A q-QUEENS PROBLEM
IV. QUEENS, BISHOPS, NIGHTRIDERS (AND ROOKS)

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Abstract. Parts I–III showed that the number of ways to place q nonattacking queens or similar chess pieces on an $n \times n$ chessboard is a quasipolynomial function of n whose coefficients are essentially polynomials in q and, for pieces with some of the queen's moves, proved formulas for these counting quasipolynomials for small numbers of pieces and high-order coefficients of the general counting quasipolynomials.

In this part, we focus on the periods of those quasipolynomials by calculating explicit denominators of vertices of the inside-out polytope. We find an exact formula for the denominator when a piece has one move, give intuition for the denominator when a piece has two moves, and show that when a piece has three or more moves, geometrical constructions related to the Fibonacci numbers show that the denominators grow at least exponentially with the number of pieces.

Furthermore, we provide the current state of knowledge about the counting quasipolynomials for queens, bishops, rooks, and pieces with some of their moves. We extend these results to the nightrider and its subpieces, and we compare our results with the empirical formulas of Kotěšovec.

Contents

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1. Introduction

The famous n-Queens Problem asks for the number of arrangements of n nonattacking queens—the largest possible number—on an $n \times n$ chessboard. (See, for instance, [15].) There is no known general formula, other than the very abstract, that is to say impractical one we obtained in Part II. Solutions have been found only by individual analyses for small n.

In this series of five papers [**?**, 8, 19] we treat the problem by separating the board size, n, from the number of queens, q, and rephrasing the whole problem in geometry. We also generalize to every piece $\mathbb{P}$ whose moves are, like those of the queen, rook, and bishop, unlimited in length. Such pieces are known as "riders" in fairy chess (chess with modified rules, moves, or boards); an example is the nightrider, whose moves are those of the knight extended to any distance. The problem then, given a fixed rider $\mathbb{P}$, is:

Problem 1. Find an explicit, easily evaluated formula for $u_{\mathbb{P}}(q; n)$, the number of nonattacking configurations of q unlabelled pieces $\mathbb{P}$ on an $n \times n$ board.

Two kinds of piece were previously solved: the rook, which is elementary, and the bishop, for which Arshon and Kotěšovec found a single formula. Aside from those two, formulas in terms of n have been found for only a few riders and only for small numbers of pieces—for instance, up to 6 queens or 3 nightriders—and mostly heuristically, without rigorous proof. (Given the power of computers, proving a formula is more difficult than making an educated guess, which itself is by no means easy even for an expert like Václav Kotěšovec [10, 11].) Finding a single comprehensive formula, for all numbers of pieces q and all board sizes n, for any piece other than the rook and bishop—especially for the queen, the original problem of this type—looks impossible.

One difficulty is not being certain what such a formula should look like.

Problem 2. Describe the nature of a formula for $u_{\mathbb{P}}(q; n)$ for an arbitrary piece.

One would wish there to be one style of formula that applies to all riders. This ideal is realized to an extent. We proved in Part I that for each rider $\mathbb{P}$, $u_{\mathbb{P}}(q;n)$ is a quasipolynomial function of n of degree $2q$ and that the coefficients of powers of n are given by polynomials in q, up to a simple normalization; for instance, the leading term is $n^{2q}/q!$. Being a quasipolynomial means that for each fixed q, $u_{\mathbb{P}}(q;n)$ is given by a cyclically repeating sequence of polynomials in n (called the *constituents* of the quasipolynomial); the shortest length of such a cycle is the period of $u_{\mathbb{P}}(q;n)$. That raises a fundamental question.

Problem 3. What is the period p of the quasipolynomial formula for $u_{\mathbb{P}}(q;n)$? (The period, which pertains to the variable n, may depend on q.)

The period tells us how much data is needed to rigorously determine the complete formula; $2qp$ values of the counting function determine it completely, since the degree is $2q$ and the leading coefficient is known. The difficulty with this computational approach is that, in general, p is hard to determine and seems usually to explode with increasing q. (Indeed, we have reason to believe the period increases at least exponentially for any rider with at least three moves; see Theorem 5.10.) A better way would be to find information about the $u_{\mathbb{P}}(q;n)$ that is valid for all q.

Problem 4. For a given piece $\mathbb{P}$, find explicit, easily evaluated formulas for the coefficients of powers of n in the quasipolynomials $u_{\mathbb{P}}(q;n)$, valid for all values of q.

A complete solution to Problem 4 would solve Problem 1. We think that is unrealistic but we have achieved some results. In Part I we took a first step: in each constituent polynomial, the coefficient γ_i of n^{2q-i} is (neglecting a denominator of $q!$) itself a polynomial in q of degree $2i$, that varies with the residue class of n modulo a period p_i that is independent of q. In other words, if we count down from the leading term there is a general formula for the ith coefficient as a function of q that has its own intrinsic period; the coefficient is independent of the overall period p. This opens the way to explicit formulas and in Part II we found such a formula for the second leading coefficient as well as the complete quasipolynomial for an arbitrary rider with only one move—an unrealistic game piece but mathematically informative. In Part III we found the third and fourth coefficients by concentrating on *partial queens*, whose moves are a subset of the queen's.

Still, that is only scratching at the surface; we want to go further. In this part we present our current state of knowledge about specific pieces: the queen, rook, bishop, and nightrider and the pieces that have a subset of their moves. Our goal is to prove exact quasipolynomial formulas for a fixed number q of each piece, where q is (unavoidably) small, and along the way to see how many complete formulas we can prove for coefficients of high powers of n.

How large a number q and how many coefficients we can handle depends on the piece. For the rook, naturally, we get well-known formulas for all q. At the other extreme we have only very partially solved three nightriders, for which the formula was previously found heuristically, without proof, by Kotěšovec. No formula for four nightriders has even been guessed; it is conceivable, but judging by the 11-digit denominator we computed (see Table 9.1) not probable, that it could be obtained by a painstaking analysis using our method. We offer no hope for five.

One reason we want a quasipolynomial is to substitute $n = -1$ to get the number of combinatorially distinct types of configuration, as explained in Section I.5. The Arshon–Kotěšovec formula for bishops is not a quasipolynomial and does not help us find the bishops counting quasipolynomials. For that reason we consider the bishops problem only partially solved and we give it close attention in Part V.

We summarize our geometrical approach: We create a configuration of points and lines to represent attacks of specified slopes. The boundaries of the square determine a hypercube in $\mathbb{R}^{2q}$ and the attack lines determine hyperplanes whose $1/(n+1)$-fractional points within the hypercube represent attacking configurations, which must be excluded; the nonattacking configurations are the integral points inside the hypercube and outside every hyperplane, so it is they we want to count. The combination of the hypercube and the hyperplanes is an inside-out polytope [2]. The Ehrhart theory of inside-out polytopes implies quasipolynomiality of the counting function and that the period divides the *denominator* D, defined as the least common multiple of the denominators of all coordinates of vertices of the inside-out polytope. Then we apply the inside-out adaptation of Ehrhart lattice-point counting theory, in which we combine by Möbius inversion the numbers of lattice points in the polytope that are in each intersection subspace of the hyperplanes.

We also investigate the denominators of individual vertices, which provide a better understanding of the period because the overall denominator bounds it. In general in Ehrhart theory the period and the denominator need not be equal and often are not, so it is surprising that in all our examples, and for any rider with only one move, they are. We cannot prove that is always true for the inside-out polytopes arising from the problem of nonattacking riders, but this observation suggests that our approach to that problem may be a good test case for understanding the relationship between denominators and periods.

We find an exact formula for the denominator of a one-move rider in Proposition 4.1, and we introduce a notion of trajectories to give intuition about finding the denominator for a piece with two moves (Section 4.2). We show that when a piece has three or more moves, by letting the number q of pieces increase we obtain a sequence of inside-out polytope vertices with denominators that increase exponentially, and the polytope denominators may increase even faster. These vertices arise from geometrical constructions related to Fibonacci numbers.

A summary of this paper: Section 2 recalls some essential notation and formulas from Parts I–III. Section 3 describes the concepts we use to analyze the periods. We turn to the theory of attacking configurations of pieces with small numbers of moves in Sections 4 and 5, partly to establish formulas and conjectural bounds for the denominators of their inside-out polytopes, especially for partial queens and nightriders, and partly to support the exponential lower bound on periods and our many conjectures. After connecting to our theory the known results on rooks in Section 6, we discuss the current state of knowledge and ignorance about bishops and semi-bishops in Section 7. Section 8 treats the queen as well as the partial queens that are not the rook, bishop, and semi-bishop. Section 9 concerns the nightrider and sub-nightriders, whose nonattacking placements have not been the topic of any previous theoretical discussion that we are aware of.

We conclude with questions related to these ideas and with proposals for research. For example, since counting nonattacking rider placements is an accessible topic in Ehrhart theory, we suggest in Section 10.1 fairy chess pieces with relatively simple behavior that might provide insight into the central open problem of a good general bound on the period of the counting quasipolynomial in terms of q and the set of moves. In future work we will prove stronger properties, such as that every positive integer Δ appears as a denominator given enough pieces; specifically, about $C \log \Delta$ pieces where C depends on the piece's set of moves.

We append a dictionary of notation for the benefit of the authors and readers.

We must mention Kotěšovec's book [11], replete with formulas, mostly generated by himself, for all kinds of nonattacking chess problems. We learned of this work after beginning our research; then properties of Kotěšovec's bishops and queens formulas inspired much of our detailed results. For instance, we saw that the coefficients of highest degree are constant (independent of n); then we proved most of the observed constancies. Kotěšovec conjectured some formulas for high-degree coefficients; we prove some of those conjectures. We saw that the bishops quasipolynomials (for $q \geq 3$) all have period 2; in Part V we prove that is true for every number (at least 3) of bishops. Kotěšovec conjectured that the period of a queens quasipolynomial is a product of Fibonacci numbers; we take a step toward a proof.

Anyone who wants to know the actual number of nonattacking placements of q of our four principal pieces will find answers in the Online Encyclopedia of Integer Sequences [16]. Table 1.1 gives sequence numbers in the OEIS. The first sequence in each column is the sequence of square numbers. After that it gets interesting.

q	Rooks	Bishops	Queens	Nightriders
1	A000290	A000290	A000290	A000290
2	A163102*	A172123	A036464	A172141
3	A179058	A172124	A047659	A173429
4	A179059	A172127	A061994	—
5	A179060	A172129	A108792	—
6	A179061	A176886	A176186	—
7	A179062	A187239	A178721	—
8	A179063	A187240	—	—
9	A179064	A187241	—	—
10	A179065	A187242	—	—

Table 1.1. Sequence numbers in the OEIS for nonattacking placements of q rooks, bishops, queens, and nightriders. In each sequence the board size n varies from 1 to (usually) 1000. * means n in the OEIS is offset from our value.

2. Essentials from Before

Each configuration-counting problem arises from making two choices: a chess piece, and the number of pieces. (The size of the board is considered a variable within the problem.) The pieces are placed on the integral points, (x, y) for $x, y \in [n] := \{1, \ldots, n\}$, in the interior of an integral dilation $(n + 1)[0, 1]^2$ of the unit square. We call the set

$$[n]^2 = (n + 1)(0, 1)^2 \cap \mathbb{Z}^2,$$

whose dilation factor is $n + 1$, the *board*, in full the *integral square board*. We also call the open or closed unit square the "(square) board"; it will always be clear which board we mean.

We sometimes consider a general board $\mathcal{B}$, which is any rational convex polygon, i.e., it has rational corners. (We call the vertices of $\mathcal{B}$ its *corners* to avoid confusion with other points called vertices.) When we do not mention $\mathcal{B}$ or a polygonal board, our board will always be square.

The piece $\mathbb{P}$ has *moves* defined as all integral multiples of a finite set $\mathbf{M}$ of non-zero, non-parallel integral vectors $m = (c, d) \in \mathbb{Z}^2$, which we call the *basic moves*. Each one must be reduced to lowest terms; that is, its two coordinates need to be relatively prime; and no basic move may be a scalar multiple of any other. Thus, the slope of m contains all necessary information and can be specified instead of m itself. We say two distinct pieces *attack* each other if the difference of their locations is a move. In other words, if a piece is in position $z := (x, y) \in \mathbb{Z}^2$, it attacks any other piece in the lines $z+rm$ for $r \in \mathbb{Z}$ and $m \in \mathbf{M}$. Attacks are not blocked by a piece in between, and they include the case where two pieces occupy the same location. (The set $\mathbf{M}$ is $\{(1, 1), (1, -1)\}$ for the bishop, $\{(1, 0), (1, 1), (0, 1), (1, -1)\}$ for the queen, $\{(2, 1), (1, 2), (2, -1), (1, -2)\}$ for the nightrider, and of course $\{(1, 0), (0, 1)\}$ for the rook.) The number q is the number of pieces that are to occupy places on the board; we assume $q > 0$.

A *configuration* $\mathbf{z} = (z_1, \dots, z_q)$ is any choice of locations for the q pieces, including on the board's boundary, where $z_i := (x_i, y_i)$ denotes the position of the ith piece $\mathbb{P}_i$. (The boundary, while not part of the board proper, is necessary in our counting method.) Therefore, $\mathbf{z}$ is an integral point in the $(n + 1)$-fold dilation of the $2q$-dimensional closed, convex polytope $\mathcal{P} = \mathcal{B}^q$.. If we are considering the undilated board, $\mathbf{z}$ is a fractional point in $\mathcal{B}^q$. We consider these two points of view equivalent; it will always be clear which kind of board or configuration we are dealing with. Any integral point $\mathbf{z}$ in the dilated polytope, or its fractional representative $\frac{1}{n+1}\mathbf{z}$ in the undilated board, represents a placement of pieces on the board, and vice versa; thus we use the same term "configuration" for the point and the placement. In this part $\mathcal{B}$ is usually the square board; then the closed and open polytopes are $[0, 1]^{2q}$ and $(0, 1)^{2q}$.

