}\right) \end{aligned}$$

$$\begin{aligned}
&= (-1)^{n-1} \frac{1}{2} i x^{-\frac{2\pi i k}{\log(a_m)}} \left(\frac{S_1 \left(a_1^{\frac{2\pi i k}{\log(a_m)}}, a_2^{\frac{2\pi i k}{\log(a_m)}}, a_3^{\frac{2\pi i k}{\log(a_m)}}, \dots, \widehat{a_m^{\frac{2\pi i k}{\log(a_m)}}}, \dots, a_n^{\frac{2\pi i k}{\log(a_m)}} \right)}{\prod_{\substack{l=1 \\ l \neq m}}^n \left(a_l^{\frac{2\pi i k}{\log(a_m)}} - 1 \right)} \right) \\
&\quad - \frac{1}{2} i x^{\frac{2\pi i k}{\log(a_m)}} \left(\frac{S_2 \left(a_1^{\frac{2\pi i k}{\log(a_m)}}, a_2^{\frac{2\pi i k}{\log(a_m)}}, a_3^{\frac{2\pi i k}{\log(a_m)}}, \dots, \widehat{a_m^{\frac{2\pi i k}{\log(a_m)}}}, \dots, a_n^{\frac{2\pi i k}{\log(a_m)}} \right)}{\prod_{\substack{l=1 \\ l \neq m}}^n \left(a_l^{\frac{2\pi i k}{\log(a_m)}} - 1 \right)} \right) \\
&= (-1)^{n-1} \frac{1}{2} i x^{-\frac{2\pi i k}{\log(a_m)}} \left(\frac{2^{n-1}}{\prod_{\substack{l=1 \\ l \neq m}}^n \left(a_l^{\frac{2\pi i k}{\log(a_m)}} - 1 \right)} \right) - \frac{1}{2} i x^{\frac{2\pi i k}{\log(a_m)}} \left(\frac{2^{n-1} \prod_{\substack{l=1 \\ l \neq m}}^n a_l^{\frac{2\pi i k}{\log(a_m)}}}{\prod_{\substack{l=1 \\ l \neq m}}^n \left(a_l^{\frac{2\pi i k}{\log(a_m)}} - 1 \right)} \right) \\
&= (-1)^{n+1} \frac{2^{n-2} i \left(x^{-\frac{2\pi i k}{\log(a_m)}} + (-1)^n \prod_{\substack{l=1 \\ l \neq m}}^n a_l^{\frac{2\pi i k}{\log(a_m)}} x^{\frac{2\pi i k}{\log(a_m)}} \right)}{\prod_{\substack{l=1 \\ l \neq m}}^n \left(a_l^{\frac{2\pi i k}{\log(a_m)}} - 1 \right)}.
\end{aligned}$$

In the above calculation, we have used the two algebraic identities

$$\begin{aligned}
S_1(x_1, x_2, x_3, \dots, x_n) &= 2^n, \\
S_2(x_1, x_2, x_3, \dots, x_n) &= 2^n \prod_{k=1}^n x_k,
\end{aligned}$$

where

$$\begin{aligned}
S_1(x_1, x_2, x_3, \dots, x_n) &= \sum_{\substack{\{\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_n\} \\ \subset \{\pm 1, \pm 1, \pm 1, \dots, \pm 1\}}} \prod_{k=1}^n \epsilon_k (x_k + \epsilon_k) \\
S_2(x_1, x_2, x_3, \dots, x_n) &= \sum_{\substack{\{\epsilon_1, \epsilon_2, \epsilon_3, \dots, \epsilon_n\} \\ \subset \{\pm 1, \pm 1, \pm 1, \dots, \pm 1\}}} \prod_{k=1}^n (x_k + \epsilon_k).
\end{aligned}$$

These two identities follow by induction, because for $n = 1$, we have that

$$\begin{aligned}
S_1(x_1) &= (x_1 + 1) - (x_1 - 1) = 2 \\
S_2(x_1) &= (x_1 + 1) + (x_1 - 1) = 2x_1.
\end{aligned}$$

These two identities are exactly the claimed formulas for $S_1(x_1)$ and $S_2(x_1)$.

Supposing now that the statement is also true for $S_1(x_1, x_2, x_3, \dots, x_{n-1})$ and $S_2(x_1, x_2, x_3, \dots, x_{n-1})$,

we prove by induction

$$\begin{aligned}
S_1(x_1, x_2, x_3, \dots, x_{n-1}, x_n) &= (x_n + 1)S_1(x_1, x_2, x_3, \dots, x_{n-1}) - (x_n - 1)S_1(x_1, x_2, x_3, \dots, x_{n-1}) \\
&= (x_n + 1 - x_n + 1)S_1(x_1, x_2, x_3, \dots, x_{n-1}) \\
&= 2S_1(x_1, x_2, x_3, \dots, x_{n-1}) \\
&= 2 \cdot 2^{n-1} \\
&= 2^n, \\
S_2(x_1, x_2, x_3, \dots, x_{n-1}, x_n) &= (x_n + 1)S_2(x_1, x_2, x_3, \dots, x_{n-1}) + (x_n - 1)S_2(x_1, x_2, x_3, \dots, x_{n-1}) \\
&= (x_n + 1 + x_n - 1)S_2(x_1, x_2, x_3, \dots, x_{n-1}) \\
&= 2x_n S_2(x_1, x_2, x_3, \dots, x_{n-1}) \\
&= 2x_n 2^{n-1} \prod_{k=1}^{n-1} x_k \\
&= 2^n \prod_{k=1}^n x_k.
\end{aligned}$$

These are the claimed statements for $S_1(x_1, x_2, x_3, \dots, x_{n-1}, x_n)$ and $S_2(x_1, x_2, x_3, \dots, x_{n-1}, x_n)$. Therefore, the inductive proof is finished.

The above established identity implies that

$$\begin{aligned}
&\operatorname{Res}_{s=\frac{2\pi ik}{\log(a_m)}} \left(\frac{x^s}{s \prod_{k=1}^n \left(1 - \frac{1}{a_k^s}\right)} \right) + \operatorname{Res}_{s=-\frac{2\pi ik}{\log(a_m)}} \left(\frac{x^s}{s \prod_{k=1}^n \left(1 - \frac{1}{a_k^s}\right)} \right) \\
&= (-1)^{n+1} \frac{i \left(x^{-\frac{2\pi ik}{\log(a_m)}} + (-1)^n \prod_{\substack{l=1 \\ l \neq m}}^n a_l^{\frac{2\pi ik}{\log(a_m)}} x^{\frac{2\pi ik}{\log(a_m)}} \right)}{2\pi k \prod_{\substack{l=1 \\ l \neq m}}^n \left(a_l^{\frac{2\pi ik}{\log(a_m)}} - 1 \right)} \\
&= \frac{1}{2^{n-1} \pi k} \left(\sum_{r=1}^{n-1} \sum_{\substack{i_1 < i_2 < i_3 < \dots < i_r \\ \{i_1, i_2, i_3, \dots, i_r\} \\ \subset \{a_1, a_2, a_3, \dots, a_m, \dots, a_n\}}} \sin \left(2\pi k \frac{\log(x)}{\log(a_m)} - \frac{\pi r}{2} \right) \prod_{l=1}^r \cot \left(\frac{\pi k \log(i_l)}{\log(a_m)} \right) + \sin \left(2\pi k \frac{\log(x)}{\log(a_m)} \right) \right) \\
&\quad \text{for all } k \in \mathbb{N} \text{ and for all } 1 \leq m \leq n.
\end{aligned}$$

Summing up all the Residues, we get our formula for $N_{a_1, a_2, a_3, \dots, a_n}(x)$. \square

Remark 12. The first few identities of the family of identities, which we encountered in the

above proof, are

$$S_1(x_1) = (x_1 + 1) - (x_1 - 1) = 2$$

$$S_2(x_1) = (x_1 + 1) + (x_1 - 1) = 2x_1$$

$$S_1(x_1, x_2) = (x_1 + 1)(x_2 + 1) - (x_1 - 1)(x_2 + 1) - (x_1 + 1)(x_2 - 1) + (x_1 - 1)(x_2 - 1) = 4$$

$$S_2(x_1, x_2) = (x_1 + 1)(x_2 + 1) + (x_1 - 1)(x_2 + 1) + (x_1 + 1)(x_2 - 1) + (x_1 - 1)(x_2 - 1) = 4x_1x_2$$

$$\begin{aligned} S_1(x_1, x_2, x_3) &= (x_1 + 1)(x_2 + 1)(x_3 + 1) - (x_1 - 1)(x_2 + 1)(x_3 + 1) - (x_1 + 1)(x_2 - 1)(x_3 + 1) \\ &\quad - (x_1 + 1)(x_2 + 1)(x_3 - 1) + (x_1 + 1)(x_2 - 1)(x_3 - 1) + (x_1 - 1)(x_2 + 1)(x_3 - 1) \\ &\quad + (x_1 - 1)(x_2 - 1)(x_3 + 1) - (x_1 - 1)(x_2 - 1)(x_3 - 1) = 8 \end{aligned}$$

$$\begin{aligned} S_2(x_1, x_2, x_3) &= (x_1 + 1)(x_2 + 1)(x_3 + 1) + (x_1 - 1)(x_2 + 1)(x_3 + 1) + (x_1 + 1)(x_2 - 1)(x_3 + 1) \\ &\quad + (x_1 + 1)(x_2 + 1)(x_3 - 1) + (x_1 + 1)(x_2 - 1)(x_3 - 1) + (x_1 - 1)(x_2 + 1)(x_3 - 1) \\ &\quad + (x_1 - 1)(x_2 - 1)(x_3 + 1) + (x_1 - 1)(x_2 - 1)(x_3 - 1) = 8x_1x_2x_3. \end{aligned}$$

And so on.

