THE RANGE OF A STEINER OPERATION

L. H. HARPER

Abstract. This paper answers a fundamental question in the theory of Steiner operations (StOps) as defined and studied in the monograph, [21]. StOps are morphisms for combinatorial isoperimetric problems, analogous to Steiner symmetrization for continuous isoperimetric problems. The usefulness of a StOp, $\varphi : \mathbf{2}^V \to \mathbf{2}^V$, V a finite set, depends on having an efficient representation of its range. In [21] the problem was treated case-by-case. In each case the StOp induced a partial order, $\mathcal{P}$, on V so that $Range(\varphi) = \mathcal{I}(\mathcal{P})$, the set of all order ideals of $\mathcal{P}$. Here we show (directly from the axioms for a StOp) that every idempotent StOp admits such a representation of its range ($\mathcal{P}$ is then called the StOp-order of φ). That result leads to another question: What additional structure does $Range(\varphi)$ have? The answer is none. We show that every finite poset is the StOp-order of some idempotent Steiner operation.

1. Background

1.1. **Combinatorial Isoperimetric Problems.** Certain combinatorial optimization problems are analogous to the classical isoperimetric problem of plane geometry. The simplest example is the *edge-isoperimetric problem* (EIP) on a graph, $G = (V; E)$. V is the (finite) set of *vertices* of G and $E \subseteq \binom{V}{2}$, (unordered pairs of (distinct) vertices) is its set of *edges*. The *edge-boundary* of a set $S \subseteq V$ is $\Theta(S) = \{e \in E : e = \{u, v\}, u \in S, v \notin S\}$, the edges having one end in S and the other end in $V - S$. The EIP on G is to minimize $|\Theta(S)|$ given $|S|$. In general the EIP is a hard problem (NP-complete), but certain special cases of interest for computer science and engineering have been solved. The author's first paper, written fifty years ago (1962), solved the EIP for the graph of the n-cube.

1.2. **Steiner Operations.**

1.2.1. *Axioms.* The concept of Steiner operation (*StOp*) was the result of a deliberate effort, beginning around 1976, to identify morphisms for combinatorial isoperimetric problems (see [20]). The basic theory and results that followed were surveyed in [21], published in 2004. There, on pages 27-28 a StOp is characterized as a set-map, $\varphi : \mathbf{2}^V \to \mathbf{2}^V$, V being a finite set with *boundary function*, Ω, having the following properties:

(1) φ preserves the size of subsets: $\forall S \subseteq V$, $|\varphi(S)| = |S|$,
(2) φ does not increase the size of their boundaries: $|\Omega(\varphi(S))| \leq |\Omega(S)|$,
(3) φ preserves the structure of $\mathbf{2}^V$: $\forall S \subseteq T \subseteq V$, $\varphi(S) \subseteq \varphi(T)$.

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The original family of StOps [20], called *stabilization*, was derived from reflective symmetry of a graph embedded in Euclidean space, $\mathbb{R}^n$ (*e.g.* the graph of the n-cube). G.-C. Rota's observation that stabilization is a discrete analog of Steiner symmetrization (See [25] and [10] for background on Steiner symmetrization or Google "Steiner symmetrization") added considerable *gravitas* to the project. Later it was discovered that stabilization is also a StOp for the vertex-isoperimetric problem (VIP) on the graph, $G = (V; E)$ (the boundary, $\Phi(S)$, is the set of vertices in $V - S$ with neighbors in S). Also there is another whole class of systematic StOps, called *compression*. Compression is based on a product decomposition of G, one of the factors having nested solutions for the isoperimetric problem. Unlike stabilization, compression had been discovered independently many times, appearing in the majority of papers on combinatorial isoperimetric problems (See [21] or [16] for more background and details). Furthermore, Steiner symmetrization and most of its variants (such as Schwartz symmetrization) are compressions.

Properties (1), (2) & (3) above may be regarded as axioms for StOps. However, in the light of experience and expedience, we wish to extend these axioms a bit. First by allowing the domain and codomain of a StOp to be the more general $\mathcal{I}(\mathcal{P})$, where $\mathcal{P} = (V; \leq)$, $\leq$ being a partial order relation on V and $\mathcal{I}(\mathcal{P})$ the set of all ideals of $\mathcal{P}$ (for definitions see Sec. 2.1). In scheduling problems, where V is a set of tasks, $\mathcal{P}$ represents precedence constraints under which the tasks must be performed. No task may be worked on until all its antecedents have been completed. Also, as mentioned in our abstract and demonstrated in [21], such restrictions on the domain of a StOp arise from Steiner operations themselves.

StOps were created to help solve combinatorial isoperimetric problems. These problems may, in principle, be solved by Brute Force, trying out all possibilities in the domain, $\mathcal{I}(\mathcal{P})$. However, if $\mathcal{I}(\mathcal{P})$ is large, the cost may be prohibitive. The range of φ will generally be much smaller than its domain, but to be able to take advantage of the reduction in size we must have a simple way to identify sets in

$$Range\,(\varphi) = \{T \in \mathcal{I}(\mathcal{P}) : \exists S \in \mathcal{I}(\mathcal{P}) \text{ such that } \varphi(S) = T\},$$

and to generate them all efficiently. This fundamental technical problem in the theory of StOps was apparent right from the start, with stabilization, and an effective answer was found (see [20]). For the edge- and vertex-isoperimetric problems on graphs, the initial domain is $\mathbf{2}^V$, all subsets, so $\mathcal{P} = \Delta$, the discrete order on V. It was observed that each of the basic stabilization operations, defined by a reflective symmetry and a point not on the fixed hyperplane of the reflection (the Fricke-Klein point), induced a simple partial order on V. The basic stabilizations, with a common Fricke-Klein point so that they all had a common total extension (are *consistent*), could be composed, giving a StOp that combined the simplifications of its constituents. Surprisingly, although basic stabilizations were idempotent (*i.e.* $\varphi^2 = \varphi$), their compositions were generally not. However, if repeatedly (cyclically) composed, they would eventually become constant and therefore idempotent. The range of this superstabilization was then characterized (Theorem 4 of [20]) as $\mathcal{I}(\mathcal{Q})$, $\mathcal{Q}$ being the transitive closure of the union of all the basic partial orders.

Later on the author was pleasantly surprised to note that compression fit into the same theoretical framework: Basic compressions are idempotent Steiner operations, each defining a partial order on V. If consistent, compressions could be cyclically composed to give an idempotent StOp combining all their simplifications into one. The range of that supercompression is exactly the set of all ideals of the transitive closure of the union of the basic partial orders (see [21], Section 3.3.4). Also, all *ad hoc* (unsystematic) Steiner operations have been found to have partial orders representing their ranges.