The constraint for a *nonattacking configuration* is that the pieces must be in the board proper (so $\mathbf{z} \in (\mathcal{B}^\circ)^q$ or its dilation) and that no two pieces may attack each other. In other words, if there are pieces at positions z_i and z_j, then $z_j - z_i$ is not a multiple of any $m \in \mathbf{M}$; equivalently, $(z_j - z_i) \cdot m^\perp \neq 0$ for each $m \in \mathbf{M}$, where $m^\perp := (d, -c)$.

For counting we treat nonattacking configurations as interior integral lattice points in the dilation of an inside-out polytope $(\mathcal{P}, \mathscr{A}_\mathbb{P})$, where $\mathcal{P} = \mathcal{B}^q$ and $\mathscr{A}_\mathbb{P}$ is the *move arrangement*, whose members are the *move hyperplanes* (or *attack hyperplanes*)

$$\mathcal{H}^{d/c}_{ij} := \{(z_1, \dots, z_q) \in \mathbb{R}^{2q} : (z_j - z_i) \cdot m^\perp = 0\}$$

for $m = (c, d) \in \mathbf{M}$; the equations of these hyperplanes are called the *move equations* (or *attack equations*) of $\mathbb{P}$. Thus (by the definition of "interior" of an inside-out polytope [2]), a configuration $\mathbf{z} \in \mathcal{P}$ is nonattacking if and only if it is in $\mathcal{P}^\circ$ and not in any of the hyperplanes $\mathcal{H}^{d/c}_{ij}$. The *intersection lattice* $\mathscr{L}(\mathscr{A}_\mathbb{P})$ is the lattice of all intersections of subsets of the move arrangement, ordered by reverse inclusion. The Möbius function of $\mathscr{L}(\mathscr{A}_\mathbb{P})$ is denoted by μ. A *vertex* of $(\mathcal{P}, \mathscr{A}_\mathbb{P})$ is any point in $\mathcal{P}$ that is the intersection of facets of $\mathcal{P}$ and hyperplanes of $\mathscr{A}_\mathbb{P}$. For instance, it may be a vertex of $\mathcal{P}$, or it may be the intersection point of hyperplanes if that point is in $\mathcal{P}$, or it may be the intersection of some facets and some hyperplanes.

Each subspace $\mathcal{U} \in \mathscr{L}(\mathscr{A}_\mathbb{P})$ is the intersection of hyperplanes involving a set I consisting of κ of the q pieces. The *essential part* of $\mathcal{U}$ is the subspace $\widetilde{\mathcal{U}}$ of $\mathbb{R}^{2\kappa}$ that satisfies the same attack equations as $\mathcal{U}$. Define $\alpha(\mathcal{U};n)$ to be the number of integral points in the dilation of $(\mathcal{B}^\circ)^\kappa \cap \widetilde{\mathcal{U}}$, i.e.,

$$\alpha(\mathcal{U};n) := E_{(0,1)^{2\kappa}\cap\widetilde{\mathcal{U}}}(n+1).$$

(The utility of this quantity is that it is independent of q, because $\widetilde{\mathcal{U}}$ is independent of the value of q used to construct it from $\mathcal{U}$.) By Ehrhart theory $\alpha(\mathcal{U};n)$ is a quasipolynomial of degree $2\kappa - \operatorname{codim}\mathcal{U}$. Since $\mathcal{U} \cong \mathbb{R}^{2(q-\kappa)} \times \widetilde{\mathcal{U}}$, the number of lattice points in $\mathcal{U}\cap\mathcal{P}^\circ$ is $n^{2(q-\kappa)}\alpha(\mathcal{U};n)$.

The Parity Theorem (Theorem II.4.1) tells us that $\alpha(\mathcal{U};n)$ is an even or odd function of n (depending on the codimension of $\mathcal{U}$). What it does not say is how that affects the number of undetermined coefficients in computing $\alpha(\mathcal{U};n)$, which is, in particular, the number of values of the function we need to interpolate all the coefficients. In general, an Ehrhart quasipolynomial of degree d with period p has $pd+1$ coefficients that have to be computed. (The leading coefficient is the same for all constituents; it is the volume of $\mathcal{U}\cap\mathcal{P}$.) The full theorem, then, should be this:

Theorem 2.1 (Strong Parity Theorem)**.** *For a subspace $\mathcal{U}\in\mathscr{L}(\mathscr{A}_\mathbb{P})$ whose equations involve κ pieces, for which $\alpha(\mathcal{U};n)$ has period p, the number of values of $\alpha(\mathcal{U};n)$ that are sufficient to determine all the coefficients in all constituents is $\lceil p(\kappa - \frac{1}{2}\operatorname{codim}\mathcal{U})\rceil + \varepsilon$, where $\varepsilon = 1$ if $\operatorname{codim}\mathcal{U}$ is even and 0 if it is odd.*

Proof. Let $\alpha(n) := \alpha(\mathcal{U};n)$ and $\nu := \operatorname{codim}\mathcal{U}$. Thus, α has degree $d := 2\kappa - \nu$.

Let the constituents of α be $\alpha_0, \alpha_1, \dots, \alpha_{p-1}$; that means $\alpha(n) = \alpha_{n \bmod p}(n)$. We take subscripts modulo p so that, e.g., $\alpha_{-1} = \alpha_{p-1}$. Write $\alpha_i(n) = a_d n^d + a_{i,d-1}n^{d-1} + \cdots + a_{i,0}n^0$. Since $\alpha_{-i}(-n) = (-1)^d\alpha_i(n)$ (Corollary II.4.1),

$$\begin{aligned}\alpha_{-i}(-n) = a_d n^d + a_{-i,d-1}n^{d-1} + \cdots + a_{-i,0}n^0 = \\ (-1)^d\alpha_i(n) = a_d n^d(-1)^0 + a_{i,d-1}n^{d-1}(-1)^1 + \cdots + a_{i,0}n^0(-1)^d.\end{aligned}$$

Subtracting, $\sum_{j=0}^{d-1}[a_{i,j}(-1)^{d-j} - a_{-i,j}]n^j = 0$, which implies that $a_{-i,j} = (-1)^{d-j}a_{i,j}$ for $j<d$. It follows that only the coefficients for $0 \le i \le p/2$ need to be computed. There are $d\lfloor (p-1)/2 \rfloor$ coefficients with $j<d$ for $0<i<p/2$. For $i=0$, Corollary II.4.1 says that α_0 is an even or odd polynomial (depending on d) and so is $\alpha_{p/2}$ if the period is even. The number of coefficients to determine, other than α_d, is therefore $\lfloor d/2 \rfloor$ for α_0 and the same for $\alpha_{p/2}$ if it exists. Summing these up, there are

$$\frac{pd}{2} + \begin{cases} 1 & \text{if } d \text{ is even,} \\ 0 & \text{if } d \text{ is odd and } p \text{ is even,} \\ \frac{1}{2} & \text{if } pd \text{ is odd} \end{cases}$$

independent coefficients to be computed in α. □

The quasipolynomial for the number of nonattacking configurations of q unlabelled pieces on an $n\times n$ board expands in powers of n in the form

$$u_\mathbb{P}(q;n) = \gamma_0(n)n^{2q} + \gamma_1(n)n^{2q-1} + \gamma_2(n)n^{2q-2} + \cdots + \gamma_{2q}(n)n^0.$$

With labelled pieces the number is $o_\mathbb{P}(q;n)$, which equals $q!u_\mathbb{P}(q;n)$. Our task is to find the coefficients $\gamma_i(n)$, or in practice $q!\gamma_i(n)$, which we know to be polynomials in q that may differ for each residue class of n modulo the period p (Theorem I.4.2). Ehrhart theory says that the leading coefficient of $o_{\mathbb{P}}(q;n)$ is the volume of the polytope $[0,1]^{2q}$, i.e., 1; so

$$\gamma_0 = 1/q!$$

for every piece.

In the proofs we assume acquaintance with the counting theory of Part II for the square board, $[0,1]^2$ in the notation of Part II. For a basic move $m = (c,d)$, we define $\hat{c} := \min(|c|,|d|)$, $\hat{d} := \max(|c|,|d|)$, and $\bar{n} := (n \bmod d) \in \{0,\ldots,n-1\}$.

In Part II we defined

$$\alpha^{d/c}(n) := \alpha(\mathcal{H}^{d/c}_{12}; n),$$

the number of ordered pairs of positions that attack each other along slope d/c (they may occupy the same position; that is considered attacking). Similarly,

$$\beta^{d/c}(n) := \alpha(\mathcal{H}^{d/c}_{12} \cap \mathcal{H}^{d/c}_{13}; n),$$

the number of ordered triples that are collinear along slope d/c. Proposition II.3.1 gives general formulas for α and β. We need only a few examples for later use:

$$\begin{aligned}
\alpha^{0/1}(n) &= \alpha^{1/0}(n) = n^3, \qquad \alpha^{\pm 1/1}(n) = \frac{2n^3+n}{3},\\
\beta^{0/1}(n) &= \beta^{1/0}(n) = n^4, \qquad \beta^{\pm 1/1}(n) = \frac{n^4+n^2}{2},\\
\alpha^{\pm 2/1}(n) &= \alpha^{\pm 1/2}(n) = \begin{cases} \frac{5}{12}n^3 + \frac{1}{3}n & \text{for } n \text{ even},\\ \frac{5}{12}n^3 + \frac{7}{12}n & \text{for } n \text{ odd} \end{cases}\\
&= \frac{5}{12}n^3 + \frac{11}{24}n + (-1)^n \left\{\frac{1}{8}n\right\},\\
\beta^{\pm 2/1}(n) &= \beta^{\pm 1/2}(n) = \begin{cases} \frac{3}{16}n^4 + \frac{1}{4}n^2 & \text{for } n \text{ even},\\ \frac{3}{16}n^4 + \frac{5}{8}n^2 + \frac{3}{16} & \text{for } n \text{ odd}. \end{cases}\\
&= \frac{3}{16}n^4 + \frac{7}{16}n^2 + \frac{3}{32} - (-1)^n \left\{\frac{3}{16}n^2 + \frac{3}{32}\right\}.
\end{aligned} \tag{2.1}$$

3. Periods and Denominators

3.1. Vertex Configurations and the Denominator.

A point $\mathbf{z} = (z_1,\ldots,z_q)$ of an inside-out polytope $(\mathcal{B}^q, \mathscr{A}_{\mathbb{P}})$, associated with q copies of a piece $\mathbb{P}$ on a board $\mathcal{B}$, represents a configuration of q pieces on the board, with piece $\mathbb{P}_i$ located at position $z_i \in \mathcal{B}$.

A vertex $\mathbf{z}$ of $(\mathcal{B}^q, \mathscr{A}_{\mathbb{P}})$ is determined by $2q$ equations that are either *move equations*, associated to hyperplanes $\mathcal{H}^{d/c}_{ij} \in \mathscr{A}_{\mathbb{P}}$, or *boundary equations*, also called *fixations* in this part, of the form $z_i \in$ an edge line $\mathcal{E}$ of $\mathcal{B}$; for the square board a fixation is one of $x_i = 0$, $y_i = 0$, $x_i = 1$, and $y_i = 1$. (For a configuration in an N-fold dilation $N \cdot \mathcal{B}$, a fixation has the form $z_i \in N \cdot \mathcal{E}$.) The vertex $\mathbf{z}$ represents a configuration with $\mathbb{P}_i$ on the boundary of the board (if it has a fixation) or attacking one or more other pieces (if in a hyperplane). We call the configuration of pieces that corresponds to a vertex $\mathbf{z}$ a *vertex configuration*.

Let us write $\Delta(\mathbf{z})$ for the least common denominator of a fractional point $\mathbf{z} \in \mathbb{R}^{2q}$ and call it the *denominator of* $\mathbf{z}$. The denominator $D = D(\mathcal{B}^q, \mathscr{A}_{\mathbb{P}})$ of the inside-out polytope is the least common multiple of the denominators $\Delta(\mathbf{z})$ of the individual vertices.

One way to find $\Delta(\mathbf{z})$ for a vertex $\mathbf{z}$ is to find its coordinates by intersecting move hyperplanes of $\mathbb{P}$ and facet hyperplanes of $\mathcal{B}^q$. There is an equivalent method to find $\Delta(\mathbf{z})$. For a set of move equations and fixations producing a vertex configuration $\mathbf{z}$, notice that the $\Delta(\mathbf{z})$-multiple of $\mathbf{z}$ has integer coordinates and no smaller multiple of $\mathbf{z}$ does. This proves:

Lemma 3.1. *For a vertex $\mathbf{z}$ of $(\mathcal{B}^q, \mathscr{A}_{\mathbb{P}})$, $\Delta(\mathbf{z})$ equals the smallest integer N such that a configuration $N \cdot \mathbf{z}$ satisfying the move equations and fixations $z_i \in N \cdot \mathcal{E}$ for edge lines $\mathcal{E}$ ($x_i = 0$, $y_i = 0$, $x_i = N$, or $y_i = N$ on the square board) has integral coordinates.*

We define Δ_q to be the maximum value of the denominator $\Delta(\mathbf{z})$ over all vertices $\mathbf{z}$ of the inside-out polytope for q. (It is not to be confused with D_q, which means the least common multiple of vertex denominators.)

