1700 FORESTS

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Arvoles lloran por lluvia
y montañas por aire
Ansí lloran mis ojos
por ti querido amante

Ladino folk song

Abstract. Since ordered trees and Dyck paths are equinumerous, so are ordered forests and grand-Dyck paths that start with an upwards step.

1. Introduction

We are interested in the number of ordered forests (that is, sequences of non-trivial ordered rooted trees) with a total of n edges. Every tree in the forest must have at least one edge (in that sense they are non-trivial), or else there would be infinitely many forests for every n. For example, there are $\binom{5}{3} = 10$ such forests with $n = 3$ edges, as depicted in Figure 1.

It turns out—easily enough—that these forests are counted by

$$F_n = \frac{1}{2}\binom{2n}{n} = \binom{2n-1}{n} \tag{1}$$

This enumeration is sequence A001700 in Neil Sloane's *On-Line Encyclopedia of Integer Sequences (OEIS)*:[1]

n	0	1	2	3	4	5	6	7	8	$\cdots$
F_{n+1}	1	3	10	35	126	462	1716	6435	24310	$\cdots$

In other words, sequence A001700$(n) = F_{n+1}$ also counts the number of ordered forests with $n + 1$ edges (and no trivial trees).

Compare this forest enumeration with the Catalan numbers, $C(n) = \frac{1}{n+1}\binom{2n}{n}$, which count (among many combinatorial objects) ordered

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[1] `http://oeis.org`. Previously in print form: Neil J. A. Sloane, *A Handbook of Integer Sequences*, Academic Press, NY, 1973.

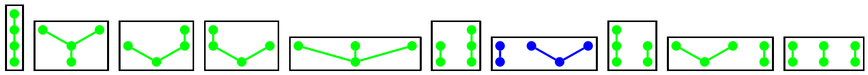

FIGURE 1. Ten triple-edge forests ($n = 3$).

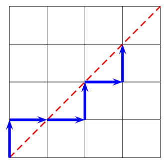

FIGURE 2. The grand-Dyck path corresponding to the 7th (blue) forest in Figure 1.

forests with n *nodes*, allowing for trivial (leaf-only, edgeless) trees. They form sequence A000108 in the *OEIS*.

Perhaps it is because parameterizing by the number of edges is less common than the use of a node parameter that this enumeration of ordered forests has not appeared in the literature until now.

2. PATHS AND FORESTS

The justification for enumeration (1) follows from the standard correspondence between trees and lattice paths. A *Dyck path* is a (monotonic, "staircase") lattice path (consisting of a mix of $\uparrow$ and $\rightarrow$ steps) beginning and ending on the diagonal and never venturing below; a *grand-Dyck* path may go both above and below the diagonal but must end on it.[2]

Ordered (rooted plane) trees with n edges are well-known to be in bijection with Dyck paths of length $2n$.[3] So a forest, which is a sequence of trees, corresponds to a sequence of Dyck paths. Every grand-Dyck can be interpreted as a sequence of Dyck paths, one per (non-trivial) tree, delineated by the points at which the path crosses the diagonal. See Figure 2 (left). There are $\binom{2n}{n}$ such paths (since they must have $n \uparrow$ steps and $n \rightarrow$ steps). But a path and its mirror image (reflected about the diagonal) correspond to the same forest. The equation follows.

[2]Grand-Dyck paths are classified as "bridges" in Cyril Banderier and Philippe Flajolet, "Basic analytic combinatorics of directed lattice paths", *Theoretical Computer Science* **281** (2002): 37–80.

[3]David A. Klarner, "Correspondence between plane trees and binary sequences", *Journal of Combinatorial Theory* **9** (1970) 401–411.

3. Height Restrictions

The correspondence between forests and grand-Dyck paths applies equally to trees of restricted height and paths within a band, the latter analyzed by Mohanty.[4] It follows that the number of n-edge forests whose trees are all of height at most h is

$$F_n^h = \frac{1}{2}\sum_{k\in\mathbb{Z}}\left[\binom{2n}{n+2k(h+1)} - \binom{2n}{n+(2k+1)(h+1)}\right] \tag{2}$$

For example, there are

$$\begin{aligned} F_3^1 &= \frac{1}{2}\sum_{k}\left[\binom{6}{3+4k} - \binom{6}{5+4k}\right] \\ &= \frac{1}{2}\left[\binom{6}{-1} - \binom{6}{1} + \binom{6}{3} - \binom{6}{5}\right] \\ &= \frac{1}{2}[0-6+20-6] = 4 \end{aligned}$$

forests in Figure 1 with trees of height 1.

[4] Sri Gopal Mohanty, *Lattice Path Counting and Applications*, volume 37 of *Probability and Mathematical Statistics*, Academic Press, New York, 1979, pp. 6–7.