All of these StOps and their associated StOp-orders share the property that $\varphi(S)$ is "lower" (wrt the StOp-order) than S. In each case this is justified by a one-to-one function, $f_S : S \to \varphi(S)$ such that $\forall x \in S, f_S(x) \leq x$, the definition of f_S depending on the definition of $\varphi(S)$. This leads us to add a fourth axiom:

(4) $\exists$a total extension

$$\tau : \mathcal{P} \to \mathbf{n} = \{0 < 1 < ... < n-1\},$$

(τ one-to-one and onto so $|V| = n$ and $x \leq y \Rightarrow \tau(x) \leq \tau(y)$). Also, with $\tau(S)$ defined to be $\sum_{x\in S} \tau(x)$, $\forall S \in \mathcal{I}(\mathcal{P})$, $\tau(\varphi(S)) \leq \tau(S)$ and $\tau(\varphi(S)) = \tau(S) \Rightarrow \varphi(S) = S$.

For stabilizations, τ is the order induced by proximity to the Fricke-Klein point. For compressions it is the total order of the inductive hypothesis. If a set of StOps share a common τ they are called *consistent*. Axiom 4 makes the repeated compositions of consistent StOps eventually constant since $\tau(\varphi(S))$ can only decrease a finite number of times. The requirement of consistency is a pragmatic one. Compositions of StOps that are not consistent would still be StOps but their composition could cycle and may not become constant. The reduction in size of the range achieved by the composition of consistent StOps is seems to be more than that achieved by identifying equivalence classes of sets. A good example of that is the solution of the edge-isoperimetric problem on V_{600}, the graph of the 600-vertex regular solid in 4 dimensions (See [21]). The Brute Force solution would generate all $2^{600} \gtrsim 10^{180}$subsets of vertices, an impossible task. The symmetry group of V_{600} is of order $14,400$ (See Coxeter's classic monograph [12]) so there are at least $10^{180}/14,400 \gtrsim 6.9 \times 10^{177}$ equivalence classes of sets of vertices. That is still a huge number and it is not even clear how to generate those equivalence classes efficiently. However, V_{600} has 60 reflective symmetries and the superstabilization they generate has about 10^{10} sets in its range. Those sets are ideals in the stabilization-order (aka the Bruhat order) of V_{600} and can be recursively generated in lexicographic order very efficiently. A Brute Force solution of the edge-isoperimetric problem on V_{600} (generating all 10^{10} sets in the range of its superstabilization) was carried out on a 3 MgHz PC in one day.

We also modify the second axiom by extending "size of the boundary" to any functional, $\partial : \mathbf{2}^V \to \mathbb{R}$, requiring that

(2') φ does not increase the size of their boundaries: $\partial(\varphi(S)) \leq \partial(S)$.

The author's monograph [21] treats concrete Steiner operations (StOps), principally stabilization & compression, and the partial orders (StOp-orders) that characterize their ranges. These concepts systematically simplify hard problems, pointing the way to subsequent developments such as passage to a continuous limit. Rather than survey the whole book here, we just give a glimpse of the culminating application, the solution of a problem posed by A. A. Sapozhenko. It demonstrates the power and efficiency of Steiner operations and their StOp-orders. Sapozhenko asked about the VIP on the Johnson graph, $J(d,n)$. The vertices of $J(d,n)$ are n-tuples of 0s & 1s with exactly d 1s. Two such vertices are neighbors if they differ in exactly two places. $J(d,n)$ does not have nested solutions for $d > 1$. $J(d,n)$ is not factorable as a product, so even the nested solutions of $J(1,n)$ cannot be used for compression. The most successful strategy for solving combinatorial isoperimetric problems without nested solutions has been to pass to a continuous limit and apply calculus (See Chapter 10 of [21]). This works for several other problems lacking nested solutions: The EIP on $(\mathbb{Z}_n)^d$ (the d-fold product of n-cycles) and the VIP on $(\mathbb{K}_n)^d$, (the d-fold product of complete graphs on n vertices). The (1-dimensional) compression-order for both $(\mathbb{Z}_n)^d$ & $(\mathbb{K}_n)^d$ is $\boldsymbol{n}^d$ and the limit of $(\boldsymbol{n}/n)^d$ as $n \to \infty$ is $[0,1]^d$, the unit d-cube. Of course $[0,1]^d$ has different boundaries for the limits of the EIP & VIP. Bollobas & Leader solved the EIP on $[0,1]^d$ with a discontinuous modification of compression (relaxation of Axiom 3 and induction on d). The present author solved the VIP on $[0,1]^d$ with a discontinuous modification of stabilization. It is not apparent how to pass to a continuous limit with $J(d,n)$ but stabilization with respect to the symmetric group acting on its coordinates transforms it so that the limit becomes obvious. The limit of the stabilization-order of $J(d,n)$ (as $n \to \infty$) is the continuous poset,

$$\mathcal{L}(d) = \left\{ x \in [0,1]^d : x_1 \leq x_2 \leq ... \leq x_d \right\},$$

ordered coordinatewise (See Section 10 of [21] for details). The solution of Sapozhenko's problem follows immediately from the symmetry of the solution of the VIP on $[0,1]^d$. The logic of the solution is the same as that given in elementary books for the reduction of Dido's problem to the classical isoperimetric problem in the plane, a "Didonean embedding" of $\mathcal{L}(d)$ into $[0,1]^d$ (See [21], Chapter 10 for details). One might expect that the EIP on $J(d,n)$ could be similarly solved. However, the solution of the EIP on $J(d,n)$ is not symmetric under interchange of coordinates, so the embedding is not Didonean. Despite considerable effort, the author has not been able to adapt the techniques that solved the EIP & VIP on $[0,1]^d$ to solve the EIP on $\mathcal{L}(d)$ (*i.e.* $\lim_{n\to\infty} J(d,n)$). It remains an open problem.

1.2.2. *Is there a theorem here?* With all those "coincidences", the range of so many different Steiner operations, $\varphi : \mathcal{I}(\mathcal{P}) \to \mathcal{I}(\mathcal{P})$ being represented as $\mathcal{I}(\mathcal{Q})$ for some $\mathcal{Q} \supseteq \mathcal{P}$, it was natural to wonder if *every* StOp has such a partial order characterizing its range? We kept coming back to this question because of the efficacy and power of StOp-orders, but each time were brought up short by the lack of any obvious source for the additional order relations. Where could they possibly come from? Then one day we realized that there already was a precedent in the literature for just such spontaneous creation of order: Garret Birkhoff's characterization of finite distributive lattices. (See Birkhoff's classic monograph, Lattice Theory [8], Section III.3).

2. The Ideal Transform and its Ramifications

Birkhoff's theorem requires some background. We now summarize the definitions and basic results for it. We have taken these from the monograph by Davey & Priestley [14] to which the reader may refer for proofs and additional theory. Also see Gratzer's more recent monograph [18].

