

THE FULL WARD-TAKAHASHI IDENTITY FOR COLORED TENSOR MODELS

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ABSTRACT. Colored tensor models (CTM) is a random geometrical approach to quantum gravity. We scrutinize the structure of the connected correlation functions of general CTM-interactions and organize them by boundaries of Feynman graphs. For rank- D interactions including, but not restricted to, all melonic φ^4 -vertices—to wit, solely those quartic vertices that can lead to dominant spherical contributions in the large- N expansion—the aforementioned boundary graphs are shown to be precisely all (possibly disconnected) vertex-bipartite regularly edge- D -colored graphs. The concept of CTM-compatible boundary-graph automorphism is introduced and an auxiliary graph calculus is developed. With the aid of these constructs, certain $U(\infty)$ -invariance of the path integral measure is fully exploited in order to derive a strong Ward-Takahashi Identity for CTMs with a symmetry-breaking kinetic term. For the rank-3 φ^4 -theory, we get the exact integral-like equation for the 2-point function. Similarly, exact equations for higher multipoint functions can be readily obtained departing from this full Ward-Takahashi identity. Our results hold for some Group Field Theories as well. Altogether, our non-perturbative approach trades some graph theoretical methods for analytical ones. We believe that these tools can be extended to tensorial SYK-models.

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1. INTRODUCTION

The term *colored random tensor models* is a collective for random geometries obtained from quantum field theories for tensor fields. Aiming at a theory of quantum gravity in dimension $D \geq 2$, these models are machineries of weighted triangulations of piecewise linear manifolds, the weights being defined by certain path integrals. In that probabilistic ambit, which we shall leave soon, what we obtain here is, crudely, recursions for the connected correlation function $G^{(2k+2)}$ in terms of $G^{(2k)}$ —the *Ward-Takahashi Identities* (WTI)—which are a consequence of $U(\infty)$ -symmetries in the measure of their generating functional, that is to say the free energy $\log Z[J, \bar{J}]$ (see below). Here we do not specialize in the construction of those measures, nor use the probability terminology, but we adhere to the physical one (e.g. we tend to use *propagator* instead of *correlation*, etc.; this does not imply that their probabilistic meaning could not be tracked back, though). Accordingly, we drop qualificative “random” and stick to *colored tensor models* (CTM). These correlation functions reflect, as we shall prove, some of the structure of the tensor fields. The tensors have forbidden symmetries, which has been deemed *color*. In the arbitrary-dimensional setting the coloring is needed in order for the Feynman expansion to restrict to exactly those graphs one can associate a sensible Ψ -complex to [25, Lemma 1]¹. As a byproduct of this coloring, these theories might have several, say $a_k(D)$, independent correlation functions of the same number $2k$ of points: $G_1^{(2k)}, \dots, G_{a_k(D)}^{(2k)}$.

This is not a feature exclusively of the complex tensor models that we analyze, but it will also be present in the (real) tensorial SYK-models (after Sachdev-Ye-Kitaev [29, 42]) that have been studied lately [6, 47] if one considers them *not* as a $0 + 1$ field theory (as in [18]), but allows spacial degrees of freedom, e.g. as in [4]. In this sense, the present article could be useful if one wants to solve the (melonic sector of) that theories.

The initial idea in the primitive versions of random tensor models was to reproduce, in higher dimensions, the success of random matrices in modelling 2D-quantum gravity [1, 11]. The consummation of this generalization had to wait long, however, until the analogue of the large- N expansion, which, as in matrix models, is bedrock of most physical applications, was found [26]. For these higher dimensional analogues of random matrices, what empowered the $1/N$ -expansion is an integer called *Gur au’s degree*, which for rank-2 tensor models (complex matrix models), coincides with the genus (see Def. 4). Crucially, for dimensions greater than two, the degree is not a topological invariant; in particular this integer has complementary information to homology and is able to tell apart triangulations of homeomorphic spaces. Being tensor models a theory of random geometry, the fact that their large- N expansion relies on a non-topological quantity is a rather wished feature, by which the theory of random tensors gains reliability as a properly geometric quantum gravity framework for dimensions $D \geq 2$.

The *Tensor Track* [39–41] encompasses several classes of tensor models as study objects and synthesizes these random-geometry-foundations in a gravity-quantization program that has as watermark to leave the core of quantum field theory intact—whenever possible. Rooting itself in Wilson’s approach to renormalization and functional integrals, the novelty in the tensor track is trading the locality of interactions for invariance under certain large unitary groups (Sec. 2). The origins of the Tensor Track are also amends to the renormalization of Group Field Theory (GFT). In [39], Rivasseau stated Osterwalder-Schrader-like rules that

¹ Pseudosimplicial or Ψ -complexes allow simplices to have more than a common face. Moreover, ostensibly, the coloring of GFTs is not absolutely necessary [44], but we stick in this paper to colors, as they more easily permit a systematic identification of graphs as spaces. Later on, we discuss models which drop coloring or part of.

tensor models should satisfy. One of the principal frameworks in the Tensor Track is precisely that of CTM. We provide in the next paragraph an encapsulated description of alike settings sometimes evoked by the name “tensor model” and studied also in the Tensor Track.

Belonging to the clade of Group Field Theories [16], colored tensor models were propelled by Gurău. Roughly, GFTs are scalar field theories on D -fold products of compact Lie groups (see e.g. [33] for their relation to Loop Quantum Gravity, and [38] for the origin of the group manifold in the context of spin foams). Their Feynman diagrams encode simplicial complexes: fields (interpreted as $D - 1$ simplices) are paired by propagators (see Sec. 6.3). Monomials in the interaction part of the action S_{int} , typically of degree $D + 1$ in the fields, are understood as D -simplices, so Feynman graphs are gluings of these. It was in that framework where the idea of coloring, which facilitated the large- N expansion, emerged [25]. Ever since, the auspicious tensor model family has dramatically grown: GFTs with other unitary groups like $SU(2)$ [7] and, recently, with orthogonal groups [8]. Another framework, not addressed here, but which our results might be extended to, are *multi-orientable tensor models* [45], which have some symmetry of the $U(N) \times O(N) \times U(N)$ -hybrid type (assuming rank 3). They retain still some of the graphs forbidden by coloring and are still treatable with the large- N expansion [46]. Tensor Group Field Theories is another GFT-related setting to which actually some of our results are extended (Section 6.3 for the $U(1)^D$ -group). Concerning renormalization of TGFTs a good deal of results pertaining the classification of these models, has been undertaken specially by Ben Geloun and Rivasseau [3] in $D = 4$, and Ousmane Samary, Vignes-Tourneret in $D = 3$. The former model (BGR), a TGFT on $U(1)^4$, is one of the prominent 4-dimensional models which, moreover, as its authors themselves proved, is a renormalizable field theory to all orders in perturbation theory. Among all its relatives, CTM render the best-behaved spaces. We choose to temporarily constrain to this framework because geometric notions become more transparent there. We also ought to show the surjectivity of certain map having as domain the Feynman graphs of a fixed tensor model action. That set is meager in the CTM-framework, where the result becomes then stronger.

Relying on it, the main result of the present work, the *full Ward-Takahashi Identity* (Theorem 2), is non-perturbative QFT for tensor models, in essence. Historically, the WTI appeared in matrix models in order to show the vanishing of the β -function of the Grosse-Wulkenhaar model—which had been already accomplished by other methods at one [19] and three loops [13]—to all orders in perturbation theory. The ultimate proof [12], by Disertori, Gurău, Magnen and Rivasseau, still perturbatively, was based on a Ward Identity (WI) also derived by them there [12, Sec. 3]. Later on, Grosse and Wulkenhaar [21] retook the WI for their self-dual φ_4^{*4} -model (see (36) with $\Omega = 1$) to give a non-perturbative proof that *any* quartic matrix model has a vanishing β -function. We adapt the non-perturbative matrix model approach of [21] to colored tensor models. The full WTI (Theorem 2) is proven for an arbitrary rank and for absolutely general CTM-interactions. It holds for $U(1)$ -Group Field Theories (GFTs) as well, by Fourier-transforming them.

The strategy. We closely follow the treatment given in [21, Sec. 2] to the Grosse-Wulkenhaar ($\Omega = 1$)-model, a φ^{*4} -theory in Moyal (\mathbb{R}^4, \star) which becomes, in the Moyal matrix basis, a matrix model [20, Sec. 2]. In [21] the Ward identities are used to decouple the tower of Schwinger-Dyson equations (SDE), which results in an integro-differential equation for the two-point function, in terms of which, via algebraic recursions, the theory can be solved, i.e. all $2k$ -point-functions (which are the non-vanishing ones) are thus determined. Simply stated, the strategy can be split in two tasks. First, to expand the free energy $W[J, \bar{J}] = \log Z[J, \bar{J}] \sim \sum_{\mathbf{p}} \sum_{\partial\mathcal{F}} (1/\sigma(\partial\mathcal{F})) G_{\partial\mathcal{F}}(\mathbf{p}) \cdot \partial\mathcal{F}(J, \bar{J})(\mathbf{p})$ in boundaries $\partial\mathcal{F}$ of Feynman graphs

\mathcal{F} of a specific model, with source-variables J and \bar{J} and \mathbf{p} being momenta and $\sigma(\partial\mathcal{F})$ a symmetry factor. For matrix models this approach has an astonishing result and needs, in our setting, mainly three steps:

- Finding the right symmetry factors $\sigma(\partial\mathcal{F})$, which in turn requires the CTM-compatible concept of *automorphism of colored graphs*. This new concept, contrary to the existent in the literature of graph encoded manifolds, precisely exhibits compatibility with the CTM-structure (see Sec. 3.1). Automorphism groups are also computed.
- *Non-triviality*. Since $W[J, \bar{J}] = \log Z[J, \bar{J}]$ cancels out the disconnected Feynman graphs, one has to construct *connected* Feynman graphs with possibly disconnected, arbitrary boundary graph \mathcal{B} . This would ensure that each introduced correlation function $G_{\mathcal{B}}^{(n)}$ describes indeed a process in the model under study. We develop first, in Section 3, an operation introduced in [35] for rank 2 and interpreted there as the connected sum, and take it further to arbitrary rank D . This operation sends two Feynman graphs of a fixed model to a Feynman graph of the same model (Prop. 1). Furthermore, the divergence degree that controls the large- N expansion behaves additively with respect to it (Prop. 1).
- *Completeness*. The exact set of boundary graphs is expected to be model-dependent. We determine it for *quartic* (for $D \geq 4$ *quartic melonic*) interactions and show that it is the *whole* set of D -colored graphs (see Section 4).

The second task is to actually derive the WTI from these constructs. In order to be able to read off from W any correlation function, a graph calculus is developed in Section 5.3.

The results. For tensor models, a version of the WTI was obtained in [43], with emphasis on ranks 3 and 4. Here we go a different, considerably longer way that has the following advantages:

- it is a *non-perturbative* treatment. This approach shows a way out of treating single Feynman graphs in tensor models and proposes analytic methods instead. We prove that the correlation functions are indexed by boundary graphs, though, so graph theory cannot be fully circumvented.
- it exhibits the intricate, so far unknown structure of the Green’s functions. That the structure of the boundary sector of *single* models had not been studied underlies this shortcoming. Green’s functions are indexed by all boundary graphs; for quartic interactions, namely by all D -colored graphs. Using [2] (see eqs. (25) and (26) below) there are then in rank-3, four 4-point functions, eight 6-point functions; for $D = 4$, eight 4-point, forty nine 6-point functions and so on.
- it is the *full* WTI. Roughly speaking, the Ward-Takahashi identities contain a skew-symmetric tensor E_{mn} times a double derivative on the partition function. This double derivative splits in a part proportional to δ_{mn} , which is annihilated by E_{mn} , and the rest. The existing WTI in [43] does not contain the former term. It was enough for successfully treating a “melonic-approximation” [34] and writing down a closed integro-differential equations for the lower-order correlation functions. Our aim, on the other hand, is the full theory. Accordingly, we compute here *all terms*: non-planar contributions, in the matrix case, and non-melonic terms—their tensor-model counterpart—are all recovered.

After succinctly introducing the general setting of CTMs in next section, we recap in Section 3 the main graph theory of colored tensor models² but adding some new definitions and results

²A much more thorough exposition is given in [35] (keeping a very similar notation).

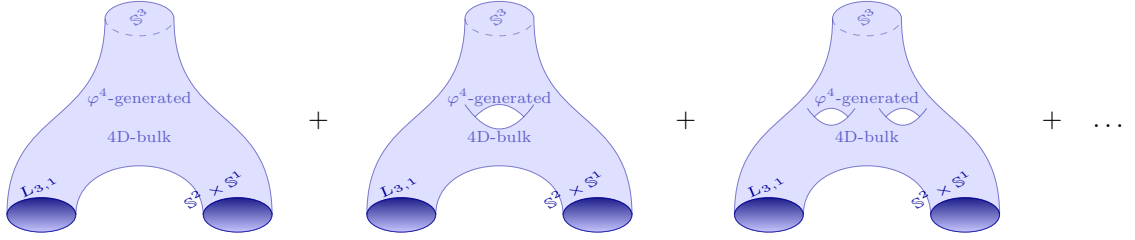


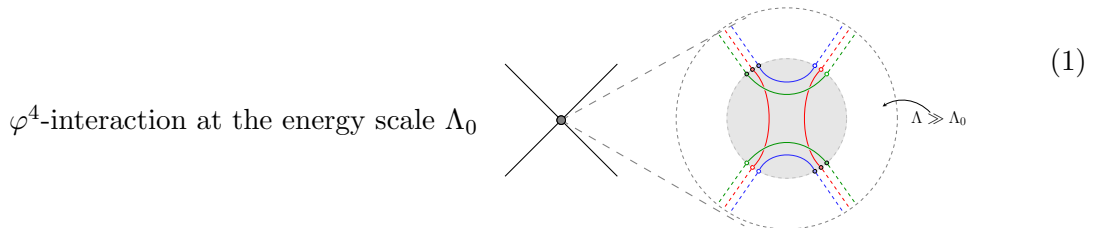
Fig. 1 Geometric picture of the expansion of a concrete Green’s function in Gurău’s degree for a particular correlation function. Gurău’s degree is depicted by a handle (it is not a topological invariant, though)

useful in order to find, in Section 4, the boundary sector of quartic theories. This has a twofold application. On the one hand it is basis for the expansion of the free energy in boundary graphs (Section 5) which we use in Section 6 to obtain the full WTI. On the other hand, it is useful in finding the spectrum of manifolds that a specific CTM is able to generate. We offer some non-sphere examples of prime factors graphs generated by boundaries of quartic CTMs—here a lens space and $\mathbb{S}^2 \times \mathbb{S}^1$. Section 4.3 serves to emphasize this and following aspect about the results of Section 4: If \mathcal{B} is a graph with n vertices representing a manifold M , then the multi-point function $G_{\mathcal{B}}^{(n)}$ is expected to have geometrical information about all compact, oriented 4-manifolds bounded by M . In this bordism picture—here including the vacuum graphs to the picture, for which M is empty—some manifolds cannot be obtained from tensor model Feynman graphs, independently of the particular model, e.g. from the onset, Freedman’s E_8 manifold cannot appear [15]. Notice that in dimension 4, the categories of topological and PL-manifolds (PL_4) are not equivalent, so manifolds with non-trivial Kirby-Siebenmann class [28] cannot be tensor model graphs. Nevertheless³, the PL_4 category is the same as the category of smooth 4-manifolds [9]. Therefore, in dimension 4, tensor models still can in principle access all smooth structures, and which of them are obtained, is model dependent. (It is likely that the model given by the four “pillow-like” invariants in $D = 4$ colors, what we here call the $\varphi_{4,m}^4$ -theory, suffices to generate them all.)

Each Green’s function can be expanded in subsectors determined by common value of Gurău’s degree ω , symbolically represented as in Figure 1 for $M = L_{3,1} \sqcup (\mathbb{S}^1 \times \mathbb{S}^2) \sqcup \mathbb{S}^3$ (see ex. 9). That expansion, as in the matrix case, can lead to closed integro-differential equations for sectors such sectors. In particular, this paper provides techniques to find integro-differential equations that these Green’s functions obey.

2. COLORED TENSORS MODELS

The next setting describes a theory that works in certain high-energy scale Λ . With that resolution, an ordinary scalar vertex shows more structure. For instance, this one:



At the energy scale Λ there is a $U(N_1 \cdots N_D)$ -symmetry that is broken into $U(N_1) \times \cdots \times U(N_D)$ giving rise to more invariants. One postulates tensor fields φ and $\bar{\varphi}$ that transform independently

³I thank the referee for the comments concerning the 4-dimensional case

Objects	Stranded representation	Bipartite representation	Geometric realization
Field $\varphi_{a_1 a_2 \dots a_D}$			$\sigma_{\varphi_{\mathbf{a}}} =$
Field $\bar{\varphi}_{p_1 p_2 \dots p_D}$			$\sigma_{\bar{\varphi}_{\mathbf{p}}} =$
Delta $\delta_{a_k p_k}$			face identification
Propagator-attachment to $\psi \in \{\varphi_{\mathbf{a}}, \bar{\varphi}_{\mathbf{p}}\}$			$\sigma_{\psi} \mapsto C(\sigma_{\psi})$
$\text{Tr}_{\mathcal{V}_3}(\varphi, \bar{\varphi})$ with ext. legs, $D = 3$			a ball D^3
$\text{Tr}_{\mathcal{V}_3}(\varphi, \bar{\varphi})$, $D = 3$			a sphere \mathbb{S}^2

Table 1 A dictionary between two equivalent representations of graphs and their associated geometric realization is shown. A more detailed construction is exposed in Section 4.3. Here C denotes the cone of a simplex

under each unitary group factor. Concretely, being \mathcal{H}_c Hilbert spaces, usually $\ell_2([1, N])$, but also $\ell_2([-N, N])$, our fields are tensors $\varphi, \bar{\varphi} : \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_D \rightarrow \mathbb{C}$ that transform like

$$\begin{aligned} \varphi_{a_1 a_2 \dots a_D} &\mapsto \varphi'_{a_1 a_2 \dots a_D} = \sum_{b_k} W_{a_k b_k}^{(k)} \varphi_{a_1 a_2 \dots b_k \dots a_D}, \\ \bar{\varphi}_{p_1 p_2 \dots p_D} &\mapsto \bar{\varphi}'_{p_1 p_2 \dots p_D} = \sum_{q_k} \bar{W}_{p_k q_k}^{(k)} \bar{\varphi}_{p_1 p_2 \dots q_k \dots p_D}, \end{aligned}$$

for every $W^{(k)} \in \text{U}(N_k)$ and for each one of the so-called *colors* $k = 1, \dots, D$. Here, the rank of the tensors, $D \geq 2$, is the dimension of the random geometry we want to generate. For sake of simplicity, one sets $N_k = N$, for each color k , but one insists in distinguishing each factor of the group $\text{U}(N)^D$. Each such factor acts independently on a single index of both φ and $\bar{\varphi}$, which is referred to as *tensor-coloring*. The energy scale Λ can be seen as (a monotone increasing function of) this large integer N . Symbolically we write the indices of each tensor in \mathbb{Z}^D , but one should think of it as a cutoff-lattice $(\mathbb{Z}_N)^D$.

The classical action functional is build from a selection of connected $\text{U}(N)^{\otimes D}$ -invariants, which are given by traces $\{\text{Tr}_{\mathcal{B}_\alpha}(\varphi, \bar{\varphi})\}_\alpha$ indexed by regularly D -edge colored, vertex-bipartite graphs. We shorten this term simply to D -colored graphs (see Sec. 3 for details). There is, in any rank, only one quadratic invariant, $\text{Tr}_2(\varphi, \bar{\varphi}) = \sum_{\mathbf{a} \in \mathbb{Z}^D} \bar{\varphi}_{\mathbf{a}} \varphi_{\mathbf{a}}$, which is, as always, understood as the kinetic part. Higher order invariants as

$$\begin{aligned} \text{Tr}_{K_c(3,3)}(\varphi, \bar{\varphi}) &= \sum_{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{p}, \mathbf{q}, \mathbf{r}} (\bar{\varphi}_{r_1 r_2 r_3} \bar{\varphi}_{q_1 q_2 q_3} \bar{\varphi}_{p_1 p_2 p_3}) \cdot \\ &\quad (\delta_{a_1 p_1} \delta_{a_2 r_2} \delta_{a_3 q_3} \delta_{b_1 q_1} \delta_{b_2 p_2} \delta_{b_3 r_3} \delta_{c_1 r_1} \delta_{c_2 q_2} \delta_{c_3 p_3}) \cdot (\varphi_{a_1 a_2 a_3} \varphi_{b_1 b_2 b_3} \varphi_{c_1 c_2 c_3}), \end{aligned} \quad (2)$$

are the *interaction vertices*⁴, in this rank-3 example $\text{Tr}_{K_c(3,3)}(\varphi, \bar{\varphi})$ being of sixth degree, and the sum being carried over *momenta* $\mathbf{a}, \dots, \mathbf{r} \in \mathbb{Z}^3$. The D -colored graph \mathcal{B} that indexes a generic interaction vertex $\text{Tr}_{\mathcal{B}}$ is obtained by the prescription in Table 1. Thus, for instance in

⁴ Due to the common occurrence of the word *vertex* both by field theory and graph theory, we cannot opt, unfortunately, for a concise terminology.

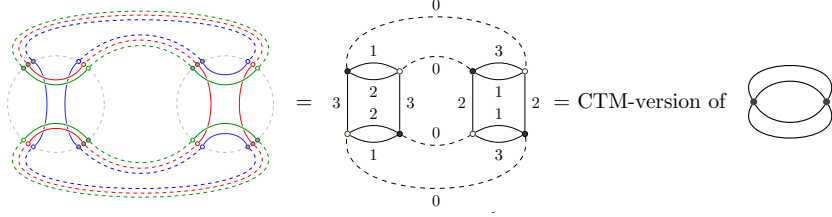
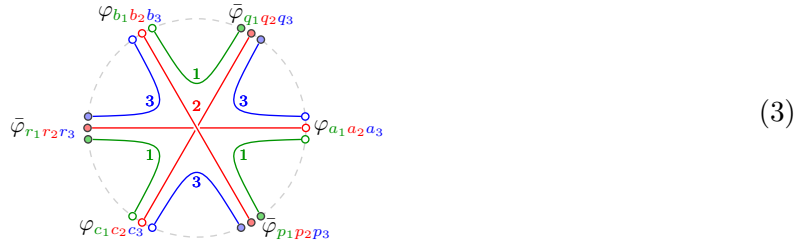


Fig. 2 Change of notations. Forgetting the tensor structure these both are an enriched version of the leftmost Feynman diagram in an ordinary scalar theory

$D = 3$ colors, the colored graph $K_c(3, 3)$ that indexes the interaction vertex (2) is



This somehow obsolete notation is the so-called *stranded representation*. We shall now use an equivalent, simpler notation of these graphs: the *bipartite representation*. This transition is summarized in Table 1 and allows a connection with the graph theoretical representation of piecewise-linear manifolds [14], as we explain later in Section 4.3, which is the main link to the geometry of CTMs. However, the graphs one actually associates a (pseudo)manifold-meaning to arise in the Feynman expansion of

$$Z[J, \bar{J}] = \frac{\int \mathcal{D}[\varphi, \bar{\varphi}] e^{\text{Tr}(\bar{J}\varphi) + \text{Tr}(\bar{\varphi}J) - N^{D-1}S[\varphi, \bar{\varphi}]}{\int \mathcal{D}[\varphi, \bar{\varphi}] e^{-N^{D-1}S[\varphi, \bar{\varphi}]}} , \quad \text{where } \mathcal{D}[\varphi, \bar{\varphi}] := \prod_{\mathbf{a} \in \mathbb{Z}^D} N^{D-1} \frac{d\varphi_{\mathbf{a}} d\bar{\varphi}_{\mathbf{a}}}{2\pi i} e^{-\text{Tr}_2(\varphi, \bar{\varphi})},$$

and have one extra edge between any pair of Wick-contracted fields. Associated to these Wick's contractions is the *0-color*, drawn always dashed (or in the stranded representation, D parallel lines as reads in Table 1) and the graph one remains with turns out to be $(D + 1)$ -colored (open or closed) as explained in the next example.

Example 1. We will study a particular model: *the $(\varphi_{D=3}^4)$ -theory*. Its interaction vertices are

$$\mathcal{V}_1 = \lambda \cdot \begin{array}{c} 2 \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ 2 \end{array} \begin{array}{c} 3 \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ 1 \end{array} , \quad \mathcal{V}_2 = \lambda \cdot \begin{array}{c} 3 \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ 3 \end{array} \begin{array}{c} 1 \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ 1 \end{array} \begin{array}{c} 2 \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ 2 \end{array} , \quad \mathcal{V}_3 = \lambda \cdot \begin{array}{c} 1 \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ 1 \end{array} \begin{array}{c} 2 \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ 2 \end{array} \begin{array}{c} 3 \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ 3 \end{array} . \quad (4)$$

We have chosen directly the bipartite representation but, in order to clarify the switch of notations explained in Table 1, we consider one of the $\mathcal{O}(\lambda^2)$ -vacuum-graph contributions to the integral $\int \mathcal{D}[\varphi, \bar{\varphi}] \exp(-S_0)(\text{Tr}_{\mathcal{V}_3}(\varphi, \bar{\varphi})\text{Tr}_{\mathcal{V}_1}(\varphi, \bar{\varphi}))$, given in Figure 2. It will be seen thereafter that this graph is a (pseudo)simplicial complex that triangulates the sphere \mathbb{S}^3 with eight 3-simplices.