We expect the period to be weakly increasing with q and also with the set of moves; that is, if $q' > q$, the period for q' pieces should be a multiple of that for q; and if $\mathbb{P}'$ has move set containing that of $\mathbb{P}$, then the period of $\mathbb{P}'$ should be a multiple of that of $\mathbb{P}$. (The pieces need not be partial queens.) We cannot prove either property, but they are obvious for denominators, and we see in examples that the period equals the denominator. We write $D_q(\mathbb{P})$ for the denominator of the inside-out polytope $([0,1]^{2q}, \mathscr{A}_{\mathbb{P}}^q)$. (The optional superscript in $\mathscr{A}_{\mathbb{P}}^q$ shows the number of pieces.)

Proposition 3.2. *Let $\mathcal{B}$ be any board, let $q' > q > 0$, and suppose $\mathbb{P}$ and $\mathbb{P}'$ are pieces such that every basic move of $\mathbb{P}$ is also a basic move of $\mathbb{P}'$. Then the denominators satisfy $D_q(\mathbb{P}) | D_{q'}(\mathbb{P})$ and $D_q(\mathbb{P}) | D_q(\mathbb{P}')$.*

Proof. The first part is clear if we embed $\mathbb{R}^{2q}$ into $\mathbb{R}^{2q'}$ as the subspace of the first $2q$ coordinates, so the polytope $\mathcal{B}^q$ is a face of $\mathcal{B}^{2q'}$ and the move arrangement $\mathscr{A}_{\mathbb{P}}^q$ in $\mathbb{R}^{2q}$ is a subarrangement of the arrangement induced in $\mathbb{R}^{2q}$ by $\mathscr{A}_{\mathbb{P}}^{q'}$.

The second part is obvious since $\mathscr{A}_{\mathbb{P}}^q \subseteq \mathscr{A}_{\mathbb{P}'}^q$. ☐

Although in Ehrhart theory periods often are less than denominators, we observe that not to be true for our solved chess problems. We believe that will some day become a theorem.

Conjecture 3.3. For every rider piece $\mathbb{P}$ and every number of pieces $q \geq 1$, the period of the counting quasipolynomial $u_{\mathbb{P}}(q; n)$ equals the denominator $D([0,1]^{2q}, \mathscr{A}_{\mathbb{P}})$ of the inside-out polytope for q pieces $\mathbb{P}$.

3.2. **Partial queens and polynomials.** A *partial queen* $\mathbb{Q}^{hk}$ is a piece with h basic moves that are horizontal or vertical (obviously, $h \leq 2$) and k basic moves at $\pm 45°$ to the horizontal (also, $k \leq 2$). We studied partial queens in Part III. Table 3.1 contains a list of the partial queens with their names and what we know or believe about the periods of their counting quasipolynomials. In particular, we think four of the partial queens are uniquely special.

Conjecture 3.4. The rook and semirook, the semibishop, and the subqueen are the only four pieces that have period 1—that is, whose counting functions are polynomials in n.

Conjecture 3.4 is supported by the fact that, by Theorem 4.2, the only one-move pieces with period 1 are the semirook and semibishop. The rook is obvious. If Conjecture 4.3 is true Corollary 4.5 establishes the same for the subqueen. Any other one-move piece has larger

Name	(h,k)	Periods					
		$q=2$	$q=3$	$q=4$	$q=5$	$q=6$	$q>6$
Semirook	$(1,0)$	1	1	1	1	1	1
Rook	$(2,0)$	1	1	1	1	1	1
Semibishop	$(0,1)$	1	1	1	1	1	1
Subqueen	$(1,1)$	1	1	1	1	1	1* $(q=7,8)$
Semiqueen	$(2,1)$	1	1	2*	2*	6*	12° $(q=7)$
Bishop	$(0,2)$	1	2	2	2	2	2
Frontal queen	$(1,2)$	1	2	6*	12°	60°	420° $(q=7)$
Queen	$(2,2)$	1	2	6*	60*	840†	360360* $(q=7)$

TABLE 3.1. The quasipolynomial periods for partial queens $\mathbb{Q}^{hk}$.
* is a number deduced from a formula in [11].
† is deduced from the formula of Karavaev; see [9, 11].
° is a value we conjecture.

period and denominator. Conjecture 3.4 follows from Proposition 3.2 and Conjecture 3.3, if the latter is true.

4. PIECES WITH FEWER MOVES

4.1. **One-move riders.** The denominator of the inside-out polytope of a one-move rider can be explicitly determined for arbitrary boards $\mathcal{B}$.

Given a move $m=(c,d)$, the line parallel to m through a corner z of $\mathcal{B}$ may pass through another point on the boundary of $\mathcal{B}$. Call that point the *antipode* of z. The antipode may be another corner of $\mathcal{B}$. When m is parallel to an edge z_iz_j of $\mathcal{B}$, we consider z_i and z_j to be each other's antipodes.

Proposition 4.1. *For a one-move rider $\mathbb{P}$ with move (c,d), the denominator of the inside-out polytope $(\mathcal{B}^q, \mathscr{A}_{\mathbb{P}})$ equals the least common denominator of the corners of $\mathcal{B}$ when $q=1$, and when $q\geq 2$ it equals the least common denominator of the corners of $\mathcal{B}$ and their antipodes.*

Proof. A vertex of $(\mathcal{B}^q, \mathscr{A}_{\mathbb{P}})$ is generated by some set of hyperplanes, possibly empty, and a set of fixations. The total number of hyperplanes and fixations required is $2q$. When $q=1$, because there are no move equations involved, a vertex of the inside-out polytope is a corner of $\mathcal{B}$.

When $q\geq 2$, a vertex is determined by its fixations and the intersection $\mathcal{U}$ of the move hyperplanes it lies in. Let π be the partition of $[q]$ into blocks for which i and j are in the same block if $\mathcal{H}^{d/c}_{ij}$ is one of the hyperplanes containing $\mathcal{U}$. The number of hyperplanes necessary to determine $\mathcal{U}$ is q minus the number of blocks of π. ($\mathcal{U}$ will be contained in additional, unnecessary hyperplanes if a block of π has three or more members; we do not count those.)

Consider a particular block of π, which we may suppose to be $[k]$ for some $k \geq 1$. We need $k+1$ fixations on the k pieces to specify a vertex, so there must be two fixations that apply to the same $i \in [k]$, anchoring z_i to a corner of $\mathcal{B}$. The remaining $k-1$ fixations fix the other values z_j for $j \in [k]$ to either z_i's corner or its antipode.

It follows that all vertices $(z_1, \ldots, z_q)$ of the inside-out polytope satisfy that each z_i is either a corner or a corner's antipode for all i. Furthermore, with at least two pieces and for every corner z, it is possible to create a vertex containing z and its antipode as components, from which the proposition follows. □

For the square board, the corners are $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$, and the antipodes have denominator $\max(|c|,|d|)$. Alongside Proposition II.6.2, this proves Conjecture II.6.1.

Theorem 4.2. *On the square board with $q \geq 2$ copies of a one-move rider with basic move (c,d), the period of $u_{\mathbb{P}}(q;n)$ is $\max(|c|,|d|)$.*

This is an example where Conjecture 3.3 is true: the period agrees with the denominator.

4.2. **Two-move riders.** We continue with a general board $\mathcal{B}$.

One fruitful kind of configuration of two-move riders with moves (c_1,d_1) and (c_2,d_2) involves a *trajectory*, composed of a sequence of distinct pieces in which the first piece $\mathbb{P}_{t_1}$ is anchored at a corner of $\mathcal{B}$ by two fixations, and the position of each subsequent piece $\mathbb{P}_{t_i}$ is determined by one move hyperplane $\mathcal{H}^{d_i'/c_i'}_{t_{i-1},t_i}$ involving $\mathbb{P}_{t_{i-1}}$ and by one fixation. The *basic moves* of a trajectory are the moves (c_i',d_i'), each of which equals either (c_1,d_1) or (c_2,d_2). We can regard a trajectory as a sequence of points in the board instead of pieces; the position of $\mathbb{P}_{t_i}$ is denoted by z_{t_i}.

A trajectory is *primitive* if any piece after $\mathbb{P}_{t_1}$ that is placed at a corner of $\mathcal{B}$ is its last. Any trajectory can be decomposed into primitive trajectories in the following manner. Suppose $T = (\mathbb{P}_{t_1}, \ldots, \mathbb{P}_{t_k}, \ldots, \mathbb{P}_{t_l})$ is a trajectory, perhaps accompanied by an additional set of trajectories in order to completely determine a vertex $\mathbf{z}$ of the inside-out polytope, in which $k < l$ and z_{t_k} is a corner. We break T into $T_1 = (\mathbb{P}_{t_1}, \ldots, \mathbb{P}_{t_{k-1}})$ and $T_2 = (\mathbb{P}_{t_k}, \ldots, \mathbb{P}_{t_l})$ with $\mathbb{P}_{t_k}$ anchored at its corner by two fixations. We have replaced the move hyperplane $\mathcal{H}^{d_k'/c_k'}_{t_{k-1},t_k}$ by a new fixation that had been implied by this move hyperplane because it had forced $\mathbb{P}_{t_k}$ to be in a corner. Conversely, any equation, whether from a move hyperplane or a fixation, that fixes $\mathbb{P}_{t_k}$ in its corner is sufficient to give the same vertex $\mathbf{z}$ as we get from T and the other accompanying trajectories. Therefore T can be replaced by T_1 and T_2 in determining $\mathbf{z}$.

A *simple trajectory* is a primitive trajectory in which no two consecutive slopes are equal and no segment $z_{t_{i-1}}z_{t_i}$ lies on the boundary of the board.

Figure 4.1 shows that primitive trajectories can involve complex dynamics. The pattern of piece placements depends on the range into which the slopes fall (less than -1, between -1 and 0, between 0 and 1, and greater than 1). In most cases, the piece positions and the denominator of the corresponding vertex follow a pattern that is difficult to describe completely.

We believe that for two-move riders simple trajectories encompass the full scope of possible denominators, as conjectured here.

Conjecture 4.3. The points $z_i \in \mathbb{R}^2$ that occur as components of a vertex $\mathbf{z}$ of an inside-out polytope of q two-move riders are determined by simple trajectories. Such points can only occur as points along a k-point simple trajectory where $k \leq q$, as intersection points of two

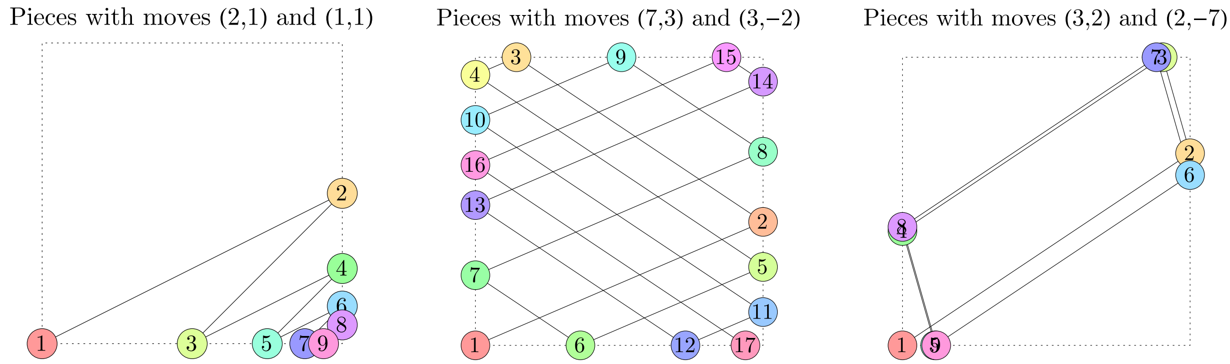

FIGURE 4.1. A two-move rider with diagonal slopes can produce configurations with arbitrarily large denominators. The coordinates of the ninth pieces are, from left to right, $(15/16, 0)$, $(32/63, 1)$, and $(22850/194481, 0)$.

simple trajectories of i and j points where $i + j \leq q + 1$, or as self-intersection point of a k-point simple trajectory where $k \leq q$. As a consequence, denominators of $\mathbf{z}$ only arise from such points.

Intersection points have to be included in the conjecture. If z is the intersection of segments $\mathbb{P}_i\mathbb{P}_j$ and $\mathbb{P}_k\mathbb{P}_l$ of the trajectory or trajectories, segments parallel respectively to moves m and m', $\mathbf{z}$ may also have a piece $\mathbb{P}_h$ located at z (so $z = z_h$), which is fixed in place by the move equations connecting it to $\mathbb{P}_i$ by move m and $\mathbb{P}_k$ by move m'. The length limits on the trajectories come from the fact that there are only q pieces. For an intersection point, one piece $\mathbb{P}_h$ has to be reserved to place on the intersection in order to generate that point's denominator. With $\mathbb{P}_h$ on the intersection it is no longer necessary to have a piece on the final points of the intersecting trajectory or trajectories.

Example 5.2 demonstrates that the conjecture is false for three or more moves.

Proposition 4.4. *Let $\mathcal{B}$ be the square board and let c and d be relatively prime positive integers. If Conjecture 4.3 is true, then the denominator of the inside-out polytope for q two-move riders with moves $(1,0)$ and $(\pm c, \pm d)$ is*

$$D = \begin{cases} 1 & \text{if } q = 1, \\ d & \text{if } d \geq c \text{ and } q > 1, \\ c & \text{if } d < c \text{ and } 1 < q \leq 2\lfloor c/d \rfloor + 1, \\ cd & \text{if } d < c \text{ and } q \geq 2\lfloor c/d \rfloor + 2. \end{cases}$$

Proof. We consider the two-move rider with move (c, d); the argument is similar for the other signs. We assume $q > 1$ since otherwise the period is 1 and the problem is trivial.