2.1. **Posets and Ideals.** A *partial order*, $\leq$, on a set, V, is a binary relation, $\leq \subseteq V \times V$, which is

(1) Reflexive: $\forall x \in V, x \leq x$,

(2) Antisymmetric: $\forall x, y \in V$, $(x \leq y \;\&\; y \leq x) \Rightarrow (x = y)$,
(3) Transitive: $\forall x, y, z \in V$, $(x \leq y \;\&\; y \leq z) \Rightarrow (x \leq z)$.

A *partially ordered set* (*poset*), $\mathcal{P} = (V; \leq)$, consists of a set, V, with a partial order, $\leq$, on V.

Example 1. $\boldsymbol{n} = \{0 < 1 < ... < n-1\}$ *is a total order (chain) of size* n*.*

Example 2. $n = \{0, 1, ..., n-1\}$ *is a discrete order (antichain) of size* n*. In this case the partial order is* $\Delta_n = \{(0,0), (1,1), ..., (n-1, n-1)\}$*, the identity relation.*

Example 3. *The Boolean lattice,* $\mathcal{B}_n$*, with* n *generators is* $\mathbf{2}^n = \{0 < 1\} \times \{0 < 1\} \times ... \times \{0 < 1\}$*, ordered coordinatewise.* $\mathcal{B}_n$ *is isomorphic to the power set of an* n*-set.*

A set, $I \subseteq V$, is called an (order) *ideal* of $\mathcal{P} = (V; \leq)$, if $(y \in I \;\&\; x \leq y) \Rightarrow (x \in I)$. In [14] these are called down-sets, in [8] hereditary sets.

Example 4. $\boldsymbol{m}$ *is an ideal of* $\boldsymbol{n} \Leftrightarrow m \leq n$*.*

$\mathcal{I}(\mathcal{P}) = \{I \subseteq V : I \text{ is an ideal of } \mathcal{P}\}$ is called the *ideal-set* or *ideal transform* of $\mathcal{P}$. In [14] our $\mathcal{I}(\mathcal{P})$ is denoted $\mathcal{O}(\mathcal{P})$, $\mathcal{I}$ being reserved for the ideal-sets of lattices, which have additional structure. We use $\mathcal{I}$ for both, feeling that the concept of ideal for posets, lattices (and rings) are essentially the same, differing only by context (different categories).

Example 5. $\mathcal{I}(\boldsymbol{n}) \simeq \boldsymbol{n+1}$

Example 6. $\mathcal{I}(n) \simeq \mathcal{B}_n$*, since every subset of* n *is an ideal.*

If in a poset, $\mathcal{P} = (V; \leq)$, every pair of elements, $\{x, y\}$ has a least upper bound (greatest lower bound), it is denoted $x \vee y$ $(x \wedge y)$ and called the *join* (*meet*) of x and y. $\mathcal{L} = (V; \vee, \wedge)$ is then a *lattice*. Note that if $\mathcal{L}$ is finite it must have a least element, $\perp$, and a greatest element, $\top$. Because of the roles they play in the algebra of lattices, $\perp$ is often denoted as 0, and $\top$ as 1. We prefer $\perp, \top$ because $0, 1$ are already overloaded.

An element, $x \neq \perp$, in a lattice $\mathcal{L} = (V; \vee, \wedge)$, is called *join-irreducible* if

$$\nexists y, z < x \text{ such that } y \vee z = x.$$

That is, x has exactly one immediate predecessor.

Example 7. *In the chain,* $\boldsymbol{n = \{0 < 1 < ... < n-1\}}$ *, every element, except* 0*, is join-irreducible.*

Example 8. *In* $\mathbf{2}^V$*, the join-irreducible elements are exactly the generators (singleton sets, elements of rank 1).*

For a lattice, $\mathcal{L} = (V; \vee, \wedge)$ define $\mathcal{J}(\mathcal{L})$ to be $\{x \in V : x \text{ is join-irreducible in } \mathcal{L}\}$, partially ordered by its induced order in $\mathcal{L}$.

A lattice, $\mathcal{L} = (V; \vee, \wedge)$ is called *distributive* if $\forall x, y, z \in V$ it satisfies the

Distributive Laws: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Example 9. *The Boolean lattice,* $\mathcal{B}_n \simeq (2^n; \cup, \cap)$ *is distributive, in fact for any finite* $\mathcal{P}$*,* $\mathcal{I}(\mathcal{P})$ *is closed under* $\cup$ *&* $\cap$ *and inherits the distributive laws from* $\mathcal{B}_n$*, where* $n = |\mathcal{P}|$*.*

The following theorem is Birkhoff's fundamental result characterizing finite distributive lattices. In [14] it is Theorem 5.12.

Theorem 1. *Let* $\mathcal{L} = (V; \vee, \wedge)$ *be a (finite) distributive lattice. Then the map* $\eta : \mathcal{L} \to \mathcal{I}(\mathcal{J}(\mathcal{L}))$ *defined by*

$$\eta(x) = \{y \in \mathcal{J}(\mathcal{L}) : y \leq x\}$$

is an isomorphism of $\mathcal{L}$ *onto* $\mathcal{I}(\mathcal{J}(\mathcal{L}))$.

For a finite distributive lattice, $\mathcal{L}$, we call $\mathcal{I}(\mathcal{P})$ (with $\mathcal{P} = \mathcal{J}(\mathcal{L})$) its *Birkhoff representation.* Note that $\mathcal{I}(\mathcal{J}(\mathcal{L}))$ is a sublattice of $\mathbf{2}^V$. Furthermore,

Theorem 2. $\mathcal{P} \subseteq \mathcal{Q} \Leftrightarrow \mathcal{I}(\mathcal{Q}) \subseteq \mathcal{I}(\mathcal{P})$.

This is a special case of Theorem 5.19 of [14].

3. General Derivation of StOp-order

3.1. **The range of** $\varphi : \mathcal{I}(\mathcal{P}) \to \mathcal{I}(\mathcal{P})$**.** Now we come to the main result of this paper. Having stated Birkhoff's theorem (Theorem 1 above) we can reveal the insight that lead to our result: The range of φ is a subposet of $\mathcal{I}(\mathcal{P})$. $\mathcal{I}(\mathcal{P})$ in turn is a sublattice of the Boolean lattice, $\mathbf{2}^V$ (closed under $\cup$ and $\cap$ and therefore distributive). If *Range*(φ) is representable as $\mathcal{I}(\mathcal{Q})$ for some extension $\mathcal{Q}$ of $\mathcal{P}$, then it would be closed under $\cup$ and $\cap$ and again, distributive. But if we can just show that *Range*(φ) is closed under $\cup$ and $\cap$ then it will be a sublattice of $\mathcal{I}(\mathcal{P})$, must be distributive and, by Theorems 1&2, isomorphic to $\mathcal{I}(\mathcal{Q})$ for some extension $\mathcal{Q}$ of $\mathcal{P}$.

First a preliminary result. We have observed that repeated composition of a StOp with itself will produce an idempotent StOp with a smaller range so we need only consider idempotent StOps.

