Minimal number of points on a grid forming patterns of blocks

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Abstract

We consider the minimal number of points on a regular grid on the plane that generates n blocks of points of exactly length k. We illustrate how this is related to the n-queens problem on the toroidal chessboard and show that this number is upper bounded by $kn/3$ and approaches $kn/4$ as $n \to \infty$ when $k+1$ is coprime with 6 or when k is large.

1 Introduction

We consider points on a regular grid on the plane which form horizontal, vertical or diagonal blocks of exactly k points (which we will call *patterns*)[1]. For example, the set of points in Fig. 1 shows 12 points forming 3 patterns of length 5. Note that since a pattern of length k has to have exactly k points flanked by empty grid locations, the set of points in Fig. 1 contains 4 patterns of length 2 and does not contain any patterns of length 4 or of length 3. Our motivation for studying this problem is the Bingo-4 problem proposed by Sun et al. and described in OEIS[1] sequence A273916 where the case $k = 4$ is considered. Let $a_k(n)$ denote the minimal number of points needed to form n patterns of length k, i.e. Fig. 1 shows that $a_5(3) = 12$. Finding the exact value of $a_k(n)$ appears to be difficult and not feasible for large n. The purpose of this note is to provide an analysis on the asymptotic behavior of $a_k(n)$.

2 Bounds and asymptotic behavior of $a_k(n)$

It is easy to see that $a_k(1) = k$, $a_k(2) = 2k-1$ and $a_k(3) = 3(k-1)$. Next, consider Fekete's subadditive Lemma [2] which is applicable to subadditive sequences.

[1] We use the convention that an isolated point corresponds to 4 patterns of length 1; a horizontal, a vertical and 2 diagonal patterns.

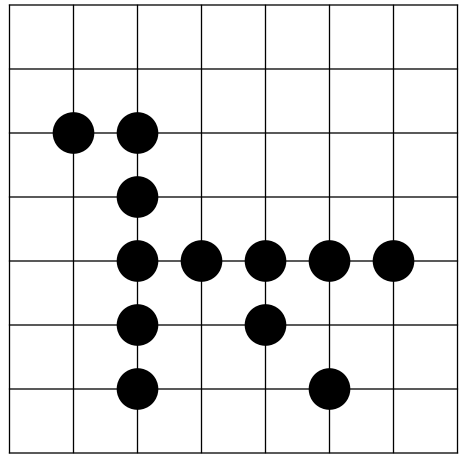

Figure 1: 12 points on a grid forming 3 patterns of length 5.

Lemma 1 (Fekete's subadditive Lemma)**.** *If the sequence $a(n)$ is subadditive, i.e. $a(n+m) \leq a(n) + a(m)$, then $\lim_{n\to\infty} \frac{a_n}{n}$ exists and is equal to $\inf \frac{a_n}{n}$.*

Theorem 1. *For all k, $a_k(n)$ is subadditive, and $f(k) = \lim_{n\to\infty} \frac{a_k(n)}{n}$ exists and satisfies $\frac{k}{4} \leq f(k) \leq \frac{k}{3}$.*

Proof. Since each pattern takes k points and each point can be part of at most 4 patterns, $a_k(n) \geq \frac{kn}{4}$. It is clear that $a_k(n)$ is subadditive. Lemma 1 implies that $f(k)$ exists and is equal to $\inf_n \frac{a_k(n)}{n}$. Consider a k by m rectangular array of points with $k \leq m$. It is easy to see that there are $3m-2k+2$ length k patterns there. This shows that $a_k(3m-2k+2) \leq km$ which implies that $\frac{k}{4} \leq f(k) \leq \frac{k}{3}$. □

3 Constellations where each point is part of 4 different patterns

The upper bound $\frac{k}{3}$ on $f(k)$ in Theorem 1 shows that for large n we can construct a constellation of n points such that most points are part of 3 different patterns. Is it possible to construct a constellation such that most points are part of 4 different patterns (a horizontal, a vertical and two diagonal patterns) and thus achieve the lower bound $\frac{k}{4}$? The case $k = 1$ is simple. Since $a_1(4n) = n$ as exhibited by the constellation of n isolated points, this implies that $f(1) = \frac{1}{4}$.