Remark 1. Tensor field theory also has propagators that break the invariance in the action, in this case under the unitary groups. It is therefore sensible to consider a slightly modified trace with a symmetry-breaking term E in the quadratic term: $S[\varphi, \bar{\varphi}] = \text{Tr}_2(\bar{\varphi}, E\varphi) + \sum_{\alpha} \text{Tr}_{\mathcal{B}_{\alpha}}(\varphi, \bar{\varphi})$, with $E : \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ *self-adjoint*, $\text{Tr}_2(\bar{\varphi}, E\varphi) = \text{Tr}_2(\bar{E}\bar{\varphi}, \varphi)$. The first term is distinguished, and represents the kinetic part of the action, where E could be interpreted as the Laplacian.

3. COLORED GRAPH THEORY

In this section we intersperse examples aimed at explaining a series of definitions that concern the CTM-graphs. Each Feynman graph will be taken connected, but boundary graphs of these need not to be so, whence the occurrence of the disconnected graphs in our definitions.

Definition 1. A D -colored graph is a finite graph $\mathcal{G} = (\mathcal{G}^{(0)}, \mathcal{G}^{(1)})$ that is vertex-bipartite and regularly edge- D -colored in the following sense:

- the vertex-set of \mathcal{G} , denoted by $\mathcal{G}^{(0)}$, is composed by *black* $\mathcal{G}_b^{(0)}$ and *white* vertices $\mathcal{G}_w^{(0)}$:
 $\mathcal{G}^{(0)} = \mathcal{G}_w^{(0)} \cup \mathcal{G}_b^{(0)}$,
- any edge $e \in \mathcal{G}^{(1)}$ is attached to precisely one white vertex a and one black one w , which we denote by $t(e) = a, s(e) = w$ or, alternatively, $e = \overline{aw}$ (thus the number of white and black vertices is the same; loops are forbidden),
- the edge set is regularly D -colored, i.e. $\mathcal{G}^{(1)} = \cup_{k=1}^D \mathcal{G}_k^{(1)}$, where $\mathcal{G}_k^{(1)}$ are the *color- k edges*. Moreover at each vertex there are D differently colored incident edges.

We write $\text{Grph}_{c,D}$ for the set of all *connected* D -colored graphs and $\text{IIGrph}_{c,D}$ for the set of (possibly) disconnected graphs with finite number of connected components. Each connected component of the subgraph of \mathcal{G} with edges colored by a subset $I = \{i_1, \dots, i_q\} \subset \{1, \dots, D\}$ of cardinality q is called q -*bubble*. On top of the edge-color set I , one needs a vertex v or edge e of \mathcal{G} that specifies the connected component. The notation for a bubble is therefore $\mathcal{G}_v^I, \mathcal{G}_e^I$ or, if specifying the colors that do not appear is easier, $I = \{1, \dots, D\} \setminus \{c_1, \dots, c_r\}$, say, $\mathcal{G}_v^{\hat{c}_1, \dots, \hat{c}_r}$. We write $\mathcal{G}^{(q)}$ for the set of q -bubbles of \mathcal{G} ; in particular, $\mathcal{G}^{(2)}$ is the set of the *faces* of \mathcal{G} .

Graphs in either set $\text{Grph}_{c,D}$ or $\text{IIGrph}_{c,D}$ are said to be *closed*, in contrast to:

Definition 2. A graph \mathcal{G} is an *open* $(D+1)$ -colored graph if, first, its vertex-set is bipartite in the sense of (i) and (ii) below and if the edge set $\mathcal{G}^{(1)} = \cup_{c=0}^D \mathcal{G}_c^{(1)}$ is quasi-regularly (up to the 0 color) $(D+1)$ -colored in the sense of (a) and (b):

- (i) the vertex-set is bipartite, $\mathcal{G}^{(0)} = \mathcal{G}_w^{(0)} \cup \mathcal{G}_b^{(0)}$, where $\mathcal{G}_w^{(0)}$ are the *white*, and $\mathcal{G}_b^{(0)}$ the *black* vertices, and any edge e is adjacent to precisely one vertex in $\mathcal{G}_b^{(0)}$ and a vertex in $\mathcal{G}_w^{(0)}$. Therefore one has the same number of black and white vertices,
- (ii) any vertex is either *internal* or *external*, $\mathcal{G}^{(0)} = \mathcal{G}_{\text{inn}}^{(0)} \cup \mathcal{G}_{\text{out}}^{(0)}$; moreover, the set $\mathcal{G}_{\text{inn}}^{(0)}$ of internal vertices is regular with valence $D+1$ and external vertices have valence 1,

and, denoting by $\mathcal{G}_c^{(1)}$ the edge-set of color- c :

- (a) for each color c and each inner vertex $v \in \mathcal{G}_{\text{inn}}^{(0)}$, there is exactly one color- c edge, $e \in \mathcal{G}_c^{(1)}$, attached to v ,
- (b) external vertices $v \in \mathcal{G}_{\text{out}}^{(0)}$ are attached only to color-0 edges. We call both a vertex in $\mathcal{G}_{\text{out}}^{(0)}$ and the edge attached to it an *external leg*.

As is common in QFT, we sometimes drop the external vertices and keep only the external (in this case 0-colored) edge. Given an open graph \mathcal{G} one can extract a (in general non-regularly) colored graph $\text{inn}(\mathcal{G})$ defined by $\text{inn}(\mathcal{G})^{(0)} = \mathcal{G}_{\text{inn}}^{(0)}$ and $\text{inn}(\mathcal{G})^{(1)} = \mathcal{G}^{(1)} \setminus \{\text{external legs of } \mathcal{G}\}$. The graph $\text{inn}(\mathcal{G})$ is called *amputated* graph. For any $p \in \mathbb{Z}_{\geq 0}$, we set

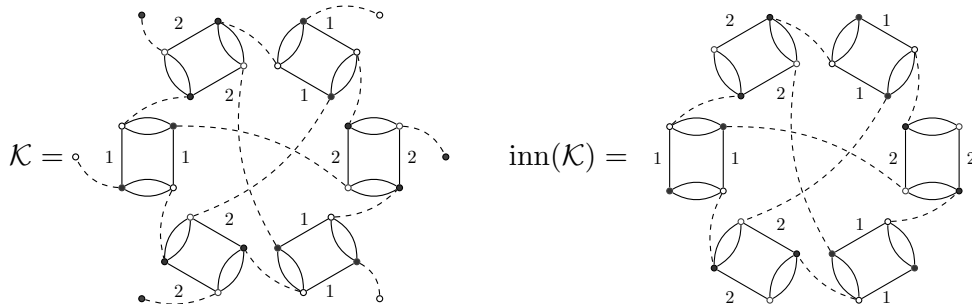
$$\text{Grph}_{c,D+1}^{(2p)} := \{ \mathcal{G} \text{ open } (D+1)\text{-colored} \mid \#(\mathcal{G}_{\text{out}}^{(0)}) = 2p \}. \quad (5)$$

The factor 2 arises from vertex-bipartiteness. Here for $p = 0$, of course $\text{Grph}_{c,D+1}^{(0)} := \text{Grph}_{c,D+1}$.

Example 2. In this example $K_c(3, 3)$ is the colored utility graph, which, tangentially, is the “bipartite” version of the stranded representation of (3). One has:

$$K_c(3, 3) = \begin{array}{c} \text{1} \\ \text{2} \quad \text{2} \\ \text{3} \quad \text{3} \\ \text{1} \quad \text{1} \\ \text{2} \end{array} \in \text{Grph}_{c,3} \quad \text{and} \quad \begin{array}{c} \text{1} \\ \text{2} \quad \text{2} \\ \text{3} \quad \text{3} \\ \text{1} \quad \text{1} \\ \text{2} \end{array} \sqcup \begin{array}{c} \text{1} \\ \text{1} \\ \text{1} \end{array} \in \text{IIGrph}_{c,3}.$$

Example 3. The graph \mathcal{K} below is open and lies in $\text{Grph}_{c,3+1}^{(6)}$. We depict also its amputation, $\text{inn}(\mathcal{K})$:

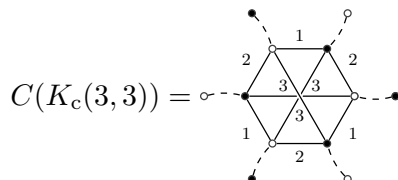


Definition 3. The *boundary graph* $\partial\mathcal{G}$ of a $(D + 1)$ -colored graph $\mathcal{G} \in \text{Grph}_{c,D+1}^{(2k)}$ defined by:

- its vertex set is $(\partial\mathcal{G})^{(0)} = \mathcal{G}_{\text{out}}$, inheriting the bipartiteness of the \mathcal{G}_{out} .
- the edge set $(\partial\mathcal{G})^{(1)}$ is partitioned by colors $k \in \{1, \dots, D\}$. For each color k , one sets $(\partial\mathcal{G})_k^{(1)} := \{(0k)\text{-colored paths in } \mathcal{G}\}$. The incidence relations are given by the following rule: a white vertex $a \in (\partial\mathcal{G})_{\text{w}}^{(0)}$ is connected to a black vertex $x \in (\partial\mathcal{G})_{\text{b}}^{(0)}$ by a k -colored edge $e_k \in \partial\mathcal{G}_k^{(1)}$ if and only if there is a $(0k)$ -bicolored path in \mathcal{G} between the external vertices a and x .

One can easily see that $\partial\mathcal{G} \in \text{IIGrph}_{c,D}$ by identifying $(0i)$ -bicolored edges with i -colored edges, for $i = 1, \dots, D$.

Example 4. The next graph is the *cone* of $K_c(3, 3)$,



The boundary of $C(K_c(3, 3))$ is obviously $K_c(3, 3)$ itself (ex. 2). In our construct, it will be important to be able to generate arbitrary graphs $\mathcal{B} \in \text{IIGrph}_{c,D}$ as boundaries of a certain theory with *fixed* interaction vertices. Then, generating them by *coning* \mathcal{B} —that is, by adding an external color-0 leg to each vertex of \mathcal{B} —is not an option, for one would need to add to the classical action the interaction vertex given by the connected components of $\text{inn}(C\mathcal{B}) \in \text{IIGrph}_{c,D}$ (and thereby additional coupling constants should in principle be measured). The boundary graph $\partial\mathcal{K}$ of \mathcal{K} in example 3 is $K_c(3, 3)$. This is the “right” type of graph for us, e.g. obtained solely from a φ^4 -theory.

Definition 4. Given a graph $\mathcal{G} \in \text{Grph}_{c,D+1}$, each cycle $\sigma \in \mathfrak{S}_{D+1}$ a ribbon graph \mathcal{J}_σ called *jacket*, which is specified by:

$$\mathcal{J}_\sigma^{(0)} = \mathcal{G}^{(0)}, \quad \mathcal{J}_\sigma^{(1)} = \mathcal{G}^{(1)}, \quad \mathcal{J}_\sigma^{(2)} = \{f \in \mathcal{G}^{(2)} : f \text{ has colors } \sigma^q(0) \text{ and } \sigma^{q+1}(0), q \in \mathbb{Z}\}.$$

Here $\sigma^q(0)$ is the q -fold application of σ to 0. Obviously σ and σ^{-1} lead to the same jacket. Moreover each jacket, being a ribbon graph, has a genus [35] and the sum of the genera of the

$D!/2$ jackets of \mathcal{G} is called *Gurău's degree* and denoted by $\omega(\mathcal{G})$. If a graph has a vanishing degree, it is called *melon*. For $D = 2$ then Gurău's degree is the genus of the graph, as the only jacket is the graph itself; melons in rank-2 are planar ribbon graphs. In any degree, melons triangulate spheres [27].

Example 5. The necklace graph \mathcal{N} defined by eq. (13) has two spherical jackets $\mathcal{J}_{(1234)}$ and $\mathcal{J}_{(1423)}$ and a toric jacket $\mathcal{J}_{(1324)}$ (see [35] for the full computation). Jackets are the graph-version of surfaces corresponding to Heegaard splittings [27]. Hence the geometric realization of \mathcal{N} has a genus-0 Heegaard splitting and is therefore a sphere. Also $\mathcal{J}_{(1324)}$ in \mathcal{G} is the ‘‘Clifford torus’’ \mathbb{T}^2 in \mathbb{S}^3 .

One way to determine Gurău's degree [5, App. A, Prop 1] of a graph $\mathcal{G} \in \text{Grph}_{c,D+1}$ is to count its faces $\mathcal{G}^{(2)}$ and to use the formula

$$|\mathcal{G}^{(2)}| = \frac{1}{2} \binom{D}{2} \cdot |\mathcal{G}^{(0)}| + D - \frac{2\omega(\mathcal{G})}{(D-1)!}. \quad (6)$$

The relevance of this integer relies in the analytic control it gives to the theory of random tensor models. Here, the amplitude $\mathcal{A}(\mathcal{G})$ of Feynman graphs \mathcal{G} in CTMs has the following behavior $\mathcal{A}(\mathcal{G}) \sim N^{D - \frac{2\omega(\mathcal{G})}{(D-1)!}}$.

Definition 5. A *colored tensor model* $V(\varphi, \bar{\varphi})_D$ is determined by three items. First, an integer $D \geq 2$, called *dimension* of the model. This integer D is the rank the tensors. Secondly, by an *action*

$$V(\varphi, \bar{\varphi})_D = \sum_{\mathcal{B} \in \Omega} \lambda_{\mathcal{B}} \text{Tr}_{\mathcal{B}}(\varphi, \bar{\varphi}),$$

where $\Omega \subset \text{Grph}_{c,D}$, $|\Omega| < \infty$, and $\lambda_{\mathcal{B}} \in \mathbb{R}$. Finally, by a *kinetic term* $E : \bigotimes_{c=1}^D \mathcal{H}_c \rightarrow \bigotimes_{c=1}^D \mathcal{H}_c$ that is self-adjoint in the sense of $\text{Tr}_2(\overline{E\varphi}, \varphi) = \text{Tr}_2(\bar{\varphi}, E\varphi)$. Usually terms $E \neq 1$ are employed to make connection with GFTs and TGFTs, as the Laplacian boils down to such a term. We will often obviate E and specify the model only by the potential. The set of (*connected*) *Feynman diagrams* of the model $V(\varphi, \bar{\varphi})_D$ is denoted by $\mathfrak{Feyn}_D(V)$ and satisfies

$$\mathfrak{Feyn}_D(V) = \left\{ \mathcal{G} \in \bigcup_{k=0}^{\infty} \text{Grph}_{c,D+1}^{(2k)} \mid \text{inn}(\mathcal{G})^{\hat{0}} \in \Omega \text{ and } (\text{inn}(\mathcal{G}))_0^{(1)} \neq \emptyset \right\}.$$

The graphs in $\text{Grph}_{c,D+1}^{(0)} \cap \mathfrak{Feyn}_D(V)$ are called *vacuum graphs* of the model V . We are interested in honest Feynman graphs, that is, those having internal propagators (in other words, those that are not the cone of an interaction vertex). This explains the mysterious restriction $(\text{inn}(\mathcal{G}))_0^{(1)} \neq \emptyset$.

Definition 6. Let \mathcal{R} and \mathcal{Q} be connected $(D+1)$ -colored graphs, $\mathcal{R} \in \text{Grph}_{c,D+1}^{(2k)}$ and $\mathcal{Q} \in \text{Grph}_{c,D+1}^{(2l)}$. Let k be any color and let e and f be color- k edges in \mathcal{R} and \mathcal{Q} , respectively, i.e. $e \in \mathcal{R}_k^{(1)}$ and $f \in \mathcal{Q}_k^{(1)}$. We define the graph $\mathcal{R}_e \#_f \mathcal{Q}$ as follows:

$$\begin{aligned} (\mathcal{R}_e \#_f \mathcal{Q})^{(0)} &= \mathcal{R}^{(0)} \cup \mathcal{Q}^{(0)}, \\ (\mathcal{R}_e \#_f \mathcal{Q})^{(1)} &= (\mathcal{R}^{(1)} \setminus \{e\}) \cup (\mathcal{Q}^{(1)} \setminus \{f\}) \cup \{E, F\}, \end{aligned}$$

being E and F new k -colored edges defined by $s(E) = s(e)$, $t(E) = t(f)$ and $s(F) = s(f)$, $t(F) = t(e)$ (see Figure 3). Otherwise, the incidence relations and coloring are inherited from those of \mathcal{R} and \mathcal{Q} . This implies that $\mathcal{R}_e \#_f \mathcal{Q}$ is a connected graph in $\text{Grph}_{c,D}^{(2l+2k)}$.

It is obvious that if one chooses only color-0 edges e and f , one can restrict $\#$ to a well-defined binary operation on the set of Feynman graphs,

$$e \#_f : \mathfrak{Feyn}_D(V) \times \mathfrak{Feyn}_D(V) \rightarrow \mathfrak{Feyn}_D(V),$$

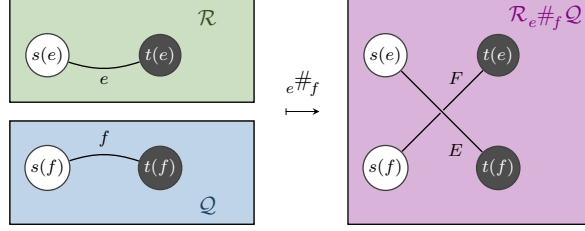


Fig. 3 On the definition of $\#$. Here s and t are source and target, respectively

for arbitrary rank- D colored (complex) tensor model $V(\varphi, \bar{\varphi})$.

This operation $\#$ was defined in [35] for 3-colored graphs that are Feynman diagrams of rank-2 tensor models. It is straightforward to check that $\#$ is associative. The notation is due to the fact that on $\text{Grph}_{c,3} \times \text{Grph}_{c,3}$, $\#$ is the graph-theoretical connected sum. We use it now in higher dimensions, but for $D \geq 3$, we (still) do not interpret $\#$ as connected sum. We have, nevertheless the following result, which for $D = 2$ has been proven in [35, Lemma 3].

Proposition 1. *For arbitrary edges $e \in \mathcal{G}_c^{(1)}$, $f \in \mathcal{K}_c^{(1)}$ of any color c , the operation $e \#_f$ behaves additively with respect to Gurău's degree i.e. $\omega(\mathcal{G}_e \#_f \mathcal{K}) = \omega(\mathcal{G}) + \omega(\mathcal{K})$, for any graph $\mathcal{G}, \mathcal{K} \in \text{Grph}_{c,D+1}$.*

Proof. We use the face-counting formula (6) to calculate Gurău's degree and compute how it changes after $e \#_f$. First, notice that the vertices of \mathcal{G} and \mathcal{K} add up exactly to those of $\mathcal{G}_e \#_f \mathcal{K}$. Concerning faces, in \mathcal{G} there are exactly D two-bubbles containing the edge e , namely the connected component $\mathcal{G}_e^{(cd)}$, where d is any color but c itself. By the same token, there are D faces of \mathcal{K} whose boundary loop contains f . Erasing e and f in favor of the edges E and F in $\mathcal{G}_e \#_f \mathcal{K}$ puts the bubbles $\mathcal{G}_e^{(cd)}$ and $\mathcal{K}_f^{(cd)}$ together in a single one. This happens for each color $d \neq c$, whence $|\mathcal{G}^{(2)}| + |\mathcal{K}^{(2)}| - D = |(\mathcal{G}_e \#_f \mathcal{K})^{(2)}| = \binom{D}{2} |(\mathcal{G}_e \#_f \mathcal{K})^{(0)}|/2 + D - 2\omega(\mathcal{G}_e \#_f \mathcal{K})/(D-1)!$. Then using formula (6) for both \mathcal{G} and \mathcal{K} yields the result. \square

Example 6. We consider two copies of the $(D+1)$ -colored graph with two vertices, \mathcal{M} . It has only planar jackets, whence its Gurău's degree is zero. Therefore, if e_i denotes the only color- i edge of \mathcal{M} , by Proposition 1, one has $\mathcal{P} = \mathcal{M}_{e_1} \#_{e_1} \mathcal{M}_{e_0} \#_{e_0} \mathcal{M}_{e_D} \#_{e_D} \mathcal{M}$ is a melon, for $\omega(\mathcal{P}) = \omega(\mathcal{M}_{e_1} \#_{e_1} \mathcal{M}) + \omega(\mathcal{M}_{e_D} \#_{e_D} \mathcal{M}) = 4\omega(\mathcal{M}) = 0$. This graph will be handy in the sequel (in Eq. (16), specifically) in order to separate boundary components (see Lemma 3). By a similar argument one can see that the vacuum graph in example 1 is a melon. Since melons triangulate spheres [27], our claim there is proven.

3.1. Colored graph automorphisms. The available concept of automorphism in the theory of manifold crystallization [14, Sec. 1] and graph-encoded manifolds of the late 70s and early 80s cannot be used here, for boundary graphs $\partial\mathcal{C}$ have a bipartite-vertex set (which is moreover labeled by the momenta corresponding to the ones carried by open legs of \mathcal{C} ; see Sec. 5.2); here we introduce the concept that discloses the compatibility with the whole CTM-structure.