Assuming Conjecture 4.3, we obtain all vertices of the inside-out polytope by combining simple trajectories.

Construct a trajectory T by fixing a piece $\mathbb{P}_1$ at the corner $(0, 0)$ and following the two moves alternately, with slope first d/c, then 0, etc. At each step, stop when the move hits an edge of the square, place the next piece there, and then begin the next move. So $\mathbb{P}_2$, $\mathbb{P}_3$, $\mathbb{P}_4$ have coordinates $(1, d/c)$, $(0, d/c)$, $(1, 2d/c)$, and so forth. If q is sufficiently large, this generates a trajectory that continues until it reaches $y = 1$, where it stops. Evidently, T is simple and has no self-intersections. The only other simple trajectory is T', a rotation of T by 180° around the center of the square. T' begins at $(1, 1)$ and is centrally symmetric to T.

If $d \geq c$, then T ends at $\mathbb{P}_2$ with coordinates $(c/d, 1)$. T and T' do not intersect, so d is the only denominator.

If $d < c$, then $\mathbb{P}_2$ in trajectory T has coordinates $(1, d/c)$ and $\mathbb{P}_3$ has coordinates $(0, d/c)$. This zigzag pattern continues up to $\mathbb{P}_{2k+1}$ at $(0, k(d/c))$, where $k = \lceil c/d \rceil - 1 = \lfloor c/d \rfloor$, and then $\mathbb{P}_{2k+2}$ on the line $y = 1$ with x-coordinate $c/d - \lfloor c/d \rfloor$. (See the illustration in Figure 4.2.) By central symmetry, the points along T and T' have the same denominators and neither T nor T' has a self-intersection. Thus, if $q \geq 2k + 2$, there is a denominator d as well as c and the overall denominator of the configuration is cd.

Pieces with moves (13,4) and (1,0)

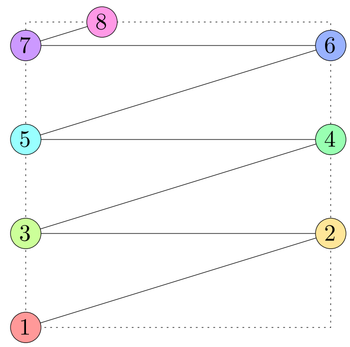

Figure 4.2. The coordinates of pieces in a trajectory of a two-move rider with a horizontal move has denominator c until a piece has y-coordinate 1. In this example $(c, d) = (13, 4)$ and the coordinates of the even pieces are $\mathbb{P}_2(1, 4/13)$, $\mathbb{P}_4(1, 8/13)$, $\mathbb{P}_6(1, 12/13)$, and $\mathbb{P}_8(1/4, 1)$.

It remains to calculate the intersections of T and T' if they are not the same trajectory and if q is big enough for them to intersect. They are the same when c/d is an integer (that means $d = 1$) and $q \geq 2c$ ($\mathbb{P}_{2c}$ has coordinates $(1, 1)$ in that case); then the denominator is c, which is also the denominator when there are q points in T alone if $1 < q < 2c$.

Now assume T and T' are not the same; that is, c/d is fractional ($d \neq 1$). Intersections occur when a sloped edge of one trajectory (say T) intersects a horizontal edge of the other trajectory (say T'). The horizontal edges of T' occur at y-coordinates $1 - id/c$, $1 \leq i \leq k$. The sloped edge joins, say, $\mathbb{P}_{2j-1}$ to $\mathbb{P}_{2j}$. The x-coordinate of the intersection point is $i + j - c/d$, whose denominator is d. Thus, points on the edges and points of intersection have denominators c and d, implying that the least common denominator of all points of the configuration is cd. For there to be an intersection, however, $\mathbb{P}_{2j}$ must have greater y-coordinate than a horizontal edge of T'; thus, $jd/c > 1 - id/c$, or $i + j > \lfloor c/d \rfloor$. As the horizontal edge in T' is $\mathbb{P}'_{2i}\mathbb{P}'_{2i+1}$, the number of pieces necessary for an intersection to occur is $2(i + j) + 1 \geq 2(\lfloor c/d \rfloor + 1) + 1$, the denominator d appears only if $q \geq 2\lfloor c/d \rfloor + 3$. Otherwise, the overall denominator in the trajectories is c. □

Proposition 4.4 applies to the subqueen, which has one diagonal move and one horizontal or vertical move. Therefore:

Corollary 4.5. *If Conjecture 4.3 is true, the denominator and period of the subqueen are* 1.

5. Pieces with More Moves

A piece with three or more moves enters a new domain of complexity that begins when we have just three copies of it. (In this section and from now on we assume the square board.)

For such a piece, the denominator D_q grows exponentially, or super-exponentially, with q (Theorem 5.10). Conjecture 3.3 would imply that the period of its counting quasipolynomial also grows at least exponentially. Two fruitful constructions that yield the largest known vertex denominators $\Delta(\mathbf{z})$ combine $2q-3$ move equations and three fixations. These constructions produce pieces arranged in a golden parallelogram configuration when the piece has at least three moves (see Example 5.4) or in a twisted Fibonacci spiral when the piece has at least four moves (see Example 5.15).

5.1. Triangle configurations.

With three or more moves, a new key configuration appears. It is a triangle of pairwise attacking pieces. We can calculate the corresponding denominator Δ.

Consider a piece with the three moves $m_1 = (c_1, d_1)$, $m_2 = (c_2, d_2)$, and $m_3 = (c_3, d_3)$. Since no move is a multiple of another, there exist nonzero integers w_1, w_2, and w_3 such that $w_1m_1 + w_2m_2 + w_3m_3 = (0, 0)$ with $\gcd(w_1, w_2, w_3) = 1$. The w_i are unique up to negating all of them.

Proposition 5.1. *For $q = 3$, a triangular configuration of three pieces on the square board, attacking pairwise along three distinct move directions $m_1 = (c_1, d_1)$, $m_2 = (c_2, d_2)$, and $m_3 = (c_3, d_3)$, together with three fixations that fix its position in the square $[0, 1]^2$, gives a vertex $\mathbf{z}$ of the inside-out polytope. Its denominator is*

$$\Delta(\mathbf{z}) = \max(|w_1c_1|, |w_1d_1|, |w_2c_2|, |w_2d_2|, |w_3c_3|, |w_3d_3|). \tag{5.1}$$

The pieces may be at corners, and there may be two pieces on the same edge. The three fixations may be choosable in more than one way but they will give the same denominator.

Proof. There is a unique similarity class of triangles with edge directions m_1, m_2, and m_3, if we define triangles with opposite orientations to be similar. We can assume the the pieces are located at coordinates z_1, z_2, z_3 with $\max y_i - \min y_i \leq \max x_i - \max x_i$ (by diagonal reflection), with $x_1 \leq x_2 \leq x_3$ (by suitably numbering the pieces), with $y_1 \leq y_3$ (by horizontal reflection), and with z_2 below the line z_1z_3 (by a half-circle rotation). The reflections change the move vectors m_i by negating or interchanging components; that makes no change in Equation (5.1). We number the slopes so that m_1, m_2, and m_3 are, respectively, the directions of z_1z_2, z_1z_3, and z_2z_3.

Given these assumptions the triangle must have width $x_3 - x_1 = 1$, since otherwise it will be possible to enlarge it by a similarity transformation while keeping it in the square $[0, 1]^2$; consequently $x_1 = 0$ and $x_3 = 1$. Furthermore, the slopes satisfy $d_1/c_1 < d_2/c_2 < d_3/c_3$. (If $c_3 = 0$ we say the slope $d_3/c_3 = +\infty$ and treat it as greater than all real numbers. If $c_1 = 0$ we say $d_1/c_1 = -\infty$ and treat it as less than all real numbers. c_2 cannot be 0.) Our configuration has $d_2/c_2 \geq 0$ so two slopes are nonnegative but d_1/c_1 may be negative. That gives two cases.

If $d_1/c_1 \leq 0$, we choose fixations $x_1 = 0$, $y_2 = 0$, and $x_3 = 1$. (A different choice of fixations is possible if z_1z_2 is horizontal or vertical, if z_2z_3 is horizontal or vertical, or if z_1z_3 is horizontal, not to mention combinations of those cases. Note that the denominator computation depends on the differences of coordinates rather than their values. In each horizontal or vertical case the choice of fixations affects only the triangle's location in the square, not its size or orientation.)

If $d_1/c_1 > 0$, we choose fixations $x_1 = y_1 = 0$ and $x_3 = 1$.

The rest of the proof is the same for both cases. First we prove that the configuration is a vertex. That means the locations of the three pieces are completely determined by the fixations and the fact that $\mathbf{z} = (z_1, z_2, z_3) \in \mathcal{H}_{12}^{m_1} \cap \mathcal{H}_{13}^{m_2} \cap \mathcal{H}_{23}^{m_3}$. We know the similarity class of $\triangle z_1z_2z_3$ and its orientation. The fixations of $\mathbb{P}_1$ and $\mathbb{P}_3$ determine the length of the segment z_1z_3. That determines the congruence class of $\triangle z_1z_2z_3$, and the fixations determine its position. Thus, $\mathbf{z}$ is a vertex.

We now aim to find the smallest integer N such that $N \cdot \triangle z_1z_2z_3$ has integral coordinates, i.e., it embeds in the integral lattice $[0, N] \times [0, N]$. By the definition of w_1, w_2, and w_3, we know that $z_1' = (0, 0)$, $z_2' = -w_1m_1$, and $z_3' = w_2m_2$ gives an integral triangle that is similar to $\triangle z_1z_2z_3$ and similarly or oppositely oriented, because its sides have the same slopes. If $\triangle z_1'z_2'z_3'$ is oppositely oriented to $\triangle z_1z_2z_3$ (that means z_2' is above the line $z_1'z_3'$), we can make the orientations the same by negating all w_i. Given these restrictions $\triangle z_1'z_2'z_3'$ is as compact as possible, for if some multiple $\nu\triangle z_1'z_2'z_3'$ were smaller ($0 < \nu < 1$) and integral, then $(\nu w_1)m_1 + (\nu w_2)m_2 + (\nu w_3)m_3 = 0$ with integers $\nu w_1, \nu w_2, \nu w_3$, so ν would be a proper divisor of 1. By Lemma 3.1, $N = \Delta(\mathbf{z})$. We can now translate $\triangle z_1'z_2'z_3'$ to the box $[0, N] \times [0, N]$ where

$$N = \max(|w_1c_1|, |w_1d_1|, |w_2c_2|, |w_2d_2|, |w_3c_3|, |w_3d_3|). \qquad \square$$

Example 5.2. For the three-move partial nightrider with move set $\mathbf{M} = \{(2, -1), (2, 1), (1, 2)\}$, since $3 \cdot (2, -1) - 5 \cdot (2, 1) + 4 \cdot (1, 2) = (0, 0)$ the denominator is

$$\max(|6|, |-3|, |-10|, |-5|, |4|, |8|) = 10,$$

as shown in Figure 5.1.

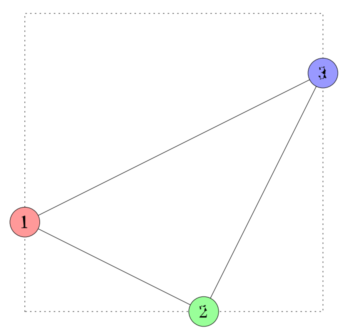

Figure 5.1. The integral configuration determined by $\mathcal{H}_{12}^{-1/2}$, $\mathcal{H}_{13}^{1/2}$, $\mathcal{H}_{23}^{2/1}$, $x_1 = 0$, $y_2 = 0$, and $x_3 = 10$. The coordinates are $(0, 3)$, $(6, 0)$, and $(10, 8)$. This illustrates Proposition 5.1 when $d_1/c_1 < 0$. The value of N is 10.

5.2. **Three-move configurations.**

Now, for pieces with three or more moves, we explore vertex configurations that use only three moves. We motivate the general case by studying the semiqueen $\mathbb{Q}^{21}$ which has a horizontal, vertical, and diagonal move: $\mathbf{M} = \{(1, 0), (0, 1), (1, 1)\}$. To supplement the standard move-hyperplane notation we define $\mathcal{X}_{ij} := \mathcal{H}_{ij}^{1/0} : x_i = x_j$ and $\mathcal{Y}_{ij} := \mathcal{H}_{ij}^{0/1} : y_i = y_j$ for the hyperplanes that express an attack along a file (column) or rank (row).

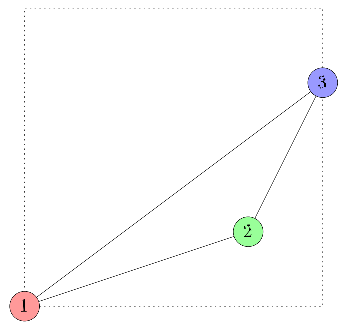

FIGURE 5.2. The integral configuration determined by $\mathcal{H}_{12}^{1/3}$, $\mathcal{H}_{13}^{3/4}$, $\mathcal{H}_{23}^{2/1}$, $x_1 = 0$, $y_2 = 0$, and $x_3 = 10$. The coordinates are $(0,0)$, $(3,1)$, and $(4,3)$. This illustrates Proposition 5.1 when $d_1/c_1 > 0$. The value of N is 4.