Setting $a_1 = 2, a_2 = 3, a_3 = 5, a_4 = 7, \dots, a_k = p_k = k$ -th prime number, $\dots, a_n = p_n = n$ -th prime number in the above theorem, we get for the sequence

$$S_{2,3,5,7,\dots,p_n} := \{2^{q_1} 3^{q_2} 5^{q_3} 7^{q_4} \dots p_n^{q_n} : q_1 \in \mathbb{N}_0, q_2 \in \mathbb{N}_0, q_3 \in \mathbb{N}_0, \dots, q_n \in \mathbb{N}_0\},$$

of p_n -smooth numbers [1, 2], immediately the following

Corollary 13. (Formula for the p_n -Smooth Numbers Counting Function $N_{2,3,5,7,\dots,p_n}(x)$)
For every real number $x \geq 1$, we have that

$$\begin{aligned} N_{2,3,5,7,\dots,p_n}(x) &= \operatorname{Res}_{s=0} \left(\frac{x^s}{s \prod_{k=1}^n \left(1 - \frac{1}{p_k^s}\right)} \right) - \frac{1}{2^{n-1}} \sum_{k=1}^n B_1^* \left(\left\{ \frac{\log(x)}{\log(p_k)} \right\} \right) \\ &\quad + \frac{1}{2^{n-1}\pi} \sum_{m=1}^n \sum_{r=1}^{n-1} \sum_{\substack{i_1 < i_2 < i_3 < \dots < i_r \\ \{i_1, i_2, i_3, \dots, i_r\} \\ \subset \{2, 3, 5, 7, \dots, \widehat{p_m}, \dots, p_n\}}} \sum_{k=1}^{\infty} \frac{\sin \left(2\pi k \frac{\log(x)}{\log(p_m)} - \frac{\pi r}{2} \right)}{k} \prod_{l=1}^r \cot \left(\frac{\pi k \log(i_l)}{\log(p_m)} \right) \\ &\quad + \frac{1}{2} \chi_{S_{2,3,5,7,\dots,p_n}}(x), \end{aligned}$$

where the series are to be interpreted as meaning

$$\begin{aligned} & \sum_{k=1}^{\infty} \frac{\sin\left(2\pi k \frac{\log(x)}{\log(p_m)} - \frac{\pi r}{2}\right)}{k} \prod_{l=1}^r \cot\left(\frac{\pi k \log(i_l)}{\log(p_m)}\right) \\ &= \lim_{R \rightarrow \infty} \left(\sum_{k=1}^{\lfloor R \log(p_m) \rfloor} \frac{\sin\left(2\pi k \frac{\log(x)}{\log(p_m)} - \frac{\pi r}{2}\right)}{k} \prod_{l=1}^r \cot\left(\frac{\pi k \log(i_l)}{\log(p_m)}\right) \right), \end{aligned}$$

when $R \rightarrow \infty$ in an appropriate manner.

This formula converges also very rapidly.

Therefore, we have

Corollary 14. (The Hardy-Littlewood formula for $N_{a,b}(x)$ and $N_{2,3}(x)$) [11, 12]

For every real number $x \geq 1$, we have that

$$\begin{aligned} N_{a,b}(x) &= \frac{\log(x)^2}{2 \log(a) \log(b)} + \frac{\log(x)}{2 \log(a)} + \frac{\log(x)}{2 \log(b)} + \frac{1}{4} + \frac{\log(a)}{12 \log(b)} + \frac{\log(b)}{12 \log(a)} \\ &\quad - \frac{1}{2} B_1^* \left(\left\{ \frac{\log(x)}{\log(a)} \right\} \right) - \frac{1}{2} B_1^* \left(\left\{ \frac{\log(x)}{\log(b)} \right\} \right) - \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(b)}{\log(a)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(a)}\right)}{k \sin\left(\frac{\pi k \log(b)}{\log(a)}\right)} \\ &\quad - \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(a)}{\log(b)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(b)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(b)}\right)} + \frac{1}{2} \chi_{S_{a,b}}(x) \end{aligned}$$

and

$$\begin{aligned} N_{2,3}(x) &= \frac{\log(x)^2}{2 \log(2) \log(3)} + \frac{\log(x)}{2 \log(2)} + \frac{\log(x)}{2 \log(3)} + \frac{1}{4} + \frac{\log(2)}{12 \log(3)} + \frac{\log(3)}{12 \log(2)} \\ &\quad - \frac{1}{2} B_1^* \left(\left\{ \frac{\log(x)}{\log(2)} \right\} \right) - \frac{1}{2} B_1^* \left(\left\{ \frac{\log(x)}{\log(3)} \right\} \right) - \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(3)}{\log(2)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(2)}\right)}{k \sin\left(\frac{\pi k \log(3)}{\log(2)}\right)} \\ &\quad - \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(2)}{\log(3)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(3)}\right)}{k \sin\left(\frac{\pi k \log(2)}{\log(3)}\right)} + \frac{1}{2} \chi_{S_{2,3}}(x). \end{aligned}$$

Proof. The proof that we give here is Hardy's proof [11] of the formula for $N_{a,b}(x)$.

We have that

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\chi_{S_{a,b}}(k)}{k^s} &= \left(\sum_{m_1=0}^{\infty} \frac{1}{a^{m_1 s}} \right) \left(\sum_{m_2=0}^{\infty} \frac{1}{b^{m_2 s}} \right) \\ &= \frac{1}{(1 - e^{-\log(a)s}) (1 - e^{-\log(b)s})}. \end{aligned}$$

Therefore, by Perron's formula, we get that

$$\begin{aligned}
N_{a,b}(x) &= \frac{1}{2\pi i} \int_{\gamma} \frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})} ds \\
&= \operatorname{Res}_{s=0} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})} \right) + \sum_{k=1}^{\infty} \operatorname{Res}_{s=\frac{2\pi ik}{\log(a)}} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})} \right) \\
&\quad + \sum_{k=1}^{\infty} \operatorname{Res}_{s=-\frac{2\pi ik}{\log(a)}} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})} \right) \\
&\quad + \sum_{k=1}^{\infty} \operatorname{Res}_{s=\frac{2\pi ik}{\log(b)}} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})} \right) \\
&\quad + \sum_{k=1}^{\infty} \operatorname{Res}_{s=-\frac{2\pi ik}{\log(b)}} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})} \right) + \frac{1}{2} \chi_{S_{a,b}}(x),
\end{aligned}$$

where $\gamma =$ line from $1 - i\infty$ to $1 + i\infty$.

Moreover, we have that

$$\begin{aligned}
\operatorname{Res}_{s=0} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})} \right) &= \frac{\log(x)^2}{2\log(a)\log(b)} + \frac{\log(x)}{2\log(a)} + \frac{\log(x)}{2\log(b)} \\
&\quad + \frac{1}{4} + \frac{\log(a)}{12\log(b)} + \frac{\log(b)}{12\log(a)}
\end{aligned}$$

and that

$$\begin{aligned}
\operatorname{Res}_{s=\frac{2\pi ik}{\log(a)}} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})} \right) &= -\frac{ib^{\frac{2\pi ik}{\log(a)}} x^{\frac{2\pi ik}{\log(a)}}}{2\pi k \left(b^{\frac{2\pi ik}{\log(a)}} - 1 \right)} \quad \text{for all } k \in \mathbb{N} \\
\operatorname{Res}_{s=-\frac{2\pi ik}{\log(a)}} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})} \right) &= -\frac{ix^{-\frac{2\pi ik}{\log(a)}}}{2\pi k \left(b^{\frac{2\pi ik}{\log(a)}} - 1 \right)} \quad \text{for all } k \in \mathbb{N} \\
\operatorname{Res}_{s=\frac{2\pi ik}{\log(b)}} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})} \right) &= -\frac{ia^{\frac{2\pi ik}{\log(b)}} x^{\frac{2\pi ik}{\log(b)}}}{2\pi k \left(a^{\frac{2\pi ik}{\log(b)}} - 1 \right)} \quad \text{for all } k \in \mathbb{N} \\
\operatorname{Res}_{s=-\frac{2\pi ik}{\log(b)}} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})} \right) &= -\frac{ix^{-\frac{2\pi ik}{\log(b)}}}{2\pi k \left(a^{\frac{2\pi ik}{\log(b)}} - 1 \right)} \quad \text{for all } k \in \mathbb{N}.
\end{aligned}$$

Using the relations

$$\begin{aligned}\sin\left(2\pi k \frac{\log(x)}{\log(a)}\right) &= \frac{1}{2}ix^{-\frac{2\pi ik}{\log(a)}} - \frac{1}{2}ix^{\frac{2\pi ik}{\log(a)}} \\ \cos\left(2\pi k \frac{\log(x)}{\log(a)}\right) &= \frac{1}{2}x^{-\frac{2\pi ik}{\log(a)}} + \frac{1}{2}x^{\frac{2\pi ik}{\log(a)}} \\ \cot\left(\frac{\pi k \log(b)}{\log(a)}\right) &= i \frac{b^{\frac{2\pi ik}{\log(a)}} + 1}{b^{\frac{2\pi ik}{\log(a)}} - 1},\end{aligned}$$

we establish (by expanding everything out) the following identity

$$\begin{aligned}& -\cot\left(\frac{\pi k \log(b)}{\log(a)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(a)}\right) + \sin\left(2\pi k \frac{\log(x)}{\log(a)}\right) \\ &= -\left(i \frac{b^{\frac{2\pi ik}{\log(a)}} + 1}{b^{\frac{2\pi ik}{\log(a)}} - 1}\right) \left(\frac{1}{2}x^{-\frac{2\pi ik}{\log(a)}} + \frac{1}{2}x^{\frac{2\pi ik}{\log(a)}}\right) + \left(\frac{1}{2}ix^{-\frac{2\pi ik}{\log(a)}} - \frac{1}{2}ix^{\frac{2\pi ik}{\log(a)}}\right) \\ &= -\frac{i \left(x^{-\frac{2\pi ik}{\log(a)}} + b^{\frac{2\pi ik}{\log(a)}} x^{\frac{2\pi ik}{\log(a)}}\right)}{b^{\frac{2\pi ik}{\log(a)}} - 1}.\end{aligned}$$