Lemma 1. *For an idempotent StOp* φ*, Range*$(\varphi) = \{T \in \mathcal{I}(\mathcal{P}) : \varphi(T) = T\}$*, the set of fixpoints of* φ.

Proof. By definition $\{T \in \mathcal{I}(\mathcal{P}) : \varphi(T) = T\}$ is a subset of *Range*(φ). Conversely, $T \in Range(\varphi) \Leftrightarrow \exists S \in \mathcal{I}(\mathcal{P})$ such that $\varphi(S) = T$. But then

$$\begin{aligned} \varphi(T) &= \varphi(\varphi(S)) \\ &= \varphi^2(S) \\ &= \varphi(S), \text{ since } \varphi \text{ is idempotent,} \\ &= T. \end{aligned}$$

□

Theorem 3. *For every idempotent Steiner operation,* $\varphi : \mathcal{I}(\mathcal{P}) \to \mathcal{I}(\mathcal{P})$*, there exists a unique partial order,* $\mathcal{Q}$ *on* V *with* $\mathcal{P} \subseteq \mathcal{Q}$*, such that Range*$(\varphi) = \mathcal{I}(\mathcal{Q})$.

Proof. As remarked above, we need only show that the range of φ is closed under $\cup$ & $\cap$:

$$\begin{aligned} S, T &\in Range(\varphi) \\ &\Rightarrow S, T \in \mathcal{I}(\mathcal{P}) \\ &\Rightarrow S \cup T \in \mathcal{I}(\mathcal{P}), \text{ since } \mathcal{I}(\mathcal{P}) \text{ is closed under } \cup. \end{aligned}$$

Also,

$$\begin{aligned}(S,T\subseteq S\cup T) &\Rightarrow \varphi(S),\varphi(T)\subseteq\varphi(S\cup T)\,,\text{ by Axiom 3,}\\ &\Rightarrow S\cup T\subseteq\varphi(S\cup T)\,,\text{ since }\varphi(S)=S,\ \varphi(T)=T\\ &\&\ \ (S,T\subseteq W\Rightarrow S\cup T\subseteq W)\,.\end{aligned}$$

But by Axiom 1, $(|\varphi(S\cup T)|=|S\cup T|)$ so $\varphi(S\cup T)=S\cup T$. Therefore $S\cup T\in$ *Range*(φ).

By duality, $S\cap T\in$ *Range*(φ). $\square$

4. The Range of Ranges

We wish to investigate the structure of all possible ranges of (finite) Steiner operations. To this end we return to the strategy that led us to StOps in the first place: We study morphisms for finite distributive lattices and the resulting category. Morphisms for lattices are easy to define, they are maps, $\varphi:\mathcal{L}\to\mathcal{M}$ that preserve the lattice operations: $\forall x,y\in\mathcal{L}$,

$$\varphi(x\wedge y)=\varphi(x)\wedge\varphi(y)\,,$$
$$\varphi(x\vee y)=\varphi(x)\vee\varphi(y)\,.$$

If $\mathcal{L}$ is distributive and φ is epi (onto), $\mathcal{M}$ must also be distributive. $\mathcal{L}$ is *complete* means that $\forall S\subseteq\mathcal{L}$, $\bigvee S=x_1\vee x_2\vee\ldots$ (for all $x_{i\in S}$) is defined and $\bigwedge S$ is also defined (Definition 2.4(ii) of [14]). Corollary 2.25 of [14] states that every finite lattice is complete. If $\mathcal{L}$ is a complete lattice $\bigvee\mathcal{L}=\top$ and $\bigwedge\mathcal{L}=\bot$ so it has a top and bottom. We require that our lattice morphisms preserve all existing meets and joins. And they must also preserve top and bottom,

$$\varphi(\top)=\top,$$
$$\varphi(\bot)=\bot.$$

In [14] these are denoted $\{0,1\}$-homomorphisms, so we call them $\{\bot,\top\}$-morphisms.

4.1. The Category of Finite Distributive Lattices.

Definition 1. *The category of distributive lattices with* $\{\bot,\top\}$*-morphisms will be denoted DL.*

Note that $\mathbf{1}$ is the unique lattice with $\bot=\top$.

Definition 2. *POSET is the category (see* [23]*) whose objects are posets,* $\mathcal{P}=(V,\leq)$*, and whose morphisms are monotone (order-preserving) maps* $\varphi:\mathcal{P}\to\mathcal{Q}$*. I.e.* φ *is a function from* $V_{\mathcal{P}}$ *to* $V_{\mathcal{Q}}$ *such that* $x\leq_{\mathcal{P}}y\Rightarrow\varphi(x)\leq_{\mathcal{Q}}\varphi(y)$*.*

By the Connecting Lemma of [14], DL is (isomorphic to) a subcategory of POSET but it is not full: Every lattice-morphism is order-preserving, but not every order-preserving function between lattices preserves the lattice operations (See Sections 2.16 to 2.19 of [14]).

The restriction of POSET to finite posets is denoted POSET_F. Then an extension of Birkhoff's theorem (Theorem 5.19 in [14]) states that the category DL_F is isomorphic to $\mathrm{POSET}^*{}_F$, the dual of POSET_F. Davey & Priestley [14] point out that the Birkhoff representation acts a lot like the logarithm function of arithmetic, replacing large, apparently complex structures in DL_F by smaller ones in $\mathrm{POSET}^*{}_F$

but maintaining their essential relationship. DL is a subcategory of POSET. The two categories appear to differ considerably. However, Birkhoff's Theorem tells us that, when restricted to finite posets and finite distributive lattices, they are anti-isomorphic: $\mathcal{I}$: $\mathrm{POSET}_F \to \mathrm{DL}_F$ and $\mathcal{J} : \mathrm{DL}_F \to \mathrm{POSET}_F$ are contravariant functors, in fact $\mathcal{I}(\mathcal{P}) \simeq Hom_{POSET_F}(\mathcal{P}, \mathbf{2})$ by $I = \varphi^{-1}(0)$, is a representable functor, and $\mathcal{J}$ is forgetful (of the join-reducible elements and the lattice operations). $\mathcal{J}$ is essentially the inverse of $\mathcal{I}$, *i.e.* $\mathcal{J} \circ \mathcal{I} = I_{POSET_F}$, the identity functor on POSET_F and $\mathcal{I} \circ \mathcal{J}$ is naturally isomorphic to I_{DL_F}, the identity functor on the category of finite distributive lattices (See [18], Section II.1.3 for details).

4.2. **The Structure of $\mathbf{DL}_F$.** We wish to determine universal constructions (limits and colimits) on the category of distributive lattices, particularly the finite ones. As observed by Davey & Priestley [14], we need only study the category of posets. In looking for a finite limit or colimit in POSET, a standard strategy is to restrict the limit (or colimit) diagram to SET, in which all finite limits exist, and try to show that the resulting limit in SET is actually a limit (colimit) in POSET. This works for the basic limits, initial object, product and equalizer. It also works for terminal object and coproduct, but not coequalizer where a bit of tweeking is required (See [17], [2] (p. 126), and the Introduction of [3]). Anyway, POSET has all finite limits and colimits.