Definition 7. An *automorphism* Θ of a graph $\mathcal{G} \in \text{Grph}_{c,D}$ is a couple of permutations $\Theta = (\theta, \tilde{\theta})$ of the set of vertices $\theta \in \text{Sym}(\mathcal{G}^{(0)})$ and the set of edges $\tilde{\theta} \in \text{Sym}(\mathcal{G}^{(1)})$ that respects

- *bipartiteness:* $\theta|_{\mathcal{G}_w^{(0)}} \in \text{Sym}(\mathcal{G}_w^{(0)})$ and $\theta|_{\mathcal{G}_b^{(0)}} \in \text{Sym}(\mathcal{G}_b^{(0)})$,
- *edge-coloring:* for any color c and $e_c \in \mathcal{G}_c^{(1)}$, then $\tilde{\theta}(e_c) \in \mathcal{G}_c^{(1)}$,

- *adjacency*: let $s : \mathcal{G}^{(1)} \rightarrow \mathcal{G}_w^{(0)}$ and $t : \mathcal{G}^{(1)} \rightarrow \mathcal{G}_b^{(0)}$ respectively denote the source and target maps. Then the following diagrams are commutative:

$$\begin{array}{ccc} \mathcal{G}^{(1)} & \xrightarrow{\tilde{\theta}} & \mathcal{G}^{(1)} \\ \downarrow s & & \downarrow s \\ \mathcal{G}_w^{(0)} & \xrightarrow{\theta|} & \mathcal{G}_w^{(0)} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{G}^{(1)} & \xrightarrow{\tilde{\theta}} & \mathcal{G}^{(1)} \\ \downarrow t & & \downarrow t \\ \mathcal{G}_b^{(0)} & \xrightarrow{\theta|} & \mathcal{G}_b^{(0)} \end{array}$$

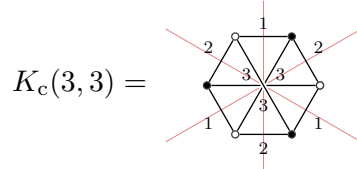
We denote by $\text{Aut}_c(\mathcal{G})$ the group of automorphisms of the colored graph \mathcal{G} . Notice that $\Theta \in \text{Aut}_c(\mathcal{G})$ has no more information than a permutation of white (or black) vertices plus ‘‘preserving the structure of colored graph’’. That is to say, let $r = |\mathcal{G}^{(0)}|/2$ and suppose that $\tau \in \mathfrak{S}_r$ is such that there exists an automorphism $\Theta = (\theta, \tilde{\theta}) \in \text{Aut}_c(\mathcal{G})$ that restricts to τ , $\theta|_{\mathcal{G}_w^{(0)}} = \tau$. We construct the other pieces of Θ , beginning with $\tilde{\theta}$. For an arbitrary color j , let e_j be an edge in $\mathcal{G}_j^{(1)}$. Set then

$$\tilde{\theta}(e_j) := \text{the only } j\text{-colored edge in } s^{-1}(\tau(s(e_j))) .$$

In terms of $\tilde{\theta}$, we define θ for black vertices: let $p \in \mathcal{G}_b^{(0)}$ and let, for arbitrary color j , $f_j \in \mathcal{G}_j^{(1)}$ be the edge with $p = t(f_j)$. Then set $\theta(p) := t(\tilde{\theta}(f_j))$. That is, θ and $\tilde{\theta}$ can be constructed from τ . We conclude that for connected graphs $\mathcal{G} \in \text{Grph}_{c,D}$, if τ can be lifted to a $\Theta \in \text{Aut}_c(\mathcal{G})$, then Θ is unique and (whenever it exists) it will be denoted by $\hat{\tau}$. This way we can see $\text{Aut}_c(\mathcal{G})$ as a subgroup of $\mathfrak{S}_r = \text{Sym}(\mathcal{G}_w^{(0)})$. In particular, the following bound holds:

$$|\text{Aut}_c(\mathcal{G})| \leq (|\mathcal{G}_w^{(0)}|)! = (|\mathcal{G}^{(0)}|/2)! . \quad (7)$$

Example 7. By contrast with the ‘uncolored’ utility graph $K(3, 3)$, for which $|\text{Aut}(K(3, 3))| = 2(3!)^2$, one has for its color version $K_c(3, 3)$ a quite modest $\text{Aut}_c(K_c(3, 3)) \cong \mathbb{Z}_3$. The two non trivial elements of $\text{Aut}_c(K_c(3, 3))$ are rotations by $\pm 2\pi/3$. The rotations by $\pm\pi/3, \pi$ are forbidden by edge-coloring preservation. On the other hand, reflections about the depicted axes do preserve edge-coloring but not the bipartiteness of the edges:



The following lines complete the short list of automorphism groups of connected graphs in ≤ 6 vertices; there $d = 1, 2, 3$ and R_θ means anti-clockwise rotation by θ :

$$\begin{aligned} \text{Aut}_c(\text{circle with arrow}) &= \{*\}, & \text{Aut}_c\left(\begin{array}{c} \text{---}^d \text{---} \\ \text{---}^d \text{---} \end{array}\right) &= \langle R_\pi \rangle \simeq \mathbb{Z}_2, \\ \text{Aut}_c\left(\begin{array}{c} \text{---}^d \text{---} \\ \text{---}^d \text{---} \end{array}\right) &= \langle R_{2\pi/3} \rangle \simeq \mathbb{Z}_3, & \text{Aut}_c\left(\begin{array}{c} \text{---}^d \text{---} \\ \text{---}^d \text{---} \end{array}\right) &= \{*\}. \end{aligned}$$

More extensive tables of automorphism groups of connected colored graphs, as well as their Gurău’s degree, can be found in [36]. If $\mathcal{G} \in \text{IIGrph}_{c,D}$ is the disjoint union of m_i copies of pairwise distinct types of connected graphs $\{\Gamma_i\}_{i=1}^s \subset \text{Grph}_{c,D}$, $\mathcal{G} = (\Gamma_1 \sqcup \dots \sqcup \Gamma_1) \sqcup \dots \sqcup (\Gamma_s \sqcup \dots \sqcup \Gamma_s)$, then

$$\text{Aut}_c(\mathcal{G}) = (\text{Aut}_c(\Gamma_1) \wr \mathfrak{S}_{m_1}) \times (\text{Aut}_c(\Gamma_2) \wr \mathfrak{S}_{m_2}) \times \dots \times (\text{Aut}_c(\Gamma_s) \wr \mathfrak{S}_{m_s}), \quad (8)$$

where \wr is the wreath product of groups. Hence $|\text{Aut}_c(\mathcal{G})| = \prod_{i=1}^s (m_i)! \cdot |\text{Aut}_c(\Gamma_i)|^{m_i}$.

4. COMPLETENESS OF THE BOUNDARY SECTOR FOR QUARTIC INTERACTIONS

Definition 8. The *boundary sector* of a rank- D colored tensor model $V(\varphi, \bar{\varphi})$ is the image of the map $\partial : \mathfrak{Fen}_D(V) \rightarrow \mathbb{H}\text{Grph}_{c,D}$.

In [35] it has been shown, constructively, that the geometric realization of the boundary sector of the φ_3^4 -theory is enough to reconstruct all orientable, closed (possibly disconnected) surfaces. Here we present, first, a stronger result in Section 4.1 for $D = 3$. A similar statement with a similar proof for the rank $D > 3$ case follows in Section 4.2. Both results are needed for the Ward-Takahashi Identity.

4.1. The boundary sector of the φ_3^4 -theory.

Lemma 1. *Every connected 3-colored graph is the boundary of (at least) one Feynman diagram of the φ_3^4 -theory. In other words, the boundary sector contains $\text{Grph}_{c,3}$.*

Proof. Let \mathcal{R} be a connected 3-colored graph. If $\mathcal{R} = \emptyset$ is the empty graph, trivially, one can pick any closed (or vacuum) graph $\tilde{\mathcal{R}}$ of the model. Assume then, that \mathcal{R} is not the empty graph. We construct $\tilde{\mathcal{R}}$ so that $\partial\tilde{\mathcal{R}} = \mathcal{R}$. To each white (resp. black) vertex $d \in \mathcal{R}_w^{(0)}$ (resp. $x \in \mathcal{R}_b^{(0)}$) we associate the following contractions:

$$\tilde{d}(c_1, q_1, c_3, c_2, q_2) = \begin{array}{c} \text{---} \circ \text{---} \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ \text{---} \end{array} \quad , \quad \tilde{x}(b_2, p_2, p_3, b_1, p_1) = \begin{array}{c} \text{---} \circ \text{---} \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \text{---} \end{array}$$

The edges of any of the three colors are associated the following Wick-contractions. For $e_1 \in \mathcal{R}_1^{(1)}$

$$e_1 \mapsto \tilde{d}(c_1, q_1, c_3, c_2, q_2) \tilde{x}(b_2, p_2, p_3, b_1, p_1) = \begin{array}{c} \text{---} \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \end{array}$$

Similarly to given $f_2 \in \mathcal{R}_2^{(1)}$ and $g_3 \in \mathcal{R}_3^{(1)}$, one associates, respectively, the following graphs:

$$f_2 \mapsto \tilde{d}(c_1, q_1, c_3, c_2, q_2) \tilde{x}(b_2, p_2, p_3, b_1, p_1) = \begin{array}{c} \text{---} \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \end{array}$$

$$g_3 \mapsto \tilde{d}(c_1, q_1, c_3, c_2, q_2) \tilde{x}(b_2, p_2, p_3, b_1, p_1) = \begin{array}{c} \text{---} \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \text{---} \end{array}$$

Since each vertex $v \in \mathcal{R}$ is regularly 3-colored, the five Wick contractions added to \tilde{v} saturate all but one irregularly colored vertices in \tilde{v} and make them regularly colored. The only one that remains is a leaf and will be an open leg. Of course, connectedness of two vertices $d, x \in \mathcal{R}^{(0)}$ by an edge of color i (with $i = 1, 2, 3$) is transferred to the connectedness of the (unmarked) external vertices of \tilde{d} and \tilde{x} in $\tilde{\mathcal{R}}$ by a $(0i)$ -colored path in that graph. Thus, by construction, $\partial\tilde{\mathcal{R}} = \mathcal{R}$. \square

Remark 2. In the proof of Lemma 1 the vertex \mathcal{V}_3 has been used. We suspect, there is an optimal construction, which it only uses \mathcal{V}_1 and \mathcal{V}_2 . The optimization of this proof would use the dipole contraction [3, Lemma 4] (in that setting for rank-4 TGFTs) but we defer this proof.

4.2. The boundary sector of the $\varphi_{D,m}^4$ -theory. In two dimensions there is a single (complex) quartic model; in three dimensions, there are three interaction vertices. Both in two and three dimensions quartic vertices are all melonic. The situation changes from 4 dimensions on. For instance, in $D = 4$, the interaction vertex \mathcal{N} given by eq. (13) and \mathfrak{S}_4 permutations thereof are not melonic, for their Gurău's degree is $\omega(\mathcal{N}) = 1$ (see computation [35, Sec 2]). For arbitrary rank, $D \geq 2$, we use the following shortcut: the $\varphi_{D,m}^4$ -theory denotes the model with the following D melonic vertices $\{\text{Tr}_{\mathcal{V}_k}(\varphi, \bar{\varphi})\}_{k=1,\dots,D}$, being

$$\text{Tr}_{\mathcal{V}_k}(\varphi, \bar{\varphi}) = \begin{array}{c} \textcircled{\hat{k}} \\ \vdots \\ \textcircled{k} \quad \textcircled{k} \\ \vdots \\ \textcircled{\hat{k}} \end{array} .$$

Here some of the edges with colors $\hat{k} = \{1, 2, \dots, D\} \setminus \{k\}$ are shortened by dots. We also abbreviate the Feynman diagrams of the $\varphi_{D,m}^4$ -theory as $\mathfrak{Feyn}_D(\varphi_m^4)$. For $D = 3$ the subindex m denoting melonicity is redundant. There, the $\varphi_{3,m}^4$ -theory is the φ_3^4 -theory and $\mathfrak{Feyn}_3(\varphi_m^4) = \mathfrak{Feyn}_3(\varphi^4)$, according to previous remarks.

Theorem 1. *For arbitrary rank D , the boundary sector $\partial\mathfrak{Feyn}_D(\varphi_m^4)$ of the $\varphi_{D,m}^4$ -theory is all of $\text{IIGrph}_{c,D}$.*

We need first two lemmas. The first one is most of the work and concerns the connected case. The second lemma tells how glue $\varphi_{D,m}^4$ -Feynman graphs into a connected $\varphi_{D,m}^4$ -Feynman that has a custom (disconnected) boundary.

Lemma 2. *The boundary sector of the $\varphi_{D,m}^4$ -theory contains $\text{Grph}_{c,D}$.*

The idea is to associate, to each vertex v of \mathcal{B} , a partially Wick-contracted ‘‘raceme’’, \tilde{v} , of interaction-vertices of the $\varphi_{D,m}^4$ -theory. Each raceme has a marked (graph-theoretical) vertex. Among all the associated racemes, one contracts with a 0-color all but the marked vertex, in such a way that one has a 0i-bicolored path in $\tilde{\mathcal{B}}$ between two such preferred vertices at racemes \tilde{x} and \tilde{d} , whenever there is an i -colored edge in \mathcal{B} between x and d .

Proof. Let $\mathcal{B} \in \text{Grph}_{c,D}$. We construct a graph $\tilde{\mathcal{B}} \in \mathfrak{Feyn}_D(\varphi_m^4)$ with $\partial\tilde{\mathcal{B}} = \mathcal{B}$. Concretely, we assemble $\tilde{\mathcal{B}}$ from \mathcal{B} as follows. Only after *Step 2* we will have a well-defined Feynman graph. *Step 1:* Replace any black vertex $x \in \mathcal{B}_b^{(0)}$ and any white vertex $d \in \mathcal{B}_w^{(0)}$ by \tilde{x} and \tilde{d} , respectively:

$$x \mapsto \tilde{x} = \begin{array}{c} \begin{array}{cccc} b_1 & p_D & b_{D-1} & p_{D-1} & b_3 & p_3 & b_2 & p_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \textcircled{D} & \textcircled{D} & \textcircled{D-1} & \textcircled{D-1} & \textcircled{3} & \textcircled{3} & \textcircled{2} & \textcircled{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_1 & & & & & & & r \end{array} \end{array} \quad (9)$$

$$d \mapsto \tilde{d} = \begin{array}{c} \begin{array}{cccc} c_1 & q_1 & c_3 & q_3 & c_{D-1} & q_{D-1} & c_D & q_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \textcircled{1} & \textcircled{1} & \textcircled{3} & \textcircled{3} & \textcircled{D-1} & \textcircled{D-1} & \textcircled{D} & \textcircled{D} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a & & & & & & & c_2 \end{array} \end{array} \quad (10)$$

At this stage, $\tilde{\mathcal{B}}$ consists of the following connected components $\{\tilde{d}\}_{d \in \mathcal{B}_w^{(0)}} \cup \{\tilde{x}\}_{x \in \mathcal{B}_b^{(0)}}$, which, altogether, have the following set of vertices that are *not* contracted with the 0-color:

$$\{b_1^x, p_1^x, \dots, b_{D-1}^x, p_{D-1}^x, p_D^x\}_{x \in \mathcal{B}_b^{(0)}} \cup \{c_1^d, q_1^d, \dots, c_{D-1}^d, q_{D-1}^d, c_D^d\}_{d \in \mathcal{B}_w^{(0)}} . \quad (11)$$

Step 2: We shall contract all open vertices (11) as follows: Whenever $x = t(e_i)$ and $d = s(e_i)$, for e_i an edge of color $i \neq D$, $e_i \in \mathcal{B}_i^{(1)}$, one Wick-contracts b_i^x with q_i^d and p_i^x with c_i^d :

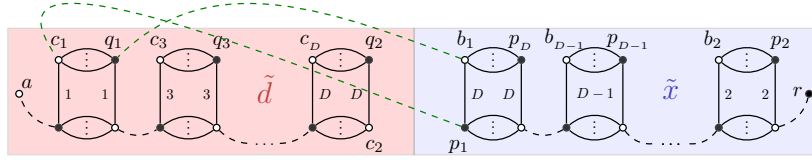
$$\tilde{x}(\dots, b_i^x, p_i^x, \dots, b_{D-1}^x, p_{D-1}^x, p_D^x, r^x) \tilde{d}(\dots, c_i^d, q_i^d, c_{D-1}^d, q_{D-1}^d, c_D^d, a^d) \quad (12)$$

Whenever $x = t(e_D)$ and $d = s(e_D)$ for $e_D \in \mathcal{B}_D^{(1)}$, contract p_D^x with c_D^d :

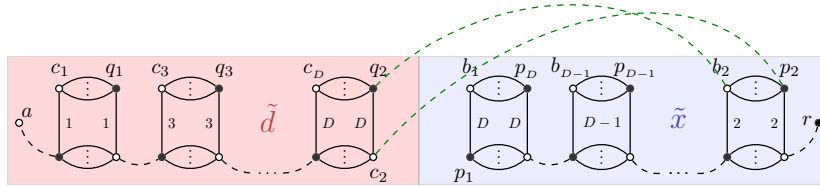
$$\tilde{x}(\dots, b_i^x, p_i^x, \dots, b_{D-1}^x, p_{D-1}^x, p_D^x, r^x) \tilde{d}(\dots, c_i^d, q_i^d, c_{D-1}^d, q_{D-1}^d, c_D^d, a^d)$$

The regularity and the bipartiteness of \mathcal{B} imply the well-definedness of $\tilde{\mathcal{B}}$ as open $(D+1)$ -colored graph. We now see that $\partial\tilde{\mathcal{B}} = \mathcal{B}$. Indeed, for each black vertex x (resp. white vertex d) in \mathcal{B} , there exactly is a black (resp. white) external leg, namely r^x (resp. a^d) which is mapped by ∂ to a black vertex ∂r^x (resp. white vertex ∂a^d). Therefore, \mathcal{B} and $\partial\tilde{\mathcal{B}}$ have the same bipartite vertex set. To conclude, we remark that for every k -colored edge e_k in \mathcal{B} between x and d , there is indeed a $(0k)$ -bicolored path in $\tilde{\mathcal{B}}$ between r^x and a^d , and this ensures that there is a k -colored edge between ∂a^d and ∂r^x , by the mere definition of the boundary graph:

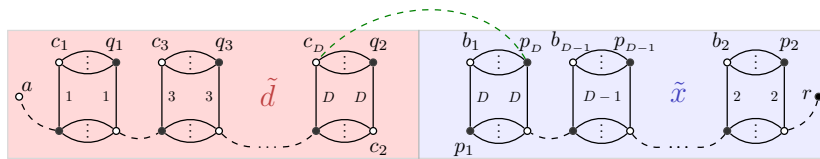
- $k = 1$: From right to left in the following graph, notice that since the vertex \mathcal{V}_1 does not appear in \tilde{x} , there is (in the bottom part) there is a (01) -bicolored path between r and p_1 . That path can be concatenated with $\overline{p_1 c_1 a}$, which is also (01) -bicolored. (Notice that from the two Wick-contractions, only one lies on such a path. The other is secondary.)



- $k = 2$: By a similar token, there is a (02) -colored path between a and c_2 .



- $k = 3, \dots, D-1$: The Wick-contraction (12) connects c_k and p_k with a color-0 edge. It is evident that from graphs (9) and (10), that there is a $0k$ -bicolored path through it that connects r and a .
- $k = D$. There are \mathcal{V}_D -vertices neither to the left of c_D nor to the right of p_D , so there is a $(0D)$ -bicolored path $\overline{a c_D}$, which can be concatenated with $\overline{c_D p_D}$ and subsequently with $\overline{p_D r}$.

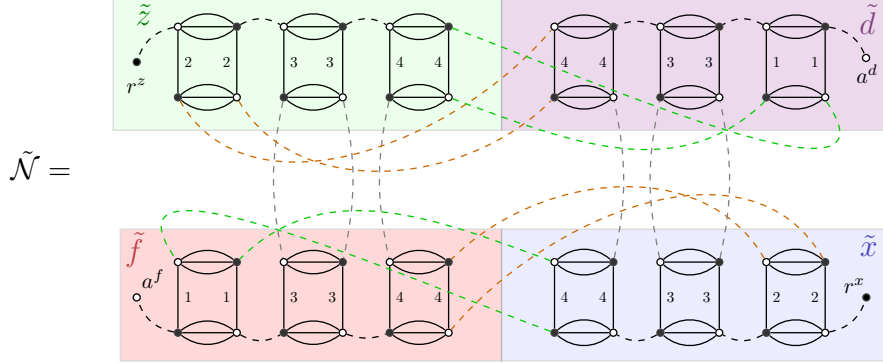


□

Example 8. In rank-4, the construction in Theorem 2 (Step 1) associates to each vertex $\{f, d, x, z\}$ of the necklace-graph

$$\mathcal{N} = \begin{array}{c} \begin{array}{ccc} & 1 & \\ & \circ & \circ \\ & \text{---} & \text{---} \\ & \circ & \circ \\ & 2 & \\ \circ & & \circ \\ \text{---} & & \text{---} \\ \circ & & \circ \\ & 3 & \\ & \circ & \circ \\ & \text{---} & \text{---} \\ & \circ & \circ \\ & 2 & \\ \circ & & \circ \\ & 4 & \\ & \circ & \circ \\ & \text{---} & \text{---} \\ & \circ & \circ \\ & 1 & \end{array} \end{array} \quad (13)$$

“racemes” $\{\tilde{f}, \tilde{d}, \tilde{x}, \tilde{z}\}$. According to Step 2, they are contracted by 0-colored edges to form the following φ_m^4 -Feynman diagram $\tilde{\mathcal{N}}$, which obviously satisfies $\partial\tilde{\mathcal{N}} = \mathcal{N}$.



Lemma 3. The φ_m^4 -graph \mathcal{S} given by

$$\mathcal{S}(g, v; h, w) := \begin{array}{c} \begin{array}{ccccccc} & g & & 1 & & 0 & & D & & 0 & & w \\ & \circ & & \circ & & \circ & & \circ & & \circ & & \circ \\ & \text{---} & & \text{---} & & \text{---} & & \text{---} & & \text{---} & & \text{---} \\ & \circ & & \circ & & \circ & & \circ & & \circ & & \circ \\ & 0 & & 1 & & 0 & & D & & 0 & & h \\ & \circ & & \circ & & \circ & & \circ & & \circ & & \circ \end{array} \end{array} \quad (14)$$

separates boundary components. More precisely: Given two open graphs $\mathcal{G}_1, \mathcal{G}_2 \in \mathfrak{Feyn}_D(\varphi_m^4)$, with $2p_1$ and $2p_2$ external legs, respectively,

$$\mathcal{G}_1(c^{(1)}, c^{(2)}, \dots, c^{(p_1)}; r^{(1)}, r^{(2)}, \dots, r^{(p_1)}) \quad \text{and} \quad \mathcal{G}_2(d^{(1)}, d^{(2)}, \dots, d^{(p_2)}; s^{(1)}, s^{(2)}, \dots, s^{(p_2)}),$$

for any $1 \leq i, j \leq p_1$ and $1 \leq k, l \leq p_2$, being $c^{(i)}$ (resp. $r^{(j)}$) any outer white (resp. black) vertex of \mathcal{G}_1 and $d^{(i)}$ (resp. $s^{(j)}$) any outer white (resp. black) vertex of \mathcal{G}_2 , we claim that

$$\mathcal{C} := \mathcal{G}_1(\dots, \overbrace{c^{(i)}, \dots, r^{(j)}, \dots}^{\mathcal{S}(g, v; h, w)}, \dots) \mathcal{G}_2(\dots, \overbrace{d^{(k)}, \dots, s^{(l)}, \dots}^{\mathcal{S}(g, v; h, w)}, \dots)$$

is a Feynman graph in $\mathfrak{Feyn}_D(\varphi_m^4)$, whose boundary is given by

$$\partial(\mathcal{G}_1(c^{(1)}, \dots, \overbrace{c^{(i)}, \dots, r^{(j)}, \dots}^{\mathcal{S}(g, v; h, w)}, \dots)) \sqcup \partial(\mathcal{G}_2(d^{(1)}, \dots, \overbrace{d^{(k)}, \dots, s^{(l)}, \dots}^{\mathcal{S}(g, v; h, w)}, \dots)). \quad (15)$$

Thus, if \mathcal{C} is given by



the dots listing uncontracted external legs, Lemma 3 says that

$$\partial\mathcal{C} = \partial \left(\begin{array}{c} \vdots \\ \circ \\ \text{---} \\ \circ \\ \vdots \end{array} \mathcal{G}_1 \begin{array}{c} \text{---} \\ \circ \\ \text{---} \\ \circ \\ \vdots \end{array} \right) \sqcup \partial \left(\begin{array}{c} \text{---} \\ \circ \\ \text{---} \\ \circ \\ \vdots \end{array} \mathcal{G}_2 \begin{array}{c} \vdots \\ \circ \\ \text{---} \\ \circ \\ \vdots \end{array} \right)$$

Proof. This is a restatement of [35, Lemma 6]. □

One can restate a more general result by considering $\mathcal{P} = \mathcal{S}(\overline{g}, \overline{v}; \overline{h}, \overline{w})$ and taking $(D+1)$ -colored graphs \mathcal{K} and \mathcal{G} that might even be closed. By taking edges $e \in \mathcal{K}_0^{(1)}$ and $f \in \mathcal{G}_0^{(1)}$ and letting $k = \overline{g\overline{w}}$ and $l = \overline{h\overline{w}}$ one has

$$\partial(\mathcal{K}_e \#_k \mathcal{P}_l \#_f \mathcal{G}) = \partial\mathcal{K} \sqcup \partial\mathcal{G}. \quad (16)$$

of theorem 1. Let \mathcal{B} be an arbitrary graph in $\text{IIGrph}_{c,D}$. We decompose \mathcal{B} in its connected components $\{\mathcal{R}^\alpha\}_{\alpha=1}^B \subset \text{Grph}_{c,D}$, $\mathcal{B} = \sqcup_{\alpha=1}^B \mathcal{R}^\alpha$. For each connected component α , we consider the graphs $\tilde{\mathcal{R}}^\alpha$ given by Lemma 1 if $D = 3$ or by Lemma 2 if $D \geq 4$. Fix two arbitrary vertices $d^\alpha \in (\mathcal{R}^\alpha)_w^{(0)}$ and $x^\alpha \in (\mathcal{R}^\alpha)_b^{(0)}$ and consider the vertices c_1^α, q_1^α and b_1^α, p_1^α that lie on the racemes \tilde{d}^α and \tilde{x}^α of $\tilde{\mathcal{R}}^\alpha$ respectively. One considers also the 0-colored edges $e_\alpha = \overline{c_1^\alpha p_1^\alpha}$ and $f_\alpha = \overline{q_1^\alpha b_1^\alpha}$ that connect the racemes \tilde{d}^α with \tilde{x}^α . Consider, $B-1$ copies of \mathcal{P} ,

$$\mathcal{P}^i = \mathcal{S}(\overline{g_i}, \overline{v_i}; \overline{h_i}, \overline{w_i}) \quad (i = 1, \dots, B-1),$$

and denote by $k_i = \overline{g_i \overline{v_i}}$ and $l_i = \overline{h_i \overline{w_i}}$ the 0-edges arising from the Wick-contracting. Then

$$\mathcal{T} = (\tilde{\mathcal{R}}^1)_{e_1} \#_{k_1} (\mathcal{P}^1)_{l_1} \#_{f_2} (\tilde{\mathcal{R}}^2)_{e_2} \#_{k_2} (\mathcal{P}^2)_{l_2} \#_{f_3} (\tilde{\mathcal{R}}^3)_{e_3} \#_{k_3} \cdots (\mathcal{P}^{B-1})_{l_{B-1}} \#_{f_B} (\tilde{\mathcal{R}}^B)$$

implies, after repetitively using eq. (16),

$$\partial\mathcal{T} = \partial\tilde{\mathcal{R}}^1 \sqcup \dots \sqcup \partial\tilde{\mathcal{R}}^B = \mathcal{R}^1 \sqcup \dots \sqcup \mathcal{R}^B = \mathcal{B}. \quad \square$$

4.3. Geometric interpretation. Graphs in $\text{IIGrph}_{c,D+1}$ serve to construct triangulations $\Delta(\mathcal{G})$ of D -(pseudo)manifolds, i.e. a (pseudo)complex as stated in [14]:

- for each vertex $v \in \mathcal{G}^{(0)}$, add a D -simplex σ_v to $\Delta(\mathcal{G})$
- one labels the vertices of σ_v by the colors $\{0, 1, \dots, D\}$
- for each edge $e_c \in \mathcal{G}_c^{(1)}$ of arbitrary color c , one identifies the two $(D-1)$ -simplices $\sigma_{s(e_c)}$ and $\sigma_{t(e_c)}$ that do not contain the color c .