Therefore, for the first Residues (the "a -Residues"), we have that

$$\begin{aligned}\text{Res}_{s=\frac{2\pi ik}{\log(a)}}\left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})}\right) &+ \text{Res}_{s=-\frac{2\pi ik}{\log(a)}}\left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})}\right) \\ &= -\frac{i \left(x^{-\frac{2\pi ik}{\log(a)}} + b^{\frac{2\pi ik}{\log(a)}} x^{\frac{2\pi ik}{\log(a)}}\right)}{2\pi k \left(b^{\frac{2\pi ik}{\log(a)}} - 1\right)} \\ &= \frac{1}{2\pi k} \left(-\cot\left(\frac{\pi k \log(b)}{\log(a)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(a)}\right) + \sin\left(2\pi k \frac{\log(x)}{\log(a)}\right)\right) \text{ for all } k \in \mathbb{N}.\end{aligned}$$

Exchanging a and b ("permuting a and b "), we get also the other Residues (the "b -Residues"), namely

$$\begin{aligned}\text{Res}_{s=\frac{2\pi ik}{\log(b)}}\left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})}\right) &+ \text{Res}_{s=-\frac{2\pi ik}{\log(b)}}\left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})}\right) \\ &= -\frac{i \left(x^{-\frac{2\pi ik}{\log(b)}} + a^{\frac{2\pi ik}{\log(b)}} x^{\frac{2\pi ik}{\log(b)}}\right)}{2\pi k \left(a^{\frac{2\pi ik}{\log(b)}} - 1\right)} \\ &= \frac{1}{2\pi k} \left(-\cot\left(\frac{\pi k \log(a)}{\log(b)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(b)}\right) + \sin\left(2\pi k \frac{\log(x)}{\log(b)}\right)\right) \text{ for all } k \in \mathbb{N}.\end{aligned}$$

Summing everything up, we get our formula for $N_{a,b}(x)$. Setting $a = 2$ and $b = 3$, we get also the formula for $N_{2,3}(x)$. \square

Corollary 15. (The Formulas for $N_{a,b,c}(x)$ and $N_{2,3,5}(x)$)

For every real number $x \geq 1$, we have that

$$\begin{aligned}
N_{a,b,c}(x) = & \frac{\log(x)^3}{6 \log(a) \log(b) \log(c)} + \frac{\log(x)^2}{4 \log(a) \log(b)} + \frac{\log(x)^2}{4 \log(a) \log(c)} + \frac{\log(x)^2}{4 \log(b) \log(c)} + \frac{\log(x)}{4 \log(a)} \\
& + \frac{\log(x)}{4 \log(b)} + \frac{\log(x)}{4 \log(c)} + \frac{\log(a) \log(x)}{12 \log(b) \log(c)} + \frac{\log(b) \log(x)}{12 \log(a) \log(c)} + \frac{\log(c) \log(x)}{12 \log(a) \log(b)} \\
& + \frac{\log(a)}{24 \log(b)} + \frac{\log(a)}{24 \log(c)} + \frac{\log(b)}{24 \log(a)} + \frac{\log(b)}{24 \log(c)} + \frac{\log(c)}{24 \log(a)} + \frac{\log(c)}{24 \log(b)} + \frac{1}{8} \\
& - \frac{1}{4} B_1^* \left(\left\{ \frac{\log(x)}{\log(a)} \right\} \right) - \frac{1}{4} B_1^* \left(\left\{ \frac{\log(x)}{\log(b)} \right\} \right) - \frac{1}{4} B_1^* \left(\left\{ \frac{\log(x)}{\log(c)} \right\} \right) \\
& - \frac{1}{4\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(b)}{\log(a)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(a)}\right)}{k \sin\left(\frac{\pi k \log(b)}{\log(a)}\right)} - \frac{1}{4\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(a)}{\log(b)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(b)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(b)}\right)} \\
& - \frac{1}{4\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(c)}{\log(b)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(b)}\right)}{k \sin\left(\frac{\pi k \log(c)}{\log(b)}\right)} - \frac{1}{4\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(b)}{\log(c)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(c)}\right)}{k \sin\left(\frac{\pi k \log(b)}{\log(c)}\right)} \\
& - \frac{1}{4\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(a)}{\log(c)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(c)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(c)}\right)} - \frac{1}{4\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(c)}{\log(a)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(a)}\right)}{k \sin\left(\frac{\pi k \log(c)}{\log(a)}\right)} \\
& - \frac{1}{4\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(b)}{\log(a)}\right) \cos\left(\frac{\pi k \log(c)}{\log(a)}\right) \sin\left(2\pi k \frac{\log(x)}{\log(a)}\right)}{k \sin\left(\frac{\pi k \log(b)}{\log(a)}\right) \sin\left(\frac{\pi k \log(c)}{\log(a)}\right)} \\
& - \frac{1}{4\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(a)}{\log(b)}\right) \cos\left(\frac{\pi k \log(c)}{\log(b)}\right) \sin\left(2\pi k \frac{\log(x)}{\log(b)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(b)}\right) \sin\left(\frac{\pi k \log(c)}{\log(b)}\right)} \\
& - \frac{1}{4\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(a)}{\log(c)}\right) \cos\left(\frac{\pi k \log(b)}{\log(c)}\right) \sin\left(2\pi k \frac{\log(x)}{\log(c)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(c)}\right) \sin\left(\frac{\pi k \log(b)}{\log(c)}\right)} + \frac{1}{2} \chi_{S_{a,b,c}}(x)
\end{aligned}$$

and

$$\begin{aligned}
N_{2,3,5}(x) &= \frac{\log(x)^3}{6 \log(2) \log(3) \log(5)} + \frac{\log(x)^2}{4 \log(2) \log(3)} + \frac{\log(x)^2}{4 \log(2) \log(5)} + \frac{\log(x)^2}{4 \log(3) \log(5)} + \frac{\log(x)}{4 \log(2)} \\
&+ \frac{\log(x)}{4 \log(3)} + \frac{\log(x)}{4 \log(5)} + \frac{\log(2) \log(x)}{12 \log(3) \log(5)} + \frac{\log(3) \log(x)}{12 \log(2) \log(5)} + \frac{\log(5) \log(x)}{12 \log(2) \log(3)} \\
&+ \frac{\log(2)}{24 \log(3)} + \frac{\log(2)}{24 \log(5)} + \frac{\log(3)}{24 \log(2)} + \frac{\log(3)}{24 \log(5)} + \frac{\log(5)}{24 \log(2)} + \frac{\log(5)}{24 \log(3)} + \frac{1}{8} \\
&- \frac{1}{4} B_1^* \left(\left\{ \frac{\log(x)}{\log(2)} \right\} \right) - \frac{1}{4} B_1^* \left(\left\{ \frac{\log(x)}{\log(3)} \right\} \right) - \frac{1}{4} B_1^* \left(\left\{ \frac{\log(x)}{\log(5)} \right\} \right) \\
&- \frac{1}{4\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(3)}{\log(2)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(2)}\right)}{k \sin\left(\frac{\pi k \log(3)}{\log(2)}\right)} - \frac{1}{4\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(2)}{\log(3)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(3)}\right)}{k \sin\left(\frac{\pi k \log(2)}{\log(3)}\right)} \\
&- \frac{1}{4\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(5)}{\log(3)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(3)}\right)}{k \sin\left(\frac{\pi k \log(5)}{\log(3)}\right)} - \frac{1}{4\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(3)}{\log(5)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(5)}\right)}{k \sin\left(\frac{\pi k \log(3)}{\log(5)}\right)} \\
&- \frac{1}{4\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(2)}{\log(5)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(5)}\right)}{k \sin\left(\frac{\pi k \log(2)}{\log(5)}\right)} - \frac{1}{4\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(5)}{\log(2)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(2)}\right)}{k \sin\left(\frac{\pi k \log(5)}{\log(2)}\right)} \\
&- \frac{1}{4\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(3)}{\log(2)}\right) \cos\left(\frac{\pi k \log(5)}{\log(2)}\right) \sin\left(2\pi k \frac{\log(x)}{\log(2)}\right)}{k \sin\left(\frac{\pi k \log(3)}{\log(2)}\right) \sin\left(\frac{\pi k \log(5)}{\log(2)}\right)} \\
&- \frac{1}{4\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(2)}{\log(3)}\right) \cos\left(\frac{\pi k \log(5)}{\log(3)}\right) \sin\left(2\pi k \frac{\log(x)}{\log(3)}\right)}{k \sin\left(\frac{\pi k \log(2)}{\log(3)}\right) \sin\left(\frac{\pi k \log(5)}{\log(3)}\right)} \\
&- \frac{1}{4\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(2)}{\log(5)}\right) \cos\left(\frac{\pi k \log(3)}{\log(5)}\right) \sin\left(2\pi k \frac{\log(x)}{\log(5)}\right)}{k \sin\left(\frac{\pi k \log(2)}{\log(5)}\right) \sin\left(\frac{\pi k \log(3)}{\log(5)}\right)} + \frac{1}{2} \chi_{S_{2,3,5}}(x).
\end{aligned}$$

Proof. We have that

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{\chi_{S_{a,b,c}}(k)}{k^s} &= \left(\sum_{m_1=0}^{\infty} \frac{1}{a^{m_1 s}} \right) \left(\sum_{m_2=0}^{\infty} \frac{1}{b^{m_2 s}} \right) \left(\sum_{m_3=0}^{\infty} \frac{1}{c^{m_3 s}} \right) \\
&= \frac{1}{(1 - e^{-\log(a)s}) (1 - e^{-\log(b)s}) (1 - e^{-\log(c)s})}.
\end{aligned}$$

Therefore, by Perron's formula, we get that

$$\begin{aligned}
N_{a,b,c}(x) &= \frac{1}{2\pi i} \int_{\gamma} \frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})(1-e^{-\log(c)s})} ds \\
&= \text{Res}_{s=0} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})(1-e^{-\log(c)s})} \right) \\
&\quad + \sum_{k=1}^{\infty} \text{Res}_{s=\frac{2\pi ik}{\log(a)}} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})(1-e^{-\log(c)s})} \right) \\
&\quad + \sum_{k=1}^{\infty} \text{Res}_{s=-\frac{2\pi ik}{\log(a)}} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})(1-e^{-\log(c)s})} \right) \\
&\quad + \sum_{k=1}^{\infty} \text{Res}_{s=\frac{2\pi ik}{\log(b)}} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})(1-e^{-\log(c)s})} \right) \\
&\quad + \sum_{k=1}^{\infty} \text{Res}_{s=-\frac{2\pi ik}{\log(b)}} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})(1-e^{-\log(c)s})} \right) \\
&\quad + \sum_{k=1}^{\infty} \text{Res}_{s=\frac{2\pi ik}{\log(c)}} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})(1-e^{-\log(c)s})} \right) \\
&\quad + \sum_{k=1}^{\infty} \text{Res}_{s=-\frac{2\pi ik}{\log(c)}} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})(1-e^{-\log(c)s})} \right) + \frac{1}{2} \chi_{S_{a,b,c}}(x),
\end{aligned}$$

where $\gamma =$ line from $1 - i\infty$ to $1 + i\infty$.