Example 10. *Let $\mathcal{P} = \mathbf{1}$, $\mathcal{Q} = \mathbf{3}$ and define $\varphi_1, \varphi_2 : \mathcal{P} \to \mathcal{Q}$ by $\varphi_1(0) = 0$ and $\varphi_2(0) = 2$. Then in POSET their coequalizer $\mathcal{C}$ has only one equivalence class ($\{0, 1, 2\}$) whereas in SET it has two ($\{0, 2\}$ & $\{1\}$).*

However, in many cases, particularly those that arise in applications, coequalizers in POSET can be constructed as though they were in SET. Poset morphisms that preserve the covering relation ($x \lessdot y \Rightarrow \varphi(x) \lessdot \varphi(y)$) are equivalent to digraph morphisms. DIGRAPH, the category of directed graphs, is a functor category, DIGRAPH $\simeq$ FUNCT(D, SET) where D is the diagram category in Figure 1 (appended). As a functor category, DIGRAPH inherits all limits from its range, SET (Theorem 1, p. 115 of [23]). If the limit is acyclic, then it is the Hasse diagram for a partial order, the limit in POSET.

4.2.1. *Limits and Colimits in DL_F.* As in POSET, we can try to construct a given limit (or colimit) directly from the corresponding limit (colimit) in SET. If that does not work we can construct it as the image under $\mathcal{I} : \mathrm{POSET}_F \to \mathrm{DL}_F$ of the colimit (limit) of the image under the forgetful functor $\mathcal{J} : \mathrm{DL}_F \to \mathrm{POSET}_F$.

Initial Object: $\mathbf{2} = \{0 < 1\}$. Given any finite lattice $\mathcal{L} \in$ DL, there is a unique lattice-morphism $\varphi : \mathbf{2} \to \mathcal{L}$ defined by $\varphi\{0\} = \perp$ & $\varphi\{1\} = \top$. Note that $\mathbf{2}$ is not the initial object of POSET ($\mathbf{0}$ is) but $\mathbf{1}$ is the terminal object of POSET and $\mathbf{2} = \mathcal{I}(\mathbf{1})$.

Products: Given $\mathcal{L}$, $\mathcal{M} \in$ DL,

$$\mathcal{L} \times \mathcal{M} = \left(L_{\mathcal{L}} \times L_{\mathcal{M}}; \wedge_{\mathcal{L}\times\mathcal{M}}, \vee_{\mathcal{L}\times\mathcal{M}}; \perp_{\mathcal{L}\times\mathcal{M}}, \top_{\mathcal{L}\times\mathcal{M}}\right),$$

with lattice operations defined coordinatewise. *I.e.*

$$\begin{aligned}(x_1, y_1) \wedge_{\mathcal{L}\times\mathcal{M}} (x_2, y_2) &= (x_1 \wedge_{\mathcal{L}} y_1, x_2 \wedge_{\mathcal{M}} y_2)\,,\\ (x_1, y_1) \vee_{\mathcal{L}\times\mathcal{M}} (x_2, y_2) &= (x_1 \vee_{\mathcal{L}} y_1, x_2 \vee_{\mathcal{M}} y_2)\,,\\ \bot_{\mathcal{L}\times\mathcal{M}} &= (\bot_{\mathcal{L}}, \bot_{\mathcal{M}}) \text{ and}\\ \top_{\mathcal{L}\times\mathcal{M}} &= (\top_{\mathcal{L}}, \top_{\mathcal{M}})\,.\end{aligned}$$

The unique projection maps, $\pi_1 : \mathcal{L} \times \mathcal{M} \to \mathcal{L}$ and $\pi_2 : \mathcal{L} \times \mathcal{M} \to \mathcal{M}$ are then DL-morphisms.

Equalizers: Given parallel DL-morphisms $\varphi_1 : \mathcal{L} \to \mathcal{M}$ and $\varphi_2 : \mathcal{L} \to \mathcal{M}$, $\mathcal{E} = \{x \in \mathcal{P} : \varphi_1(x) = \varphi_2(x)\}$ with the embedding map, $\varphi : \mathcal{E} \to \mathcal{L}$ is their equalizer. It is trivially a DL-morphism.

Once again, by Mac Lane, DL_F has all finite limits. Note that products and equalizers are inherited from SET, but the initial object is not.

Terminal Object: The singleton lattice, $\mathbf{1} = \{0\}$. Given any $\mathcal{L} \in \mathrm{DL}$, there is a unique DL-morphism $\mathbf{1} : \mathcal{L} \to \mathbf{1}$ defined by $\mathbf{1}(x) = 0$. Note that $\mathbf{1}$ is the terminal object in POSET, but also $\mathbf{1} = \mathcal{I}(\mathbf{0})$ and $\mathbf{0}$ is the initial object in POSET.

Coproducts: Given $\mathcal{L}$, $\mathcal{M} \in \mathrm{DL}$, their coproduct, in SET (and POSET) is the disjoint union of $\mathcal{L}$, $\mathcal{M}$. But the disjoint union is not closed under meets and joins and there is no obvious way to tweek it. However, if $\mathcal{L}$, $\mathcal{M}$ are finite and distributive, Birkhoff's theorem gives $\mathcal{I}(\mathcal{J}(\mathcal{L}) \times \mathcal{J}(\mathcal{M}))$ as their coproduct in DL_F . We denote it by $\mathcal{L} + \mathcal{M}$.

Coequalizers: Given parallel DL-morphisms $\varphi_1, \varphi_2 : \mathcal{L} \to \mathcal{M}$, we have the same difficulty extending the coequalizer from SET that we had in POSET. However, we have the equalizer,

$$\mathcal{E} = \{x \in \mathcal{J}(\mathcal{M}) : \mathcal{J}(\varphi_1)(x) = \mathcal{J}(\varphi_2)(x)\}$$

in POSET and Birkhoff's theorem guarantees that $\mathcal{I}(\mathcal{E})$ will be their coequalizer in DL_F.

So again, by Mac Lane, DL_F has all finite colimits, but this time only the terminal object is inherited from SET.