Corollary 1. *The boundary sector of the $\varphi_{D,m}^4$ -model generates all orientable, closed piecewise linear manifolds. Thus, for $D = 4$, it generates all orientable, closed 3-manifolds.*

Proof. By Pezzana's theorem [14, 37], all compact, connected PL- $(D-1)$ -manifolds possess a suitable crystallization. Crystallizations are, in particular, D -colored graphs, all of which are generated by certain boundary $\partial\mathcal{G} \in \partial\mathfrak{Feyn}_D(\varphi_m^4)$, by Theorem 1. The second statement follows from Moise's theorem [31] on equivalence of topological and PL 3-manifolds. \square

Example 9. This result implies that there exist a 4-dimensional Ψ -manifold that is represented by the φ_m^4 -theory, whose boundary is any closed, orientable (honest) 3-manifold. In particular, for instance, the 3-manifold with the following, say, three connected components: a lens space, $L_{3,1}$; the 3-manifold with cyclic infinite fundamental group, $\mathbb{S}^2 \times \mathbb{S}^1$; and a more common prime factor, \mathbb{S}^3 . First one needs to crystallize them. The next three are crystallizations of said manifolds, in which we represent the color 4 by a wavy line and suppress redundant labels:

$$\Gamma = \quad \mathcal{C} = \quad \mathcal{M} = \quad (17)$$

We compute the fundamental group of these crystallizations in Appendix A. Theorem 1 states that $\mathcal{G} = \tilde{\Gamma} \# \mathcal{P} \# \mathcal{M} \# \mathcal{C}$ has as boundary the disjoint union of these graphs. Therefore the geometric realization $|\Delta(\mathcal{G})|$ of \mathcal{G} has as boundary $L_{3,1} \sqcup (\mathbb{S}^2 \times \mathbb{S}^1) \sqcup \mathbb{S}^3$.

(n_1, \dots, n_5) with $\sum_k k \cdot n_k = 5$	Cycles $\mathbf{J}_{\mathbb{P}_1^1} \cdots \mathbf{J}_{\mathbb{P}_{n_5}^5}$	Green Functions $G_{ \mathbb{P}_1^1 \dots \mathbb{P}_{n_1}^1 \dots \mathbb{P}_{n_5}^5 }$	Symmetry factor $\prod_k k^{n_k} \cdot n_k!$
$(5, 0, 0, 0, 0)$	$J_{pp}J_{qq}J_{rr}J_{ss}J_{tt}$	$G_{ p q r s t }$	$5!$
$(3, 1, 0, 0, 0)$	$J_{pp}J_{qq}J_{rr}J_{st}J_{ts}$	$G_{ p q r st }$	$3! \cdot 2$
$(2, 0, 1, 0, 0)$	$J_{pp}J_{qq}J_{rs}J_{st}J_{tr}$	$G_{ p q rst }$	$2! \cdot 3$
$(1, 2, 0, 0, 0)$	$J_{pp}J_{qr}J_{rq}J_{st}J_{ts}$	$G_{ p qr st }$	$2^2 2!$
$(0, 1, 1, 0, 0)$	$J_{pq}J_{qp}J_{rs}J_{st}J_{tr}$	$G_{ pq rst }$	$2 \cdot 3$
$(1, 0, 0, 1, 0)$	$J_{pp}J_{qr}J_{rs}J_{st}J_{tq}$	$G_{ p qrst }$	4
$(0, 0, 0, 0, 1)$	$J_{pq}J_{qr}J_{rs}J_{st}J_{tp}$	$G_{ pqrst }$	5

Table 2 The boundary-graph-expansion's fifth order

For further applications it might be important to modify Gurău's degree of a graph while simultaneously sparing its boundary. This is also due to the relevance of the difference $\tilde{\omega}(\mathcal{G}) - \omega(\partial\mathcal{G})$, where $\tilde{\omega}$ is the degree for open graphs defined as the sum of the genera of its *pinched jackets* [3]. On closed graphs $\tilde{\omega}$ is the same as ω (closed jackets cannot be “pinched”). The remark is that one can modify any graph $\mathcal{G} \in \mathfrak{Feyn}_D(\varphi_m^4)$ into a graph \mathcal{G}' of the same quartic model, so that $\partial\mathcal{G}' = \partial\mathcal{G}$. The only ingredient one needs is a vacuum graph $\mathcal{L} \in \mathfrak{Feyn}_D(\varphi_m^4)$ with $\omega(\mathcal{L}) > 0$. (e.g. in $D = 3$, the necklace graph with the color 4 equal to 0, and being thus in $\mathfrak{Feyn}_3(\varphi^4)$, cf. ex. 8). Then

$$\mathcal{G}' = \mathcal{G} \# \mathcal{P} \# \mathcal{L} = \vdots \begin{array}{c} \text{---} \circlearrowleft \mathcal{G} \text{---} \\ \text{---} \circlearrowleft \mathcal{S} \text{---} \\ \text{---} \circlearrowleft \mathcal{L} \text{---} \end{array}$$

has degree $\tilde{\omega}(\mathcal{G}') = \tilde{\omega}(\mathcal{G}) + \omega(\mathcal{P}) + \tilde{\omega}(\mathcal{L}) = \tilde{\omega}(\mathcal{G}) + \tilde{\omega}(\mathcal{L}) > \tilde{\omega}(\mathcal{G})$, by Theorem 1, and $\partial(\mathcal{G}') = \partial\mathcal{G} \sqcup \partial\mathcal{L} = \partial\mathcal{G}$ by Theorem 3 and because \mathcal{L} is a vacuum graph. Notice that the degree cannot be increased by an arbitrary amount, but only by multiples of $2/(D-1)!$.

5. THE EXPANSION OF THE FREE ENERGY IN BOUNDARY GRAPHS

Before tackling the main problem, it will be useful to recall the expansion of the free energy for real matrix models. The reader in a hurry might accept eq. (23) and go to eq. (22) for notation.

5.1. The free energy expansion for a general real matrix model. As background, consider the following model, whose objects are compact operators $M : \mathcal{H} \rightarrow \mathcal{H}$ (“matrices”), with \mathcal{H} a separable Hilbert space. The interactions are described by a polynomial potential, $P(M)$. The partition function reads

$$\frac{Z[J]}{Z[0]} = \frac{\int \mathcal{D}M e^{\text{Tr}(JM) - \text{Tr}(EM^2) - \text{Tr} P(M)}}{\int \mathcal{D}M e^{-\text{Tr}(EM^2) - \text{Tr} P(M)}}, \quad (18)$$

where E is a Hermitian operator on \mathcal{H} . The free energy, $W_{\text{matrix}}[J] \propto \log(Z[J]/Z[0])$, generates the connected Green's functions. To expand in terms of the combinatorics of the sources' indices, we shall use a multi-index notation, with \mathbb{P}^m having *length* $m = |\mathbb{P}^m|$. This just means that \mathbb{P}^m is an m -tuple $\mathbb{P}^m = (p_1 p_2 \dots p_m) \in I^m$ for given index set I . I^m is often the integer lattice, and m will not be a fixed integer, but we will deal with multi-indices of arbitrary length. To enumerate multi-indices of the same length we use a subindex, so $\mathbb{P}_1^m, \mathbb{P}_2^m, \dots, \mathbb{P}_{n_m}^m$ are all length- m cycles. Sums over multiple multi-indices are understood as follows:

$$\sum_{\mathbb{P}^k, \dots, \mathbb{Q}^m} = \sum_{p_1} \cdots \sum_{p_k} \cdots \sum_{q_1} \cdots \sum_{q_m} \quad \text{with } \mathbb{P}^k = (p_1 \dots p_k) \quad \text{and} \quad \mathbb{Q}^m = (q_1 \dots q_m).$$

The J -cycles of size ℓ , namely $J_{p_1 p_2} J_{p_2 p_3} \cdots J_{p_{\ell-1} p_\ell} J_{p_\ell p_1}$, are here for sake of brevity denoted by $\mathbf{J}^{\mathbb{P}^\ell} := \prod_{i=1}^{\ell} J_{p_i p_{i+1}}$ with $\mathbb{P}^\ell = (p_1 \dots p_\ell)$ and $p_{\ell+1} := p_1$. With that notation, the free energy can be expanded [21, Sec. 2.3] in length- ℓ cycles, with ℓ variable, as follows:

$$\sum_{\ell=1}^{\infty} \sum_{n_\ell \geq 1} \sum_{\substack{n_1=0 \\ \dots \\ n_{\ell-1}=0}}^{\infty} \left[\prod_{j=1}^{\ell} \frac{1}{n_j! j^{n_j}} \right] \sum_{\substack{\mathbb{P}_1^1, \dots, \mathbb{P}_{n_1}^1 \\ \dots \\ \mathbb{P}_1^\ell, \dots, \mathbb{P}_{n_\ell}^\ell}} \left\{ G_{|\mathbb{P}_1^1| \dots |\mathbb{P}_{n_1}^1| \dots |\mathbb{P}_1^\ell| \dots |\mathbb{P}_{n_\ell}^\ell|}^{(\mathcal{N}_{\text{matrix}})} \prod_{k=1}^{\ell} (\mathbf{J}^{\mathbb{P}_1^k} \cdots \mathbf{J}^{\mathbb{P}_{n_k}^k}) \right\}. \quad (19)$$

One word more on notation: Fixed the ℓ by the first sum, for $1 \leq k \leq \ell$, the non-negative integer n_k stands for the number of boundary components with k sources (whence $n_\ell \neq 0$ in the second sum is precisely a way to paraphrase the decomposition in the longest cycle). The number of boundary components B_{matrix} , and the number of sources, $\mathcal{N}_{\text{matrix}}$ (i.e. the order of the Green's function) are $B_{\text{matrix}} = \sum_{j=1}^{\ell} n_j$ and $\mathcal{N}_{\text{matrix}} = \sum_{j=1}^{\ell} j \cdot n_j$. Instead of expanding by longest-cycles, we can also rephrase (19) as an explicit Taylor expansion to $\mathcal{O}(J^6)$,

$$\begin{aligned} W[J] = & \sum_p G_{|p|} J_{pp} + \frac{1}{2} \sum_{p,q} (G_{|pq|} J_{pq} J_{qp} + G_{|p|q|} J_{pp} J_{qq}) \\ & + \sum_{p,q,r} \left(\frac{1}{3} G_{|pqr|} J_{pq} J_{qr} J_{rp} + \frac{1}{2} G_{|pq|r|} J_{pq} J_{qp} J_{rr} + \frac{1}{3!} G_{|p|q|r|} J_{pp} J_{qq} J_{rr} \right) \\ & + \sum_{p,q,r,s} \left(\frac{1}{4} G_{|pqr s|} J_{pq} J_{qr} J_{rs} J_{sp} + \frac{1}{3} G_{|pqr|s|} J_{pq} J_{qr} J_{rp} J_{ss} \right. \\ & + \frac{1}{8} G_{|p|q|r s|} J_{pq} J_{qp} J_{rs} J_{sr} + \frac{1}{4} G_{|p|q|r s|} J_{pp} J_{qq} J_{rs} J_{sr} + \frac{1}{4!} G_{|p|q|r|s|} J_{pp} J_{qq} J_{rr} J_{ss} \left. \right) \\ & + \sum_{p,q,r,s,t} \left(\frac{1}{5} G_{|pqr st|} J_{pq} J_{qr} J_{rs} J_{st} J_{tp} + \frac{1}{4} G_{|p|q r s t|} J_{pp} J_{qr} J_{rs} J_{st} J_{tq} \right. \\ & + \frac{1}{2 \cdot 3} G_{|p q r s t|} J_{pq} J_{qp} J_{rs} J_{st} J_{tr} + \frac{1}{2 \cdot 2 \cdot 2!} G_{|p|q r|s t|} J_{pp} J_{qr} J_{rq} J_{st} J_{ts} \\ & + \frac{1}{2! \cdot 3} G_{|p|q|r s t|} J_{pp} J_{qq} J_{rs} J_{st} J_{tr} + \frac{1}{3! \cdot 2} G_{|p|q|r|s t|} J_{pp} J_{qq} J_{rr} J_{st} J_{ts} \\ & \left. + \frac{1}{5!} G_{|p|q|r|s|t|} J_{pp} J_{qq} J_{rr} J_{ss} J_{tt} \right) + \mathcal{O}(J^6). \end{aligned}$$

Table 2 shows how to read off from (19), say, the fifth power in J . The Green's function for a fixed cycle can be furthermore expanded in genus- g sectors:

$$G_{|\mathbb{P}_1^1| \dots |\mathbb{P}_{n_1}^1| \dots |\mathbb{P}_1^\ell| \dots |\mathbb{P}_{n_\ell}^\ell|}^{(\mathcal{N}_{\text{matrix}})} = \sum_{g \geq 0} G_{|\mathbb{P}_1^1| \dots |\mathbb{P}_{n_1}^1| \dots |\mathbb{P}_1^\ell| \dots |\mathbb{P}_{n_\ell}^\ell|}^{(\mathcal{N}_{\text{matrix}}, g)}. \quad (20)$$

For the 5-tuple $(n_1, \dots, n_5) = (3, 1, 0, 0, 0)$, here chosen only to exemplify the genus expansion's meaning, $G_{|p|q|r|s t|}^{(5)}$ reads

$$G_{|p|q|r|s t|} J_{pp} J_{qq} J_{rr} J_{st} J_{ts} =$$

5.2. The general expansion for rank-3 CTMs. The combinatorics of the matrix-sources just shown in Section 5.1 gets modified for the rank-3 colored tensors because of the coloring; moreover, because the theory is now complex the non vertex-bipartite graphs are forbidden. The first implication of coloring is that the sources do not exhibit repeated indices in the same source, e.g. none of the following terms is allowed in the expansion of $W[J, \bar{J}] = \log(Z[J, \bar{J}])$:

$$J\dots a\dots a\dots, \bar{J}\dots a\dots a\dots, \bar{J}aaa, Jaaa, \bar{J}aabJbcc, Jaab\bar{J}bcc, \dots \quad (\text{terms forbidden by coloring}).$$

Whilst for the lowest order correlation functions this seems to be quite restrictive, the expansion shows intricacy as one goes to higher order ones.

We now consider a graph $\mathcal{G} \in \mathfrak{F}\epsilon\eta\mathfrak{n}_3(\varphi^4)$ and set the first convention. We fix the indices of the J -sources (the external lines connected to the black vertices) and let \mathcal{G} yield the indices of the \bar{J} -sources. For any i , both index types $\mathbf{a}^i, \mathbf{p}^i \in \mathbb{Z}^3$, are known as *momenta*.



$$\mathcal{G} = \begin{array}{c} J_{\mathbf{a}^1} \\ \vdots \\ J_{\mathbf{a}^k} \end{array} \begin{array}{c} \bar{J}_{\mathbf{p}^1} \\ \vdots \\ \bar{J}_{\mathbf{p}^k} \end{array} \quad (21)$$

We let the notation for the $2k$ -point function $G_{\dots}^{(2k)}$ that describes the “process” \mathcal{G} reflect this combinatorics via another graph \mathcal{B} to be constructed shortly. The resulting $G_{\mathcal{B}}^{(2k)}$ ought to encompass all graphs in $\mathfrak{F}\epsilon\eta\mathfrak{n}_3(\varphi^4)$ that lead to the same combination of indices in the \bar{J} -sources. From the \bar{J} -sources, for each $\alpha = 1, \dots, k$, $\mathbf{p}^\alpha = \mathbf{p}^\alpha(\mathbf{a}^1, \dots, \mathbf{a}^k)$ is a triple index that depends on $\mathbf{a}^1, \dots, \mathbf{a}^k$. The j -th color of \mathbf{p}^α will be denoted by p_j^α and to fix the enumeration of \mathbf{p}^α , we will ask $p_1^\alpha := a_1^\alpha$, for each $\alpha = 1, \dots, k$. Moreover, regularity and coloring of the graph implies that $\{p_j^\alpha\}_{\alpha=1}^k$ and $\{a_j^1, a_j^2, \dots, a_j^k\}$ coincide *as sets, also* for the colors $j = 2, 3$.

A crucial step in order to find the generalization of the expansion (19), is to notice that⁵ that very equation is a sum over boundaries of $\mathfrak{F}\epsilon\eta\mathfrak{n}_2^{\mathbb{R}}(\varphi^4)$. In order to adapt (19) to $\mathfrak{F}\epsilon\eta\mathfrak{n}_3(\varphi^4)$, we take each monomial $G_{\mathcal{B}}^{(2k)}(\mathbf{a}^1, \dots, \mathbf{a}^k) J_{\mathbf{a}^1} \cdots J_{\mathbf{a}^k} \bar{J}_{\mathbf{p}^1} \cdots \bar{J}_{\mathbf{p}^k}$, which in all generality looks like in eq. (21) and notice that the structure of the sources is, of course, encoded by the boundary graph $\mathcal{B} = \partial\mathcal{G}$. Parenthetically, this is not an uncommon practice in (scalar) QFT, where the boundary graph is just a graph without edges, i.e. a finite set whose cardinality gives the number of points of the correlation function. The graph \mathcal{B} and said monomial are uniquely, mutually determined as follows:

- a source $J_{\mathbf{a}^s}$ determines a white vertex in \mathcal{B} ; a source $\bar{J}_{\mathbf{p}^j}$, a black vertex in \mathcal{B} ;
- two vertices are joined by a c -colored edge in \mathcal{B} if and only if there exists a $(0c)$ -bicolored path *in* \mathcal{G} between the (vertices associated to the) external lines $J_{\mathbf{a}^s}$ and $\bar{J}_{\mathbf{p}^j}$. Then set

$$(\mathbb{J}(\mathcal{B}))(\mathbf{a}^1, \dots, \mathbf{a}^k) := J_{\mathbf{a}^1} \cdots J_{\mathbf{a}^k} \bar{J}_{\mathbf{p}^1} \cdots \bar{J}_{\mathbf{p}^k} = J_{a_1^1 a_2^1 a_3^1} \cdots J_{a_1^k a_2^k a_3^k} \bar{J}_{a_1^1 p_2^1 p_3^1} \cdots \bar{J}_{a_1^k p_2^k p_3^k}. \quad (22)$$

Here the momenta \mathbf{p}^α are determined as in the graph (21) and the convention below it, and $\mathbb{J}(\mathcal{B})$ is a function of the momenta $\{\mathbf{a}\} = (\mathbf{a}^1, \dots, \mathbf{a}^k) \in (\mathbb{Z}^3)^k$. Thus, the expansion can be recast as

$$W[J, \bar{J}] = \sum_{k=1}^{\infty} \sum_{\mathcal{B} \in \partial(\mathfrak{F}\epsilon\eta\mathfrak{n}_3(\varphi^4))} \sum_{\{\mathbf{a}\}} \frac{1}{|\text{Aut}_c(\mathcal{B})|} G_{\mathcal{B}}^{(2k)}(\{\mathbf{a}\}) \cdot \mathbb{J}(\mathcal{B})(\{\mathbf{a}\}).$$

⁵The author is indebted to Raimar Wulkenhaar for this valuable remark.

It will be convenient to define a pairing \star of functions $F : (\mathbb{Z}^3)^k \rightarrow \mathbb{C}$ with boundary graphs $\mathcal{B} \in \partial(\mathfrak{Fen}_3(\varphi^4))$:

$$F \star \mathbb{J}(\mathcal{B}) := \sum_{\mathbf{a}^1, \dots, \mathbf{a}^k} F(\mathbf{a}^1, \dots, \mathbf{a}^k) \mathbb{J}(\mathcal{B}(\mathbf{a}^1, \dots, \mathbf{a}^k)).$$

With this notation, W takes the neater form:

$$W[J, \bar{J}] = \sum_{k=1}^{\infty} \sum_{\substack{\mathcal{B} \in \text{IIGrph}_{c,3} \\ 2k = |\mathcal{B}^{(0)}|}} \frac{1}{|\text{Aut}_c(\mathcal{B})|} G_{\mathcal{B}}^{(2k)} \star \mathbb{J}(\mathcal{B}). \quad (23)$$

The fact that *all 3-colored graphs* appear listed in eq. (23) is consequence of Theorem 1.

Remark 3. A conspicuous difference with matrix models' expansion (19) —where the boundary of each graph is topologically “uniform”, all being triangulations of $\sqcup^B \mathbb{S}^1$ — is that in rank-3 tensor field theories, the analogous connected components of the boundary have a non-trivial topology, since these are 3-colored graphs and therefore [35] define closed orientable surfaces, $\Delta_{\mathcal{B}} \cong \sqcup_{\beta=1}^B \Sigma^{g_{\beta}}$ with $g_{\beta} \in \mathbb{Z}_{\geq 0}$ and $\Sigma^g = \#^g \mathbb{T}^2$ (being $\Sigma^0 := \mathbb{S}^2$ for the empty connected sum $g = 0$). As shown here, an analogous result holds for higher dimensions. Details on the expansion of W in disconnected boundary graphs are presented in Appendix B.

To illustrate the expansion, we derive the first terms in powers of the sources:

$$\begin{aligned} W_{D=3}[J, \bar{J}] &= G_{\ominus}^{(2)} \star \mathbb{J}(\ominus) + \frac{1}{2!} G_{|\ominus|_{\ominus}}^{(4)} \star \mathbb{J}(\ominus \sqcup \ominus) + \frac{1}{2} G_{\sqcup_1}^{(4)} \star \mathbb{J}(\sqcup_1) \\ &+ \frac{1}{2} G_{\sqcup_2}^{(4)} \star \mathbb{J}(\sqcup_2) + \frac{1}{2} G_{\sqcup_3}^{(4)} \star \mathbb{J}(\sqcup_3) + \sum_{c=1}^3 \frac{1}{3} G_{\text{cyc}_c}^{(6)} \star \mathbb{J}(\text{cyc}_c) \\ &+ \frac{1}{3} G_{\text{tet}}^{(6)} \star \mathbb{J}(\text{tet}) + \sum_{c=1}^3 G_{\text{cyl}_c}^{(6)} \star \mathbb{J}(\text{cyl}_c) + \frac{1}{3!} G_{|\ominus|_{\ominus}^3}^{(6)} \star \mathbb{J}(\ominus^{\sqcup 3}) \\ &+ \sum_{c=1}^3 \frac{1}{2} G_{|\ominus|_{\sqcup^c}}^{(6)} \star \mathbb{J}(\ominus \sqcup \sqcup^c) + \mathcal{O}(8). \end{aligned} \quad (24)$$

In this expansion, the monomial $\mathbb{J}(\mathcal{B})$ in the sources J and \bar{J} is defined by formula (22). Thus, for instance the term in $W[J, \bar{J}]$ for the trace indexed by the colored complete graph $K_c(3, 3)$ is

$$\frac{1}{3} G_{\text{tet}}^{(6)} \star \mathbb{J}(\text{tet}) = \frac{1}{3} \sum_{\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{Z}^3} G_{\text{tet}}^{(6)}(\mathbf{a}, \mathbf{b}, \mathbf{c}) J_{a_1 a_2 a_3} \bar{J}_{a_1 b_2 c_3} J_{b_1 b_2 b_3} \bar{J}_{a_3 b_1 c_2} J_{c_1 c_2 c_3} \bar{J}_{a_2 b_3 c_1}.$$

This monomial $\mathbb{J}(\mathcal{B})$ should not be confused with the trace $\text{Tr}_{\mathcal{B}}(J, \bar{J})$, which would imply sums over all indices inside the graph. Actually $\text{Tr}_{\mathcal{B}}(\varphi, \bar{\varphi}) = \mathbf{1} \star \mathbb{J}(\mathcal{B})$ holds, being $\mathbf{1}$ the constant function $\mathbb{Z}^{|\mathcal{B}^{(0)}|/2} \rightarrow \mathbb{C}$, $\mathbf{a} \mapsto 1$. Notice that formula (24) pairs the momenta indices⁶ of $\mathbb{J}(\mathcal{B})$ with those of the corresponding Green's function $G_{\mathcal{B}}^{(2k)}$. This seemingly redundant notation will pay off not before the WTI below. The next short section explains why those factors have been chosen, and how to recover each Green's functions $G_{\mathcal{B}}^{(2k)}$ in the expansion of $W[J, \bar{J}]$.