In this subsection and the next we use the terminology "golden rectangle configuration" and "discrete Fibonacci spiral", which is inspired by the following two concepts. (We index the Fibonacci numbers F_i so that $F_0 = F_1 = 1$.) A *golden rectangle* is a rectangle whose sides are in the ratio $1{:}\varphi$, φ being the golden ratio $\frac{1+\sqrt{5}}{2}$. The rectangle that has side lengths F_i and F_{i+1} is a close approximation to such a rectangle. The *Fibonacci spiral* is an approximation to the *golden spiral* (the logarithmic spiral with growth factor φ) where squares of Fibonacci side length are arranged in an outwardly spiraling manner and each has a quarter circle inscribed, as shown in Figure 5.3.

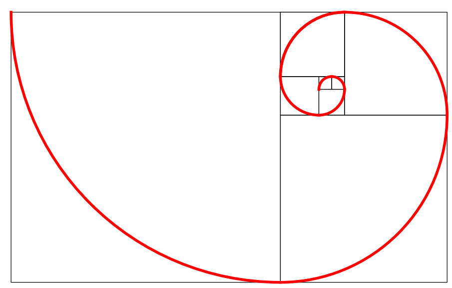

FIGURE 5.3. The Fibonacci spiral

There are multiple vertex configurations of q semiqueens that have denominator $F_{\lfloor q/2 \rfloor}$. In this analysis we take the diagonal move to be $(-1, 1)$.

The *golden rectangle configuration* with q semiqueens is defined by the move equations

$$\mathcal{X}_{4i,4i+1}, \quad \mathcal{X}_{4i+2,4i+6}, \quad \mathcal{X}_{4i+1,4i+3},$$
$$\mathcal{Y}_{14}, \quad \mathcal{Y}_{4i,4i+4}, \quad \mathcal{Y}_{4i+2,4i+3}, \quad \mathcal{Y}_{4i+3,4i+5},$$
$$\mathcal{H}_{2i+1,2i+2}^{-1/1},$$

for all i such that both indices fall between 1 and q, inclusive, and fixations $y_1 = 0$, $x_2 = 0$, and either $x_q = F_{\lfloor q/2 \rfloor}$ if $\lfloor q/2 \rfloor$ is even or $y_q = F_{\lfloor q/2 \rfloor}$ if $\lfloor q/2 \rfloor$ is odd. These fixations define the smallest square box that contains all pieces in the configuration. They also serve to locate the configuration in the unit-square board, by giving the unique positive integer N such that dividing by N fits the shrunken configuration $\mathbf{z}$ into the square board with three queens fixed on its boundary; thus $\mathbf{z}$ is a vertex (and the shrinkage justifies the use of the term *fixation* for equations like $x_q = F_{\lfloor q/2 \rfloor}$). The denominator $\Delta(\mathbf{z})$ is that integer N. (See Lemma 3.1.) The denominator of this configuration for every value of q is therefore

$$\Delta(\mathbf{z}) = F_{\lfloor q/2 \rfloor}.$$

Figure 5.4(a) shows the golden rectangle configuration of 12 semiqueens; the configuration fits in an 8×13 rectangle. We note that Figure 5.4(b) is a related expanding discrete Fibonacci spiral that has the same denominator; a similar spiral will figure more prominently in configurations with four moves.

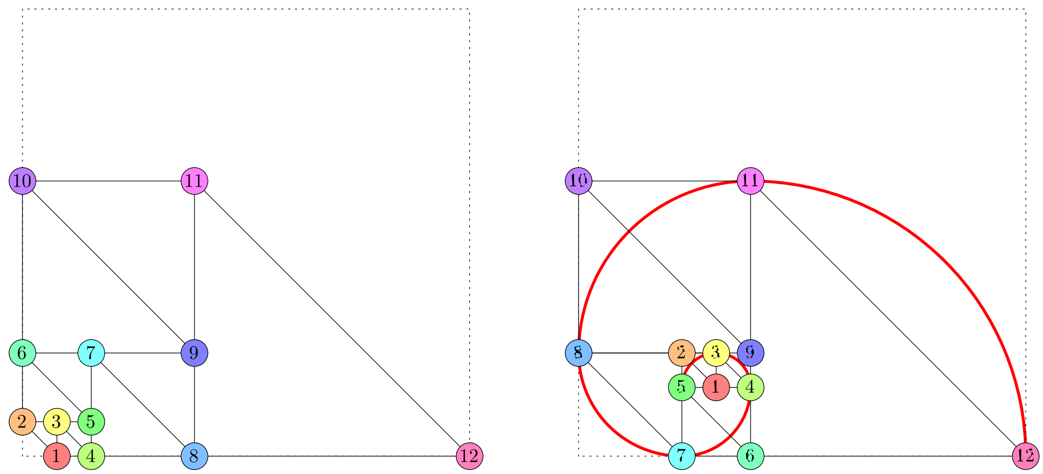

Figure 5.4. (a) The golden rectangle configuration. (b) The discrete Fibonacci spiral.

It is straightforward to find the coordinates of $\mathbb{P}_i$, which we present without proof. We assume coordinates with origin in the lower left corner of Figure 5.4(a).

Proposition 5.3. *For the semiqueen $\mathbb{P} = \mathbb{Q}^{21}$, when the pieces are arranged in the golden rectangle configuration, $\mathbb{P}_1$ is in position $(1,0)$ and for $i \geq 2$, $\mathbb{P}_i$ is in position*

$$\begin{aligned} (F_{\lfloor i/2 \rfloor}, 0) \qquad & \textit{if } i \equiv 0 \bmod 4, \\ (F_{\lfloor i/2 \rfloor}, F_{\lfloor i/2 \rfloor - 1}) \qquad & \textit{if } i \equiv 1 \bmod 4, \\ (0, F_{\lfloor i/2 \rfloor}) \qquad & \textit{if } i \equiv 2 \bmod 4, \\ (F_{\lfloor i/2 \rfloor - 1}, F_{\lfloor i/2 \rfloor}) \qquad & \textit{if } i \equiv 3 \bmod 4. \end{aligned}$$

The step from $\mathbb{P}_{i-1}$ to $\mathbb{P}_i$ is

$$\begin{aligned} F_{\lfloor i/2 \rfloor - 1}(1, -1) \qquad & \textit{if } i \equiv 0 \bmod 4, \\ F_{\lfloor i/2 \rfloor - 1}(0, 1) \qquad & \textit{if } i \equiv 1 \bmod 4, \\ F_{\lfloor i/2 \rfloor - 1}(-1, 1) \qquad & \textit{if } i \equiv 2 \bmod 4, \\ F_{\lfloor i/2 \rfloor - 1}(1, 0) \qquad & \textit{if } i \equiv 3 \bmod 4. \end{aligned}$$

A key idea is that we can apply a linear transformation to the golden rectangle configuration to create **six** *golden parallelogram configurations* for any piece with three moves, some of which may coincide if there is symmetry in the move set. To define the golden parallelogram, in the golden rectangle configuration consider the semiqueens $\mathbb{Q}_1^{21}$ at position $(1,0)$, $\mathbb{Q}_2^{21}$ at $(0,1)$, and $\mathbb{Q}_3^{21}$ at $(1,1)$. They form the smallest possible triangle; they and the construction rule determine the positions of the remaining pieces. For an arbitrary piece $\mathbb{P}$ with moves m_1, m_2, and m_3, we consider the smallest integral triangle involving three copies of $\mathbb{P}$, which we discussed in Proposition 5.1. We apply to the golden rectangle configuration a linear transformation that takes vectors $(1, 0)$ and $(0, 1)$ to any ordered choice of two of the vectors w_1m_1, w_2m_2, and w_3m_3, with a minus sign on one of them if needed to ensure that the third side of the triangle has the correct orientation; that transforms the golden rectangle with the $\mathbb{Q}_i^{21}$ in their locations to a golden parallelogram with pieces $\mathbb{P}_i$ in the transformed locations and with $\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3$ forming the aforementioned smallest triangle; hence, there are six possible golden parallelograms.

Example 5.4. For the three-move partial nightrider (defined in Section 9) the vectors are $w_1m_1 = (6, -3)$, $w_2m_2 = (-10, -5)$, and $w_3m_3 = (4, 8)$. The corresponding six distinct golden parallelogram configurations are presented visually in Figure 5.5. The precise linear transformations are given in Table 5.2. We see that of these six parallelograms, the one yielding the largest denominator is that in the upper left.

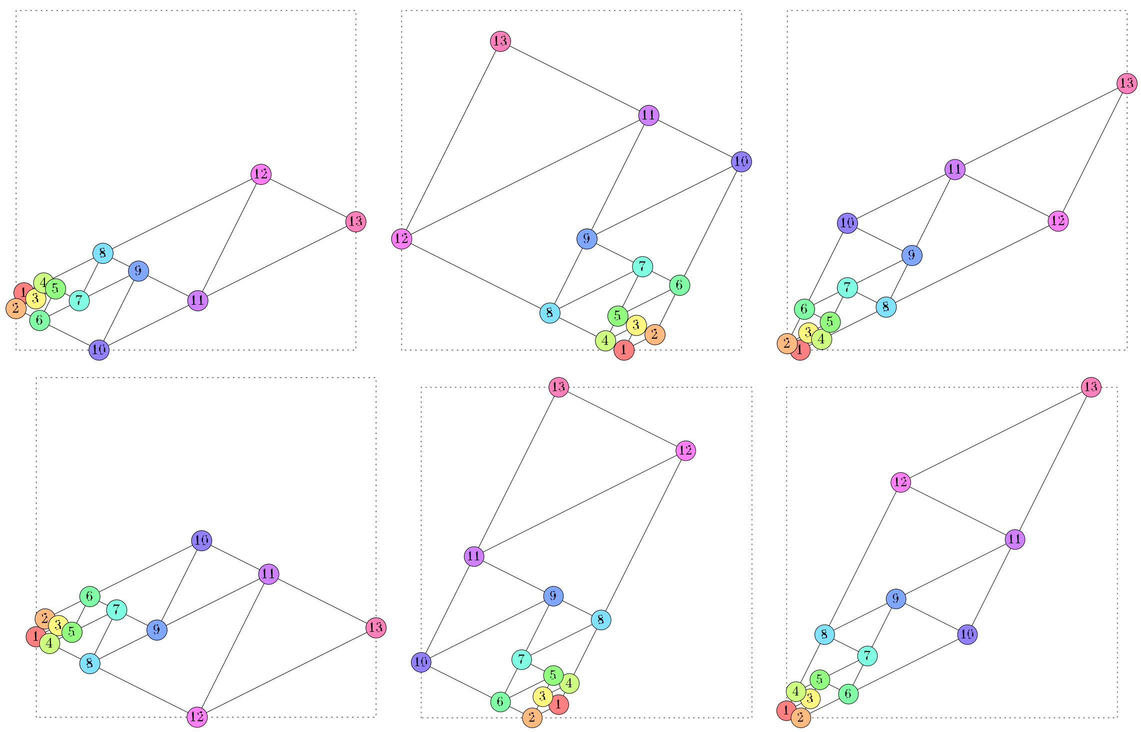

FIGURE 5.5. The six golden parallelograms for 13 three-move partial nightriders. The corresponding linear transformations are given in Table 5.2.

These golden parallelograms appear to maximize the denominator, from which we infer formulas for the largest denominators for various pieces of interest.

Conjecture 5.5. For a piece with exactly three moves, the vertex configuration giving the largest denominator is one of the golden parallelogram configurations.

Example 5.6. The semiqueen has (up to symmetry) only one other golden parallelogram besides the golden rectangle; it is shown in Figure 5.6. It has a larger denominator than the golden rectangle configuration when q is odd and $q \geq 7$. Here, if we consider $\mathbb{P}_2$ to be in

Transformation	$(1,0) \mapsto (10,5)$ $(0,1) \mapsto (6,-3)$	$(1,0) \mapsto (-6,3)$ $(0,1) \mapsto (4,8)$	$(1,0) \mapsto (10,5)$ $(0,1) \mapsto (4,8)$
Δ	172	110	158
Transformation	$(1,0) \mapsto (6,-3)$ $(0,1) \mapsto (10,5)$	$(1,0) \mapsto (4,8)$ $(0,1) \mapsto (-6,3)$	$(1,0) \mapsto (4,8)$ $(0,1) \mapsto (10,5)$
Δ	152	125	139

TABLE 5.1. The linear transformations corresponding to the golden parallelogram configurations of 13 pieces in Figure 5.5, along with the denominator Δ for each configuration.