Let σ be a permutation on the integers $\{0, 1, \cdots, k\}$. Consider a $k+1$ by $k+1$ square grid and place a point on each position (i, j) except when it is of the form $(i, \sigma(i))$. It is clear that tiling this grid on the plane results in a constellation that have horizontal and vertical patterns of length k. In order for the diagonals to also have a block of exactly k points, $\{i+\sigma(i) \mod k+1\}$ and $\{i-\sigma(i) \mod k+1\}$ need to be permutations of $\{0,1,\cdots,k\}$ as well. Consider a N by N subgrid of this tiling. Except for points near the edges which is on the order of $kN \propto k\sqrt{n}$, all points belong to 4 patterns of length k. Thus we have proved the following:

Theorem 2. *If there is a permutation σ of the numbers $\{0,1,\cdots,k\}$ such that $\sigma_1 = \{i+\sigma(i) \mod k+1\}$ and $\sigma_2 = \{i-\sigma(i) \mod k+1\}$ are both permutations, then $f(k) = \frac{k}{4}$. In particular, $\frac{a_k(n)}{n}$ converges to $f(k)$ on the order of $O\left(\frac{1}{\sqrt{n}}\right)$.*

If σ satisfies the conditions of Theorem 2, then so does σ^{-1}. For a fixed integer m, the permutation $\sigma(i)+m \mod k+1$ also satisfies these conditions. We will use this to partition the set of admissible permutations into equivalent classes. More specifically,

Definition 1. *Let S_{k+1} be the set of permutations on $\{0,1,\cdots,k\}$. $T_{k+1} \subset S_{k+1}$ is defined as the set of permutations σ such that $\{i+\sigma(i) \mod k+1\}$ and $\{i-\sigma(i) \mod k+1\}$ are in S_{k+1}. The equivalence relation $\sim$ is defined as follows. If $\sigma, \tau \in T_{k+1}$, then $\sigma \sim \tau$ if $\tau = \sigma^{-1}$ or there exist an integer m such that $\sigma(i) = \tau(i) + m \mod k+1$ for all i.*

Thus if $T_{k+1} \neq \emptyset$, then $f(k) = \frac{k}{4}$.

4 Modular n-queens problem

The n-queens problem asks whether n nonattacking queens can be placed on an n by n chessboard. The answer is yes and is first shown by Pauls [3, 4]. Next consider a toroidal n by n chessboard, where the top edge is connected to the bottom edge and the left edge is connected to the right edge. Polya [5] showed that a solution to the corresponding modular n-queens problem exists if and only if n is coprime with 6. It is clear that a permutation in T_{k+1} corresponds to a solution of the modular $(k+1)$-queens problem. Thus Polya's result is equivalent to the following result:

Theorem 3. *$T_{k+1} \neq \emptyset$ if and only if $k+1$ is coprime with 6.*

Corollary 1. *If $k+1$ is coprime with 6, then $f(k) = \frac{k}{4}$.*

Monsky [6] shows that $n-2$ nonattacking queens can be placed on an n by n toroidal chess board and $n-1$ queens can be placed if n is not divisible by 3 or 4. This implies the following which shows that for k large, $f(k)$ approaches the lower bound $\frac{k}{4}$:

Theorem 4. *$f(k) \leq \frac{k(k+1)+2}{4(k-1)}$. If $k+1$ is not divisible by 3 or 4, then $f(k) \leq \frac{k(k+1)+1}{4k}$.*

Proof. Consider a $k+1$ by $k+1$ array with $k+1-r$ nonattacking queens. By placing a point on the location where there are no queens we obtain a constellation with $(k+1)^2-(k+1-r)$ points. Each queen position corresponds to 4 patterns. Thus when this array is tiled, we get for a large number of points a ratio $\frac{a_k(n)}{n}$ approaching $\frac{(k+1)^2-(k+1-r)}{4(k+1-r)} = \frac{k(k+1)+r}{4(k+1-r)}$. The conclusion follows by setting $r=1$ or $r=2$. □

Corollary 2. $\lim_{k\to\infty} \frac{f(k)}{k} = \frac{1}{4}$.