5. Natural Distributive Lattices

We have shown that for a Steiner operation, $\varphi : \mathcal{I}(\mathcal{P}) \to \mathcal{I}(\mathcal{P})$, $Range(\varphi)$ is a distributive lattice, $\mathcal{I}(\mathcal{Q})$ for some poset, $\mathcal{Q}$, that extends $\mathcal{P}$. We have argued that one should only consider consistent StOps since they can be systematically combined to get an even better StOp. Thus, given an isoperimetric problem on a set, V, the possible ranges for StOps under consideration will be sublattices of 2^V whose representing posets are suborders of a fixed total order, τ, of V. For purposes of study, we may take $V = n$ and the total order to be the natural one, $\boldsymbol{n} = \{0 < 1 < ... < n-1\}$. This means that all the sublattices of 2^n will contain $\mathcal{I}(\boldsymbol{n}) = \boldsymbol{n+1}$. Let us call these the *natural distributive lattices (of order* n*)*. Ordered by $\subseteq$ they form a poset, $\mathcal{NDL}(n)$. It is not hard to see that $\mathcal{NDL}(n)$ is a lattice with $\mathcal{L} \wedge \mathcal{M} = \mathcal{L} \cap \mathcal{M}$ and $\mathcal{L} \vee \mathcal{M} = \overline{\mathcal{L} \cup \mathcal{M}}$, the closure of $\mathcal{L} \cup \mathcal{M}$ under unions and intersections. Its minimum element $\bot = \boldsymbol{n+1}$ and its maximum element $\top = 2^n$. What else can we say about the structure of $\mathcal{NDL}(n)$?:

Question 1: Does it satisfy the Jordan-Dedekind chain condition?

Question 2: If so, is it distributive or modular?

These look like challenging questions. Fortunately, following the recipe of Davey & Priestley in Section 4 of [14], we have an easy way to answer them: A *natural partial order (of order n)* is any suborder of $\boldsymbol{n}$. Let $\mathcal{NPO}(n)$ be the set of all natural partial orders (suborders of n) ordered by $\subseteq$. Birkhoff's Theorem tells us that $\mathcal{NDL}(n)$ is isomorphic to $\mathcal{NPO}^*(n)$, the dual of $\mathcal{NPO}(n)$. $\mathcal{NPO}(n)$ has already been investigated by S. P. Avann [1] and R. A. Dean & G. Keller [15]. We can translate their findings into theorems about $\mathcal{NDL}(n)$:

Answer 1: $\mathcal{NPO}(n)$ satisfies the Jordan-Dedekind chain condition. The rank of $\mathcal{P}$ is $r_{\mathcal{NPO}(n)}(\mathcal{P}) = |<_{\mathcal{P}}|$, so $0 \leq r_{\mathcal{NPO}(n)}(\mathcal{P}) \leq \binom{n}{2}$ [1]. Therefore $\mathcal{NDL}(n)$ satisfies the Jordan-Dedekind chain condition and its rank function is $r_{\mathcal{NDL}(n)}(\mathcal{I}(\mathcal{P})) = r^*_{\mathcal{NPO}(n)}(\mathcal{P}) = \binom{n}{2} - r_{\mathcal{NPO}(n)}(\mathcal{P})$.

Answer 2: $\mathcal{NPO}(n)$ is not distributive or even modular. However, it is lower semimodular, *i.e.* $\forall \mathcal{P}, \mathcal{Q} \in \mathcal{NPO}(n)$,

$$r_{\mathcal{NPO}(n)}(\mathcal{P}) + r_{\mathcal{NPO}(n)}(\mathcal{Q}) \leq r_{\mathcal{NPO}(n)}(\mathcal{P} \wedge \mathcal{Q}) + r_{\mathcal{NPO}(n)}(\mathcal{P} \vee \mathcal{Q}) \text{ [1].}$$

Therefore $\mathcal{NDL}(n)$ is upper semimodular, *i.e.* the inequality above is reversed for $r_{\mathcal{NDL}(n)}$.

There are many more fascinating facts about natural partial orders (and thus natural distributive lattices) in [1] and [15]. Also, the sequence $|\mathcal{NPO}(n)| = |\mathcal{NDL}(n)|$ is A006455 in The On-Line Encyclopedia of Integer Sequences (OEIS [24]). The second column of the following table contains all known values:

n	$\lvert\mathcal{NPO}(n)\rvert$	$BPS(n)$	$BPS(n) / \lvert\mathcal{NPO}(n)\rvert$
0	1	0.0	0.0
1	1	15.179	15.179
2	2	51.055	25.528
3	7	182.14	26.02
4	40	816.87	20.422
5	357	4857.1	13.605
6	4824	39210.	8.1281
7	96428	4.3520×10^5	4.5132
8	2800472	6.6918×10^6	2.3895
9	116473461	1.4324×10^8	1.2298
10	6855780268	4.2828×10^9	0.62470
11	565505147444	1.7928×10^{11}	0.31703
12	64824245807684	1.0525×10^{13}	0.16236

.

Brightwell, Prömel & Steger [9] give a beautifully simple formula,

$$BPS(n) = C_n n 2^{\frac{n^2}{4}} \text{ with}$$

$$C_n = \begin{cases} 12.7636300... & \text{if } n \text{ is even} \\ \\ 12.7635965... & \text{if } n \text{ is odd} \end{cases},$$

and show that $BPS(n) / |\mathcal{NPO}(n)| \to 1$ as $n \to \infty$. Since C_n is the same in the first 5 decimal places whether n is even or odd, the difference does not effect the values given in the table above. It is strange then, to see how poor the approximation is for the known values (ratios are given in the fourth column). This is not unprecedented however. It takes awhile for some of these asymptotic sequences to settle down.

5.1. **What Finite Orders are StOp-Orders?** We have shown that for any (finite) Steiner operation φ, *Range*(φ) is closed under $\cup$ & $\cap$ and is therefore a distributive lattice. What other structure might *Range*(φ) have? All the StOp-orders in [21] satisfy the Jordan-Dedekind chain condition. Could that be a theorem? The following results answers this question in the negative.

5.1.1. *MWI Problems.* An interesting class of combinatorial isoperimetric problems is those for which the boundary functional, $\omega : \mathcal{I}(\mathcal{P}) \to \mathbb{R}$, is additive:

$$\omega(S) = \sum_{v \in S} \omega(v).$$

Such an additive function is called a *weight* and finding

$$\min_{\substack{I \in \mathcal{I}(\mathcal{P}) \\ |I|=k}} \omega(I)$$

is called the *minimum weight ideal* (MWI) problem. Note that $\omega(v)$ may be negative as well as positive, so

$$\min_{\substack{I \in \mathcal{I}(\mathcal{P}) \\ |I|=k}} \omega(I) = -\max_{\substack{I \in \mathcal{I}(\mathcal{P}) \\ |I|=k}} (-\omega(I))$$

and minimizing or maximimizing are equivalent problems. If $\omega(v) < 0$ for some $v \in V$ and $\min_{v \in V} \omega(v) = -C$ then $\omega^+(v) = \omega(v) + C \geq 0$, and

$$\min_{\substack{I \in \mathcal{I}(\mathcal{P}) \\ |I|=k}} \omega^+(I) = \min_{\substack{I \in \mathcal{I}(\mathcal{P}) \\ |I|=k}} \omega(I) + kC$$

so restricting ω to be positive makes no essential difference. The MWI problem is trivial if $\mathcal{P} = \Delta_V$, the discrete order on V ($\mathcal{I}(\Delta) = 2^V$): If we number the elements of V, $\tau : V \to \{1, 2, ..., n\}$, one-to-one and onto, in increasing order of their weight, $\tau(u) < \tau(v) \Rightarrow \omega(u) \leq \omega(v)$, then $S_m = \left\{\tau^{-1}(1), \tau^{-1}(2), ..., \tau^{-1}(m)\right\}$ will be a solution of the MWI problem. However, the general problem is NP-complete and many challenging edge-isoperimetric and vertex-isoperimetric problems reduce to MWI problems (see Section 6.2 of [21]).