The number of correlation functions in rank $D = 3, 4$ theories are counted. In [2], Ben Geloun and Ramgoolam found the generating function $Z_{D, \text{conn.}}(x) = \sum_p a_{p, \text{conn.}}^{(D)} x^p$ of the number $a_{p, \text{conn.}}^{(D)}$ of *connected* graphs $\text{Grph}_{c, D}$ of a fixed number of vertices $2p$. It has the

⁶Recall that we give the white indices and let the graph determine the black ones (see above eq. (21)).

following behavior⁷:

$$Z_{3,\text{conn.}}(x) = \sum_p a_{p,\text{conn.}}^{(3)} x^p = x + 3x^2 + 7x^3 + 26x^4 + 97x^5 + 624x^6 + 4163x^7 + 34470x^8 + \dots \quad (25)$$

The first three terms of this series are evident in eq. (24). For $D = 4$, they also computed

$$Z_{4,\text{conn.}}(x) = x + 7x^2 + 41x^3 + 604x^4 + 13753x^5 + 504243x^6 + 24824785x^7 + 1598346352x^8 + \dots \quad (26)$$

From those expressions one can readily compute the number $a_p(D)$ of disconnected D -colored graphs in $2p$ vertices. That integer is the number of correlation $2p$ -point functions.

5.3. Graph calculus. Let $\mathcal{R}, \mathcal{Q} \in \text{Grph}_{c,3}$ and $\mathbf{a}^1, \dots, \mathbf{a}^r, \mathbf{c}^1, \dots, \mathbf{c}^q \in \mathbb{Z}^3$. In view of the discussion above, we associate to those graphs, respectively, the monomials $\mathbb{J}(\mathcal{R})(\mathbf{a}^1, \dots, \mathbf{a}^r)$ and $\mathbb{J}(\mathcal{Q})(\mathbf{c}^1, \dots, \mathbf{c}^q)$. Here, the white vertices of the graph \mathcal{R} have incoming momenta $\mathbf{a}^1, \dots, \mathbf{a}^r$ and similarly for \mathcal{Q} . So we can derive one with respect to the other:

$$\frac{\partial \mathcal{R}(\mathbf{c}^1, \dots, \mathbf{c}^q)}{\partial \mathcal{Q}(\mathbf{a}^1, \dots, \mathbf{a}^r)} := \frac{\partial \mathbb{J}(\mathcal{R})(\mathbf{a}^1, \dots, \mathbf{a}^r)}{\partial \mathbb{J}(\mathcal{Q})(\mathbf{c}^1, \dots, \mathbf{c}^q)} \Big|_{J=0=\bar{J}}.$$

This can be straightforwardly computed. First notice that trivially, if $p \neq q$, automatically $\partial \mathcal{R} / \partial \mathcal{Q} \equiv 0$. Otherwise we have:

Lemma 4. *Let $\mathbf{a}^1, \dots, \mathbf{a}^r \in \mathbb{Z}^3$ be colorwise, pairwise different, i.e. such that for each $\alpha, \beta = 1, \dots, r$, and for each color $c = 1, 2, 3$, $a_c^\alpha \neq a_c^\beta$ holds whenever $\alpha \neq \beta$. Then for connected graphs $\mathcal{R}, \mathcal{Q} \in \text{Grph}_{c,3}$,*

$$\frac{\partial \mathcal{R}(\mathbf{c}^1, \dots, \mathbf{c}^r)}{\partial \mathcal{Q}(\mathbf{a}^1, \dots, \mathbf{a}^r)} = \begin{cases} \sum_{\hat{\sigma} \in \text{Aut}_c(\mathcal{R})} \delta_{\mathbf{a}^1, \mathbf{a}^2, \dots, \mathbf{a}^r}^{\mathbf{c}^{\sigma(1)}, \dots, \mathbf{c}^{\sigma(r)}} & \text{if } \mathcal{R} \cong \mathcal{Q}, \\ 0 & \text{if } \mathcal{R} \not\cong \mathcal{Q}. \end{cases} \quad (27)$$

Here $\hat{\sigma} \in \text{Aut}_c(\mathcal{R})$ means the automorphism $\hat{\sigma} : \mathcal{R} \rightarrow \mathcal{Q}$ whose restriction to white vertices satisfies $\hat{\sigma}|_{\mathcal{R}_w^{(0)}} = \sigma \in \text{Sym}(\mathcal{R}_w^{(0)}) = \mathfrak{S}_r$. Also “ \cong ” denotes isomorphism in the sense of colored graphs, and the δ -function is shorthand for the following product of $3r$ Kronecker-deltas: $\delta_{c_1, c_2, c_3, \dots, d_1, d_2, d_3}^{a_1, a_2, a_3, \dots, b_1, b_2, b_3} = \delta_{c_1}^{a_1} \delta_{c_2}^{a_2} \delta_{c_3}^{a_3} \dots \delta_{d_1}^{b_1} \delta_{d_2}^{b_2} \delta_{d_3}^{b_3}$.

Proof. If we compute directly using

$$\frac{\partial J_{\mathbf{u}}^\#}{\partial J_{\mathbf{w}}^\#} = \delta_{\#}^{\#} \delta_{w_1}^{u_1} \delta_{w_2}^{u_2} \delta_{w_3}^{u_3} \quad \text{where } J^\#, J^\# \in \{J, \bar{J}\}, \quad (28)$$

then one splits this in the J -derivatives and the \bar{J} -terms

$$\frac{\partial \mathcal{R}(\mathbf{c}^1, \dots, \mathbf{c}^r)}{\partial \mathcal{Q}(\mathbf{a}^1, \dots, \mathbf{a}^r)} = \frac{\partial^r (J_{\mathbf{c}^1} \dots J_{\mathbf{c}^r})}{\partial J_{\mathbf{p}^1} \dots \partial J_{\mathbf{p}^r}} \Big|_{J=0} \frac{\partial^r (\bar{J}_{\mathbf{q}^1} \dots \bar{J}_{\mathbf{q}^r})}{\partial \bar{J}_{\mathbf{p}^1} \dots \partial \bar{J}_{\mathbf{p}^r}} \Big|_{\bar{J}=0},$$

being the labels of the sources fully determined by

$$p_1^\alpha = a_1^\alpha \quad \text{and} \quad q_1^\alpha = c_1^\alpha \quad \text{for all } \alpha = 1, \dots, r. \quad (29)$$

One can again use eq. (28) and compute each of these terms:

$$\frac{\partial^r (J_{\mathbf{c}^1} \dots J_{\mathbf{c}^r})}{\partial J_{\mathbf{a}^1} \dots \partial J_{\mathbf{a}^r}} \Big|_{J=0} = \sum_{\sigma \in \mathfrak{S}_p} \delta_{\mathbf{a}^1}^{\mathbf{c}^{\sigma(1)}} \delta_{\mathbf{a}^2}^{\mathbf{c}^{\sigma(2)}} \dots \delta_{\mathbf{a}^r}^{\mathbf{c}^{\sigma(r)}},$$

and

$$\frac{\partial^r (\bar{J}_{\mathbf{q}^1} \dots \bar{J}_{\mathbf{q}^r})}{\partial \bar{J}_{\mathbf{p}^1} \dots \partial \bar{J}_{\mathbf{p}^r}} \Big|_{\bar{J}=0} = \sum_{\sigma \in \mathfrak{S}_p} \delta_{\mathbf{p}^1}^{\mathbf{q}^{\sigma(1)}} \delta_{\mathbf{p}^2}^{\mathbf{q}^{\sigma(2)}} \dots \delta_{\mathbf{p}^r}^{\mathbf{q}^{\sigma(r)}}.$$

⁷The OEIS series numbers [32] for $Z_{3,\text{conn.}}$ and $Z_{4,\text{conn.}}$ are A057005 and A057006, respectively.

Then

$$\begin{aligned} \frac{\partial \mathcal{R}(\mathbf{c}^1, \dots, \mathbf{c}^r)}{\partial \mathcal{Q}(\mathbf{a}^1, \dots, \mathbf{a}^r)} &= \sum_{\sigma \in \mathfrak{S}_p} \sum_{\tau \in \mathfrak{S}_p} \delta_{\mathbf{a}^1}^{\mathbf{c}^{\sigma(1)}} \delta_{\mathbf{p}^1}^{\mathbf{q}^{\tau(1)}} \delta_{\mathbf{a}^2}^{\mathbf{c}^{\sigma(2)}} \delta_{\mathbf{p}^2}^{\mathbf{q}^{\tau(2)}} \dots \delta_{\mathbf{a}^r}^{\mathbf{c}^{\sigma(r)}} \delta_{\mathbf{p}^r}^{\mathbf{q}^{\tau(r)}} \\ &= \sum_{\substack{\sigma, \tau \in \mathfrak{S}_r \\ \sigma = \tau}} \delta_{\mathbf{a}^1}^{\mathbf{c}^{\sigma(1)}} \delta_{\mathbf{p}^1}^{\mathbf{q}^{\sigma(1)}} \delta_{\mathbf{a}^2}^{\mathbf{c}^{\sigma(2)}} \delta_{\mathbf{p}^2}^{\mathbf{q}^{\sigma(2)}} \dots \delta_{\mathbf{a}^r}^{\mathbf{c}^{\sigma(r)}} \delta_{\mathbf{p}^r}^{\mathbf{q}^{\sigma(r)}} \end{aligned} \quad (30)$$

where the restriction to sum only over the diagonal $\tau = \sigma$ is derived from the color-1 deltas by using the index-definition (29): $\delta_{\mathbf{a}_i^\alpha}^{\mathbf{c}_1^{\sigma(\alpha)}} \delta_{\mathbf{p}_i^\alpha}^{\mathbf{q}_1^{\tau(\alpha)}} = \delta_{\mathbf{a}_i^\alpha}^{\mathbf{c}_1^{\sigma(\alpha)}} \delta_{\mathbf{a}_i^\alpha}^{\mathbf{c}_1^{\tau(\alpha)}} = \delta_{\sigma(\alpha)}^{\tau(\alpha)}$ for arbitrary $\alpha = 1, \dots, r$. The second equality follows from the condition $c_j^\alpha \neq c_j^\gamma$ if $\alpha \neq \gamma$, for each color $j = 1, 2, 3$. Now suppose that $\mathcal{R} \not\cong \mathcal{Q}$ and consider, for an arbitrary $\sigma \in \mathfrak{S}_r$, the following term in the sum:

$$\delta_{\mathbf{a}^1}^{\mathbf{c}^{\sigma(1)}} \delta_{\mathbf{p}^1}^{\mathbf{q}^{\sigma(1)}} \delta_{\mathbf{a}^2}^{\mathbf{c}^{\sigma(2)}} \delta_{\mathbf{p}^2}^{\mathbf{q}^{\sigma(2)}} \dots \delta_{\mathbf{a}^r}^{\mathbf{c}^{\sigma(r)}} \delta_{\mathbf{p}^r}^{\mathbf{q}^{\sigma(r)}} \quad (31)$$

By assumption $\mathcal{Q} \neq \hat{\sigma}(\mathcal{R})$. That is, there is a white vertex (marked by) \mathbf{a}^α , and a color $j \neq 1$, with the following property:

- if $\mathbf{p}^\nu \in \overline{\mathcal{Q}}_b^{(0)}$ denotes the black vertex where the j -colored edge e_j beginning at \mathbf{a}^α ends (i.e. $t(e_j) = \mathbf{p}^\nu$); and, moreover, if $\mathbf{q}^\gamma \in \mathcal{R}_b^{(0)}$ denotes the vertex where the j -colored edge at $\mathbf{c}^{\sigma(\alpha)}$ ends; then $\hat{\sigma}^{-1}(\mathbf{q}^\gamma) \neq \mathbf{p}^\nu$.

This means that the following deltas are contained in the term (31):

$$\delta_{\mathbf{p}_j^\nu}^{\mathbf{a}_j^\alpha} \delta_{\mathbf{c}^{\sigma(\alpha)}}^{\mathbf{a}^\alpha} \delta_{\mathbf{q}_j^\gamma}^{\mathbf{c}_j^{\sigma(\alpha)}} \delta_{\mathbf{q}^\gamma}^{\hat{\sigma}^{-1}(\mathbf{q}^\gamma)} \quad (32)$$

On the other hand, consider the j -colored edge g_j with $t(g_j) = \hat{\sigma}^{-1}(\mathbf{q}^\gamma)$ and the vertex \mathbf{a}^μ with $s(g_j) = \mathbf{a}^\mu$. Because of $\hat{\sigma}^{-1}(\mathbf{q}^\gamma) \neq \mathbf{p}^\nu$ and as consequence of the regularity of the coloring of the graph one has $\mu \neq \alpha$. Thus, the term (31) contains, on top of (32), $\delta_{\hat{\sigma}^{-1}(\mathbf{q}^\gamma)}^{\mathbf{a}^\mu}$. By using the assumption, $\mathbf{a}_j^\alpha \neq \mathbf{a}_j^\mu$ one gets $\delta_{\mathbf{p}_j^\nu}^{\mathbf{a}_j^\alpha} \delta_{\mathbf{c}^{\sigma(\alpha)}}^{\mathbf{a}^\alpha} \delta_{\mathbf{q}_j^\gamma}^{\mathbf{c}_j^{\sigma(\alpha)}} \delta_{\mathbf{q}^\gamma}^{\hat{\sigma}^{-1}(\mathbf{q}^\gamma)} \delta_{\hat{\sigma}^{-1}(\mathbf{q}^\gamma)}^{\mathbf{a}^\mu} = 0$. Since this holds for arbitrary σ , then $\mathcal{R} \not\cong \mathcal{Q}$ implies that $\partial \mathcal{R} / \partial \mathcal{Q} \equiv 0$. Hence

$$\frac{\partial \mathcal{R}(\mathbf{c}^1, \dots, \mathbf{c}^r)}{\partial \mathcal{Q}(\mathbf{a}^1, \dots, \mathbf{a}^r)} = \sum_{\sigma \in \mathfrak{S}_r} \delta(\hat{\sigma}(\mathcal{R}), \mathcal{Q}) \delta_{\mathbf{a}^1, \mathbf{a}^2, \dots, \mathbf{a}^r}^{\mathbf{c}^{\sigma(1)}, \dots, \mathbf{c}^{\sigma(r)}}, \quad \text{with } \delta(\hat{\sigma}(\mathcal{R}), \mathcal{Q}) := \begin{cases} 1 & \text{if } \hat{\sigma}(\mathcal{R}) = \mathcal{Q}, \\ 0 & \text{if } \hat{\sigma}(\mathcal{R}) \neq \mathcal{Q}. \end{cases}$$

The sole non-vanishing terms are precisely the automorphisms of \mathcal{R} and the result follows. \square

To better comprehend this formula, notice that the derivative $\partial / \partial \mathcal{Q}$ still has momentum dependence, for \mathcal{Q} has external lines as vertices. For instance,

$$\frac{\partial}{\partial \begin{array}{c} \mathbf{a} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \mathbf{c} \end{array}} = \frac{\partial^6}{\partial J_{a_1 a_2 a_3} \partial \bar{J}_{a_1 b_2 c_3} \partial J_{b_1 b_2 b_3} \partial \bar{J}_{a_3 b_1 c_2} \partial J_{c_1 c_2 c_3} \partial \bar{J}_{a_2 b_3 c_1}}.$$

For $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ and $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ subsets of \mathbb{Z}^3 satisfying the hypothesis of Lemma 4,

$$\frac{\partial}{\partial \begin{array}{c} \mathbf{a} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \mathbf{c} \end{array}} \left(\begin{array}{c} \mathbf{e} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \mathbf{g} \\ \text{---} \\ \mathbf{f} \end{array} \right) = \delta_{\mathbf{a}}^{\mathbf{e}} \delta_{\mathbf{b}}^{\mathbf{f}} \delta_{\mathbf{c}}^{\mathbf{g}} + \delta_{\mathbf{a}}^{\mathbf{g}} \delta_{\mathbf{b}}^{\mathbf{e}} \delta_{\mathbf{c}}^{\mathbf{f}} + \delta_{\mathbf{a}}^{\mathbf{f}} \delta_{\mathbf{b}}^{\mathbf{g}} \delta_{\mathbf{c}}^{\mathbf{e}},$$

holds for $\{a_d \neq b_d \neq c_d \neq a_d\}_{d=1,2,3}$ and has the same information as $\text{Aut}_c(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array}) \simeq \mathbb{Z}_3$.

Formula (27) can be directly generalized to non-connected graphs. Any graph $\mathcal{R} \in \text{IIGrph}_{c,3}$ that has s different connected components, each of multiplicity m_i , $i = 1, \dots, s$, can be split according to

$$\mathcal{R} = \mathcal{R}_1^1 \sqcup \mathcal{R}_2^1 \sqcup \dots \sqcup \mathcal{R}_{m_1}^1 \sqcup \dots \sqcup \mathcal{R}_1^s \sqcup \mathcal{R}_2^s \sqcup \dots \sqcup \mathcal{R}_{m_s}^s, \quad (33)$$

where the subindices only label copies of the same graph \mathcal{R}_* . Using a similar expression for $\mathcal{Q} \in \text{IIGrph}_{c,3}$, one finds,

$$\frac{\partial \mathcal{R}(\mathbf{c}^{(1)}, \dots, \mathbf{c}^{(r)})}{\partial \mathcal{Q}(\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(r)})} = \frac{\partial^{|\mathcal{Q}^{(0)}|}}{\partial \mathbb{J}(\mathcal{Q})} \mathbb{J}(\mathcal{R}) \Big|_{\substack{J=0 \\ \bar{J}=0}} = \begin{cases} \sum_{\sigma \in \text{Aut}_c(\mathcal{R})} \delta_{\mathbf{a}^{(1), \mathbf{a}^{(2)}, \dots, \mathbf{a}^{(r)}}}^{\sigma(\mathbf{c}^{(1)}), \dots, \sigma(\mathbf{c}^{(r)})} & \text{if } \mathcal{R} \neq \mathcal{Q}, \\ 0 & \text{if } \mathcal{R} = \mathcal{Q}, \end{cases} \quad (34)$$

which we again denote by $\partial \mathcal{R} / \partial \mathcal{Q}$. For \mathcal{R} of the type (33), this derivative contains certain number $\sigma(\mathcal{R})$ Kronecker deltas, being $\sigma(\mathcal{R}) := m_1! \dots m_s! |\text{Aut}_c(\mathcal{R}_\bullet^1)|^{m_1} \dots |\text{Aut}_c(\mathcal{R}_\bullet^s)|^{m_s}$. This explains the factors accompanying the Green's functions in the expansion (53).

Lemma 5. *For $\mathcal{B} \in \text{IIGrph}_{c,3}$, let \mathcal{N} be the number of vertices of \mathcal{B} . Then the \mathcal{N} -point function corresponding to \mathcal{B} is non-trivial and can be recovered from the free energy W as follows:*

$$G_{\mathcal{B}}^{(\mathcal{N})} = \frac{\partial}{\partial \mathcal{B}} W[J, \bar{J}] \Big|_{J=0=\bar{J}} := \frac{\partial^{|\mathcal{B}^{(0)}|}}{\partial \mathbb{J}(\mathcal{B})} W[J, \bar{J}] \Big|_{J=0=\bar{J}}.$$

Proof. In the expansion (23), we single out \mathcal{B} and derive with respect to \mathcal{B} :

$$\frac{\partial^{|\mathcal{B}^{(0)}|}}{\partial \mathbb{J}(\mathcal{B})} W[J, \bar{J}] \Big|_{J=0=\bar{J}} = \frac{\partial}{\partial \mathcal{B}} \left(W[J, \bar{J}] - \frac{1}{\sigma(\mathcal{B})} G_{\mathcal{B}}^{(\mathcal{N})} \mathbb{J}(\mathcal{B}) \right) + \frac{1}{\sigma(\mathcal{B})} \frac{\partial}{\partial \mathcal{B}} G_{\mathcal{B}}^{(\mathcal{N})} \mathcal{B}.$$

The first summand vanishes, since \mathcal{B} does not appear in that sum of terms. The second term yields, after equation (34), precisely $G_{\mathcal{B}}^{(\mathcal{N})}$. In the φ_3^4 -theory, this Green's function is non-trivial, for there exists at least one graph, whose boundary is \mathcal{B} , as stated by Lemma 1. \square

For instance, the 6-point function $G_{\text{hex}}^{(6)}$ reads in full notation

$$G_{\text{hex}}^{(6)}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \frac{\partial^6 W[J, \bar{J}]}{\partial J_{a_1 a_2 a_3} \partial \bar{J}_{a_1 b_2 c_3} \partial J_{b_1 b_2 b_3} \partial \bar{J}_{a_3 b_1 c_2} \partial J_{c_1 c_2 c_3} \partial \bar{J}_{a_2 b_3 c_1}} \Big|_{J=0=\bar{J}}.$$

5.4. Arbitrary-rank graph calculus. As is it obvious from the proofs, the results in previous section do not rely on the number of colors. In fact, we claim that for any rank- D model $V(\varphi, \bar{\varphi})$, the following expansion in boundary-graphs holds:

$$W[J, \bar{J}] = \sum_{k=1}^{\infty} \sum_{\substack{\mathcal{B} \in \partial \mathfrak{Feyn}_D(V(\varphi, \bar{\varphi})) \\ k = \frac{1}{2} \#(\mathcal{B}^{(0)})}} \frac{1}{|\text{Aut}_c(\mathcal{B})|} G_{\mathcal{B}}^{(2k)} \star \mathbb{J}(\mathcal{B}). \quad (35)$$

Here, we have set $N = 1$, which can be reverted by a rescaling of the kinetic term and of each of the correlation functions $G_{\mathcal{B}}^{(2k)} \rightarrow N^{\gamma(\mathcal{B})} G_{\mathcal{B}}^{(2k)}$, and $\gamma(\mathcal{B})$ should be determined. We postpone this task and depart from the simplified version, eq. (35). For the φ_m^4 -theory, the sum is over all $\partial \mathfrak{Feyn}_D(\varphi_m^4) = \text{IIGrph}_{c,D}$, as consequence of Theorem 1. For that model the free energy $W_{D=4}[J, \bar{J}]$ to $\mathcal{O}(6)$ is then given by

$$\begin{aligned} W_{D=4}[J, \bar{J}] &= G_{\text{hex}}^{(2)} \star \mathbb{J}(\text{hex}) + \frac{1}{2!} G_{\text{hex}}^{(4)} \star \mathbb{J}(\text{hex} \sqcup^2) \\ &+ \sum_{k=1}^4 \frac{1}{2} G_{\text{hex}}^{(4)} \star \mathbb{J}(\text{hex}_k) + \sum_{k=2}^4 \frac{1}{2} G_{\text{hex}}^{(4)} \star \mathbb{J}(\text{hex}_k^{\text{cylinder}}). \end{aligned}$$

We omit the next $49 = 41_{\text{conn. boundary}} + 8_{\text{disconn. boundary}}$ connected $\mathcal{O}(6)$ -multipoint functions. The way to recover the correlation function $\mathcal{G}_{\mathcal{B}}$ from the free energy, in arbitrary rank is described by an obvious generalization of Lemma 5.

6. THE FULL WARD-TAKAHASHI IDENTITY

The Ward-Takahashi Identities for tensor models are inspired by those found for the Grosse-Wulkenhaar model given by the action

$$\int d^4x \left(\frac{1}{2} (\partial_\mu \varphi) \star (\partial^\mu \varphi) + \frac{\Omega^2}{2} (\tilde{x}_\mu \varphi) \star (\tilde{x}^\mu \varphi) + \frac{\mu^2}{2} \varphi \star \varphi + \frac{\lambda}{4!} \varphi \star \varphi \star \varphi \star \varphi \right) (x), \quad (36)$$

on Moyal \mathbb{R}^4 . Here $x_\mu = 2\Theta_{\mu\nu}^{-1}x^\nu$, being Θ a 4×4 skew-symmetric matrix, that also parametrizes the Moyal product $(f \star g)(x) = \int \frac{d^4k}{(2\pi)^4} \int d^4y f(x + \frac{1}{2}\Theta \cdot k) g(x+y) e^{ik \cdot y}$ on the Schwartz space, $f, g \in \mathcal{S}(\mathbb{R}^4, \mathbb{C})$. The modification by a harmonic oscillator term makes the theory dual under certain “position-momentum”-duality, also known as Langmann-Szabo-duality [30]. The authors of the model have shown that their action (36) with $\Omega = 1$ can be grasped as a matrix model in such a way that it fits in the setting of (18) by using the Moyal matrix base [20]. We generalize the existent WTI in [43] by following the non-perturbative strategy by [21, Sec. 2-3].