4.1 Lattice construction

As in the n-queens problem, we can construct permutations in T_{k+1} via a lattice construction. In particular, we construct a constellation of points by placing a point on the grid if and only if it is not a point on a lattice spanned by two vectors v_1 and v_2. For instance with the lattice points generated by the vectors $(1,2)$ and $(2,-1)$, the set of points with $N = 15$ is shown in Fig. 2. In particular, this configuration shows that $f(4) = 1$.

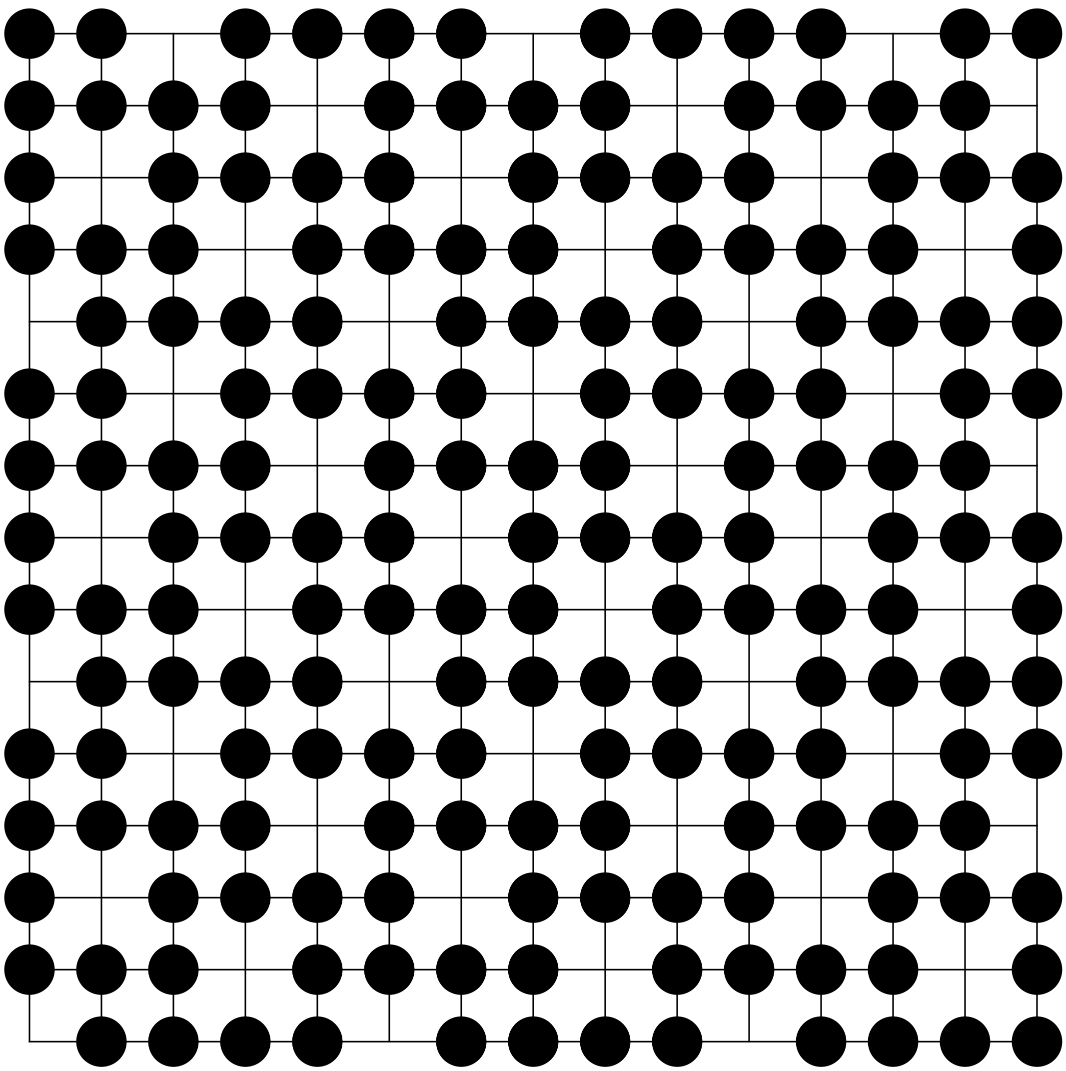

Figure 2: A lattice constellation. Points in the center of the grid are part of 4 different patterns, showing that $\frac{a_4(n)}{n} \to 1$ as $n \to \infty$.