5.1.2. *Weight-Reductions.* As a special kind of the combinatorial isoperimetric problem, the defining properties of Steiner operations apply to MWI problems. Stabilization and compression are Steiner operations for MWI, but there is another systematic family of Steiner operations that does not appear to apply to the EIP or VIP: Suppose that $\mathcal{Q}$ is an extension of $\mathcal{P} = (V, \leq)$ and that the weight function, ω, for $\mathcal{P}$ is increasing on $\mathcal{Q}$ $((u \leq_{\mathcal{Q}} v) \Rightarrow (\omega(u) \leq_{\mathcal{Q}} \omega(v)))$. Let $\tau : \mathcal{Q} \to [n]$ be any one-to-one & onto total extension of $\mathcal{Q}$, and a any member of V. Then define $\varphi_{a,\tau} : \mathcal{I}(\mathcal{P}) \to \mathcal{I}(\mathcal{P})$ by

$$\varphi_{a,\tau}(I) = I - v_{\max} + v_{\min},$$

where $v_{\max} = \tau^{-1}(\max\{\tau(v) : a \leq_{\mathcal{Q}} v \in I\})$ and $v_{\min} = \tau^{-1}(\min\{\tau(u) : a >_{\mathcal{Q}} u \notin I\})$ if both sets are nonempty (if not, $\varphi_{a,\tau}(I) = I$). $I - v_{\max} \in \mathcal{I}(\mathcal{P})$ because $v_{\max}$ is maximal wrt $\mathcal{Q}$ and therefore wrt $\mathcal{P}$. By the dual arguement, $(I - v_{\max}) + v_{\min} \in \mathcal{I}(\mathcal{P})$.

Theorem 4. *$\varphi_{a,\tau}$ is a StOp for the MWI problem on $(\mathcal{P}; w)$.*

Proof. We apply the definition of a StOp in Section 1.2:

(1) $|\varphi_{a,\tau}(I)| = |I| - 1 + 1 = |I|$.
(2) $\omega(\varphi_{a,\tau}(I)) = \omega(I) - \omega(v_{\max}) + w(v_{\min}) \leq \omega(I)$ since $v_{\min} <_{\mathcal{Q}} a \leq_{\mathcal{Q}} v_{\max} \Rightarrow \omega(v_{\min}) \leq \omega(v_{\max})$.
(3) If $I \subseteq J$, then $\max\{\tau(v) : a \leq_{\mathcal{Q}} v \in I\} \leq \max\{\tau(v) : a \leq_{\mathcal{Q}} v \in J\}$. If $=$ holds then $v_{\max}(I) = v_{\max}(J)$ and the same element is removed. from I, J. If $<$ holds then an element not in I will be removed from J. Also $\min\{\tau(u) : a >_{\mathcal{Q}} v \notin I\} \leq \min\{\tau(u) : a >_{\mathcal{Q}} u \notin J\}$. If $=$ holds then $v_{\min}(I) = v_{\min}(J)$ and the same element is added to I, J. If $<$ holds then $v_{\min}(I) \in J$ already.
(4) $\tau(\varphi_{a,\tau}(I)) = \tau(I) - \tau(v_{\max}) + \tau(v_{\min}) \leq \tau(I)$, by the definition of $v_{\max}$, $v_{\min}$ and $=$ holds iff $\varphi_{a,\tau}(I) = I$.

□

Since $\varphi_{a,\tau}$ reduces (or at least does not increase) the weight of an ideal, we call it a *"reduction"*.

Theorem 5. *Every finite poset, $\mathcal{Q}$, is a StOp-order.*

Proof. For a fixed τ the $\varphi_{a,\tau}$'s are consistent so the superreduction, $\varphi_{\infty,\tau}$, defined by their cyclic composition will be an idempotent StOp. $\varphi_{\infty,\tau}$ will determine a StOp-order by Theorem 3 and it is easily seen that the StOp-order is $\mathcal{Q}$. □

Example 11. *Many of the StOps that we called "ad hoc" in* [21] *are actually reductions. Their definition seems superficial but the circumstances under which they arise are still mysterious and they were useful in administering the "coup de grace" after stabilization and compression had done the heavy lifting. Anyway, those applications and Theorem 5 show that reductions are not ad hoc but members of a rich systematic family of StOps.*

6. Conclusions & Comments

6.1. **Towards a Theory of StOp-orders.** Theorem 5 shows that in general StOp-orders have no additional structure. However, many StOp-orders that occur in applications do have additional structure. Some are distributive lattices themselves. Can such structure be used to simplify their calculation? The Matsumoto-Verma theory of Bruhat orders (the stabilization-orders derived from Coxeter groups), based on the fact that Coxeter groups are generated by a relatively small subset of its reflections (a basis) and that Bruhat orders have the Jordan-Dedekind chain condition, is a great help in calculating Bruhat orders. Is there an extension of those results to compression-orders or other families of Stop-orders?

6.2. **Do Continuous Steiner Operations Induce StOp-Orders?** In 1966 [19] the author labeled a combinatorial optimization problem as "isoperimetric" because of its similarity with the classical isoperimetric problem in the plane. The hope was that the analogy would guide intuition and that techniques for classical (continuous) isoperimetric problems could be extended to their combinatorial analogs. That hope has been fulfilled with the theory of Steiner operations [21], applications of spectral theory [11] and abstract harmonic analysis [13]. With Theorems 3 & 5 it may now be possible for the combinatorial theory of Steiner operations to repay something of its debt to classical analysis! Is there an analog of the Birkhoff representation for the range of Steiner operations on continuous measure spaces? For Steiner symmetrization the answer is, "yes, but the order is not very interesting": The supersymmetrization of any bounded measureable set in $\mathbb{R}^n$ is a sphere, centered at the origin, of the same volume. Ordered by $\subseteq$, these spheres form a chain, isomorphic to $\mathbb{R}_+$, which constitutes the symmetrization-order.

We began the search for a nontrivial StOp-order with Antonio Ros's survey of classical isoperimetric problems [26]. His paper was based on lectures given at the Clay Mathematical Institute in 2001. In Section 1.6 Ros writes, "The explicit description of the solutions of the isoperimetric problem in flat 3-tori (C_1^3, the 3-fold product of unit circles) is one of the nicest open problems in classical geometry". This is intriguing because it is the L_2 analog of an L_1 problem solved by Bollobas & Leader in 1991 ([5] or see [21] Section 10.1). The Bollobas-Leader problem is the continuous limit of the EIP on $\mathbb{Z}_n^d$ as $n \to \infty$. They solved it in all dimensions d, even though it does not have nested solutions for $d > 1$, with a discontinuous variant of compression that makes clever use of the convexity of the local solutions in dimension $d-1$. Can the same strategy work for Ros's problem?