6.1. Derivation of the Ward-Takahashi Identity. We set $N = 1$ from now on, which does not affect our analysis. We consider an arbitrary colored tensor model in D colors $V(\varphi, \bar{\varphi}) = \sum_{\mathcal{B}} \lambda_{\mathcal{B}} \text{Tr}_{\mathcal{B}}(\varphi, \bar{\varphi})$, as pointed out in Definition 5, with non-trivial kinetic form $S_0(\varphi, \bar{\varphi}) = \sum_{p \in \mathbb{Z}^D} \bar{\varphi}_{p_1 \dots p_D} E_{p_1 \dots p_D} \varphi_{p_1 \dots p_D}$, $E \neq \text{id}$, with E self-adjoint. The measure $\mathcal{D}[\varphi, \bar{\varphi}]$ in the path integral of such a model is invariant under the action of each factor of the group $U(N) \times \dots \times U(N)$. We take an infinitesimal transformation in its a -th factor

$$W_a \in U(N), \quad W_a = 1 + i\alpha T_a + \mathcal{O}(\alpha^2), \quad T_a^\dagger = T_a,$$

for any $a = 1, \dots, D$ and recast the invariance of the partition function with respect to this group action as the following matrix equation:

$$\frac{\delta \log Z[J, \bar{J}]}{\delta T_a} = 0. \quad (37)$$

In the sequel, we drop the N^{D-1} prefactors, which can be restored by rescaling E and the coupling constant(s). Denote by F the source term $\text{Tr}_2(\bar{\varphi}, J) + \text{Tr}_2(\bar{J}, \varphi)$. One finds by using

$$\frac{\delta F(J, \bar{J})}{\delta (T_a)_{m_a n_a}} = \sum_{p_i \in \mathbb{Z}} [\bar{J}_{p_1 \dots p_{a-1} m_a \dots p_D} \varphi_{p_1 \dots p_{a-1} n_a \dots p_D} - \bar{\varphi}_{p_1 \dots p_{a-1} n_a \dots p_D} J_{p_1 \dots p_{a-1} m_a p_{a+1} \dots p_D}]$$

and

$$\begin{aligned} \frac{\delta S(\varphi, \bar{\varphi})}{\delta (T_a)_{m_a n_a}} &= \frac{\delta S_0(\varphi, \bar{\varphi})}{\delta (T_a)_{m_a n_a}} = \sum_{p_i \in \mathbb{Z}} [\bar{\varphi}_{p_1 \dots p_{a-1} m_a p_{a+1} \dots p_D} E_{p_1 \dots p_{a-1} n_a p_{a+1} \dots p_D} \varphi_{p_1 \dots p_{a-1} n_a p_{a+1} \dots p_D} \\ &\quad - \varphi_{p_1 \dots p_{a-1} n_a p_{a+1} \dots p_D} E_{p_1 \dots p_{a-1} m_a p_{a+1} \dots p_D} \bar{\varphi}_{p_1 \dots p_{a-1} m_a p_{a+1} \dots p_D}], \end{aligned}$$

that eq. (37) implies

$$\begin{aligned} &\int \mathcal{D}[\varphi, \bar{\varphi}] e^{-S+F} \sum_{p_i \in \mathbb{Z}} [(E_{p_1 \dots m_a \dots p_D} - E_{p_1 \dots n_a \dots p_D}) \bar{\varphi}_{p_1 \dots p_{a-1} m_a p_{a+1} \dots p_D} \varphi_{p_1 \dots p_{a-1} n_a p_{a+1} \dots p_D}] \\ &= \int \mathcal{D}[\varphi, \bar{\varphi}] e^{-S+F} \sum_{p_i \in \mathbb{Z}} (\bar{J}_{p_1 \dots m_a \dots p_D} \varphi_{p_1 \dots n_a \dots p_D} - \bar{\varphi}_{p_1 \dots m_a \dots p_D} J_{p_1 \dots n_a \dots p_D}). \end{aligned}$$

Hence, after functional integration, one gets

$$\begin{aligned}
& \sum_{p_i \in \mathbb{Z}} (E_{p_1 \dots p_{a-1} m_a p_{a+1} \dots p_D} - E_{p_1 \dots p_{a-1} n_a p_{a+1} \dots p_D}) \times \\
& \quad \frac{\delta}{\delta \bar{J}_{p_1 \dots p_{a-1} m_a p_{a+1} \dots p_D}} \frac{\delta}{\delta \bar{J}_{p_1 \dots p_{a-1} n_a p_{a+1} \dots p_D}} \exp(-S_{\text{int}}(\delta/\delta \bar{J}, \delta/\delta J)) e^{\sum_{\mathbf{a}} \bar{J}_{\mathbf{a}} E_{\mathbf{a}}^{-1} J_{\mathbf{a}}} \\
& = \sum_{p_i \in \mathbb{Z}} \left(\bar{J}_{p_1 \dots p_{a-1} m_a p_{a+1} \dots p_D} \frac{\delta}{\delta \bar{J}_{p_1 \dots p_{a-1} n_a p_{a+1} \dots p_D}} \right. \\
& \quad \left. - J_{p_1 \dots p_{a-1} n_a p_{a+1} \dots p_D} \frac{\delta}{\delta J_{p_1 \dots p_{a-1} m_a p_{a+1} \dots p_D}} \right) \exp(-S_{\text{int}}(\delta/\delta \bar{J}, \delta/\delta J)) e^{\sum_{\mathbf{a}} \bar{J}_{\mathbf{a}} E_{\mathbf{a}}^{-1} J_{\mathbf{a}}}.
\end{aligned}$$

We have identified $\delta/\delta J_{\mathbf{p}}$ with $\bar{\varphi}_{\mathbf{p}}$ and $\delta/\delta \bar{J}_{\mathbf{p}}$ with $\varphi_{\mathbf{p}}$. Thus, the preliminary WTI reads

$$\begin{aligned}
& \sum_{p_i \in \mathbb{Z}} (E_{p_1 \dots p_{a-1} m_a p_{a+1} \dots p_D} - E_{p_1 \dots p_{a-1} n_a p_{a+1} \dots p_D}) \frac{\delta^2 Z[J, \bar{J}]}{\delta J_{p_1 \dots p_{a-1} m_a p_{a+1} \dots p_D} \bar{J}_{p_1 \dots p_{a-1} n_a p_{a+1} \dots p_D}} \\
& = \sum_{p_i \in \mathbb{Z}} \left\{ \bar{J}_{p_1 \dots p_{a-1} m_a p_{a+1} \dots p_D} \frac{\delta}{\delta \bar{J}_{p_1 \dots p_{a-1} n_a p_{a+1} \dots p_D}} - J_{p_1 \dots p_{a-1} n_a p_{a+1} \dots p_D} \frac{\delta}{\delta J_{p_1 \dots p_{a-1} m_a p_{a+1} \dots p_D}} \right\} Z[J, \bar{J}].
\end{aligned} \tag{38}$$

As mentioned in the introduction, a Ward-Takahashi Identity was obtained already in [34,43]. Namely, if we derive eq. (38) with respect to

$$\frac{\delta}{\delta \bar{J}_{q_1 \dots q_{a-1} n_a q_{a+1} \dots q_D}} \frac{\delta}{\delta \bar{J}_{q_1 \dots q_{a-1} m_a q_{a+1} \dots q_D}},$$

we obtain a relation between “the 4-point function” and the following difference of 2-point functions (here also expressed in graphical notation, which we wish to surrogate by $G_{\mathcal{B}}^{(2k)}$'s):

$$\begin{aligned}
& \text{Diagram 1} = \text{Diagram 2} - \text{Diagram 3}
\end{aligned} \tag{39}$$

The term in the LHS is defined as follows:

$$\sum_{p_i \in \mathbb{Z}} (E_{p_1 \dots p_{a-1} m_a p_{a+1} \dots p_D} - E_{p_1 \dots p_{a-1} n_a p_{a+1} \dots p_D}) \cdot \text{Diagram 4}$$

The issue is that for arbitrary degree D , there exist not only those 4-point functions. For instance, for $D = 3$ there are four 4-point functions and for $D = 4$ there are eight 4-point functions. So we opt for an analytical method that shows this missing structure. With that aim, we need to solve eq. (38) for the double derivative of Z . One can split $\delta^2 Z/\delta J_{p_1 \dots m_a \dots p_D} \delta \bar{J}_{p_1 \dots n_a \dots p_D}$ as a sum of a *singular* part, denoted by $Y_{m_a}^{(a)}$ and defined by being all the terms in there proportional to $\delta_{m_a n_a}$, and the “*regular contribution*”. While the latter can be read off, no vestige from $Y_{m_a}^{(a)}$ remains, for it is annihilated by $E_{p_1 \dots m_a \dots p_D} - E_{p_1 \dots n_a \dots p_D}$ in eq. (38). A direct approach with graphs does not consider those contributions.

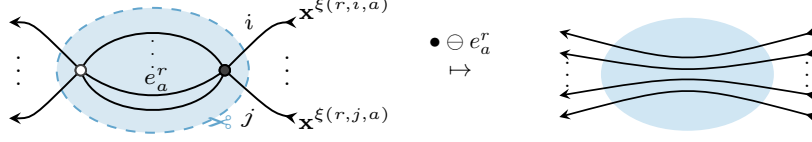


Fig. 4 On the definition of the graph $\mathcal{B} \ominus e_a^r$ in an arbitrary number of colors, being a one of them. The graph on the left locally represents the a -colored edge e_a^r and the vertices $s(e_a^r)$ and $t(e_a^r)$. The dipole that they form is removed and broken edges are colorwise glued (right graph)

Of course, our method reduces to the result to (39) when $m_a \neq n_a$. In order to find the singular contributions, we need to introduce some terminology.

Definition 9. Let $\mathcal{B} \in \text{IIGrph}_{c,D}$ and $e \in \mathcal{B}^{(1)}$. The graph $\mathcal{B} \ominus e$ is defined as the graph that is formed after removal of *all* the edges between the two vertices e is attached at, and by subsequently colorwise gluing the remaining edges. More formally, we let

$$s^{-1}(s(e)) =: (s^{-1}(s(e)) \cap t^{-1}(t(e))) \cup A_{s(e)}, \quad t^{-1}(t(e)) =: (s^{-1}(s(e)) \cap t^{-1}(t(e))) \cup A_{t(e)}.$$

We let $I(e)$ be the set of colors in of the edges $s^{-1}(s(e)) \cap t^{-1}(t(e))$. Then the coloring of $A_{s(e)}$ and of $A_{t(e)}$ agrees, both being equal to $\{1, \dots, D\} \setminus I(e)$. We define $\mathcal{B} \ominus e$ by

$$\begin{aligned} (\mathcal{B} \ominus e)^{(0)} &= \mathcal{B}^{(0)} \setminus \{s(e), t(e)\}, \\ (\mathcal{B} \ominus e)^{(1)} &= \mathcal{B}^{(1)} \setminus (s^{-1}(s(e)) \cap t^{-1}(t(e))) / (A_{s(e)} \sim_c A_{t(e)}), \end{aligned}$$

where $f \sim_c g$ iff $f \in A_{s(e)}$ and $g \in A_{t(e)}$ have the same color; see Figure 4. By definition $\mathbb{J}(\emptyset) = 1$, so $\mathbb{J}((\ominus) \ominus e) = 1$ for any edge e of \ominus .

Keeping in mind the Ward-Takahashi Identity for a fixed color a and fixing the entries $(m_a n_a)$ of the generator T_a , we shall define an operator $\Delta_{m_a, r}^{\mathcal{B}} : \mathbb{C}^{((\mathbb{Z}^D)^k)} \rightarrow \mathbb{C}^{((\mathbb{Z}^D)^{k-1})}$. In order to do so, we need first to introduce more notation concerning the edge removal $\mathcal{B} \ominus e_a^r$ as in Definition 9. Let $\mathcal{B} \in \partial \mathfrak{F}\eta\eta_D(V)$ with $|\mathcal{B}^{(0)}| \geq 4$. To stress the essence of the discussion we assume that the boundary graph \mathcal{B} is connected and leave the extension of this discussion of the full disconnected boundary sector to Appendix B. We label the (say) white vertices of a boundary graph, \mathcal{B} , and of $\mathcal{B} \ominus e_c^j$ by momenta $\mathbf{x}^i \in \mathbb{Z}^D$ and denote by e_c^j the edge of color c at the vertex $\mathbf{x}^j \in \mathbb{Z}^D$:

$$\mathcal{B}_w^{(0)} = (\mathbf{x}^1, \dots, \mathbf{x}^k), \quad (\mathcal{B} \ominus e_a^r)_w^{(0)} = (\mathbf{x}^1, \dots, \widehat{\mathbf{x}^r}, \dots, \mathbf{x}^k) = (\mathbf{y}^1, \dots, \mathbf{y}^{k-1}), \quad (\mathbf{y}^l \in \mathbb{Z}^D). \quad (40)$$

When $\mathcal{B} \ominus e_a^r$ is formed out of \mathcal{B} , one removes, in particular, a single black vertex $t(e_a^r)$. By regularity, certain edge $e_i^{\xi(r, i, a)}$ of color i is in $A_{t(e_a^r)}$, and this edge determines again a unique white vertex $\mathbf{x}^{\xi(r, i, a)}$, $1 \leq \xi(r, i, a) \leq k$, by the relation $s(e_i^{\xi(r, i, a)}) = \mathbf{x}^{\xi(r, i, a)} \in \mathcal{B}^{(0)}$. If we pick $i \in \{1, \dots, D\} \setminus I(e_a^r)$, which is the color-set of $A_{t(e_a^r)}$, one has $\xi(r, i, a) \neq r$ and, furthermore, $\mathbf{x}^{\xi(r, i, a)}$ remains in $\mathcal{B} \ominus e_a^r$. This vertex is renamed $\mathbf{y}^{\kappa(r, i, a)} \in (\mathcal{B} \ominus e_a^r)^{(0)}$ following (40), whence

$$\kappa(r, i, a) = \begin{cases} \xi(r, i, a) & \text{if } \xi(r, i, a) < r, \\ \xi(r, i, a) - 1 & \text{if } \xi(r, i, a) > r. \end{cases} \quad (41)$$

Definition 10. Keeping fixed the color a and entries $(m_a n_a)$ of a generator of the a -th summand $\mathfrak{u}(N)$ in the Lie algebra $\text{Lie}(\text{U}(N) \times \dots \times \text{U}(N))$, we consider the Green's function $G_{\mathcal{B}}^{(2k)} : (\mathbb{Z}^D)^k \rightarrow \mathbb{C}$ associated to a boundary-graph $\mathcal{B} \in \partial(\mathfrak{F}\eta\eta_D(V))$. For any integer r ,

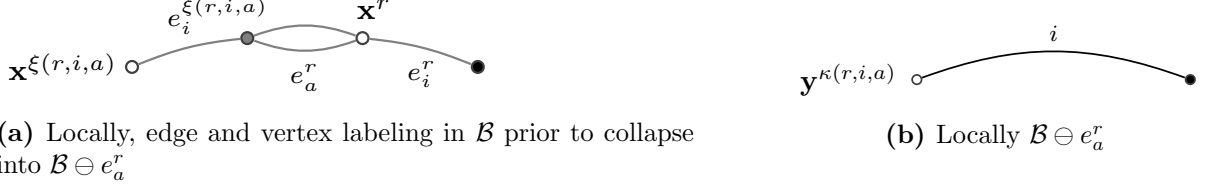


Fig. 5 On the notation for the definition of $\Delta_{m_a,r}^{\mathcal{B}}$. All gray edges in the leftmost figure disappear and merge into a single color- i edge. The two vertices $t(e_a^r)$ and \mathbf{x}^r disappear as well

$1 \leq r \leq k$, we define the function $\Delta_{m_a,r}^{\mathcal{B}} G_{\mathcal{B}}^{(2k)} : (\mathbb{Z}^D)^{k-1} \rightarrow \mathbb{C}$ by

$$(\Delta_{m_a,r}^{\mathcal{B}} G_{\mathcal{B}}^{(2k)})(\mathbf{y}^1, \dots, \mathbf{y}^k) = \begin{cases} \sum_{q_h} G_{\mathcal{B}}^{(2k)}(\mathbf{y}^1, \dots, \mathbf{y}^{r-1}, \mathbf{z}^r(m_a, \mathbf{q}), \mathbf{y}^r, \dots, \mathbf{y}^k) & \text{if } I(e_a^r) \neq \{a\}, \\ G_{\mathcal{B}}^{(2k)}(\mathbf{y}^1, \dots, \mathbf{y}^{r-1}, \mathbf{z}^r(m_a, \mathbf{q}), \mathbf{y}^r, \dots, \mathbf{y}^k) & \text{if } I(e_a^r) = \{a\}, \end{cases}$$

where $I(e_a^r)$ is the *set of colors* of the edges $s^{-1}(s(e_a^r)) \cap t^{-1}(t(e_a^r))$ and q_h , for any $h \in I(e_a^r) \setminus \{a\}$, is a dummy variable to be summed over. The momentum $\mathbf{z}^r \in \mathbb{Z}^D$ has entries defined by:

$$z_i^r = \begin{cases} m_a & \text{if } i = a, \\ q_i & \text{if } i \in I(e_a^r) \setminus \{a\}, \\ y_i^{\kappa(r,i,a)} & \text{if } i \in \text{colors of } A_{t(e_a^r)} = \{1, \dots, D\} \setminus I(e_a^r), \end{cases}$$

where $\mathbf{y}^{\kappa(r,i,a)}$ ($1 \leq \kappa(r,i,a) < k$) is the white vertex $\mathcal{B} \ominus e_a^r$ defined in (41) (see also Fig. 5). This definition depends on the labeling of the vertices. However, the pairing $\langle\langle G_{\mathcal{B}}^{(2k)}, \mathcal{B} \rangle\rangle_{m_a}$ defined as follows does not:

$$\langle\langle G_{\mathcal{B}}^{(2k)}, \mathcal{B} \rangle\rangle_{m_a} := \sum_{r=1}^k \left(\Delta_{m_a,r}^{\mathcal{B}} G_{\mathcal{B}}^{(2k)} \right) \star \mathbb{J}(\mathcal{B} \ominus e_a^r). \quad (42)$$

Example 10. We consider the *empty graph* \emptyset as colored. According to Definition 9

$$\langle\langle G_{\ominus}^{(2)}, \ominus \rangle\rangle_{m_a} = (\Delta_{a,1} G_{\ominus}^{(2)}) \star (\ominus \ominus (e_1^a)) = \sum_{\substack{q_b, q_c \in \mathbb{Z}, \\ b \neq a \neq c}} G_{\ominus}^{(2)}(\text{color-ordering of } (m_a, q_b, q_c)) \mathbb{J}(\emptyset),$$

so, for instance if $a = 2$, one has:

$$\langle\langle G_{\ominus}^{(2)}, \ominus \rangle\rangle_{m_2} = \sum_{q_1, q_3 \in \mathbb{Z}} G_{\ominus}^{(2)}(q_1, m_2, q_3).$$

To clear up the notation in $\Delta_{m_a,r}^{\mathcal{K}}$, consider the graph \mathcal{K} in Figure 6 and examples concerning the edge removal, here for, say, e_1^1 and e_2^3 . One has then $\xi(1, 3, 1) = 2$, $\xi(3, 1, 2) = 5$ and $\xi(3, 3, 2) = 1$. Therefore $\kappa(1, 3, 1) = 1$, $\kappa(3, 1, 2) = 4$ and $\kappa(3, 3, 2) = 1$. Accordingly:

$$(\Delta_{m_1, r=1}^{\mathcal{K}} G_{\mathcal{K}}^{(10)})(\mathbf{y}^1, \dots, \mathbf{y}^4) = \sum_{q_2} G_{\mathcal{K}}^{(10)}(\mathbf{z}^1, \mathbf{y}^1, \dots, \mathbf{y}^4) = \sum_{q_2} G_{\mathcal{K}}^{(10)}(m_1, q_2, y_3^1, \mathbf{y}^1, \dots, \mathbf{y}^4),$$

$$(\Delta_{m_2, r=3}^{\mathcal{K}} G_{\mathcal{K}}^{(10)})(\mathbf{y}^1, \dots, \mathbf{y}^4) = G_{\mathcal{K}}^{(10)}(\mathbf{y}^1, \mathbf{y}^2, \mathbf{z}^3, \mathbf{y}^3, \mathbf{y}^4) = G_{\mathcal{K}}^{(10)}(\mathbf{y}^1, \mathbf{y}^2, y_1^4, m_2, y_3^1, \mathbf{y}^3, \mathbf{y}^4).$$

The usefulness of this operation shall be clear in the proof of the next result.

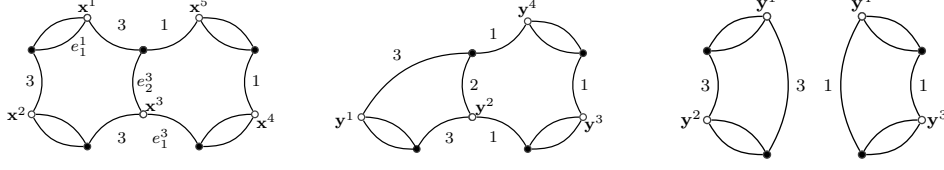


Fig. 6 Concerning example 10, from left to right: \mathcal{K} , $\mathcal{K} \ominus e_1^1$ and $\mathcal{K} \ominus e_2^3$

Theorem 2 (Full Ward-Takahashi Identity). *Consider an arbitrary rank- D tensor model whose kinetic form E in $\text{Tr}_2(\bar{\varphi}, E\varphi)$ (see Def. 5) obeys*

$$E_{p_1 \dots p_{a-1} m_a p_{a+1} \dots p_D} - E_{p_1 \dots p_{a-1} n_a p_{a+1} \dots p_D} = E(m_a, n_a) \quad \text{for each } a = 1, \dots, D$$

i.e. this difference does not depend on the momenta $p_1, \dots, \hat{p}_a, \dots, p_D$. Then the partition function $Z[J, \bar{J}]$ of that model satisfies

$$\begin{aligned} & \sum_{p_i \in \mathbb{Z}} \frac{\delta^2 Z[J, \bar{J}]}{\delta J_{p_1 \dots p_{a-1} m_a p_{a+1} \dots p_D} \delta \bar{J}_{p_1 \dots p_{a-1} n_a p_{a+1} \dots p_D}} - \left(\delta_{m_a n_a} Y_{m_a}^{(a)}[J, \bar{J}] \right) \cdot Z[J, \bar{J}] \\ &= \sum_{p_i \in \mathbb{Z}} \frac{1}{E_{p_1 \dots m_a \dots p_D} - E_{p_1 \dots n_a \dots p_D}} \left(\bar{J}_{p_1 \dots m_a \dots p_D} \frac{\delta}{\delta \bar{J}_{p_1 \dots n_a \dots p_D}} - J_{p_1 \dots n_a \dots p_D} \frac{\delta}{\delta J_{p_1 \dots m_a \dots p_D}} \right) Z[J, \bar{J}] \end{aligned} \quad (43)$$

where

$$\begin{aligned} Y_{m_a}^{(a)}[J, \bar{J}] &:= \sum_{k=1}^{\infty} \sum_{\substack{\mathcal{B} \in \partial \mathfrak{F} \epsilon \eta n_D(V) \\ 2k = \#(\mathcal{B}^{(0)})}} \frac{1}{|\text{Aut}_{\mathbb{C}}(\mathcal{B})|} \langle\langle G_{\mathcal{B}}^{(2k)}, \mathcal{B} \rangle\rangle_{m_a} \\ &= \sum_{k=1}^{\infty} \sum_{\substack{\mathcal{B} \in \partial \mathfrak{F} \epsilon \eta n_D(V) \\ 2k = \#(\mathcal{B}^{(0)})}} \frac{1}{|\text{Aut}_{\mathbb{C}}(\mathcal{B})|} \sum_{r=1}^k \left(\Delta_{m_a, r}^{\mathcal{B}} G_{\mathcal{B}}^{(2k)} \right) \star \mathbb{J}(\mathcal{B} \ominus e_r^r). \end{aligned} \quad (44)$$