The following result appears to be well-known [4], but we include it here for completeness.

Theorem 5. *If there exists $1 < m < k$ such that $m-1$, m and $m+1$ are all coprime with $k+1$, then the lattice construction with $v_1 = (1, m)$ and $(k+1, 0)$ generates a permutation σ in T_{k+1}.*

Proof. Consider the lattice generated with the vectors $(1, m)$ and $(0, k+1)$. Clearly, if m is coprime with $k+1$, then we find in a $k+1$ by $k+1$ subarray locations which do not have a point of the form $(i, \sigma(i))$ with σ a permutation. The lattice points have coordinates $(a, ma+(k+1)b)$ which lie on the 2 main diagonals if $a = ma+(k+1)b$ or $-a = ma+(k+1)b$. In the first case $-(m-1)a = (k+1)b$. Since $m-1$ is coprime with $k+1$, this means that a is a multiple of $k+1$, i.e., a diagonal pattern must have length k. In the second case $-(m+1)a = (k+1)b$. Since $k+1$ is coprime with $m+1$, again this means that a is a multiple of $k+1$. □

Theorem 5 also provides a proof of Corollary 1 since if $k+1$ is coprime with 6, then 1, 2 and 3 are all coprime with $k+1$. In particular the lattice construction with $v_1 = (1, 2)$ and $(k+1, 0)$ generates a permutation σ in T_{k+1}. Fig. 3 shows the construction for $k = 12$.

For $k = 4$, there is only one equivalence class $(0, 2, 4, 1, 3)$ in T_{k+1} that satisfies the conditions of Theorem 2. For $k = 6$, there are two equivalent classes $(0, 2, 4, 6, 1, 3, 5)$ and $(0, 3, 6, 2, 5, 1, 4)$. For $k = 10$, there are 4 equivalent classes. In particular, Theorem 5 shows that if $k+1 > 4$ is prime, then there are at least $\frac{k-2}{2}$ equivalent classes in T_{k+1}. This is because each $2 \leq m \leq k-1$ is coprime with $k+1$ and the permutation generated by m is the inverse of the permutation generated by $k-1-m$ which are equivalent[2]. It is possible to have more than $\frac{k-2}{2}$ equivalent classes as there are permutations in T_{k+1} not generated by a lattice. For $k+1$ coprime with 6, if $k = 4, 6$ and 10, all permutations in T_{k+1} are generated by a lattice. For $k = 12$, there are permutations in T_{k+1} that are not generated by a lattice. One such example is shown in Fig. 4. Such solutions are referred to as *nonlinear* solutions [4].

5 Conclusions

We studied the asymptotic behavior of the minimal number of points needed to generate n patterns of length k using a construction based on permutations of $\{0, 1, \cdots, k\}$ with certain properties. We showed that this construction allows us to create patterns where asympotically most points are part of 4 patterns. This construction is equivalent to the modular $(k+1)$-queens problem and thus $f(k) = \frac{k}{4}$ for $k+1$ coprime with 6. If $k+1$ is even or $k+1$ is divisible by 3, this construction fails to provide such a constellation. However, results in the modular n-queens problem can still provide an upper bound on $f(k)$ which shows that $\lim_{k\to\infty} \frac{f(k)}{k} = \frac{1}{4}$. Even though these constructions for the modular n-queens problem provide limiting value of $\frac{a_k(n)}{n}$ as $n \to \infty$, for a fixed n the optimal constellation to achieve $a_k(n)$ can be quite different (see for example `https://oeis.org/A273916/a273916.png`) .