For C_1 ($d = 1$) the problem is trivial: The solutions are intervals of length v, which may be nested. For $d > 1$ we apply compression wrt this 1-dimensional solution and need only look at ideals in the product order of $[0,1]^d$ (note that there are just d ways to factor C_1^d as a product $C_1 \times C_1^{d-1}$). In addition we may apply stabilization, the Steiner operation based on the reflective symmetries induced by interchanging coordinates. The definition of stabilization for continuous isoperimetric problems is the same as for combinatorial ones (see Section 3.2.4 of [21]) and is closely related to Hsiang symmetrization ([26], Section 1.3). Stabilization may be made consistent with compression and the resulting StOp-order is factorable as $\mathcal{L}(d) \times Stab(Q_d)$, where $\mathcal{L}(d)$is the standard simplex, $\{0 \leq x_1 \leq x_2 \leq ... \leq x_d \leq 1\}$ ordered coordinatewise, and $Stab(Q_d)$ is the stabilization-order of the graph of the d-cube, Q_d (see Chapters 3 & 4 of [21]). The space of solutions is further limited by regularity, which partitions it into components corresponding to the ideals of $Stab(Q_d)$. When $d = 3$, $Stab(Q_3)$ has 10 ideals (Fig. 4.2 of [21]) but the empty & whole are trivial. Also, throwing out those that are dual-complements of smaller ones (and therefore redundant) we have the 5 on Ritoré's list of candidates ([26], Section 1.5) when volume $v \leq 1/2$. The same holds in any dimension but of course it gets more complex (for $d = 4$ (Fig. 4.3 of [21]) there are 14 candidates).

There are further interaction between the local (variational) and global (Steiner operational) conditions for an optimal surface: The regularity of the surface implies it has a normal (directed outward) at every point. In order for the region enclosed to be an ideal in the StOp-order, that normal vector must lie in the positive orthant. Also, where the surface meets the boundaries of the cube, the normal vector must be orthogonal to the normal of the bounding face. Where the surface intersects a hyperplane of symmetry of the d-cube, $[0,1]^d$, the dihedral angle between the tangent plane and plane of symmetry must be acute (nonnegative inner product between their normal vectors). The strongest local condition is that the mean curvature of the surface must be constant.

In combinatorial isoperimetric problems (such as the edge-isoperimetric problem on the graph of the d-cube) the notion of "nested solutions" is fundamental. If one assumes that the problem might have nested solutions, then starting with the empty set and adding in elements one at a time so as to minimize the marginal boundary, with relatively little effort one has candidates for solution sets of each cardinality. If those candidates withstand scrutiny, then one has a powerful tool (compression) for proving them optimal. The problems we are looking at now, however, are interesting just because they do not have nested solutions. But they may have them in a weaker sense. Suppose we start off with the ideal of volume 0 corresponding to one of the ideals of Stab(Q_d), thinking of it as an empty balloon. Pumping air into the balloon will increase the volume so as to minimize the marginal increase in area and should, intuitively, give a nested family of locally optimal ideals. Conjecture 1 (Ritoré) of [27] affirms this intuition in 3 dimensions. All we need to prove Ritoré's conjecture is an efficient analytic representation of all the locally optimal surfaces, but evidently they do not exist for the larger two (Lawson's and Schwarz's surfaces). One might also hope to adapt Bollobas & Leader's discontinuous compression arguement to prove the L_2 analog of their L_1 theorem. However, we have not been able to do that either, and Ros thinks there might be a counterexample in higher dimensions.

One of the notable controversies in mathematical history was Weierstrass's challenge to Jakob Steiner's claim to have given a rigorous proof (the first) of the classical isoperimetric theorem (*circa* 1836, see [4], Section V.11., p. 295). Steiner showed that symmetrization of a planar set (wrt a given line through its centroid) has the same area as the set and that the length of its boundary is less (strictly) unless the set was already symmetric (wrt the given line). Since the only planar set symmetric wrt every line through its centroid is a circle, QED (Steiner claimed). Weierstrass pointed out that Steiner was implicitly assuming that the isoperimetric problem *has* a solution and he still needed to prove it for logical completeness. A proof of existence was finally published by Schwartz in 1884. According to Berger [4], Blaschke's proof of existence, based on a compactness argument, validated Steiner's intuition. We are hopeful it can also prove that consistent Steiner operations generate "pushouts". The basic symmetrizations and their "pushouts" are idempotent. Our demonstration that the range of an idempotent Steiner operation is closed under $\cup$ & $\cap$ does not invoke finiteness, so the ranges of the basic- and super-symmetrizations will be (continuous) distributive lattices. Our hope is that a variant of the Birkhoff-Priestley representation theory for distributive lattices [14] will produce its StOp-order. It seems that there will have to be limitations on the closure of those lattices though, like the countable unions of measure theory.

For a theory of continuous Steiner operations, the role played by Coxeter groups (see [21], Chapter 5) should be taken by Lie groups. If the action of a Lie group on a manifold is

(1) Generated by reflections (order-two actions whose fixed submanifold divides the manifold into two components),

(2) Such that the stabilization it defines does not increase boundary,

then the only subsets that need be considered in solving the isoperimetric problem would be ideals in the stabilization-order (assuming that the Birkhoff & Priestley representation theories can produce a theoretical foundation for such things).

6.3. **In Retrospect.** It might seem that Birkhoff's theorem was created to prove Theorem 3. However, Birkhoff's theorem preceded Theorem 3 by at least 60 years. Also the essential idea behind the proof of Birkhoff's theorem, the encoding of partial order relations into the algebra of lattices (Theorem 2.8 (The Connecting Lemma) & Theorem 2.10 of [14]), goes back another 60 years to Dedekind. The proof of Theorem 3 is so simple (given Birkhoff's theorem and its extensions) yet gives no insight into the interaction between StOps and the elements of the underlying set (the join-irreducibles of $\mathcal{I}(\mathcal{P})$). This lack of conceptual transparency indicates opportunity for futher study.

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Department of Mathematics, University of California, Riverside, Riverside, CA 92521
E-mail address: harper@math.ucr.edu

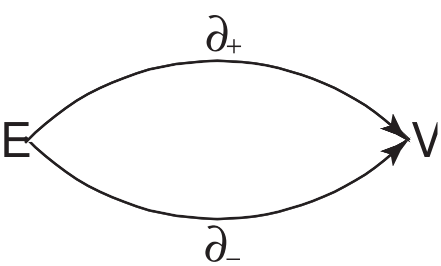

Figure 1-Diagram category