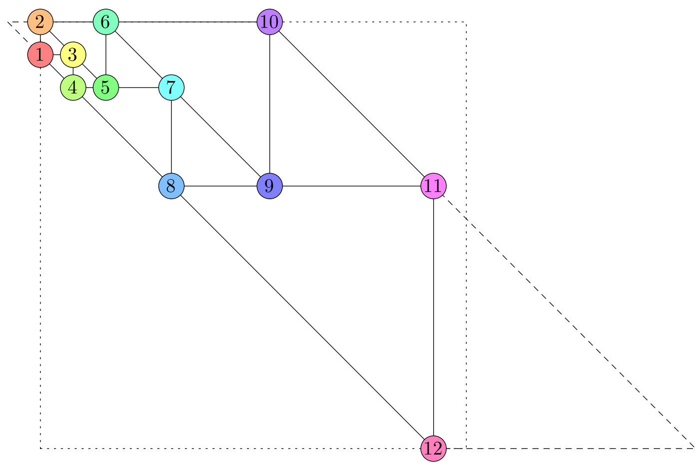

FIGURE 5.6. A golden parallelogram configuration of q semiqueens has the largest denominator when q is odd.

position $(0,0)$, then $\mathbb{P}_1$ is in position $(0,-1)$ and for $i \geq 2$, $\mathbb{P}_i$ is in position

$$\begin{aligned} &(F_{\lfloor i/2 \rfloor} - 1, -F_{\lfloor i/2 \rfloor}) && \text{if } i \equiv 0 \bmod 4,\\ &(F_{\lfloor i/2 \rfloor+1} - 1, -F_{\lfloor i/2 \rfloor}) && \text{if } i \equiv 1 \bmod 4,\\ &(F_{\lfloor i/2 \rfloor} - 1, 0) && \text{if } i \equiv 2 \bmod 4,\\ &(F_{\lfloor i/2 \rfloor+1} - 1, -F_{\lfloor i/2 \rfloor-1}) && \text{if } i \equiv 3 \bmod 4. \end{aligned}$$

Conjecture 5.7. The largest denominator of a vertex configuration for q semiqueens $\mathbb{Q}^{21}$ is $F_{\lfloor q/2 \rfloor}$ if q is even and is $F_{\lfloor q/2 \rfloor+1} - 1$ if q is odd.

Example 5.8. The frontal queen $\mathbb{Q}^{12}$ gives the three distinct golden parallelogram configurations shown in Figure 5.7. Once again, the largest denominator depends on q. Because the piece positions for $i \geq 2$ for the configuration in Figure 5.7(a) are

$$\begin{aligned} &(0, 2F_{\lfloor i/2 \rfloor} - 1) && \text{if } i \equiv 0 \bmod 4,\\ &(F_{\lfloor i/2 \rfloor-1}, F_{\lfloor i/2 \rfloor+2} - 1) && \text{if } i \equiv 1 \bmod 4,\\ &(F_{\lfloor i/2 \rfloor}, F_{\lfloor i/2 \rfloor} - 1) && \text{if } i \equiv 2 \bmod 4,\\ &(F_{\lfloor i/2 \rfloor}, F_{\lfloor i/2 \rfloor+1} + F_{\lfloor i/2 \rfloor-1} - 1) && \text{if } i \equiv 3 \bmod 4, \end{aligned}$$

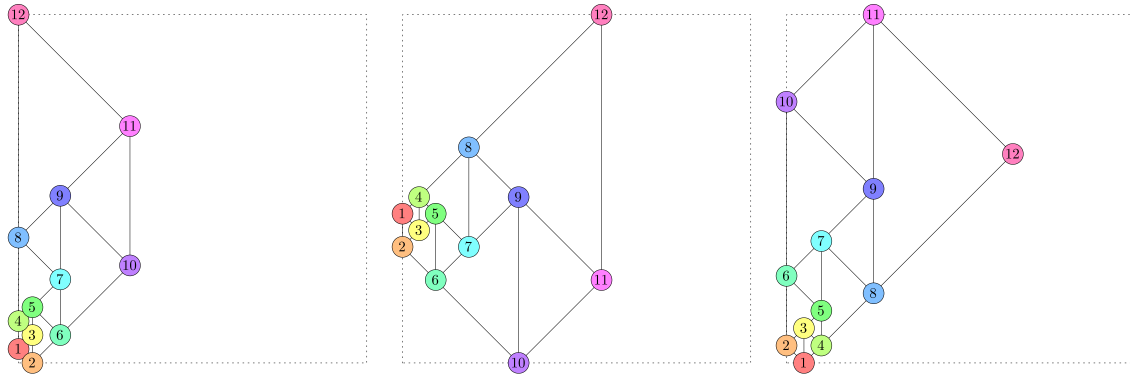

FIGURE 5.7. The three golden parallelograms for the frontal queen $\mathbb{Q}^{12}$. For twelve pieces, the denominators are 25, 21, and 20, respectively.

the largest denominator for such a configuration with q pieces is

$$\begin{aligned} 2F_{\lfloor q/2\rfloor} - 1 &\qquad \text{if } q \equiv 0 \text{ mod } 4,\\ F_{\lfloor q/2\rfloor+2} - 1 &\qquad \text{if } q \equiv 1 \text{ mod } 4,\\ F_{\lfloor q/2\rfloor+1} - 1 &\qquad \text{if } q \equiv 2 \text{ mod } 4,\\ F_{\lfloor q/2\rfloor+1} + F_{\lfloor q/2\rfloor-1} - 1 &\qquad \text{if } q \equiv 3 \text{ mod } 4. \end{aligned}$$

On the other hand, in the configuration in Figure 5.7(c), the piece positions for $i \geq 2$ are

$$\begin{aligned} (F_{\lfloor i/2\rfloor}, F_{\lfloor i/2\rfloor} - 1) &\qquad \text{if } i \equiv 0 \text{ mod } 4,\\ (F_{\lfloor i/2\rfloor}, F_{\lfloor i/2\rfloor+1} + F_{\lfloor i/2\rfloor-1} - 1) &\qquad \text{if } i \equiv 1 \text{ mod } 4,\\ (0, 2F_{\lfloor i/2\rfloor} - 1) &\qquad \text{if } i \equiv 2 \text{ mod } 4,\\ (F_{\lfloor i/2\rfloor-1}, F_{\lfloor i/2\rfloor+2} - 1) &\qquad \text{if } i \equiv 3 \text{ mod } 4, \end{aligned}$$

which yields a largest denominator of such a configuration with q pieces of

$$\begin{aligned} F_{\lfloor q/2\rfloor+1} - 1 &\qquad \text{if } q \equiv 0 \text{ mod } 4,\\ F_{\lfloor q/2\rfloor+1} + F_{\lfloor q/2\rfloor-1} - 1 &\qquad \text{if } q \equiv 1 \text{ mod } 4,\\ 2F_{\lfloor q/2\rfloor} - 1 &\qquad \text{if } q \equiv 2 \text{ mod } 4,\\ F_{\lfloor q/2\rfloor+2} - 1 &\qquad \text{if } q \equiv 3 \text{ mod } 4. \end{aligned}$$

Conjecture 5.9. The largest denominator of a vertex configuration for q frontal queens $\mathbb{Q}^{12}$ is $2F_{q/2} - 1$ if q is even and $F_{(q+3)/2} - 1$ if q is odd.

The symmetry in the piece positions for the two configurations is remarkable.

Suppose the linear transformation that creates a golden parallelogram carries $(1, 0) \mapsto w_1m_1 = (w_1c_1, w_1d_1)$ and $(0, 1) \mapsto w_2m_2 = (w_2c_2, w_2d_2)$. It is possible to write an explicit formula for the denominator of the resulting golden parallelogram configuration. We have not computed the entire formula. It has not more than $128 = 4 \cdot 2^4 \cdot 2$ cases, with one case for each value of q mod 4 and one subcase for each of the 2^4 sign patterns of the components of w_1m_1 and w_2m_2 (sign 0 can be combined with sign +), and in some of those subcases one further subcase for each of the 2^2 magnitude relations between $|w_1c_1|$ and $|w_2c_2|$ or between $|w_1d_1|$ and $|w_2d_2|$.

Theorem 5.10. *The denominators $D_q(\mathbb{P})$ of any piece that has three or more moves increase at least exponentially; specifically, they are bounded below by $\frac{1}{2}\varphi^{q/2}$ when $q \geq 12$, where φ is the golden ratio.*

Proof. To prove the theorem it suffices to produce a vertex of $([0,1]^{2q}, \mathscr{A}^q_{\mathbb{P}})$ with denominator exceeding $\varphi^{q/2}$.

First consider the semiqueen. The points $\mathbb{Q}^{21}_1$ and $\mathbb{Q}^{21}_{4j}$ of the golden rectangle have coordinates $(1,0)$ and $(F_{2j},0)$. Letting $q = 4j$ or $4j+1$ gives an x-difference of $F_{2j}-1$ for a golden rectangle of q pieces. Similarly, letting $q = 4j+2$ or $4j+3$ gives a y-difference of F_{2j+1}. The golden rectangle is a vertex configuration so it follows by Lemma 3.1 that the vertex $\mathbf{z}$ has $\Delta(\mathbf{z}) \geq F_{\lfloor q/2 \rfloor} - 1$. A calculation shows that $F_{\lfloor j \rfloor} - 1 > \frac{1}{2}\varphi^{j+\frac{1}{2}}$ for $j \geq 6$. The theorem for $\mathbb{Q}^{21}$ follows.

An arbitrary piece with three (or more) moves has a golden parallelogram configuration formed from the golden rectangle by the linear transformation $(1,0) \mapsto w_1 m_1$ and $(0,1) \mapsto w_2 m_2$. We may choose these moves from at least three, so we can select m_1 to have $c_1 \neq 0$ and m_2 to have $d_2 \neq 0$. The displacement from $\mathbb{Q}^{21}_1$ to $\mathbb{Q}^{21}_{4j}$ becomes that from $\mathbb{P}_1$ at $w_1 m_1$ to $\mathbb{P}_{4j}$ at $F_{2j} w_1 m_1$. This displacement is $(F_{2j}-1)(w_1 c_1, w_1 d_1)$. Since $c_1 \neq 0$, the x-displacement is at least that for $\mathbb{Q}^{21}$; therefore the denominator of the corresponding vertex for $\mathbb{P}$ is bounded below by $F_{2j}-1$, just as it is for $\mathbb{Q}^{21}$. Similarly, the y-displacement for $q = 4j+2$ is bounded below by F_{2j+1}. This reduces the problem to the semiqueen, which is solved. □

We know that D_q is weakly increasing, by Proposition 3.2. If, as we believe, the period equals D_q, then the period increases at least exponentially for any piece with more than two moves.

We think any board has a similar lower bound, say $C(\mathcal{B})\varphi^{q/2}$ where $C(\mathcal{B})$ is a constant depending upon $\mathcal{B}$, but we ran into technical difficulties trying to prove it.

5.3. **Four-move configurations.** When a piece has four or more moves, the diversity of vertex configurations increases dramatically and the denominators grow much more quickly. Again we start with the piece with the simplest four moves, the queen.

The *discrete Fibonacci spiral* with q queens is defined by the move hyperplanes

$$\mathcal{H}^{+1/1}_{2i,2i+1}, \quad \mathcal{H}^{-1/1}_{2i+1,2i+2}, \quad \mathcal{X}_{1,3}, \quad \mathcal{X}_{2i,2i+3}, \quad \mathcal{Y}_{2i+1,2i+4}$$

for all i such that both indices fall between 1 and q, inclusive, and fixations for pieces $\mathbb{P}_{q-2}$, $\mathbb{P}_{q-1}$, and $\mathbb{P}_q$. The fixations are

$$\begin{aligned} x_q = 0,\ y_{q-1} = 0,\ x_{q-2} = F_q & \qquad \text{if } q \equiv 0 \bmod 4, \\ x_q = 0,\ y_q = 0,\ x_{q-2} = F_q & \qquad \text{if } q \equiv 1 \bmod 4, \\ x_q = F_q,\ y_q = 0,\ x_{q-2} = 0 & \qquad \text{if } q \equiv 2 \bmod 4, \\ y_q = F_q,\ x_{q-1} = 0,\ y_{q-2} = 0 & \qquad \text{if } q \equiv 3 \bmod 4. \end{aligned}$$

Figure 5.8 shows the discrete Fibonacci spiral of 8 queens.

The bounding rectangle of the discrete Fibonacci spiral with q queens has dimensions F_q by F_{q-1} so the vertex's denominator is F_q.

Conjecture 5.11. The largest denominator that appears in any vertex configuration for q queens is F_q.

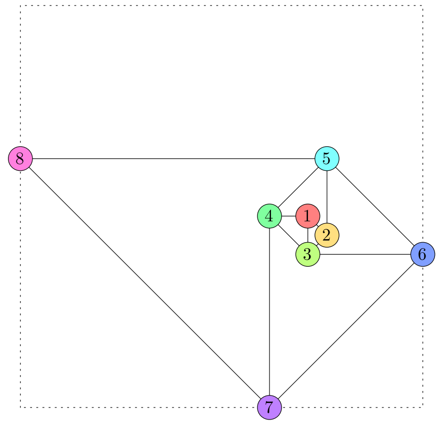

FIGURE 5.8. The discrete Fibonacci spiral for eight queens.

The queen appears to satisfy an extremely special property that is not shared with three-piece riders nor with other four-piece riders. The initial data (for $q \leq 9$) seems to indicate that it is possible to construct vertex configurations that generate *all* denominators up to F_q.

Conjecture 5.12. For every integer δ between 1 and F_q inclusive, there exists a vertex configuration of q queens with denominator δ.

Example 5.13. The eighth Fibonacci number is $F_8 = 21$. The spiral in Figure 5.8 exhibits a denominator of 21. For each $\delta \leq F_7 = 13$ there is a vertex configuration of seven or fewer queens with denominator δ (we do not show them). Figure 5.9 provides seven vertex configurations of eight queens in which the denominator ranges from 14 to 20, as one can tell from the size of the smallest enclosing square and Lemma 3.1.