Proof. In the next equation

$$\frac{1}{Z[J, \bar{J}]} \frac{\delta^2 Z[J, \bar{J}]}{\delta J_{p_1 \dots m_a \dots p_D} \delta \bar{J}_{p_1 \dots n_a \dots p_D}} = \frac{\delta^2 W[J, \bar{J}]}{\delta J_{p_1 \dots m_a \dots p_D} \delta \bar{J}_{p_1 \dots n_a \dots p_D}} + \frac{\delta W[J, \bar{J}]}{\delta J_{p_1 \dots m_a \dots p_D}} \frac{\delta W[J, \bar{J}]}{\delta \bar{J}_{p_1 \dots n_a \dots p_D}} \quad (45)$$

we want to detect the terms that provide a singular contribution $\delta_{m_a n_a}$ to the double derivative in the LHS. We group them in $\mathcal{Y}_{p_1, \dots, m_a, \dots, p_D}^{(a)}[J, \bar{J}]$. Notice that in the RHS, the product of derivatives cannot yield terms proportional to $\delta_{m_a n_a}$. Hence, all terms in $Y_{m_a}^{(a)}[J, \bar{J}]$ come from the sum over the momenta $(p_1, \dots, \hat{p}_a, \dots, p_D) \in \mathbb{Z}^{D-1}$ of the term $\mathcal{Y}_{p_1, \dots, m_a, \dots, p_D}^{(a)}$ in the double derivative of $W[J, \bar{J}]$. By definition of $\mathcal{Y}_{p_1, \dots, m_a, \dots, p_D}^{(a)}$, we can then write

$$\frac{\delta^2 W[J, \bar{J}]}{\delta J_{p_1 \dots m_a \dots p_D} \delta \bar{J}_{p_1 \dots n_a \dots p_D}} = \delta_{m_a n_a} \mathcal{Y}_{p_1, \dots, m_a, \dots, p_D}^{(a)}[J, \bar{J}] \cdot Z[J, \bar{J}] + \mathcal{X}_{p_1, \dots, [m_a n_a], \dots, p_D}^{(a)}[J, \bar{J}],$$

where $\mathcal{X}_{p_1, \dots, [m_a n_a], \dots, p_D}^{(a)}[J, \bar{J}]$ contains only regular terms. We could compute them, but we are only interested in the term proportional to $\delta_{m_a n_a}$. We let the two derivatives act on the expansion of the free energy in boundary graphs, eq. (35). Ignoring the symmetry factor, the

derivatives acting on a single term $G_{\mathcal{B}}^{(2k)} \star \mathbb{J}(\mathcal{B})$ lead to

$$\begin{aligned} & \sum_{p_i \in \mathbb{Z}} \sum_{\mathbf{x}^1, \dots, \mathbf{x}^k} G_{\mathcal{B}}^{(2k)}(\mathbf{x}^1, \dots, \mathbf{x}^k) \frac{\delta^2 \mathbb{J}(\mathcal{B}\{\mathbf{x}^1, \dots, \mathbf{x}^k\})}{\delta J_{p_1 \dots m_a \dots p_D} \delta \bar{J}_{p_1 \dots n_a \dots p_D}} \\ &= \sum_{p_i \in \mathbb{Z}} \sum_{\mathbf{x}^1, \dots, \mathbf{x}^k} G_{\mathcal{B}}^{(2k)}(\mathbf{x}^1, \dots, \mathbf{x}^k) \frac{\delta}{\delta J_{p_1 \dots m_a \dots p_D}} (J_{\mathbf{x}^1} \cdots J_{\mathbf{x}^k}) \frac{\delta}{\delta \bar{J}_{p_1 \dots n_a \dots p_D}} (\bar{J}_{\mathbf{w}^1} \cdots \bar{J}_{\mathbf{w}^k}). \end{aligned} \quad (46)$$

Here, we have the freedom to choose the order of the labels of the black vertices by the momenta \mathbf{w}^i in such a way that $w_a^r = x_a^r$. Hence $t(e_a^r)$ has the label \mathbf{w}^r , whereas $s(e_a^r)$ is labeled⁸ by \mathbf{x}^r . We now focus on the product of the two derivative terms:

$$\begin{aligned} & \left(\sum_{r=1}^k \{ \delta_{p_1}^{x_1^r} \cdots \delta_{p_{a-1}}^{x_{a-1}^r} \delta_{m_a}^{x_a^r} \delta_{p_{a+1}}^{x_{a+1}^r} \cdots \delta_{p_D}^{x_D^r} \} J_{\mathbf{x}^1} \cdots \widehat{J_{\mathbf{x}^r}} \cdots J_{\mathbf{x}^k} \right) \\ & \left(\sum_{l=1}^k \{ \delta_{p_1}^{w_1^l} \cdots \delta_{p_{a-1}}^{w_{a-1}^l} \delta_{n_a}^{w_a^l} \delta_{p_{a+1}}^{w_{a+1}^l} \cdots \delta_{p_D}^{w_D^l} \} \bar{J}_{\mathbf{w}^1} \cdots \widehat{\bar{J}_{\mathbf{w}^l}} \cdots \bar{J}_{\mathbf{w}^k} \right). \end{aligned} \quad (47)$$

For the term $(\delta_{p_1}^{x_1^r} \cdots \delta_{m_a}^{x_a^r} \cdots \delta_{p_D}^{x_D^r} J_{\mathbf{x}^1} \cdots \widehat{J_{\mathbf{x}^r}} \cdots J_{\mathbf{x}^k}) (\delta_{p_1}^{w_1^l} \cdots \delta_{n_a}^{w_a^l} \cdots \delta_{p_D}^{w_D^l} \cdots \bar{J}_{\mathbf{w}^1} \cdots \widehat{\bar{J}_{\mathbf{w}^l}} \cdots \bar{J}_{\mathbf{w}^k})$ to contribute to $\mathcal{Y}_{\dots}^{(a)}$ one requires $x_a^r = w_a^l$. But by definition of \mathbf{w}^l , $w_a^l = x_a^l$. Then from (47), the terms that contribute to $\mathcal{Y}_{\dots}^{(a)}$ are

$$\sum_{r=1}^k \left(\{ \delta_{p_1}^{x_1^r} \cdots \delta_{m_a}^{x_a^r} \cdots \delta_{p_D}^{x_D^r} \} J_{\mathbf{x}^1} \cdots \widehat{J_{\mathbf{x}^r}} \cdots J_{\mathbf{x}^k} \right) \left(\{ \delta_{p_1}^{w_1^r} \cdots \delta_{n_a}^{w_a^r} \cdots \delta_{p_D}^{w_D^r} \} \bar{J}_{\mathbf{w}^1} \cdots \widehat{\bar{J}_{\mathbf{w}^r}} \cdots \bar{J}_{\mathbf{w}^k} \right)$$

We rewrite the δ in the r -th term of this sum as

$$\delta_{m_a}^{n_a} \cdot \prod_{j \in A_t(e_a^r)} \delta_{p_j}^{x_j^r} \delta_{p_j}^{w_j^r} \prod_{i \in I} \delta_{p_i}^{x_i^r} \delta_{p_i}^{w_i^r} = \delta_{m_a}^{n_a} \cdot \prod_{j \in A_t(e_a^r)} \delta_{p_j}^{x_j^r} \delta_{p_j}^{\xi(r,j,a)} \prod_{i \in I} \delta_{p_i}^{x_i^r} \delta_{p_i}^{w_i^r}$$

(see Def. 10). Then

$$\begin{aligned} Y_{m_a}^{(a)}[J, \bar{J}] &= \sum_{p_1, \dots, \widehat{p_a}, \dots, p_D \in \mathbb{Z}} \mathcal{Y}_{p_1, \dots, m_a, \dots, p_D}^{(a)} \\ &= \sum_{k=1}^{\infty} \sum_{\mathcal{B} \in \partial(\mathfrak{F}\text{e}\eta\text{n}(V))} \frac{1}{|\text{Aut}_c(\mathcal{B})|} \sum_{p_1, \dots, \widehat{p_a}, \dots, p_D} \sum_{r=1}^k \sum_{\mathbf{x}^1, \dots, \widehat{\mathbf{x}^r}, \dots, \mathbf{x}^k} \\ & \quad \prod_{j \in A_t(e_a^r)} \delta_{p_j}^{x_j^r} \delta_{p_j}^{\xi(r,j,a)} \prod_{i \in I} \delta_{p_i}^{x_i^r} \delta_{p_i}^{w_i^r} G_{\mathcal{B}}^{(2k)}(\mathbf{x}^1, \dots, \widehat{\mathbf{x}^r}, \dots, \mathbf{x}^k) \\ & \quad \times (J_{\mathbf{x}^1} \cdots \widehat{J_{\mathbf{x}^r}} \cdots J_{\mathbf{x}^k}) \cdot (\bar{J}_{\mathbf{w}^1} \cdots \widehat{\bar{J}_{\mathbf{w}^r}} \cdots \bar{J}_{\mathbf{w}^k}) \end{aligned}$$

and by renaming the indices and using Definitions 9 and 10 one finally gets

$$\begin{aligned} Y_{m_a}^{(a)}[J, \bar{J}] &= \sum_{k=1}^{\infty} \sum_{\mathcal{B} \in \partial(\mathfrak{F}\text{e}\eta\text{n}(V))} \frac{1}{|\text{Aut}_c(\mathcal{B})|} \sum_{r=1}^k \sum_{\mathbf{y}^1, \dots, \mathbf{y}^{k-1}} G_{\mathcal{B}}^{(2k)}(\mathbf{y}^1, \dots, \mathbf{z}^r, \dots, \mathbf{y}^{k-1}) \\ & \quad \cdot \mathbb{J}(\mathcal{B} \ominus e_r^a)(\mathbf{y}^1, \dots, \mathbf{y}^{k-1}) \\ &= \sum_{k=1}^{\infty} \sum_{\mathcal{B} \in \partial(\mathfrak{F}\text{e}\eta\text{n}(V))} \frac{1}{|\text{Aut}_c(\mathcal{B})|} \sum_{r=1}^k (\Delta_{r, m_a} G_{\mathcal{B}}^{(2k)})(\mathbf{y}^1, \dots, \mathbf{y}^{k-1}) \\ & \quad \cdot \mathbb{J}((\mathcal{B} \ominus e_r^a))\{\mathbf{y}^1, \dots, \mathbf{y}^{k-1}\} \end{aligned}$$

⁸The condition $s(e_c^r) = \mathbf{x}^r$ is actually redundant, since by definition e_c^r is the edge of color c attached at \mathbf{x}^r , but we write this down for sake of clarity.

$$\begin{aligned}
&= \sum_{k=1}^{\infty} \sum_{\mathcal{B} \in \partial(\mathfrak{F}\text{en}(V))} \frac{1}{|\text{Aut}_c(\mathcal{B})|} \sum_{r=1}^k (\Delta_{r,m_a} G_{\mathcal{B}}^{(2k)}) \star (\mathcal{B} \ominus e_a^r) \\
&= \sum_{k=1}^{\infty} \sum_{\mathcal{B} \in \partial(\mathfrak{F}\text{en}(V))} \frac{1}{|\text{Aut}_c(\mathcal{B})|} \langle\langle G_{\mathcal{B}}^{(2k)}, \mathcal{B} \rangle\rangle_{m_a}. \quad \square
\end{aligned}$$

An explicit expansion of $Y_{m_a}^{(a)}[J, \bar{J}]$ is, of course, also more useful. We derive it for $D = 3$ and assume throughout that $\{a, c, d\} = \{1, 2, 3\}$ as sets; also for sake of notation, we drop \mathcal{B} in $\Delta_{m_a, r}^{\mathcal{B}}$ when this operator acts on the correlation function that already shows dependence on \mathcal{B} . The expansion⁹ reads

$$\begin{aligned}
Y_{m_a}^{(a)}[J, \bar{J}] &= \Delta_{m_a, 1} G_{\ominus}^{(2)} \star \mathbf{1} \\
&+ \frac{1}{2} \sum_{r=1}^2 (\Delta_{m_a, r} G_{|\ominus|_{\ominus}}^{(4)} + \Delta_{m_a, r} G_{1\boxplus_1}^{(4)} + \Delta_{m_a, r} G_{2\boxplus_2}^{(4)} + \Delta_{m_a, r} G_{3\boxplus_3}^{(4)}) \star \mathbb{J}(\ominus) \\
&+ \frac{1}{3} \sum_{r=1}^3 \sum_{i=1}^3 (\Delta_{m_a, r} G_{\text{hook}_i}^{(6)}) \star \mathbb{J}(\text{hook}_i) + \frac{1}{3} \sum_{r=1}^3 (\Delta_{m_a, r} G_{\boxtimes}^{(6)}) \star \mathbb{J}(\boxtimes) \\
&+ \sum_{c \neq a} (\Delta_{m_a, 1} G_{\overline{\text{hook}}_c}^{(6)}) \star \mathbb{J}(\overline{\text{hook}}_c) + (\Delta_{m_a, 2} G_{\overline{\text{hook}}_c}^{(6)}) \star \mathbb{J}(\overline{\text{hook}}_c) \tag{48} \\
&+ (\Delta_{m_a, 3} G_{\overline{\text{hook}}_c}^{(6)}) \star \mathbb{J}(\overline{\text{hook}}_c) + (\Delta_{m_a, 1} G_{c\overline{\text{hook}}}^{(6)}) \star \mathbb{J}(c\overline{\text{hook}}) + (\Delta_{m_a, 2} G_{c\overline{\text{hook}}}^{(6)}) \star \mathbb{J}(c\overline{\text{hook}}) \\
&+ (\Delta_{m_a, 3} G_{c\overline{\text{hook}}}^{(6)}) \star \mathbb{J}(c\overline{\text{hook}}) + \frac{1}{3!} \sum_{r=1}^3 (\Delta_{m_a, r} G_{|\ominus|_{\ominus}|\ominus|}^{(6)}) \star \mathbb{J}(\ominus \sqcup \ominus) \\
&+ \frac{1}{2} (\Delta_{m_a, 1} G_{|\ominus|_{\overline{\text{hook}}_c}}^{(6)}) \star \mathbb{J}(|\ominus|_{\overline{\text{hook}}_c}) + \frac{1}{2} \sum_{r=2,3} (\Delta_{m_a, r} G_{|\ominus|_{\overline{\text{hook}}_c}}^{(6)}) \star \mathbb{J}(\ominus \sqcup \ominus) + \mathcal{O}(6).
\end{aligned}$$

In this expression, for any two white vertices of a boundary graph \mathcal{B} that are not connected by an element $\tau \in \mathfrak{S}_k$ that can be lifted to $\hat{\tau} \in \text{Aut}_c(\mathcal{B})$, a convention regarding their ordering should be set. The ordering of the arguments $(\mathbf{x}^1, \dots, \mathbf{x}^k)$ of $G_{\mathcal{B}}^{(2k)}(\mathbf{x}^1, \dots, \mathbf{x}^k)$ and the labeling of $\mathcal{B}_w^{(0)}$ by these obeys in eq. (48) following convention:

if a vertex $v_i \in \mathcal{B}_w^{(0)}$ is labeled by the momentum \mathbf{x}^i and appears to the left of the vertex $v_j \in \mathcal{B}_w^{(0)}$ labeled by \mathbf{x}^j , then $i < j$.

Hence $c\overline{\text{hook}}_c$ and $\overline{\text{hook}}_c^c$ are decidedly different and attention should be paid in the order of the arguments. For boundary graphs like hook_c^c or \boxtimes (whose drawing would lead to an ambiguous rule) we can dispense with that convention, $\text{Aut}_c(\mathcal{B})$ ensures the well-definedness of $G_{\mathcal{B}}^{(2k)}$.

6.2. Two-Point function Schwinger-Dyson equations. One gets the Schwinger-Dyson equation¹⁰ for the two-point function from $G_{\ominus}^{(2)}(\mathbf{a}) = Z_0^{-1} \delta^2(Z[J, \bar{J}]) / \delta \bar{J}_a \delta J_a |_{J=\bar{J}=0}$ and by using eq. (45) with $m_a = n_a$ and $J = \bar{J} = 0$. Here $Z_0 = Z[0, 0]$. If one formally performs the

⁹In [36] we found the $\mathcal{O}(6)$ terms and treated also rank $D = 4, 5$ theories.

¹⁰Our approach to SDEs differs from that by Gurău [23]. He found constrictions in form of differential operators acting on $Z[J, \bar{J}]$ and showed that they satisfy a Lie algebra that generalizes Virasoro algebra [22] and is indexed by rooted colored trees. We here work with $\log Z[J, \bar{J}]$ and our approach is not that general but we have those concrete operators derived from Theorem 2.

functional integral, one gets for $\mathbf{a} \neq 0 \in \mathbb{Z}^D$

$$\begin{aligned}
G_{\Theta}^{(2)}(\mathbf{a}) &= \frac{1}{Z_0} \left\{ \frac{\delta}{\delta J_{\mathbf{a}}} \left[\exp(-S_{\text{int}}(\delta/\delta \bar{J}, \delta/\delta J)) \frac{1}{E_{\mathbf{a}}} J_{\mathbf{a}} e^{\sum_{\mathbf{q}} \bar{J}_{\mathbf{q}} E_{\mathbf{q}}^{-1} J_{\mathbf{q}}} \right] \right\}_{J=\bar{J}=0} \\
&= \frac{1}{Z_0 E_{\mathbf{a}}} \left[\exp(-S_{\text{int}}(\delta/\delta \bar{J}, \delta/\delta J)) e^{\sum_{\mathbf{q}} \bar{J}_{\mathbf{q}} E_{\mathbf{q}}^{-1} J_{\mathbf{q}}} \right]_{J=\bar{J}=0} \\
&\quad + \frac{1}{Z_0 E_{\mathbf{a}}} \left(\exp(-S_{\text{int}}(\delta/\delta \bar{J}, \delta/\delta J)) J_{\mathbf{a}} \frac{\delta}{\delta J_{\mathbf{a}}} e^{\sum_{\mathbf{q}} \bar{J}_{\mathbf{q}} E_{\mathbf{q}}^{-1} J_{\mathbf{q}}} \right)_{J=\bar{J}=0} \\
&= \frac{1}{E_{\mathbf{a}}} + \frac{1}{Z_0} \frac{1}{E_{\mathbf{a}}} \left(\bar{\varphi}_{\mathbf{a}} \frac{\partial}{\partial \bar{\varphi}_{\mathbf{a}}} (S_{\text{int}}(\varphi, \bar{\varphi})) \right)_{\varphi^b \rightarrow \delta/\delta J^{\sharp}} Z[J, \bar{J}],
\end{aligned} \tag{49}$$

being $\{x^b, y^{\sharp}\} = \{\bar{x}, y\}$ or $\{x, \bar{y}\}$. Here we make a crucial assumption, which is not needed in the rank-2 theory (matrix models [21, Sec. 2]). We suppose that the interaction S_{int} satisfies the following condition: *each* (graph-)vertex of *each* single interaction vertex lies, for certain color $a = 1, \dots, D$, on a subgraph of the following type (in the tensor models parlance, “melonic insertion”):



Such is the case for the melonic φ_m^4 -model in arbitrary rank. Then, whatever $S_{\text{int}}(\varphi, \bar{\varphi})$ is, in eq. (49) the term $(\bar{\varphi}_{\mathbf{a}} \frac{\partial}{\partial \bar{\varphi}_{\mathbf{a}}} S_{\text{int}}(\varphi, \bar{\varphi}))_{\varphi^b \rightarrow \delta/\delta J^{\sharp}}$ contains, for each order- $2r$ monomial, derivatives of $Z[J, \bar{J}]$ of the form

$$\frac{\delta^{2r-2}}{\delta \mathbb{J}(\mathcal{B}^{\times})(\mathbf{a}^1, \dots, \mathbf{a}^{r-1})} \left(\sum_{p_i \in \mathbb{Z}, i \neq a} \frac{\delta^2 Z[J, \bar{J}]}{\delta J_{p_1 \dots p_{a-1} m_a p_{a+1} \dots p_D} \delta \bar{J}_{p_1 \dots p_{a-1} n_a p_{a+1} \dots p_D}} \right),$$

where $\mathbb{J}(\mathcal{B}^{\times})$ is a generic broken graph, for the moment irrelevant. Here the WTI for the color a is handy.

6.3. Schwinger-Dyson equations for the φ_3^4 -model. We give an example with a concrete theory, which can be connected with Tensor Group Field Theory (TGFT). CTMs with non-trivial kinetic term usually are originated by TGFT-actions

$$\begin{aligned}
S[\varphi, \bar{\varphi}] &= \frac{1}{2} \int_{\mathbb{T}^3 \times \mathbb{T}^3} d\mathbf{g} d\mathbf{g}' \bar{\varphi}(g_1, g_2, g_3) K(\mathbf{g}, \mathbf{g}') \varphi(g'_1, g'_2, g'_3) \\
&\quad + \frac{\lambda}{4} \int_{\mathbb{T}^{12}} \prod_{\alpha} d\mathbf{g}^{(\alpha)} V(\mathbf{g}^{(0)}, \dots, \mathbf{g}^{(3)}) \varphi(\mathbf{g}^{(0)}) \bar{\varphi}(\mathbf{g}^{(1)}) \varphi(\mathbf{g}^{(2)}) \bar{\varphi}(\mathbf{g}^{(3)}),
\end{aligned}$$

by Fourier-transforming the fields φ and $\bar{\varphi}$ there. Here we will set K to be the Laplacian on $\mathbb{T}^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, but following analysis can be with little effort carried on by picking a generic diagonal operator K . The action in terms of the Fourier-modes is of the form $S[\varphi, \bar{\varphi}] = \text{Tr}_2(\varphi, E\bar{\varphi}) + \lambda(\text{Tr}_{\mathcal{V}_1}(\varphi, \bar{\varphi}) + \text{Tr}_{\mathcal{V}_2}(\varphi, \bar{\varphi}) + \text{Tr}_{\mathcal{V}_3}(\varphi, \bar{\varphi}))$, with $\text{Tr}_2(\bar{\varphi}, E\varphi) = \sum_{x_1, x_2, x_3} \bar{\varphi}_{x_1 x_2 x_3} (x_1^2 + x_2^2 + x_3^2 + m_0^2) \varphi_{x_1 x_2 x_3}$, being E thus diagonal. For this concrete theory

$$\begin{aligned}
\bar{\varphi}_{\mathbf{x}} \frac{\partial S_{\text{int}}}{\partial \bar{\varphi}_{\mathbf{x}}} \Big|_{\varphi^b \rightarrow \delta/\delta J^{\sharp}} &= 2\lambda \left\{ \frac{\delta}{\delta J_{x_1 x_2 x_3}} \sum_{b_1} \frac{\delta}{\delta \bar{J}_{b_1 x_2 x_3}} \sum_{b_2, b_3} \frac{\delta}{\delta J_{b_1 b_2 b_3}} \frac{\delta}{\delta \bar{J}_{x_1 b_2 b_3}} \right. \\
&\quad + \frac{\delta}{\delta J_{x_1 x_2 x_3}} \sum_{b_2} \frac{\delta}{\delta \bar{J}_{x_1 b_2 x_3}} \sum_{b_1, b_3} \frac{\delta}{\delta J_{b_1 b_2 b_3}} \frac{\delta}{\delta \bar{J}_{b_1 x_2 b_3}} \\
&\quad \left. + \frac{\delta}{\delta J_{x_1 x_2 x_3}} \sum_{b_3} \frac{\delta}{\delta \bar{J}_{x_1 x_2 b_3}} \sum_{b_1, b_2} \frac{\delta}{\delta J_{b_1 b_2 b_3}} \frac{\delta}{\delta \bar{J}_{b_1 b_2 x_3}} \right\} Z[J, \bar{J}] \Big|_{J=\bar{J}=0}.
\end{aligned}$$

One uses for each line the WTI for the color in question. For the first line, $a = 1$, this reads

$$2\lambda \left\{ \frac{\delta}{\delta J_{x_1 x_2 x_3}} \sum_{b_1} \frac{\delta}{\delta \bar{J}_{b_1 x_2 x_3}} \left[\delta_{x_1 b_1} Y_{x_1}^{(1)}[J, \bar{J}] \cdot \right. \right. \\ \left. \left. + \sum_{b_2, b_3} \frac{1}{|b_1|^2 - |x_1|^2} \left(\bar{J}_{b_1 b_2 b_3} \frac{\delta}{\delta \bar{J}_{x_1 b_2 b_3}} - J_{x_1 b_2 b_3} \frac{\delta}{\delta J_{b_1 b_2 b_3}} \right) \right] Z[J, \bar{J}] \right\} \Big|_{J=\bar{J}=0}.$$