Furthermore, we have that

$$\begin{aligned}
&\text{Res}_{s=0} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})(1-e^{-\log(c)s})} \right) \\
&= \frac{\log(x)^3}{6 \log(a) \log(b) \log(c)} + \frac{\log(x)^2}{4 \log(a) \log(b)} + \frac{\log(x)^2}{4 \log(a) \log(c)} + \frac{\log(x)^2}{4 \log(b) \log(c)} + \frac{\log(x)}{4 \log(a)} \\
&\quad + \frac{\log(x)}{4 \log(b)} + \frac{\log(x)}{4 \log(c)} + \frac{\log(a) \log(x)}{12 \log(b) \log(c)} + \frac{\log(b) \log(x)}{12 \log(a) \log(c)} + \frac{\log(c) \log(x)}{12 \log(a) \log(b)} \\
&\quad + \frac{\log(a)}{24 \log(b)} + \frac{\log(a)}{24 \log(c)} + \frac{\log(b)}{24 \log(a)} + \frac{\log(b)}{24 \log(c)} + \frac{\log(c)}{24 \log(a)} + \frac{\log(c)}{24 \log(b)} + \frac{1}{8}
\end{aligned}$$

and that

$$\begin{aligned}
\text{Res}_{s=\frac{2\pi ik}{\log(a)}} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})(1-e^{-\log(c)s})} \right) &= -\frac{ib^{\frac{2\pi ik}{\log(a)}} c^{\frac{2\pi ik}{\log(a)}} x^{\frac{2\pi ik}{\log(a)}}}{2\pi k \left(b^{\frac{2\pi ik}{\log(a)}} - 1 \right) \left(c^{\frac{2\pi ik}{\log(a)}} - 1 \right)} \quad \forall k \in \mathbb{N} \\
\text{Res}_{s=-\frac{2\pi ik}{\log(a)}} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})(1-e^{-\log(c)s})} \right) &= \frac{ix^{-\frac{2\pi ik}{\log(a)}}}{2\pi k \left(b^{\frac{2\pi ik}{\log(a)}} - 1 \right) \left(c^{\frac{2\pi ik}{\log(a)}} - 1 \right)} \quad \forall k \in \mathbb{N}.
\end{aligned}$$

Exactly similar expressions hold also for the other Residues under exchanging a with b , and a with c ("permuting a, b, c "). Using again the relations

$$\begin{aligned}\sin\left(2\pi k \frac{\log(x)}{\log(a)}\right) &= \frac{1}{2}ix^{-\frac{2\pi ik}{\log(a)}} - \frac{1}{2}ix^{\frac{2\pi ik}{\log(a)}} \\ \cos\left(2\pi k \frac{\log(x)}{\log(a)}\right) &= \frac{1}{2}x^{-\frac{2\pi ik}{\log(a)}} + \frac{1}{2}x^{\frac{2\pi ik}{\log(a)}} \\ \cot\left(\frac{\pi k \log(b)}{\log(a)}\right) &= i \frac{b^{\frac{2\pi ik}{\log(a)}} + 1}{b^{\frac{2\pi ik}{\log(a)}} - 1},\end{aligned}$$

we establish (by expanding everything out) the following identity

$$\begin{aligned}& -\cot\left(\frac{\pi k \log(b)}{\log(a)}\right) \cot\left(\frac{\pi k \log(c)}{\log(a)}\right) \sin\left(2\pi k \frac{\log(x)}{\log(a)}\right) - \cot\left(\frac{\pi k \log(b)}{\log(a)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(a)}\right) \\ & - \cot\left(\frac{\pi k \log(c)}{\log(a)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(a)}\right) + \sin\left(2\pi k \frac{\log(x)}{\log(a)}\right) \\ &= -\left(i \frac{b^{\frac{2\pi ik}{\log(a)}} + 1}{b^{\frac{2\pi ik}{\log(a)}} - 1}\right) \left(i \frac{c^{\frac{2\pi ik}{\log(a)}} + 1}{c^{\frac{2\pi ik}{\log(a)}} - 1}\right) \left(\frac{1}{2}ix^{-\frac{2\pi ik}{\log(a)}} - \frac{1}{2}ix^{\frac{2\pi ik}{\log(a)}}\right) - \left(i \frac{b^{\frac{2\pi ik}{\log(a)}} + 1}{b^{\frac{2\pi ik}{\log(a)}} - 1}\right) \left(\frac{1}{2}x^{-\frac{2\pi ik}{\log(a)}} + \frac{1}{2}x^{\frac{2\pi ik}{\log(a)}}\right) \\ & - \left(i \frac{c^{\frac{2\pi ik}{\log(a)}} + 1}{c^{\frac{2\pi ik}{\log(a)}} - 1}\right) \left(\frac{1}{2}x^{-\frac{2\pi ik}{\log(a)}} + \frac{1}{2}x^{\frac{2\pi ik}{\log(a)}}\right) + \left(\frac{1}{2}ix^{-\frac{2\pi ik}{\log(a)}} - \frac{1}{2}ix^{\frac{2\pi ik}{\log(a)}}\right) \\ &= \frac{2i \left(x^{-\frac{2\pi ik}{\log(a)}} - b^{\frac{2\pi ik}{\log(a)}} c^{\frac{2\pi ik}{\log(a)}} x^{\frac{2\pi ik}{\log(a)}}\right)}{\left(b^{\frac{2\pi ik}{\log(a)}} - 1\right) \left(c^{\frac{2\pi ik}{\log(a)}} - 1\right)}.\end{aligned}$$

Therefore, for the first Residues (the " a -Residues"), we have that

$$\begin{aligned}& \text{Res}_{s=\frac{2\pi ik}{\log(a)}} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})(1-e^{-\log(c)s})} \right) \\ & + \text{Res}_{s=-\frac{2\pi ik}{\log(a)}} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})(1-e^{-\log(c)s})} \right) \\ &= \frac{i \left(x^{-\frac{2\pi ik}{\log(a)}} - b^{\frac{2\pi ik}{\log(a)}} c^{\frac{2\pi ik}{\log(a)}} x^{\frac{2\pi ik}{\log(a)}}\right)}{2\pi k \left(b^{\frac{2\pi ik}{\log(a)}} - 1\right) \left(c^{\frac{2\pi ik}{\log(a)}} - 1\right)} \\ &= \frac{1}{4\pi k} \left(-\cot\left(\frac{\pi k \log(b)}{\log(a)}\right) \cot\left(\frac{\pi k \log(c)}{\log(a)}\right) \sin\left(2\pi k \frac{\log(x)}{\log(a)}\right) - \cot\left(\frac{\pi k \log(b)}{\log(a)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(a)}\right) \right. \\ & \quad \left. - \cot\left(\frac{\pi k \log(c)}{\log(a)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(a)}\right) + \sin\left(2\pi k \frac{\log(x)}{\log(a)}\right) \right) \quad \forall k \in \mathbb{N}.\end{aligned}$$

Exchanging a with b , and a with c ("permuting a, b, c "), we get also the other Residues (the " b -Residues" and the " c -Residues"), which have exactly the same structure. Summing

everything up, we get the formula for $N_{a,b,c}(x)$. Setting $a = 2$, $b = 3$ and $c = 5$, we get also the formula for $N_{2,3,5}(x)$. \square

Corollary 16. (The Formulas for $N_{a,b,c,d}(x)$ and $N_{2,3,5,7}(x)$)