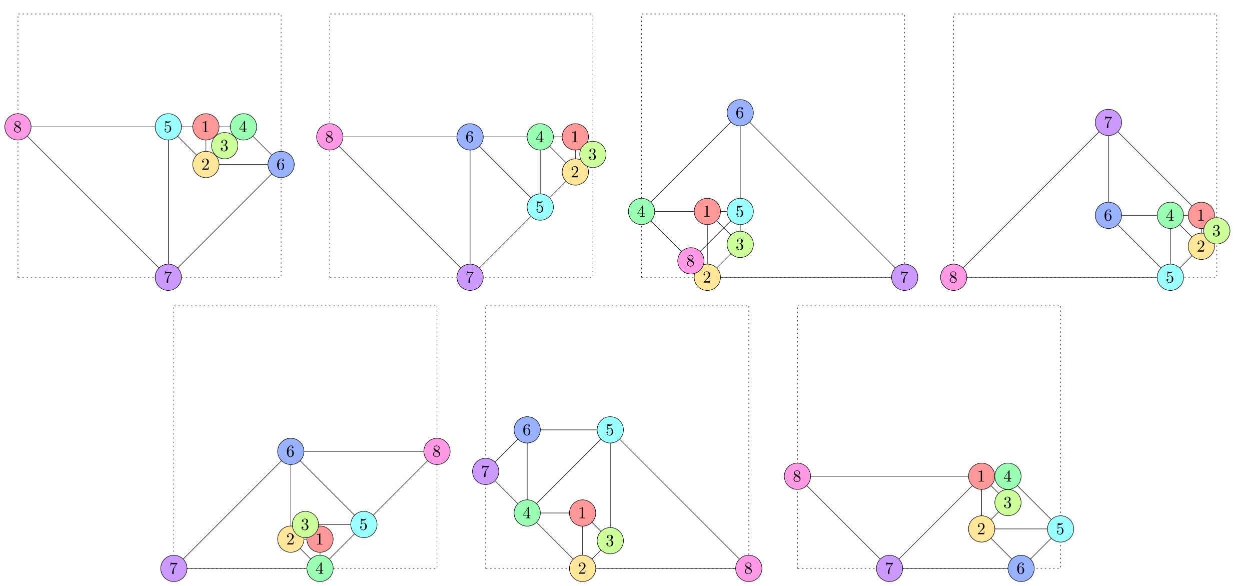

FIGURE 5.9. Vertex configurations of eight queens with denominators 14 through 20.

Conjecture 5.14. The denominator of the inside-out polytope for q queens is $\operatorname{lcm}[F_q]$, where $[F_q] = \{1, 2, \ldots, F_q\}$.

The appearance of Fibonacci numbers in Kotěšovec's Conjecture 8.2 was one of the main motivations for this line of study.

Unlike in the case of three-move pieces, a simple transformation of the queen's Fibonacci spiral to general four-move pieces does not suffice; the configuration also experiences an extra expansion as pieces are added. Consider the nightrider $\mathbb{N}$. Example 5.2 shows that the numbers w_1, w_2, and w_3 are rearrangements of the triple $(3, 4, 5)$. When we fit four nightriders in the next Fibonacci spiral, the smallest triangle must be dilated by an additional factor of four.

We define a *twisted Fibonacci spiral* of q pieces $\mathbb{P}$ with moves $\{m_1, m_2, m_3, m_4\}$ to be defined by the move equations

$$\mathcal{H}^{m_1}_{2i,2i+1}, \quad \mathcal{H}^{m_2}_{2i+1,2i+2}, \quad \mathcal{H}^{m_3}_{1,3}, \quad \mathcal{H}^{m_3}_{2i,2i+3}, \quad \mathcal{H}^{m_4}_{2i+1,2i+4},$$

for all i such that both indices fall between 1 and q, inclusive. In addition, choose the three fixations to ensure that the square box bounding all the pieces is as small as possible and so that all coordinates are integral.

By varying the choice of m_1, m_2, m_3, and m_4 we get different vertex configurations. Consider nightriders in the following example.

Example 5.15. The most obvious analog of the queens' discrete Fibonacci spiral for the nightriders is given in Figure 5.10, for which $m_1 = 1/2$, $m_2 = -2/1$, $m_3 = 2/1$, and $m_4 = -1/2$. There is an alternate vertex configuration with larger denominator, the "expanding kite" shown in Figure 5.11, that is a twisted Fibonacci spiral in which $m_1 = -2/1$, $m_2 = 1/2$, $m_3 = 2/1$, and $m_4 = -1/2$.

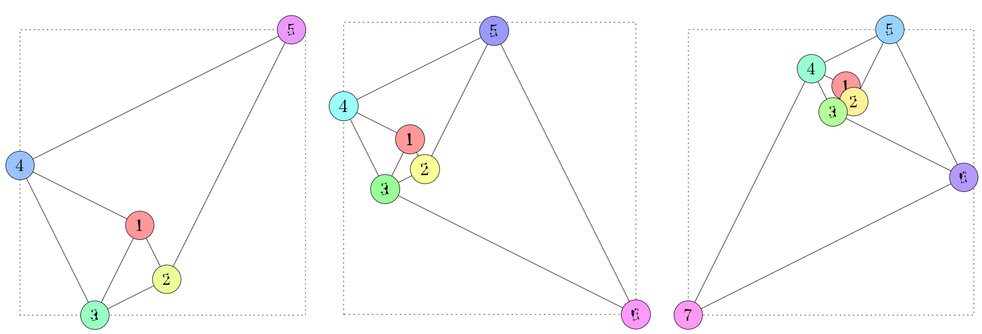

Figure 5.10. A twisted Fibonacci spiral for 5, 6, and 7 nightriders. These configurations have denominators 286, 1585, and 8914.

Conjecture 5.16. For any piece $\mathbb{P}$, there is a vertex configuration that maximizes the denominator and is a twisted Fibonacci spiral.

Unlike for queens, the maximum denominator Δ_q of a vertex configuration of q pieces $\mathbb{P}$ is difficult to compute. Furthermore, it cannot be expected that for all integers N between 1 and Δ_q inclusive, there will exist a vertex configuration of q pieces $\mathbb{P}$ with denominator N. As an example, with three nightriders the only possible denominators are $\{1, 2, 3, 4, 5, 10\}$.

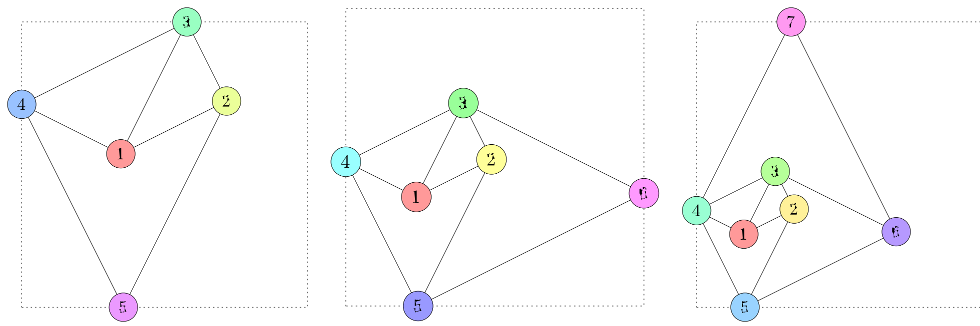

FIGURE 5.11. An expanding kite configuration for 5, 6, and 7 nightriders. These configurations have denominators 346, 2030, and 11626.

6. THE ROOK AND ITS SQUIRE

6.1. The rook.

Rooks illustrate our approach nicely because they are well understood. The well-known elementary formula is

$$u_{\mathbb{R}}(q;n) = q!\binom{n}{q}^2 = \frac{1}{q!}(n)_q^2, \tag{6.1}$$

where $(n)_q$ denotes the falling factorial. Thus, $o_{\mathbb{R}}(q;n) = q!u_{\mathbb{R}}(q;n)$ is a quasipolynomial of period 1 (that is, a polynomial) and degree $2q$, in accordance with our general theory. We want to study its coefficients.

The coefficient of n^{2q-i} is

$$q!\gamma_i = \sum_{k=0}^{i} s(q,q-k)s(q,q-(i-k)), \tag{6.2}$$

where $s(q,j)$ denotes the Stirling number of the first kind, defined as 0 if $j < 0$ or $j > q$. For instance,

$$q!\gamma_0 = 1, \qquad q!\gamma_1 = -(q)_2,$$
$$q!\gamma_2 = (q)_2\frac{3q^2-5q+1}{6}, \qquad q!\gamma_3 = -(q)_3\frac{q(q-1)^2}{6}.$$

These formulas, derived from Equation (6.2), agree with the general partial queens formulas in Theorem III.3.1. (Recall that the rook is the partial queen $\mathbb{Q}^{20}$.)

The sign of each term in the summations in (6.2) is $(-1)^i$, so that is the sign of γ_i for $0 \le i \le 2q-2$. For $i > 2q-2$, $\gamma_i = 0$ because $s(q,0) = 0$. The rook is one of few pieces for which we know the sign of every term in $o_{\mathbb{P}}(q;n)$.

Proposition 6.1. *The coefficient $q!\gamma_i$ is a polynomial in q of degree $2i$. It has a factor $(q)_{\lceil i/2\rceil+1}$. The coefficient of q^{2i} is*

$$\frac{1}{(2i)!}\sum_{k=0}^{i}\binom{2i}{2k}\sum_{r=0}^{k}(-1)^r\binom{2k}{k+r}S(k+r,r)\sum_{s=0}^{i-k}(-1)^s\binom{2(i-k)}{i-k+s}S(i-k+s,s), \tag{6.3}$$

whose sign is $(-1)^i$.

Proof. Schlömilch's formula [4, p. 216]

$$s(q,q-k) = \sum_{r=0}^{k} (-1)^r \binom{q-1+r}{k+r} \binom{q+k}{k-r} S(k+r,r)$$

(which involves the Stirling numbers $S(n,k)$ of the second kind) tells us that $s(q,q-k)$ is a polynomial in q of degree $2k$ with leading term

$$\sum_{r=0}^{k} (-1)^r \frac{q^{k+r}}{(k+r)!} \frac{q^{k-r}}{(k-r)!} S(k+r,r) = \frac{q^{2k}}{(2k)!} \sum_{r=0}^{k} (-1)^r \binom{2k}{k+r} S(k+r,r).$$

This term, as the leading term, must have the same sign as $s(q,q-k)$ for large q. So the leading coefficient of $q!\gamma_i$ is as in Equation (6.3) and the sign of this coefficient is $(-1)^i$.

It is easy to infer from (6.2) that the polynomial equals 0 if $i > 2q-2$, i.e., $q \leq \lceil i/2 \rceil$; therefore $q(q-1)\cdots(q-\lceil i/2 \rceil)$ is a factor. □

The number of combinatorial types of nonattacking configuration of q (unlabelled) rooks is $q!$. To prove it we may substitute $n = -1$ into Equation (6.1) (by Theorem I.5.3) or apply Theorem I.5.8, which says that every piece with two basic moves has $q!$ combinatorial configuration types.

6.2. **The semirook.**

The semirook has only one of the rook's moves and is consequently the least interesting of all riders. We mention it because it is a second example with period 1; also because it has no diagonal move and, as such, exemplifies Corollary III.3.2, that partial queens with at most one diagonal move have a coefficient γ_6 that is independent of n. Counting formulas for $q \leq 4$ are given in Proposition II.6.1 (where one should take $(c,d) = (1,0)$ and, for $q = 2,3$, explicitly in Tables III.4.1 and III.4.2. The leading coefficients of those polynomials are in Theorem III.3.1 and Tables III.3.1 and III.3.2. The number of combinatorial types with q of any one-move rider equals 1 (Theorem I.5.8).

7. The Bishop and Its Scion

Here we treat the bishop and its scion the semibishop.

7.1. **The bishop.**

The bishop's basic move set is $\mathbf{M}_{\mathbb{B}} = \{(1,1),(1,-1)\}$. The quasipolynomial formulas for up to 6 bishops, published by Kotěšovec in early editions of [11]—most of which were found by him—are:

$$u_{\mathbb{B}}(1;n) = n^2.$$

$$u_{\mathbb{B}}(2;n) = \frac{n^4}{2} - \frac{2n^3}{3} + \frac{n^2}{2} - \frac{n}{3}.$$

$$u_{\mathbb{B}}(3;n) = \left\{\frac{n^6}{6} - \frac{2n^5}{3} + \frac{5n^4}{4} - \frac{5n^3}{3} + \frac{4n^2}{3} - \frac{2n}{3} + \frac{1}{8}\right\} - (-1)^n\frac{1}{8}.$$

$$u_{\mathbb{B}}(4;n) = \left\{\frac{n^8}{24} - \frac{n^7}{3} + \frac{11n^6}{9} - \frac{29n^5}{10} + \frac{355n^4}{72} - \frac{35n^3}{6} + \frac{337n^2}{72} - \frac{73n}{30} + \frac{1}{2}\right\} - (-1)^n\left\{\frac{n^2}{8} - \frac{n}{2} + \frac{1}{2}\right\}.$$

(7.1)

$$u_{\mathbb{B}}(5;n) = \left\{\frac{n^{10}}{120} - \frac{n^9}{9} + \frac{49n^8}{72} - \frac{118n^7}{45} + \frac{523n^6}{72} - \frac{2731n^5}{180} + \frac{3413n^4}{144} - \frac{4853n^3}{180} + \frac{2599n^2}{120} - \frac{1321n}{120} + \frac{9}{4}\right\} - (-1)^n\left\{\frac{n^4}{16} - \frac{7n^3}{12} + \frac{17n^2}{8} - \frac{85n}{24} + \frac{9}{4}\right\}.$$

$$u_{\mathbb{B}}(6;n) = \left\{\frac{n^{12}}{720} - \frac{n^{11}}{36} + \frac{37n^{10}}{144} - \frac{4813n^9}{3240} + \frac{8819n^8}{1440} - \frac{72991n^7}{3780} + \frac{2873n^6}{60} - \frac{100459n^5}{1080} + \frac{199519n^4}{1440} - \frac{498557n^3}{3240} + \frac{14579n^2}{120} - \frac{7517n}{126} + \frac{765}{64}\right\} - (-1)^n\left\{\frac{n^6}{48} - \frac{n^5}{3} + \frac{221n^4}{96} - \frac{211n^3}{24} + \frac{467n^2}{24} - \frac{47n}{2} + \frac{765}{64}\right\}.$$

The formula for $q = 2$ is due to Dudeney [5, Problem 318—Lion-Hunting—solution] and those for $q = 3, 4$ to Fabel [6, pp. 58–62] (Kotěšovec [11, p. 234] supplies these attributions). The formulas for $q = 2, 3$ are special cases of our Theorems III.4.1and III.4.2, thereby supporting the correctness of those theorems. Kotěšovec found the formulas for $q = 5, 6$ by calculating the values $u_{\mathbb{B}}(q;n)$ for many values of n, looking for an empirical recurrence relation, deducing a generating function, and from that getting the quasipolynomial. (Reference [12] has details of his method of calculation applied to queens.) His approach, while excellent for finding formulas, does not prove their validity because it does not bound the period—though intuitively period 2 is plausible since one could guess that odd and even board sizes would have separate polynomials.