The derivatives on the $Y_{x_1}^{(1)}$ yield

$$\frac{\delta^2 Y_{x_1}^{(1)}[J, \bar{J}]}{\delta J_{x_1 x_2 x_3} \delta \bar{J}_{x_1 x_2 x_3}} \Big|_{J=\bar{J}=0} = \frac{\partial Y_{x_1}^{(1)}[J, \bar{J}]}{\partial \Theta(\mathbf{x})} \\ = (\Delta_{x_1, 1} G_{\Theta|\Theta}^{(4)} + \Delta_{x_1, 1} G_{1\mathbb{Q}_1}^{(4)} + \Delta_{x_1, 1} G_{2\mathbb{Q}_2}^{(4)} + \Delta_{x_1, 1} G_{3\mathbb{Q}_3}^{(4)}) (\mathbf{x}) \\ = \sum_{q_2, q_3} G_{\Theta|\Theta}^{(4)}(x_1, q_2, q_3; \mathbf{x}) + G_{1\mathbb{Q}_1}^{(4)}(\mathbf{x}, \mathbf{x}) \\ + \sum_{q_3} G_{2\mathbb{Q}_2}^{(4)}(x_1, x_2, q_3; \mathbf{x}) + \sum_{q_2} G_{3\mathbb{Q}_3}^{(4)}(x_1, q_2, x_3; \mathbf{x}),$$

and one obtains in similar way the terms concerning the two other colors. Straightforwardly one gets that for each $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$,

$$G_{\Theta}^{(2)}(\mathbf{x}) = \frac{1}{m^2 + |\mathbf{x}|^2} \\ + \frac{(-2\lambda)}{m^2 + |\mathbf{x}|^2} \left\{ G_{\Theta}^{(2)}(\mathbf{x}) \cdot \left[\sum_{q, r \in \mathbb{Z}} G_{\Theta}^{(2)}(x_1, q, r) + G_{\Theta}^{(2)}(q, x_2, r) + G_{\Theta}^{(2)}(q, r, x_3) \right] \right. \\ + \sum_{q \in \mathbb{Z}} \left(G_{1\mathbb{Q}_1}^{(4)}(x_1, q, x_3; \mathbf{x}) + G_{1\mathbb{Q}_1}^{(4)}(x_1, x_2, q; \mathbf{x}) + G_{2\mathbb{Q}_2}^{(4)}(q, x_2, x_3; \mathbf{x}) \right. \\ \left. + G_{2\mathbb{Q}_2}^{(4)}(x_1, x_2, q; \mathbf{x}) + G_{3\mathbb{Q}_3}^{(4)}(q, x_2, x_3; \mathbf{x}) + G_{3\mathbb{Q}_3}^{(4)}(x_1, q, x_3; \mathbf{x}) \right) \\ + \sum_{q, r \in \mathbb{Z}} \left[G_{\Theta|\Theta}^{(4)}(x_1, q, r; \mathbf{x}) + G_{\Theta|\Theta}^{(4)}(q, x_2, r; \mathbf{x}) + G_{\Theta|\Theta}^{(4)}(q, r, x_3; \mathbf{x}) \right] \quad (51) \\ - \sum_{q_1 \in \mathbb{Z}} \left[\frac{1}{q_1^2 - x_1^2} \cdot (G_{\Theta}^{(2)}(q_1, x_2, x_3) - G_{\Theta}^{(2)}(\mathbf{x})) \right] \\ - \sum_{q_2 \in \mathbb{Z}} \left[\frac{1}{q_2^2 - x_2^2} \cdot (G_{\Theta}^{(2)}(x_1, q_2, x_3) - G_{\Theta}^{(2)}(\mathbf{x})) \right] \\ \left. - \sum_{q_3 \in \mathbb{Z}} \left[\frac{1}{q_3^2 - x_3^2} \cdot (G_{\Theta}^{(2)}(x_1, x_2, q_3) - G_{\Theta}^{(2)}(\mathbf{x})) \right] + \sum_{c=1,2,3} G_{c\mathbb{Q}_c}^{(4)}(\mathbf{x}, \mathbf{x}) \right\}.$$

One can conveniently simplify the notation according to

$$G^{(2)} = G_{\Theta}^{(2)} = \frac{\partial \log Z}{\partial \Theta}, \quad G_{V_c}^{(4)} = G_{c\mathbb{Q}_c}^{(4)} = \frac{\partial \log Z}{\partial c\mathbb{Q}_c}, \quad G_{\text{mlm}}^{(4)} = G_{\Theta|\Theta}^{(4)} = \frac{\partial \log Z}{\partial (\Theta|\Theta)}, \quad (52)$$

and write the 2-point SDE in compact form:

$$\begin{aligned}
G(\mathbf{x}) = & \frac{1}{m^2 + |\mathbf{x}|^2} + \frac{(-2\lambda)}{m^2 + |\mathbf{x}|^2} \sum_{c=1}^3 \left\{ \sum_{q_a, q_b} G(q_a, q_b, x_c) \times G(\mathbf{x}) \right. \\
& + \sum_{d=a, b} \sum_{q_c} G_{V_d}^{(4)}(x_a, x_b, q_c, \mathbf{x}) + \sum_{q_a, q_b} G_{m|m}^{(4)}(q_a, q_b, x_c, \mathbf{x}) \\
& \left. - \sum_{q_c} \frac{1}{q_c^2 - x_c^2} \left[G(x_a, x_b, q_c) - G(\mathbf{x}) \right] + G_{V_c}^{(4)}(\mathbf{x}, \mathbf{x}) \right\}, \tag{51'}
\end{aligned}$$

assuming set-equality $\{a, b, c\} = \{1, 2, 3\}$. This 2-point SDE can be formally taken to a (diffeo-)integral equation, as in [34], by taking the continuum limit.

7. CONCLUSIONS

In the quest for the full Ward Takahashi Identity, we have shown that the correlation functions of rank- D CTMs are classified by boundary (D -colored) graphs. For quartic melonic models the boundary sector is the set of *all* D -colored graphs and correlation functions indexed by this set are, in their entirety, non-trivial. Concerning combinatorics, it would be also interesting to grasp, perhaps in terms of covering spaces, the counting itself of the *connected* D -colored graphs in $2k$ vertices, which, as one can extrapolate from [2, 32], gives also the number of conjugacy classes of subgroups of index k in free group F_{D-1} .

A similar, more intricate organization by boundary graphs is very likely to hold for multi-orientable tensor models; this is worth exploring, in particular if one is interested in allowing non-orientable manifolds.

The full WTI works also for U(1)-TGFT and it would be interesting to derive a Ward Takahashi Identity for the SU(2)-TGFT models, or SU(2)-related models like Boulatov's and Ooguri's.

The culmination of this work would be the construction of the φ_3^4 -theory in an Osterwalder-Schrader manner, properly along the lines of the Tensor Track. The φ_3^4 -theory is superrenormalizable and its renormalization has been studied constructively [10] using the multiscale loop vertex expansion [24]. The addition of φ^6 -interaction vertices can make the theory from the renormalization viewpoint even more interesting. In rank-4 such theory would very likely convey the interesting properties of the Ben Geloun-Rivasseau model. The new theory would have, of course, the same boundary sector and therefore the same expansion of the free energy (obviously with different solutions for the correlation functions indexed by the same boundary graph). Hence for melonic ($\varphi^4 + \varphi^6$)-theories the present results hold.

The next obvious step is to solve the equation for the 2-point function derived here. We shall begin by studying in depth, for general theories, the *discrete permutational symmetry* axiom [39, Sec. 5, Rule 2] stated by Rivasseau. Although all the models treated here are in fact \mathfrak{S}_D -invariant, there are elements of CTMs that are not manifestly \mathfrak{S}_D -invariant (e.g. the homology of graphs defined in [25]). After showing their invariance one would be entitled to state an equivalence of the form

$$G_{\square_1}^{(4)} \sim G_{\square_2}^{(4)} \sim G_{\square_3}^{(4)}, \quad G_{\begin{array}{c} c \\ \circlearrowleft \\ c \end{array}}^{(6)} \sim G_{\begin{array}{c} a \\ \circlearrowleft \\ a \end{array}}^{(6)} \quad \text{for } a \neq c,$$

and similar rules for higher multi-point functions indexed by graphs lying on the same \mathfrak{S}_D -orbit of the action $\tau : G_{\mathcal{B}}^{(2k)} \mapsto (\Delta\tau)^* G_{\tau(\mathcal{B})}^{(2k)}$ where $\Delta\tau$ is the diagonal action on $(\mathbb{Z}^D)^k$, $\tau \in \mathfrak{S}_D$, and the action of \mathfrak{S}_D on \mathbb{Z}^D and $\text{IIGrph}_{c,D}$ is in both cases permutation of colors. This equalities will noticeably simplify the SDEs, as is evident in eq. (51'). Based on the WTI

exposed here, the full¹¹ tower of Schwinger-Dyson equations for quartic theories in arbitrary rank was obtained in [36]. Their renormalized version should also be derived and we can proceed as in [34].

Nonetheless, in order to solve the equations, this approach might still not be enough and, in order to obtain a closed equation, it can be complemented as follows. Fixed a correlation function $G_{\mathcal{B}}^{(2p)}$, Gurău's degree's range of graphs contributing to $G_{\mathcal{B}}^{(2p)}$ is $\rho(\mathcal{B}) = \{t(\mathcal{B}) + n \cdot (D - 1)!/2 \mid n \in \mathbb{Z}_{\geq 0}\}$ being $t(\mathcal{B}) \in \mathbb{Z}_{\geq 0}$ a lower bound depending on \mathcal{B} . As done in [21] for matrix models, one can further possibly decouple the equation (51') for the 2-point function by expanding any correlation function occurring there in subsectors that share the same value α of the degree, $G_{\mathcal{B}}^{(2p)} = \sum_{\alpha \in \rho(\mathcal{B})} G_{\mathcal{B}}^{(2p, \alpha)}$, in order to obtain a closed equation. Furthermore, since mainly the degree conveys the geometrical information, this is a sum over all the geometries bounded by $|\Delta(\mathcal{B})|$ that a fixed model (here a quartic) triangulates. These subsectors generalize the matrix models' genus-expansion (20) and in turn justify Figure 1.

Lemmas 1 and 2, intended here first as auxiliary results, are important on their own, if one wants to understand the geometry of the spaces generated by the φ_m^4 -theories. In rank 3 and 4, for instance, they realize the triviality of Ω_2^{SO} and Ω_3^{SO} , respectively, the orientable bordism groups. This could be an accident due to the equivalence of the topological, PL and smooth categories in low dimensions. For higher dimensions one should rather compare with the piecewise linear bordism groups Ω_*^{PL} , which is beyond the scope of this study but worth analyzing.

The graph-operation $\#$ introduced in Definition 6 turned out to be optimally-behaved (additive) with respect to Gurău's degree. The fact that melonic graphs are spheres leads us to conjecture that $\#$ should be indeed the QFT-compatible graph-realization of the connected sum in arbitrary rank (a direct proof of which is found in [35]), when the colored graphs represent manifolds.

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APPENDIX A. THE FUNDAMENTAL GROUP OF CRYSTALLIZATIONS

We expose the computation of the fundamental group of two crystallizations, i.e. colored graphs whose number of $(D - 1)$ -bubbles is exactly D (for non-crystallizations, Gurău has found a representation of the fundamental group of general colored graphs [25, Eq. 40]). The next algorithm by Gagliardi [17] works *only for crystallizations*. Namely, let \mathcal{B} be a crystallization, for simplicity, of a manifold of dimension at least 3 (hence \mathcal{B} has $D > 3$ colors). The fundamental group of \mathcal{B} , $\pi_1(\mathcal{B})$, is isomorphic to the fundamental group $\pi_1(|\Delta(\mathcal{B})|)$ of the (path connected) space \mathcal{B} represents. The former group is constructed from generators X and relations R dictated by certain bubbles of \mathcal{B} as follows. Choose two arbitrary different colors i, j and consider all $D - 1$ bubbles $\{\mathcal{B}_{\alpha}^{\widehat{ij}}\}_{\alpha=1}^n$ without the two colors i, j . Let

$$X = \{x_1, \dots, x_n\}, \quad x_{\alpha} = \text{generator associated to } \mathcal{B}_{\alpha}^{\widehat{ij}}.$$

¹¹As a technicality, the SDE tower was obtained for connected correlation functions indexed by connected boundary graphs. The SDE obtained there were actually works any model having (50) as subgraph in the interaction vertices.

Consider the set $\{\mathcal{B}_\gamma^{ij}\}_{\gamma=1}^m$ of the ij -bicolored 2-bubbles of \mathcal{B} . For each γ , each vertex of the loop \mathcal{B}_γ^{ij} intersects certain \mathcal{B}_α^{ij} , and in that case write x_α^ϵ according to the following rule: set $\epsilon = 1$ if the vertex at which \mathcal{B}_γ^{ij} intersect \mathcal{B}_α^{ij} is black and $\epsilon = -1$ if it is white. For each γ let $R(\mathcal{B}_\gamma^{ij})$ be the word on X defined by juxtaposing all such elements x_α^ϵ for each vertex of \mathcal{B}_γ^{ij} , in order of occurrence. Then

$$\pi_1(\mathcal{B}) \cong \langle x_1, \dots, x_{n-1}, x_n \mid x_n, \{R(\mathcal{B}_\gamma^{ij}) : \gamma = 1, \dots, m-1\} \rangle$$

Gagliardi's algorithm [17] states that neither the choice of the $D-1$ bubble x_n that one sets to the identity is important, nor the relation $R(\mathcal{B}_m^{ij})$ that does not appear is, nor the two colors i, j are.

Example 11. We put Gagliardi's algorithm to work for the crystallization. Here we come back to the color-set $\{0, 1, 2, 3\}$, instead of $\{1, 2, 3, 4\}$ and consider $\Gamma \in \text{Grph}_{c,3+1}$ given in Figure 7 lens space $L_{3,1}$, already mentioned in Section 4.3. We choose first the two colors $i = 2, j = 3$, whose corresponding $\{2, 3\}^c$ -colored $(D-1)$ -bubbles are Γ_1^{01} and Γ_2^{01} depicted below. One associates to all but one of the 2-bubbles with chosen colors $\{2, 3\}$, a relation. There are two such 2-bubbles and we drop the inner bubble and pick the outer one Γ_1^{23} , as shown in Figure 7. The rule says that the (only non-trivial) relation $R(\Gamma_1^{23})$ corresponds to Γ_1^{23} and is given by $R(\Gamma_1^{23}) = x_1^{+1}x_2^{-1}x_1^{+1}x_2^{-1}x_1^{+1}x_2^{-1}$. Notice, incidentally, that if we had instead chosen to derive a relation for the inner bubble the corresponding relation would be $R(\Gamma_2^{23}) = x_2^{+1}x_1^{-1}x_2^{+1}x_1^{-1}x_2^{+1}x_1^{-1}$ which is just $(R(\Gamma_1^{23}))^{-1}$. Thus $\pi_1(\Gamma) \cong \langle x_1, x_2 \mid x_2, R(\Gamma_1^{23}) \rangle = \langle x_1 \mid x_1^3 \rangle \cong \mathbb{Z}_3 \cong \pi_1(L_{3,1})$.

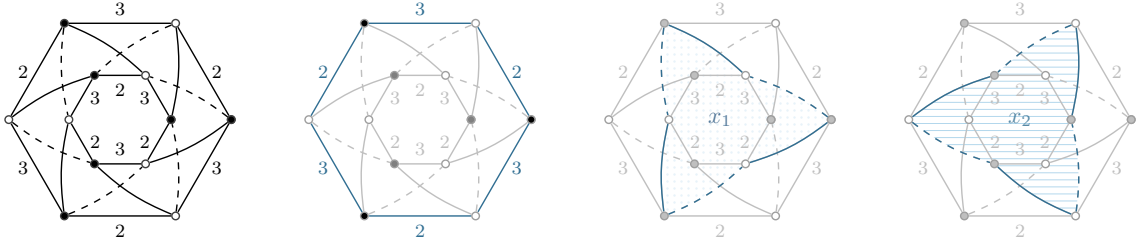
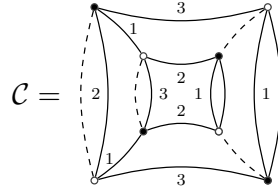


Fig. 7 From left to right: Γ and three of its bubbles Γ_1^{23} , Γ_1^{01} and Γ_2^{01} , used to compute its fundamental group. For the last two bubbles the generators x_1 and x_2 are depicted

Example 12. The same algorithm applied to the graph \mathcal{C} of example 9:



Namely, one has for the chosen colors $i = 2, j = 3$ that the relation $R(\mathcal{C}_{\text{outer}}^{23})$ corresponding to the outer 23-bicolored bubble is given by $x_2x_2^{-1}x_1x_1^{-1}$ and therefore is trivial. Here x_1 (resp. x_2) is the leftmost (resp. rightmost) 01-bicolored bubble. Thus $\pi_1(\mathcal{C}) = \langle x_1, x_2 \mid x_2, R(\mathcal{C}_{\text{outer}}^{23}) \rangle = \langle x_1 \mid \emptyset \rangle = \mathbb{Z}$.

APPENDIX B. THE TERM $Y_{m_a}^{(a)}$ FOR DISCONNECTED GRAPHS

A useful formula in order to compute higher order terms in $Y_{m_a}^{(a)}$ is presented in this last appendix. We also obtain an expression that shows that our expansion of W in boundary

graphs, eq. (23), is genuinely the generalization of the longest-cycle expansion for matrix models of Section 5.1.

We now split an arbitrary boundary graph \mathcal{B} in its (say B) connected components $\mathcal{R}^\beta \in \text{Grph}_{c,D}$, $\mathcal{B} = \prod_{\beta=1}^B \mathcal{R}^\beta$. Then there exist a non-negative integer partition $\{n_i\}_{i \geq 0}$ of B , i.e. $B = \sum_{j=1}^\ell n_j$, with $k = \sum_{j=1}^\ell j \cdot n_j$ and $\mathcal{N} = 2k$, where k is the number of J -sources (and therefore also the number of \bar{J} -sources) and \mathcal{N} the order of the correlation function, and n_k , as before, is number of boundary components with exactly k J -sources (or, equivalently, k \bar{J} -sources). We can associate to \mathcal{R}^β a product of sources, $\mathbb{J}(\mathcal{R}^\beta)$, as above. Of course $\mathbb{J}(\mathcal{B}) = \prod_{\beta} \mathbb{J}(\mathcal{R}^\beta)$. In that case

$$G_{\mathcal{B}}^{(2k)} \star \mathbb{J}(\mathcal{B}) = \sum_{\mathbf{a}, \mathbf{b}, \dots, \mathbf{c}} G_{|\mathcal{R}^1| |\mathcal{R}^2| \dots |\mathcal{R}^B|}^{(2k)}(\{\mathbf{a}\}, \{\mathbf{b}\}, \dots, \{\mathbf{c}\}) \cdot \mathbb{J}(\mathcal{R}^1)(\{\mathbf{a}\}) \mathbb{J}(\mathcal{R}^2)(\{\mathbf{b}\}) \dots \mathbb{J}(\mathcal{R}^B)(\{\mathbf{c}\}).$$

Now, some of the \mathcal{R}^β 's might be repeated. Let s be the number of *different* graphs at the disconnected components of the boundary $\{\mathcal{R}_\beta\}_{\beta=1}^B = \cup_{b=1}^s \{\mathcal{R}_1^b, \dots, \mathcal{R}_{m_b}^b\}$, where m_b is the number of copies of \mathcal{R}^b . Moreover, we order \mathcal{R}^b ascending: $\mathcal{R}^b \leq \mathcal{R}^{b'}$, if $(\mathcal{R}^b)^{(0)} \leq (\mathcal{R}^{b'})^{(0)}$ (which is equivalent to say, that the boundary component of type $\mathcal{R}^{b'}$ has the same number of external lines or more than \mathcal{R}^b). Then, of course $\sum_b m_b = B$. We rewrite then W as

$$W[J, \bar{J}] = \sum_{\ell=1}^{\infty} \sum'_{\substack{m_s \geq 1 \\ m_1, \dots, m_{\ell-1} \geq 0}} \left(\frac{1}{m_1! \dots m_s! (|\text{Aut}_c(\mathcal{R}^1)|)^{m_1} \dots (|\text{Aut}_c(\mathcal{R}^s)|)^{m_s}} \right) \quad (53) \\ \cdot G_{|\mathcal{R}_1^1| |\mathcal{R}_2^1| \dots |\mathcal{R}_{m_1}^1| \dots |\mathcal{R}_1^s| \dots |\mathcal{R}_{m_s}^s|}^{(2k)} \star \mathbb{J}((\mathcal{R}_1^1 \sqcup \mathcal{R}_2^1 \dots \sqcup \mathcal{R}_{m_1}^1) \sqcup \dots \sqcup (\mathcal{R}_1^s \sqcup \dots \sqcup \mathcal{R}_{m_s}^s)).$$

The symmetry factors are consequence of eq. (8). The prime in the sum denotes the following two restrictions:

$$2k = \sum_{p=1}^s m_p \cdot |(\mathcal{R}^p)^{(0)}| \quad \text{and} \quad 2\ell = |(\mathcal{R}^s)^{(0)}|, \quad (54)$$

so ℓ is the half of the “largest number of vertices” of the components of the boundary graph.

The sum in eq. (44) is over all possible disconnected graphs. We now obtain the expression for $Y_{m_a}^{(a)}$ in terms of the connected components of the graphs. For the boundary graph $\mathcal{B} = (\mathcal{R}_1^1 \sqcup \mathcal{R}_2^1 \dots \sqcup \mathcal{R}_{m_1}^1) \sqcup \dots \sqcup (\mathcal{R}_1^s \sqcup \dots \sqcup \mathcal{R}_{m_s}^s)$, where $\mathcal{R}_1^b, \dots, \mathcal{R}_{m_b}^b$ are copies of the same graph, having k_i white vertices. Now, the operators Δ_{r, m_a} select the r -th white vertex. We give the white vertices of \mathcal{B} the order according to the occurrence of these copies of the connected parts of \mathcal{B} , i.e. the first k_1 white vertices are in \mathcal{R}_1^1 ; the next vertices, from the $(k_1 + 1)$ -th until the $(2k_1)$ -th in \mathcal{R}_2^1 and so on. The last k_s vertices are in $\mathcal{R}_{m_s}^s$. For sake of notation $\mathcal{N} = \mathcal{N}(\{k_i\})$, the order of the Green's function, is not made explicit, given by $\mathcal{N} = 2 \cdot (\sum_{i=1}^s m_i k_i)$ one derives the following expression for $Y_{m_a}^{(a)}[J, \bar{J}]$:

$$\sum_{\ell=1}^{\infty} \sum'_{\substack{m_s \geq 1 \\ m_1, \dots, m_{\ell-1} \geq 0}} \left(\frac{1}{m_1! \dots m_s! (|\text{Aut}_c(\mathcal{R}^1)|)^{m_1} \dots (|\text{Aut}_c(\mathcal{R}^s)|)^{m_s}} \right) \quad (55) \\ \times \left[\sum_{r=1}^{k_1} (\Delta_{r, m_a} G_{|\mathcal{R}_1^1| |\mathcal{R}_2^1| \dots |\mathcal{R}_{m_1}^1| \dots |\mathcal{R}_1^s| \dots |\mathcal{R}_{m_s}^s|}^{(\mathcal{N})}) \star (\mathcal{R}_1^1 \ominus e_a^r \sqcup \mathcal{R}_2^1 \sqcup \dots \sqcup \mathcal{R}_1^s \sqcup \dots \sqcup \mathcal{R}_{m_s}^s) \right. \\ + \sum_{r=1}^{k_1} (\Delta_{k_1+r, a} G_{|\mathcal{R}_1^1| |\mathcal{R}_2^1| \dots |\mathcal{R}_{m_1}^1| \dots |\mathcal{R}_1^s| \dots |\mathcal{R}_{m_s}^s|}^{(\mathcal{N})}) \star (\mathcal{R}_1^1 \sqcup \mathcal{R}_2^1 \ominus e_a^r \sqcup \dots \sqcup \mathcal{R}_1^s \sqcup \dots \sqcup \mathcal{R}_{m_s}^s) + \dots \\ + \sum_{r=1}^{k_1} (\Delta_{m_1 \cdot k_1 + r, a} G_{|\mathcal{R}_1^1| \dots |\mathcal{R}_{m_1}^1| \dots |\mathcal{R}_1^s| \dots |\mathcal{R}_{m_s}^s|}^{(\mathcal{N})}) \star (\mathcal{R}_1^1 \sqcup \dots \sqcup \mathcal{R}_{m_1}^1 \ominus e_a^r \sqcup \dots \sqcup \mathcal{R}_1^s \dots \sqcup \mathcal{R}_{m_s}^s) \\ \left. + \sum_{r=1}^{k_s} (\Delta_{m_1 \cdot k_1 + m_2 \cdot k_2 + \dots + m_{s-1} \cdot k_{s-1} + r, a} G_{|\mathcal{R}_1^1| \dots |\mathcal{R}_{m_1}^1| \dots |\mathcal{R}_1^s| \dots |\mathcal{R}_{m_s}^s|}^{(\mathcal{N})}) \right]$$

$$\begin{aligned}
& \star (\mathcal{R}_1^1 \sqcup \dots \sqcup \mathcal{R}_{m_1}^1 \sqcup \dots \sqcup \mathcal{R}_1^s \ominus e_a^r \sqcup \dots \sqcup \mathcal{R}_{m_s}^s) \\
& + \dots + \sum_{r=1}^{k_s} (\Delta_{m_1 k_1 + m_2 k_2 \dots + m_{s-1} k_{s-1} + (m_s - 1) k_s + r, a} G_{|\mathcal{R}_1^1| |\mathcal{R}_2^1| \dots |\mathcal{R}_{m_1}^1| \dots |\mathcal{R}_1^s| \dots |\mathcal{R}_{m_s}^s|}^{(\mathcal{N})}) \\
& \star (\mathcal{R}_1^1 \sqcup \dots \sqcup \mathcal{R}_1^s \sqcup \dots \sqcup \mathcal{R}_{m_s}^s \ominus e_a^r) \Big],
\end{aligned}$$

with the prime, as before, meaning the restrictions of the sum by eqs. (54).

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