[2]For general k, see [7] for a formula of the number of such permutations.

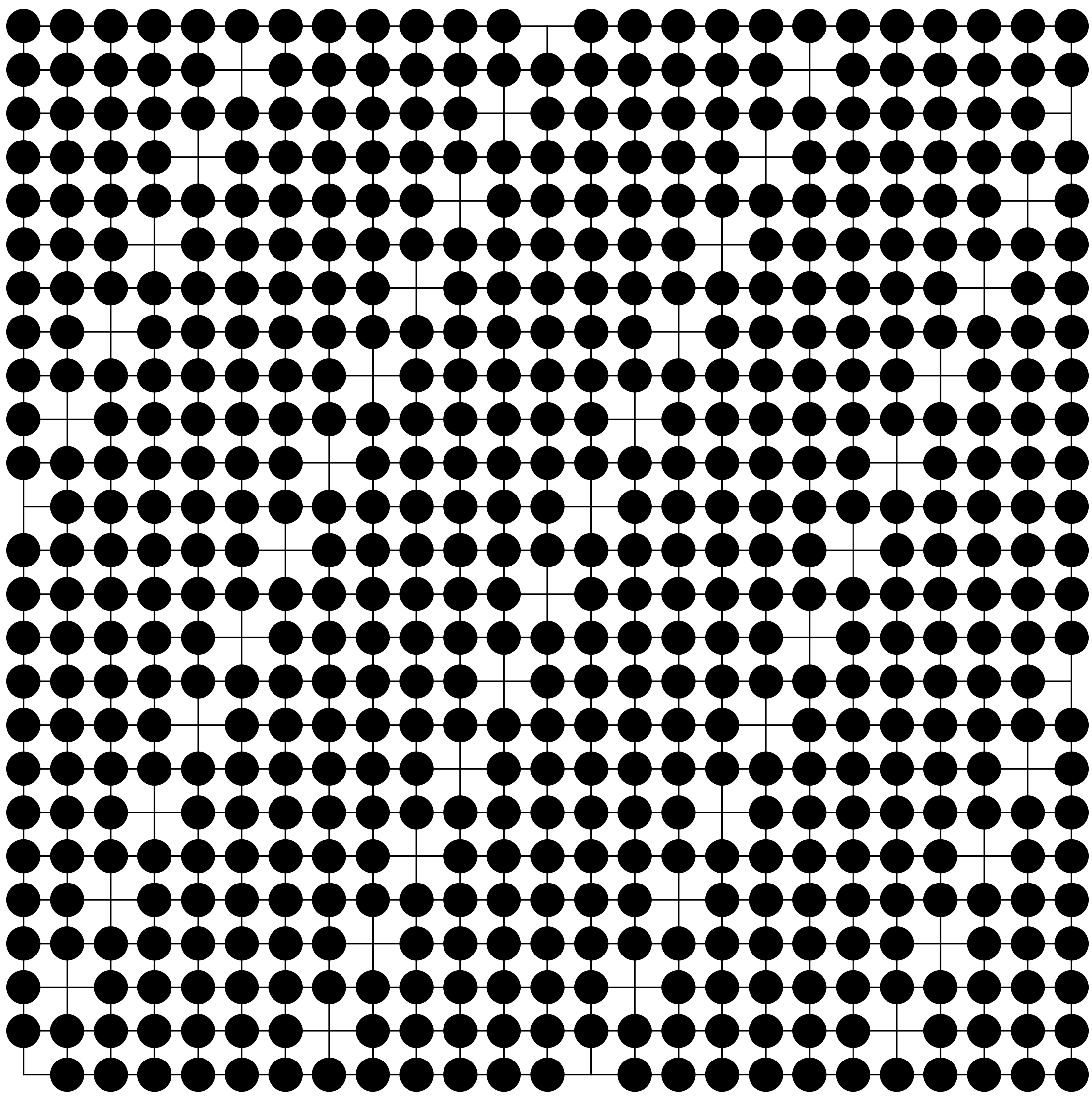

Figure 3: A lattice constellation for $k = 12$ generated by vectors $(1, 2)$ and $(0, 13)$.