For every real number $x \geq 1$, we have that

$$\begin{aligned}
N_{a,b,c,d}(x) = & \frac{\log(x)^4}{24 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(x)^3}{12 \log(a) \log(b) \log(c)} + \frac{\log(x)^3}{12 \log(a) \log(b) \log(d)} \\
& + \frac{\log(x)^3}{12 \log(a) \log(c) \log(d)} + \frac{\log(x)^3}{12 \log(b) \log(c) \log(d)} + \frac{\log(a) \log(x)^2}{24 \log(b) \log(c) \log(d)} \\
& + \frac{\log(b) \log(x)^2}{24 \log(a) \log(c) \log(d)} + \frac{\log(c) \log(x)^2}{24 \log(a) \log(b) \log(d)} + \frac{\log(d) \log(x)^2}{24 \log(a) \log(b) \log(c)} \\
& + \frac{\log(x)^2}{8 \log(a) \log(b)} + \frac{\log(x)^2}{8 \log(a) \log(c)} + \frac{\log(x)^2}{8 \log(a) \log(d)} + \frac{\log(x)^2}{8 \log(b) \log(c)} + \frac{\log(x)^2}{8 \log(b) \log(d)} \\
& + \frac{\log(x)^2}{8 \log(c) \log(d)} + \frac{\log(x)}{8 \log(a)} + \frac{\log(x)}{8 \log(b)} + \frac{\log(x)}{8 \log(c)} + \frac{\log(x)}{8 \log(d)} + \frac{\log(a) \log(x)}{24 \log(b) \log(c)} \\
& + \frac{\log(a) \log(x)}{24 \log(b) \log(d)} + \frac{\log(a) \log(x)}{24 \log(c) \log(d)} + \frac{\log(b) \log(x)}{24 \log(a) \log(c)} + \frac{\log(b) \log(x)}{24 \log(a) \log(d)} + \frac{\log(b) \log(x)}{24 \log(c) \log(d)} \\
& + \frac{\log(c) \log(x)}{24 \log(a) \log(b)} + \frac{\log(c) \log(x)}{24 \log(a) \log(d)} + \frac{\log(c) \log(x)}{24 \log(b) \log(d)} + \frac{\log(d) \log(x)}{24 \log(a) \log(b)} + \frac{\log(d) \log(x)}{24 \log(a) \log(c)} \\
& + \frac{\log(d) \log(x)}{24 \log(b) \log(c)} + \frac{1}{16} + \frac{\log(a)}{48 \log(b)} + \frac{\log(a)}{48 \log(c)} + \frac{\log(a)}{48 \log(d)} + \frac{\log(b)}{48 \log(a)} + \frac{\log(b)}{48 \log(c)} \\
& + \frac{\log(b)}{48 \log(d)} + \frac{\log(c)}{48 \log(a)} + \frac{\log(c)}{48 \log(b)} + \frac{\log(c)}{48 \log(d)} + \frac{\log(d)}{48 \log(a)} + \frac{\log(d)}{48 \log(b)} + \frac{\log(d)}{48 \log(c)} \\
& + \frac{\log(a) \log(b)}{144 \log(c) \log(d)} + \frac{\log(a) \log(c)}{144 \log(b) \log(d)} + \frac{\log(a) \log(d)}{144 \log(b) \log(c)} + \frac{\log(b) \log(c)}{144 \log(a) \log(d)} \\
& + \frac{\log(b) \log(d)}{144 \log(a) \log(c)} + \frac{\log(c) \log(d)}{144 \log(a) \log(b)} - \frac{\log(a)^3}{720 \log(b) \log(c) \log(d)} - \frac{\log(b)^3}{720 \log(a) \log(c) \log(d)} \\
& - \frac{\log(c)^3}{720 \log(a) \log(b) \log(d)} - \frac{\log(d)^3}{720 \log(a) \log(b) \log(c)} - \frac{1}{8} B_1^* \left(\left\{ \frac{\log(x)}{\log(a)} \right\} \right) \\
& - \frac{1}{8} B_1^* \left(\left\{ \frac{\log(x)}{\log(b)} \right\} \right) - \frac{1}{8} B_1^* \left(\left\{ \frac{\log(x)}{\log(c)} \right\} \right) - \frac{1}{8} B_1^* \left(\left\{ \frac{\log(x)}{\log(d)} \right\} \right) \\
& - \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(b)}{\log(a)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(a)}\right)}{k \sin\left(\frac{\pi k \log(b)}{\log(a)}\right)} - \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(a)}{\log(b)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(b)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(b)}\right)} \\
& - \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(c)}{\log(a)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(a)}\right)}{k \sin\left(\frac{\pi k \log(c)}{\log(a)}\right)} - \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(a)}{\log(c)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(c)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(c)}\right)}
\end{aligned}$$

$$\begin{aligned}
& - \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(a)}{\log(c)}\right) \cos\left(\frac{\pi k \log(d)}{\log(c)}\right) \sin\left(2\pi k \frac{\log(x)}{\log(c)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(c)}\right) \sin\left(\frac{\pi k \log(d)}{\log(c)}\right)} \\
& - \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(a)}{\log(d)}\right) \cos\left(\frac{\pi k \log(b)}{\log(d)}\right) \sin\left(2\pi k \frac{\log(x)}{\log(d)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(d)}\right) \sin\left(\frac{\pi k \log(b)}{\log(d)}\right)} \\
& - \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(b)}{\log(d)}\right) \cos\left(\frac{\pi k \log(c)}{\log(d)}\right) \sin\left(2\pi k \frac{\log(x)}{\log(d)}\right)}{k \sin\left(\frac{\pi k \log(b)}{\log(d)}\right) \sin\left(\frac{\pi k \log(c)}{\log(d)}\right)} \\
& - \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(a)}{\log(d)}\right) \cos\left(\frac{\pi k \log(c)}{\log(d)}\right) \sin\left(2\pi k \frac{\log(x)}{\log(d)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(d)}\right) \sin\left(\frac{\pi k \log(c)}{\log(d)}\right)} \\
& + \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(b)}{\log(a)}\right) \cos\left(\frac{\pi k \log(c)}{\log(a)}\right) \cos\left(\frac{\pi k \log(d)}{\log(a)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(a)}\right)}{k \sin\left(\frac{\pi k \log(b)}{\log(a)}\right) \sin\left(\frac{\pi k \log(c)}{\log(a)}\right) \sin\left(\frac{\pi k \log(d)}{\log(a)}\right)} \\
& + \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(a)}{\log(b)}\right) \cos\left(\frac{\pi k \log(c)}{\log(b)}\right) \cos\left(\frac{\pi k \log(d)}{\log(b)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(b)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(b)}\right) \sin\left(\frac{\pi k \log(c)}{\log(b)}\right) \sin\left(\frac{\pi k \log(d)}{\log(b)}\right)} \\
& + \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(a)}{\log(c)}\right) \cos\left(\frac{\pi k \log(b)}{\log(c)}\right) \cos\left(\frac{\pi k \log(d)}{\log(c)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(c)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(c)}\right) \sin\left(\frac{\pi k \log(b)}{\log(c)}\right) \sin\left(\frac{\pi k \log(d)}{\log(c)}\right)} \\
& + \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(a)}{\log(d)}\right) \cos\left(\frac{\pi k \log(b)}{\log(d)}\right) \cos\left(\frac{\pi k \log(c)}{\log(d)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(d)}\right)}{k \sin\left(\frac{\pi k \log(a)}{\log(d)}\right) \sin\left(\frac{\pi k \log(b)}{\log(d)}\right) \sin\left(\frac{\pi k \log(c)}{\log(d)}\right)} + \frac{1}{2} \chi_{S_{a,b,c,d}}(x)
\end{aligned}$$

and

$$\begin{aligned}
N_{2,3,5,7}(x) &= \frac{\log(x)^4}{24 \log(2) \log(3) \log(5) \log(7)} + \frac{\log(x)^3}{12 \log(2) \log(3) \log(5)} + \frac{\log(x)^3}{12 \log(2) \log(3) \log(7)} \\
&+ \frac{\log(x)^3}{12 \log(2) \log(5) \log(7)} + \frac{\log(x)^3}{12 \log(3) \log(5) \log(7)} + \frac{\log(2) \log(x)^2}{24 \log(3) \log(5) \log(7)} \\
&+ \frac{\log(3) \log(x)^2}{24 \log(2) \log(5) \log(7)} + \frac{\log(5) \log(x)^2}{24 \log(2) \log(3) \log(7)} + \frac{\log(7) \log(x)^2}{24 \log(2) \log(3) \log(5)} \\
&+ \frac{\log(x)^2}{8 \log(2) \log(3)} + \frac{\log(x)^2}{8 \log(2) \log(5)} + \frac{\log(x)^2}{8 \log(2) \log(7)} + \frac{\log(x)^2}{8 \log(3) \log(5)} + \frac{\log(x)^2}{8 \log(3) \log(7)} \\
&+ \frac{\log(x)^2}{8 \log(5) \log(7)} + \frac{\log(x)}{8 \log(2)} + \frac{\log(x)}{8 \log(3)} + \frac{\log(x)}{8 \log(5)} + \frac{\log(x)}{8 \log(7)} + \frac{\log(2) \log(x)}{24 \log(3) \log(5)}
\end{aligned}$$