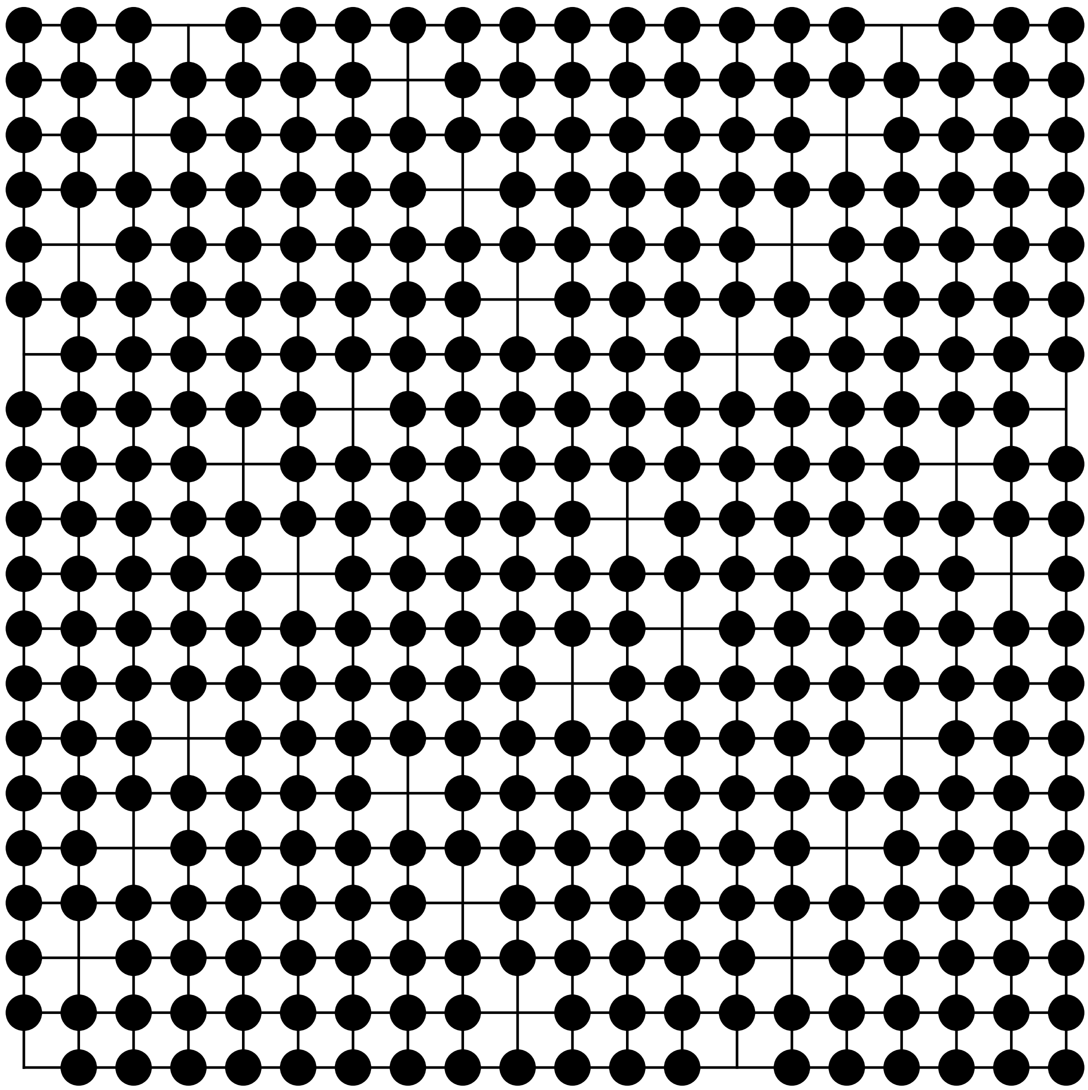

Figure 4: A constellation for $k = 12$ not generated by a lattice corresponding to the permutation $(0, 2, 4, 6, 11, 9, 12, 5, 3, 1, 7, 10, 8)$.

6 Acknowledgements

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