$$\begin{aligned}
& + \frac{\log(2)\log(x)}{24\log(3)\log(7)} + \frac{\log(2)\log(x)}{24\log(5)\log(7)} + \frac{\log(3)\log(x)}{24\log(2)\log(5)} + \frac{\log(3)\log(x)}{24\log(2)\log(7)} + \frac{\log(3)\log(x)}{24\log(5)\log(7)} \\
& + \frac{\log(5)\log(x)}{24\log(2)\log(3)} + \frac{\log(5)\log(x)}{24\log(2)\log(7)} + \frac{\log(5)\log(x)}{24\log(3)\log(7)} + \frac{\log(7)\log(x)}{24\log(2)\log(3)} + \frac{\log(7)\log(x)}{24\log(2)\log(5)} \\
& + \frac{\log(7)\log(x)}{24\log(3)\log(5)} + \frac{1}{16} + \frac{\log(2)}{48\log(3)} + \frac{\log(2)}{48\log(5)} + \frac{\log(2)}{48\log(7)} + \frac{\log(3)}{48\log(2)} + \frac{\log(3)}{48\log(5)} \\
& + \frac{\log(3)}{48\log(7)} + \frac{\log(5)}{48\log(2)} + \frac{\log(5)}{48\log(3)} + \frac{\log(5)}{48\log(7)} + \frac{\log(7)}{48\log(2)} + \frac{\log(7)}{48\log(3)} + \frac{\log(7)}{48\log(5)} \\
& + \frac{\log(2)\log(3)}{144\log(5)\log(7)} + \frac{\log(2)\log(5)}{144\log(3)\log(7)} + \frac{\log(2)\log(7)}{144\log(3)\log(5)} + \frac{\log(3)\log(5)}{144\log(2)\log(7)} \\
& + \frac{\log(3)\log(7)}{144\log(2)\log(5)} + \frac{\log(5)\log(7)}{144\log(2)\log(3)} - \frac{\log(2)^3}{720\log(3)\log(5)\log(7)} - \frac{\log(3)^3}{720\log(2)\log(5)\log(7)} \\
& - \frac{\log(5)^3}{720\log(2)\log(3)\log(7)} - \frac{\log(7)^3}{720\log(2)\log(3)\log(5)} - \frac{1}{8}B_1^* \left(\left\{ \frac{\log(x)}{\log(2)} \right\} \right) \\
& - \frac{1}{8}B_1^* \left(\left\{ \frac{\log(x)}{\log(3)} \right\} \right) - \frac{1}{8}B_1^* \left(\left\{ \frac{\log(x)}{\log(5)} \right\} \right) - \frac{1}{8}B_1^* \left(\left\{ \frac{\log(x)}{\log(7)} \right\} \right) \\
& - \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(3)}{\log(2)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(2)}\right)}{k \sin\left(\frac{\pi k \log(3)}{\log(2)}\right)} - \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(2)}{\log(3)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(3)}\right)}{k \sin\left(\frac{\pi k \log(2)}{\log(3)}\right)} \\
& - \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(5)}{\log(2)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(2)}\right)}{k \sin\left(\frac{\pi k \log(5)}{\log(2)}\right)} - \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(2)}{\log(5)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(5)}\right)}{k \sin\left(\frac{\pi k \log(2)}{\log(5)}\right)} \\
& - \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(7)}{\log(2)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(2)}\right)}{k \sin\left(\frac{\pi k \log(7)}{\log(2)}\right)} - \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(2)}{\log(7)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(7)}\right)}{k \sin\left(\frac{\pi k \log(2)}{\log(7)}\right)} \\
& - \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(5)}{\log(3)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(3)}\right)}{k \sin\left(\frac{\pi k \log(5)}{\log(3)}\right)} - \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(3)}{\log(5)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(5)}\right)}{k \sin\left(\frac{\pi k \log(3)}{\log(5)}\right)} \\
& - \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(7)}{\log(3)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(3)}\right)}{k \sin\left(\frac{\pi k \log(7)}{\log(3)}\right)} - \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(3)}{\log(7)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(7)}\right)}{k \sin\left(\frac{\pi k \log(3)}{\log(7)}\right)} \\
& - \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(7)}{\log(5)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(5)}\right)}{k \sin\left(\frac{\pi k \log(7)}{\log(5)}\right)} - \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(5)}{\log(7)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(7)}\right)}{k \sin\left(\frac{\pi k \log(5)}{\log(7)}\right)} \\
& - \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(3)}{\log(2)}\right) \cos\left(\frac{\pi k \log(5)}{\log(2)}\right) \sin\left(2\pi k \frac{\log(x)}{\log(2)}\right)}{k \sin\left(\frac{\pi k \log(3)}{\log(2)}\right) \sin\left(\frac{\pi k \log(5)}{\log(2)}\right)}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(2)}{\log(3)}\right) \cos\left(\frac{\pi k \log(5)}{\log(3)}\right) \cos\left(\frac{\pi k \log(7)}{\log(3)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(3)}\right)}{k \sin\left(\frac{\pi k \log(2)}{\log(3)}\right) \sin\left(\frac{\pi k \log(5)}{\log(3)}\right) \sin\left(\frac{\pi k \log(7)}{\log(3)}\right)} \\
& + \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(2)}{\log(5)}\right) \cos\left(\frac{\pi k \log(3)}{\log(5)}\right) \cos\left(\frac{\pi k \log(7)}{\log(5)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(5)}\right)}{k \sin\left(\frac{\pi k \log(2)}{\log(5)}\right) \sin\left(\frac{\pi k \log(3)}{\log(5)}\right) \sin\left(\frac{\pi k \log(7)}{\log(5)}\right)} \\
& + \frac{1}{8\pi} \sum_{k=1}^{\infty} \frac{\cos\left(\frac{\pi k \log(2)}{\log(7)}\right) \cos\left(\frac{\pi k \log(3)}{\log(7)}\right) \cos\left(\frac{\pi k \log(5)}{\log(7)}\right) \cos\left(2\pi k \frac{\log(x)}{\log(7)}\right)}{k \sin\left(\frac{\pi k \log(2)}{\log(7)}\right) \sin\left(\frac{\pi k \log(3)}{\log(7)}\right) \sin\left(\frac{\pi k \log(5)}{\log(7)}\right)} + \frac{1}{2} \chi_{S_{2,3,5,7}}(x).
\end{aligned}$$

Proof. We have that

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{\chi_{S_{a,b,c,d}}(k)}{k^s} &= \left(\sum_{m_1=0}^{\infty} \frac{1}{a^{m_1 s}} \right) \left(\sum_{m_2=0}^{\infty} \frac{1}{b^{m_2 s}} \right) \left(\sum_{m_3=0}^{\infty} \frac{1}{c^{m_3 s}} \right) \left(\sum_{m_4=0}^{\infty} \frac{1}{d^{m_4 s}} \right) \\
&= \frac{1}{(1 - e^{-\log(a)s}) (1 - e^{-\log(b)s}) (1 - e^{-\log(c)s}) (1 - e^{-\log(d)s})}.
\end{aligned}$$

Therefore, by Perron's formula, we get that

$$\begin{aligned}
N_{a,b,c,d}(x) &= \frac{1}{2\pi i} \int_{\gamma} \frac{x^s}{s (1 - e^{-\log(a)s}) (1 - e^{-\log(b)s}) (1 - e^{-\log(c)s}) (1 - e^{-\log(d)s})} ds \\
&= \text{Res}_{s=0} \left(\frac{x^s}{s (1 - e^{-\log(a)s}) (1 - e^{-\log(b)s}) (1 - e^{-\log(c)s}) (1 - e^{-\log(d)s})} \right) \\
&+ \sum_{k=1}^{\infty} \text{Res}_{s=\frac{2\pi i k}{\log(a)}} \left(\frac{x^s}{s (1 - e^{-\log(a)s}) (1 - e^{-\log(b)s}) (1 - e^{-\log(c)s}) (1 - e^{-\log(d)s})} \right) \\
&+ \sum_{k=1}^{\infty} \text{Res}_{s=-\frac{2\pi i k}{\log(a)}} \left(\frac{x^s}{s (1 - e^{-\log(a)s}) (1 - e^{-\log(b)s}) (1 - e^{-\log(c)s}) (1 - e^{-\log(d)s})} \right) \\
&+ \sum_{k=1}^{\infty} \text{Res}_{s=\frac{2\pi i k}{\log(b)}} \left(\frac{x^s}{s (1 - e^{-\log(a)s}) (1 - e^{-\log(b)s}) (1 - e^{-\log(c)s}) (1 - e^{-\log(d)s})} \right) \\
&+ \sum_{k=1}^{\infty} \text{Res}_{s=-\frac{2\pi i k}{\log(b)}} \left(\frac{x^s}{s (1 - e^{-\log(a)s}) (1 - e^{-\log(b)s}) (1 - e^{-\log(c)s}) (1 - e^{-\log(d)s})} \right) \\
&+ \sum_{k=1}^{\infty} \text{Res}_{s=\frac{2\pi i k}{\log(c)}} \left(\frac{x^s}{s (1 - e^{-\log(a)s}) (1 - e^{-\log(b)s}) (1 - e^{-\log(c)s}) (1 - e^{-\log(d)s})} \right) \\
&+ \sum_{k=1}^{\infty} \text{Res}_{s=-\frac{2\pi i k}{\log(c)}} \left(\frac{x^s}{s (1 - e^{-\log(a)s}) (1 - e^{-\log(b)s}) (1 - e^{-\log(c)s}) (1 - e^{-\log(d)s})} \right) \\
&+ \sum_{k=1}^{\infty} \text{Res}_{s=\frac{2\pi i k}{\log(d)}} \left(\frac{x^s}{s (1 - e^{-\log(a)s}) (1 - e^{-\log(b)s}) (1 - e^{-\log(c)s}) (1 - e^{-\log(d)s})} \right)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{k=1}^{\infty} \operatorname{Res}_{s=-\frac{2\pi ik}{\log(d)}} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})(1-e^{-\log(c)s})(1-e^{-\log(d)s})} \right) \\
& + \frac{1}{2} \chi_{S_{a,b,c,d}}(x),
\end{aligned}$$

where $\gamma =$ line from $1 - i\infty$ to $1 + i\infty$.

For the Residues, we have that

$$\begin{aligned}
& \operatorname{Res}_{s=0} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})(1-e^{-\log(c)s})(1-e^{-\log(d)s})} \right) \\
& = \frac{\log(x)^4}{24 \log(a) \log(b) \log(c) \log(d)} + \frac{\log(x)^3}{12 \log(a) \log(b) \log(c)} + \frac{\log(x)^3}{12 \log(a) \log(b) \log(d)} \\
& + \frac{\log(x)^3}{12 \log(a) \log(c) \log(d)} + \frac{\log(x)^3}{12 \log(b) \log(c) \log(d)} + \frac{\log(a) \log(x)^2}{24 \log(b) \log(c) \log(d)} \\
& + \frac{\log(b) \log(x)^2}{24 \log(a) \log(c) \log(d)} + \frac{\log(c) \log(x)^2}{24 \log(a) \log(b) \log(d)} + \frac{\log(d) \log(x)^2}{24 \log(a) \log(b) \log(c)} \\
& + \frac{\log(x)^2}{8 \log(a) \log(b)} + \frac{\log(x)^2}{8 \log(a) \log(c)} + \frac{\log(x)^2}{8 \log(a) \log(d)} + \frac{\log(x)^2}{8 \log(b) \log(c)} + \frac{\log(x)^2}{8 \log(b) \log(d)} \\
& + \frac{\log(x)^2}{8 \log(c) \log(d)} + \frac{\log(x)}{8 \log(a)} + \frac{\log(x)}{8 \log(b)} + \frac{\log(x)}{8 \log(c)} + \frac{\log(x)}{8 \log(d)} + \frac{\log(a) \log(x)}{24 \log(b) \log(c)} \\
& + \frac{\log(a) \log(x)}{24 \log(b) \log(d)} + \frac{\log(a) \log(x)}{24 \log(c) \log(d)} + \frac{\log(b) \log(x)}{24 \log(a) \log(c)} + \frac{\log(b) \log(x)}{24 \log(a) \log(d)} + \frac{\log(b) \log(x)}{24 \log(c) \log(d)} \\
& + \frac{\log(c) \log(x)}{24 \log(a) \log(b)} + \frac{\log(c) \log(x)}{24 \log(a) \log(d)} + \frac{\log(c) \log(x)}{24 \log(b) \log(d)} + \frac{\log(d) \log(x)}{24 \log(a) \log(b)} + \frac{\log(d) \log(x)}{24 \log(a) \log(c)} \\
& + \frac{\log(d) \log(x)}{24 \log(b) \log(c)} + \frac{1}{16} + \frac{\log(a)}{48 \log(b)} + \frac{\log(a)}{48 \log(c)} + \frac{\log(a)}{48 \log(d)} + \frac{\log(b)}{48 \log(a)} + \frac{\log(b)}{48 \log(c)} \\
& + \frac{\log(b)}{48 \log(d)} + \frac{\log(c)}{48 \log(a)} + \frac{\log(c)}{48 \log(b)} + \frac{\log(c)}{48 \log(d)} + \frac{\log(d)}{48 \log(a)} + \frac{\log(d)}{48 \log(b)} + \frac{\log(d)}{48 \log(c)} \\
& + \frac{\log(a) \log(b)}{144 \log(c) \log(d)} + \frac{\log(a) \log(c)}{144 \log(b) \log(d)} + \frac{\log(a) \log(d)}{144 \log(b) \log(c)} + \frac{\log(b) \log(c)}{144 \log(a) \log(d)} \\
& + \frac{\log(b) \log(d)}{144 \log(a) \log(c)} + \frac{\log(c) \log(d)}{144 \log(a) \log(b)} - \frac{\log(a)^3}{720 \log(b) \log(c) \log(d)} - \frac{\log(b)^3}{720 \log(a) \log(c) \log(d)} \\
& - \frac{\log(c)^3}{720 \log(a) \log(b) \log(d)} - \frac{\log(d)^3}{720 \log(a) \log(b) \log(c)}
\end{aligned}$$

and that

$$\begin{aligned}
& \operatorname{Res}_{s=\frac{2\pi ik}{\log(a)}} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})(1-e^{-\log(c)s})(1-e^{-\log(d)s})} \right) \\
& = - \frac{ib^{\frac{2\pi ik}{\log(a)}} c^{\frac{2\pi ik}{\log(a)}} d^{\frac{2\pi ik}{\log(a)}} x^{\frac{2\pi ik}{\log(a)}}}{2\pi k \left(b^{\frac{2\pi ik}{\log(a)}} - 1 \right) \left(c^{\frac{2\pi ik}{\log(a)}} - 1 \right) \left(d^{\frac{2\pi ik}{\log(a)}} - 1 \right)} \quad \forall k \in \mathbb{N}
\end{aligned}$$

$$\begin{aligned} & \text{Res}_{s=-\frac{2\pi ik}{\log(a)}} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})(1-e^{-\log(c)s})(1-e^{-\log(d)s})} \right) \\ &= -\frac{ix^{-\frac{2\pi ik}{\log(a)}}}{2\pi k \left(b^{\frac{2\pi ik}{\log(a)}} - 1 \right) \left(c^{\frac{2\pi ik}{\log(a)}} - 1 \right) \left(d^{\frac{2\pi ik}{\log(a)}} - 1 \right)} \quad \forall k \in \mathbb{N}. \end{aligned}$$

Exactly similar relations hold also for the other Residues under exchanging a with b , a with c , and a with d ("permuting a, b, c, d "). Using the relations

$$\begin{aligned} \sin \left(2\pi k \frac{\log(x)}{\log(a)} \right) &= \frac{1}{2} ix^{-\frac{2\pi ik}{\log(a)}} - \frac{1}{2} ix^{\frac{2\pi ik}{\log(a)}} \\ \cos \left(2\pi k \frac{\log(x)}{\log(a)} \right) &= \frac{1}{2} x^{-\frac{2\pi ik}{\log(a)}} + \frac{1}{2} x^{\frac{2\pi ik}{\log(a)}} \\ \cot \left(\frac{\pi k \log(b)}{\log(a)} \right) &= i \frac{b^{\frac{2\pi ik}{\log(a)}} + 1}{b^{\frac{2\pi ik}{\log(a)}} - 1}, \end{aligned}$$

we establish (by expanding everything out) the following identity

$$\begin{aligned} & \cot \left(\frac{\pi k \log(b)}{\log(a)} \right) \cot \left(\frac{\pi k \log(c)}{\log(a)} \right) \cot \left(\frac{\pi k \log(d)}{\log(a)} \right) \cos \left(2\pi k \frac{\log(x)}{\log(a)} \right) \\ & - \cot \left(\frac{\pi k \log(b)}{\log(a)} \right) \cot \left(\frac{\pi k \log(c)}{\log(a)} \right) \sin \left(2\pi k \frac{\log(x)}{\log(a)} \right) \\ & - \cot \left(\frac{\pi k \log(c)}{\log(a)} \right) \cot \left(\frac{\pi k \log(d)}{\log(a)} \right) \sin \left(2\pi k \frac{\log(x)}{\log(a)} \right) \\ & - \cot \left(\frac{\pi k \log(b)}{\log(a)} \right) \cot \left(\frac{\pi k \log(d)}{\log(a)} \right) \sin \left(2\pi k \frac{\log(x)}{\log(a)} \right) \\ & - \cot \left(\frac{\pi k \log(b)}{\log(a)} \right) \cos \left(2\pi k \frac{\log(x)}{\log(a)} \right) - \cot \left(\frac{\pi k \log(c)}{\log(a)} \right) \cos \left(2\pi k \frac{\log(x)}{\log(a)} \right) \\ & - \cot \left(\frac{\pi k \log(d)}{\log(a)} \right) \cos \left(2\pi k \frac{\log(x)}{\log(a)} \right) + \sin \left(2\pi k \frac{\log(x)}{\log(a)} \right) \\ &= \left(i \frac{b^{\frac{2\pi ik}{\log(a)}} + 1}{b^{\frac{2\pi ik}{\log(a)}} - 1} \right) \left(i \frac{c^{\frac{2\pi ik}{\log(a)}} + 1}{c^{\frac{2\pi ik}{\log(a)}} - 1} \right) \left(i \frac{d^{\frac{2\pi ik}{\log(a)}} + 1}{d^{\frac{2\pi ik}{\log(a)}} - 1} \right) \left(\frac{1}{2} x^{-\frac{2\pi ik}{\log(a)}} + \frac{1}{2} x^{\frac{2\pi ik}{\log(a)}} \right) \\ & - \left(i \frac{b^{\frac{2\pi ik}{\log(a)}} + 1}{b^{\frac{2\pi ik}{\log(a)}} - 1} \right) \left(i \frac{c^{\frac{2\pi ik}{\log(a)}} + 1}{c^{\frac{2\pi ik}{\log(a)}} - 1} \right) \left(\frac{1}{2} ix^{-\frac{2\pi ik}{\log(a)}} - \frac{1}{2} ix^{\frac{2\pi ik}{\log(a)}} \right) \\ & - \left(i \frac{c^{\frac{2\pi ik}{\log(a)}} + 1}{c^{\frac{2\pi ik}{\log(a)}} - 1} \right) \left(i \frac{d^{\frac{2\pi ik}{\log(a)}} + 1}{d^{\frac{2\pi ik}{\log(a)}} - 1} \right) \left(\frac{1}{2} ix^{-\frac{2\pi ik}{\log(a)}} - \frac{1}{2} ix^{\frac{2\pi ik}{\log(a)}} \right) \\ & - \left(i \frac{b^{\frac{2\pi ik}{\log(a)}} + 1}{b^{\frac{2\pi ik}{\log(a)}} - 1} \right) \left(i \frac{d^{\frac{2\pi ik}{\log(a)}} + 1}{d^{\frac{2\pi ik}{\log(a)}} - 1} \right) \left(\frac{1}{2} ix^{-\frac{2\pi ik}{\log(a)}} - \frac{1}{2} ix^{\frac{2\pi ik}{\log(a)}} \right) \end{aligned}$$

$$\begin{aligned}
& - \left(i \frac{b^{\frac{2\pi ik}{\log(a)}} + 1}{b^{\frac{2\pi ik}{\log(a)}} - 1} \right) \left(\frac{1}{2} x^{-\frac{2\pi ik}{\log(a)}} + \frac{1}{2} x^{\frac{2\pi ik}{\log(a)}} \right) - \left(i \frac{c^{\frac{2\pi ik}{\log(a)}} + 1}{c^{\frac{2\pi ik}{\log(a)}} - 1} \right) \left(\frac{1}{2} x^{-\frac{2\pi ik}{\log(a)}} + \frac{1}{2} x^{\frac{2\pi ik}{\log(a)}} \right) \\
& - \left(i \frac{d^{\frac{2\pi ik}{\log(a)}} + 1}{d^{\frac{2\pi ik}{\log(a)}} - 1} \right) \left(\frac{1}{2} x^{-\frac{2\pi ik}{\log(a)}} + \frac{1}{2} x^{\frac{2\pi ik}{\log(a)}} \right) + \left(\frac{1}{2} i x^{-\frac{2\pi ik}{\log(a)}} - \frac{1}{2} i x^{\frac{2\pi ik}{\log(a)}} \right) \\
& = - \frac{4i \left(x^{-\frac{2\pi ik}{\log(a)}} + b^{\frac{2\pi ik}{\log(a)}} c^{\frac{2\pi ik}{\log(a)}} d^{\frac{2\pi ik}{\log(a)}} x^{\frac{2\pi ik}{\log(a)}} \right)}{\left(b^{\frac{2\pi ik}{\log(a)}} - 1 \right) \left(c^{\frac{2\pi ik}{\log(a)}} - 1 \right) \left(d^{\frac{2\pi ik}{\log(a)}} - 1 \right)}.
\end{aligned}$$

This shows that for the first Residues (the "a-Residues"), we have that

$$\begin{aligned}
& \text{Res}_{s=\frac{2\pi ik}{\log(a)}} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})(1-e^{-\log(c)s})(1-e^{-\log(d)s})} \right) \\
& + \text{Res}_{s=-\frac{2\pi ik}{\log(a)}} \left(\frac{x^s}{s(1-e^{-\log(a)s})(1-e^{-\log(b)s})(1-e^{-\log(c)s})(1-e^{-\log(d)s})} \right) \\
& = - \frac{i \left(x^{-\frac{2\pi ik}{\log(a)}} + b^{\frac{2\pi ik}{\log(a)}} c^{\frac{2\pi ik}{\log(a)}} d^{\frac{2\pi ik}{\log(a)}} x^{\frac{2\pi ik}{\log(a)}} \right)}{2\pi k \left(b^{\frac{2\pi ik}{\log(a)}} - 1 \right) \left(c^{\frac{2\pi ik}{\log(a)}} - 1 \right) \left(d^{\frac{2\pi ik}{\log(a)}} - 1 \right)} \\
& = \frac{1}{8\pi k} \left(\cot \left(\frac{\pi k \log(b)}{\log(a)} \right) \cot \left(\frac{\pi k \log(c)}{\log(a)} \right) \cot \left(\frac{\pi k \log(d)}{\log(a)} \right) \cos \left(2\pi k \frac{\log(x)}{\log(a)} \right) \right. \\
& \quad - \cot \left(\frac{\pi k \log(b)}{\log(a)} \right) \cot \left(\frac{\pi k \log(c)}{\log(a)} \right) \sin \left(2\pi k \frac{\log(x)}{\log(a)} \right) \\
& \quad - \cot \left(\frac{\pi k \log(c)}{\log(a)} \right) \cot \left(\frac{\pi k \log(d)}{\log(a)} \right) \sin \left(2\pi k \frac{\log(x)}{\log(a)} \right) \\
& \quad - \cot \left(\frac{\pi k \log(b)}{\log(a)} \right) \cot \left(\frac{\pi k \log(d)}{\log(a)} \right) \sin \left(2\pi k \frac{\log(x)}{\log(a)} \right) \\
& \quad - \cot \left(\frac{\pi k \log(b)}{\log(a)} \right) \cos \left(2\pi k \frac{\log(x)}{\log(a)} \right) - \cot \left(\frac{\pi k \log(c)}{\log(a)} \right) \cos \left(2\pi k \frac{\log(x)}{\log(a)} \right) \\
& \quad \left. - \cot \left(\frac{\pi k \log(d)}{\log(a)} \right) \cos \left(2\pi k \frac{\log(x)}{\log(a)} \right) + \sin \left(2\pi k \frac{\log(x)}{\log(a)} \right) \right) \quad \forall k \in \mathbb{N}.
\end{aligned}$$

By exchanging the variable a with all other variables b, c and d ("permuting a, b, c, d "), we get all four Residues (the " a, b, c, d -Residues"), which have all the same structure. Summing everything up, we get our formula for $N_{a,b,c,d}(x)$. Setting $a = 2, b = 3, c = 5$ and $d = 7$, we get also the formula for $N_{2,3,5,7}(x)$. \square

And so on.

These formulas are exactly equivalent to the previous mentioned formulas.

6 The Formula for the Counting Function of the Natural Numbers of the Form $a^{p^2}b^{q^2}$

Let $a, b \in \mathbb{N}$ such that $a < b$ and $\gcd(a, b) = 1$.
For $x \in \mathbb{R}_0^+$, we define the function $N_{a,b}^{(2)}(x)$ by

$$N_{a,b}^{(2)}(x) := \sum_{\substack{a^{p^2}b^{q^2} \leq x \\ p \in \mathbb{N}_0, q \in \mathbb{N}_0}} 1.$$

Moreover, we define

$$S_{a,b}^{(2)} := \left\{ a^{p^2}b^{q^2} : p \in \mathbb{N}_0, q \in \mathbb{N}_0 \right\},$$

$$\chi_{S_{a,b}^{(2)}}(x) := \begin{cases} 1 & \text{if } x \in S_{a,b}^{(2)} \\ 0 & \text{if } x \notin S_{a,b}^{(2)}. \end{cases}$$

We have that

$$N_{a,b}^{(2)}(x) = 1 + \sum_{k=0}^{\lfloor \sqrt{\log_b(x)} \rfloor} \left[\sqrt{\log_a \left(\frac{x}{b^{k^2}} \right)} \right] + \left[\sqrt{\log_b(x)} \right].$$

We have also the following

Theorem 17. (Formula for $N_{a,b}^{(2)}(x)$)

For every real number $x > 1$, we have that

$$\begin{aligned} N_{a,b}^{(2)}(x) &= \frac{\pi \log(x)}{4\sqrt{\log(a)\log(b)}} + \frac{1}{2}\sqrt{\frac{\log(x)}{\log(a)}} + \frac{1}{2}\sqrt{\frac{\log(x)}{\log(b)}} + \frac{1}{4} - \frac{1}{2}B_1^* \left(\left\{ \sqrt{\frac{\log(x)}{\log(a)}} \right\} \right) \\ &\quad - \frac{1}{2}B_1^* \left(\left\{ \sqrt{\frac{\log(x)}{\log(b)}} \right\} \right) + \sqrt{\log(x)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{J_1 \left(2\pi \sqrt{\frac{n^2 \log(a) + m^2 \log(b)}{\log(a)\log(b)}} \log(x) \right)}{\sqrt{n^2 \log(a) + m^2 \log(b)}} \\ &\quad + \frac{1}{2}\sqrt{\frac{\log(x)}{\log(a)}} \sum_{k=1}^{\infty} \frac{J_1 \left(2\pi k \sqrt{\frac{\log(x)}{\log(b)}} \right)}{k} + \frac{1}{2}\sqrt{\frac{\log(x)}{\log(b)}} \sum_{k=1}^{\infty} \frac{J_1 \left(2\pi k \sqrt{\frac{\log(x)}{\log(a)}} \right)}{k} + \frac{1}{2}\chi_{S_{a,b}^{(2)}}(x). \end{aligned}$$

This formula converges very rapidly.

Setting $a = 2$ and $b = 3$, we get

Corollary 18. (Formula for $N_{2,3}^{(2)}(x)$)
For every real number $x > 1$, we have that

$$\begin{aligned}
N_{2,3}^{(2)}(x) &= \frac{\pi \log(x)}{4\sqrt{\log(2)\log(3)}} + \frac{1}{2}\sqrt{\frac{\log(x)}{\log(2)}} + \frac{1}{2}\sqrt{\frac{\log(x)}{\log(3)}} + \frac{1}{4} - \frac{1}{2}B_1^* \left(\left\{ \sqrt{\frac{\log(x)}{\log(2)}} \right\} \right) \\
&\quad - \frac{1}{2}B_1^* \left(\left\{ \sqrt{\frac{\log(x)}{\log(3)}} \right\} \right) + \sqrt{\log(x)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{J_1 \left(2\pi \sqrt{\frac{n^2 \log(2) + m^2 \log(3)}{\log(2)\log(3)}} \log(x) \right)}{\sqrt{n^2 \log(2) + m^2 \log(3)}} \\
&\quad + \frac{1}{2}\sqrt{\frac{\log(x)}{\log(2)}} \sum_{k=1}^{\infty} \frac{J_1 \left(2\pi k \sqrt{\frac{\log(x)}{\log(3)}} \right)}{k} + \frac{1}{2}\sqrt{\frac{\log(x)}{\log(3)}} \sum_{k=1}^{\infty} \frac{J_1 \left(2\pi k \sqrt{\frac{\log(x)}{\log(2)}} \right)}{k} + \frac{1}{2}\chi_{S_{2,3}^{(2)}}(x).
\end{aligned}$$

We have that

$$\begin{aligned}
S_{2,3}^{(2)} &:= \left\{ 2^{p^2} 3^{q^2} : p \in \mathbb{N}_0, q \in \mathbb{N}_0 \right\} \\
&= \{1, 2, 3, 6, 16, 48, 81, 162, \dots\},
\end{aligned}$$

and therefore we get the following table:

x	$N_{2,3}^{(2)}(x)$	Formula for $N_{2,3}^{(2)}(x)$	Number of terms (n, m) needed with $k = 400$
1	1	1.077194794603379	$(n, m) = (1, 1)$ at $x = 1.1$
10	4	4.069103424005291	$(n, m) = (1, 1)$
10^2	7	7.000949506610362	$(n, m) = (5, 5)$
10^3	9	9.086395912838084	$(n, m) = (3, 3)$
10^4	11	11.038613589829053	$(n, m) = (5, 5)$
10^5	15	15.012706923272531	$(n, m) = (5, 5)$
10^6	17	17.046462385363300	$(n, m) = (5, 5)$
10^7	18	18.408421860888305	$(n, m) = (9, 9)$
10^8	22	22.127760008955621	$(n, m) = (6, 6)$
10^9	24	24.034210155019944	$(n, m) = (8, 8)$
10^{10}	26	26.009844154207983	$(n, m) = (9, 9)$
10^{10^2}	226	226.001668111078420	$(n, m) = (39, 39)$
10^{10^3}	2122	2122.031291011313557	$(n, m) = (168, 168)$
10^{10^4}	20886	20886.032472386492101	$(n, m) = (400, 400)$
10^{10^9}	207756	207756.0303040763527672	$(n, m) = (1000, 1000)$
10^{10^6}	2074033	2074033.0733802760244109	$(n, m) = (1400, 1400)$

Table 5: Values of $N_{2,3}^{(2)}(x)$

7 Conclusion

We have presented and proved the formulas for the distribution of every smooth number sequence. This article and the proofs of these formulas will soon be published in a Journal.

References

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2010 *Mathematics Subject Classification*: Primary 40A30; Secondary 11Y55.

Keywords: distribution of smooth numbers, distribution of friable numbers, distribution of 2-smooth numbers (powers of 2), distribution of 3-smooth numbers (harmonic numbers), distribution of 5-smooth numbers (regular numbers or Hamming numbers), distribution of 7-smooth numbers (Humble numbers or "highly composite numbers"), distribution of all smooth numbers, distribution of all friable numbers, distribution of p_n -smooth numbers, distribution of the natural numbers of the form a^p less than or equal to x , distribution of the natural numbers of the form $a^p b^q$ less than or equal to x , distribution of the natural numbers of the form $a^p b^q c^l$ less than or equal to x , distribution of the natural numbers of the form $a^p b^q c^l d^f$ less than or equal to x , distribution of the natural numbers of the form $a_1^{q_1} a_2^{q_2} a_3^{q_3} \cdots a_n^{q_n}$ less than or equal to x , distribution of the natural numbers of the form $2^p 3^q$ less than or equal to x , distribution of the natural numbers of the form $2^p 3^q 5^l$ less than or equal to x , distribution of the natural numbers of the form $2^p 3^q 5^l 7^f$ less than or equal to x , distribution of the natural numbers of the form $2^{q_1} 3^{q_2} 5^{q_3} \cdots p_n^{q_n}$ less than or equal to x , distribution of the natural numbers of the form $a^{p^2} b^{q^2}$ less than or equal to x , distribution of the natural numbers of the form $2^{p^2} 3^{q^2}$ less than or equal to x .