In fact, 2 is the complete story on the period. In Theorem V.1.1 we provide the missing upper bound of 2 that rigorously establishes period 2 for every $q > 2$ and hence the correctness of Kotěšovec's quasipolynomial formulas.[1] Together with the fact that we know the degree $2q$ and the leading coefficient $1/q!$ of the constituent polynomials, this implies that, if the first $4q$ values of a candidate quasipolynomial are correct, then we have $u_{\mathbb{B}}(q;n)$. Since Kotěšovec did check those values for $q \leq 6$ [12], his formulas are proved correct.

Despite the overall period 2, in Kotěšovec's formulas (7.1) the six leading coefficients do not vary with the parity of n. Kotěšovec conjectured expressions for γ_1, γ_2, and γ_3 in terms

[1]Stanley in [18, Solution to Exercise 4.42] says that both quasipolynomiality and the period follow directly from Arshon's formulas; however, we believe such a derivation would be difficult.

of q alone, which we proved in Theorem III.3.1 (and see Tables III.3.1 and III.3.2) since the bishop is the partial queen $\mathbb{Q}^{02}$. That theorem also gives the periods of γ_4, γ_5, and γ_6.

Corollary 7.1 (of Theorem III.3.1)**.** *In $u_{\mathbb{B}}(q;n)$ the coefficients γ_i for $i \leq 5$ are constant as functions of n, while γ_6 has period 2 (if $q \geq 4$).*

The number of combinatorial types of nonattacking configuration of q unlabelled bishops is $q!$, by Theorem I.5.8. This is therefore the value of $u_{\mathbb{B}}(q;-1)$, which we know even though we do not know the general formula for $u_{\mathbb{B}}(q;n)$.

A surprising development during our work on this project was Kotěšovec's discovery that, in 1936, Arshon had solved the n-bishops problem, the number of ways to place n nonattacking bishops on an $n \times n$ board [1]. His method was to count independently the number of ways to place i nonattacking bishops on the black squares, $u_{\mathbb{B}}^B(n;i)$ (Arshon's p_i), and on the white squares, $u_{\mathbb{B}}^W(n;i)$ (Arshon's p_i'), of the $n \times n$ chessboard. When n is even,

$$u_{\mathbb{B}}^B(n;i) = u_{\mathbb{B}}^W(n;i) = \sum_{j=0}^{n-1} (-1)^j \binom{n-1-i}{j} \frac{(n+1-i-j)^{n/2}(n-i-j)^{n/2-1}}{(n-1-i)!}.$$

When n is odd,

$$u_{\mathbb{B}}^B(n;i) = \sum_{j=0}^{n} (-1)^j \binom{n-1-i}{j} \frac{(n+1-i-j)^{(n-1)/2}(n-i-j)^{(n-1)/2}}{(n-1-i)!}.$$

and

$$u_{\mathbb{B}}^W(n;i) = \sum_{j=0}^{n} (-1)^j \binom{n-1-i}{j} \frac{(n+1-i-j)^{(n+1)/2}(n-i-j)^{(n-1)/2}}{(n-1-i)!}$$

With these formulas Arshon solved the n-bishops problem.

This work was forgotten until Kotěšovec rediscovered it. It was an easy step for him to write down an explicit formula for the number of placements of q nonattacking bishops [11, fourth ed., p. 140]. Kotěšovec then restated the Arshon equations with no signed terms by using Stirling numbers of the second kind [11, fourth ed., p. 142]. His formula for q bishops is

$$u_{\mathbb{B}}(q;n) = \sum_{i=0}^{q} \sum_{j=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{\lfloor \frac{n+1}{2} \rfloor}{j} S\big(j + \lfloor \tfrac{n}{2} \rfloor, n-i\big) \cdot \sum_{h=0}^{\lfloor \frac{n}{2} \rfloor} \binom{\lfloor \frac{n}{2} \rfloor}{h} S\big(h + \lfloor \tfrac{n+1}{2} \rfloor, n-(q-i)\big).$$

Since the number of terms depends on n, these formulas do not tell us the quasipolynomial form of $u_{\mathbb{B}}(q;n)$. Neither do they allow us to substitute $n = -1$ to obtain the number of combinatorial types of nonattacking configuration (though for the bishop this number is known, obtained from Theorem I.5.8). We consequently take the point of view that bishops formulas, like those for other pieces, call for a quasipolynomial analysis via Ehrhart theory.

7.2. **The semibishop.**

The *semibishop* $\mathbb{Q}^{01}$ has just one of the bishop's moves, say $(c,d) = (1,1)$. Thus, it is an example of a one-move rider (Section II.6). As such it has counting functions

$$u_{\mathbb{Q}^{01}}(1;n) = n^2,$$
$$u_{\mathbb{Q}^{01}}(2;n) = \frac{1}{2}n^4 - \frac{1}{3}n^3 - \frac{1}{6}n,$$

$$u_{\mathbb{Q}^{01}}(3;n) = \frac{1}{6}n^6 - \frac{1}{3}n^5 + \frac{1}{6}n^4 - \frac{1}{6}n^3 + \frac{1}{6}n^2,$$
$$u_{\mathbb{Q}^{01}}(4;n) = \frac{1}{24}n^8 - \frac{1}{6}n^7 + \frac{2}{9}n^6 - \frac{11}{60}n^5 + \frac{2}{9}n^4 - \frac{1}{6}n^3 + \frac{1}{72}n^2 + \frac{1}{60}n,$$

from Proposition II.6.1 (in which we have $\bar{n} = 0$). All of these are polynomials in n. (Letting q vary gives a power series in two variables that is truncated, for each q, above the n^{2q} term. We have not investigated that power series.)

Theorem 7.2. *The counting function for nonattacking unlabelled semibishops on the square board is*

$$u_{\mathbb{Q}^{01}}(q;n) = (-1)^q \sum_{k=0}^{q} s(n+1, n+1-k)s(n, n-(q-k)),$$

which is a polynomial function of n of degree $2q$.

Proof. This is an immediate consequence of Proposition 7.4 below. Alternatively, it can be proved similarly to that proposition. □

Explicit formulas for the coefficients γ_i for $i \leq 3$ are in Theorem III.3.1 and Tables III.3.1 and III.3.2.

We prepare for the proof of Theorem 7.2 by changing the board. The *right triangle board* (*triangular board* for short) has legs parallel to the axes and hypotenuse in the direction of the semibishop's move; thus, it is the set $\mathcal{T} := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y \leq 1\}$. The $n \times n$ *triangular board* is the set of integral points in the interior of the dilation by $n+2$, i.e.,

$$(n+2)\mathcal{T}^\circ \cap \mathbb{Z}^2 = \{(x, y) \in \mathbb{Z}^2 : 1 \leq x \leq y - 1 \leq n\}.$$

Write $u^{\mathcal{T}}_{\mathbb{Q}^{01}}(q;n)$ for the counting function of nonattacking placements of q unlabelled semibishops on an $n \times n$ triangular board. Most of our theory for the square board applies equally well to the triangular board; we omit details.

Proposition 7.3. *The Stirling number of the first kind, $s(n+1, n+1-q)$, is a polynomial function of n of degree q. The coefficient of n^{2q-i} in $q!(-1)^q s(n+1, n+1-q)$ is a polynomial function of q of degree $2i$.*

The fact that $s(n+1, n+1-q)$ is a polynomial in n of degree $2q$ is well known; see e.g. [7]. We give a proof here which we believe to be new, using Ehrhart theory in the spirit of our chess series. We do not know a prior reference for the fact that the coefficients are polynomials.

Proposition 7.4. *The counting function for nonattacking unlabelled semibishops on the triangular board is $u^{\mathcal{T}}_{\mathbb{Q}^{01}}(q;n) = (-1)^q s(n+1, n+1-q)$.*

We prove both propositions together.

Proof. The $n \times n$ integral right triangle board has n diagonals (parallel to the hypotenuse) of lengths $1, 2, \ldots, n$, each of which can have at most one semibishop. The number of ways to place q labelled semibishops is the sum of all products of q of these n values, i.e., the elementary symmetric function $e_q(1, 2, \ldots, n)$, which equals $|s(n+1, n+1-q)|$. That proves the first part of the proposition.

Proposition 4.1 applies because the semibishop is a one-move rider. The corners of the triangular board $\mathcal{T}$ are $(0, 0)$, $(1, 1)$, and $(0, 1)$. For the move $(1, 1)$, the corner $(0, 1)$ has no antipode while the corners $(0,0)$ and $(1,1)$ serve as each other's antipodes. Thus, every vertex is integral and the denominator $D(\mathfrak{T}^q, \mathscr{A}_{\mathbb{Q}^{01}}) = 1$, so $q! u^{\mathfrak{T}}_{\mathbb{Q}^{01}}(q;n)$ is a polynomial in n.

Theorem I.4.2 says that the coefficients are polynomials in q with the stated degrees. ☐

8. The Queen and Its Less Hardy Sisters

There are four pieces in this section: the queen, and three others that are like defective queens without being similar to bishops or rooks, which we call the semiqueen, the frontal queen, and the subqueen. As they are all partial queens, counting formulas for $q = 2, 3$ are special cases of our Theorems III.4.1 and III.4.2 and are presented in Tables III.4.1 and III.4.2. The four leading coefficients of the general counting polynomial are implied by Theorems III.3.1, III.4.2, and III.4.2; see Tables III.3.1 and III.3.2. All our formulas that were also calculated by Kotěšovec in [11] agree with his.

Let $\zeta_r := e^{2\pi i/r}$ be a primitive r-th root of unity.

8.1. **The queen.**

The basic move set for the queen $\mathbb{Q}$ is $\mathbf{M} = \{(1,0),(0,1),(1,1),(1,-1)\}$. The quasipolynomial formulas for up to four queens are:

(8.1)

$$u_{\mathbb{Q}}(1;n) = n^2.$$

$$u_{\mathbb{Q}}(2;n) = \frac{n^4}{2} - \frac{5n^3}{3} + \frac{3n^2}{2} - \frac{n}{3} = \frac{n(n-1)(3n^2-7n+2)}{6}$$

$$u_{\mathbb{Q}}(3;n) = \left\{\frac{n^6}{6} - \frac{5n^5}{3} + \frac{79n^4}{12} - \frac{25n^3}{2} + 11n^2 - \frac{43n}{12} + \frac{1}{8}\right\} + (-1)^n \left\{\frac{n}{4} - \frac{1}{8}\right\}.$$

$$u_{\mathbb{Q}}(4;n) = \left\{\frac{n^8}{24} - \frac{5n^7}{6} + \frac{65n^6}{9} - \frac{1051n^5}{30} + \frac{817n^4}{8} - \frac{19103n^3}{108} + \frac{3989n^2}{24} - \frac{18131n}{270} + \frac{253}{54}\right\}$$
$$+ (-1)^n \left\{\frac{n^3}{4} - \frac{21n^2}{8} + 7n - \frac{7}{2}\right\} + \mathrm{Re}(\zeta_3^n)\frac{32(n-1)}{27} + \mathrm{Im}(\zeta_3^n)\frac{40\sqrt{3}}{81}.$$

The formula for two queens is given by Theorem III.4.1 and is originally due to Lucas [14, page 98, Exemple II]. The formula for three queens is implied by Theorem III.4.2 and is originally due to Landau [13]. Kotěšovec gives formulas for up to six queens, calculated by him for $q = 4, 5$ and calculated for six queens by Karavaev [9] and [16, Sequence A176186].

The number of combinatorial types of nonattacking configuration for q pieces is $u_{\mathbb{Q}}(q;-1)$. The numbers for $q \leq 6$ are in Table 8.1. For two queens the number is what one expects from two pieces with four basic moves (Proposition I.5.6).

Kotěšovec conjectured formulas for γ_1 and γ_2 based on the known and heuristically derived formulas (mostly by him) for $u_{\mathbb{Q}}(q;n)$ for small q. Theorem III.3.1 proves his conjectures along with a formula for γ_3 and further proves that γ_4 is constant as a function of n, but that the next two coefficients are not.

Corollary 8.1 (of Theorem III.3.1)**.** *In $u_{\mathbb{Q}}(q;n)$ the coefficients γ_i for $i \leq 4$ are constant as functions of n; but γ_5 has period 2 if $q \geq 3$ and γ_6 has period 2 if $q \geq 4$. Exact formulas are*

$$\gamma_0 = \frac{1}{q!}, \qquad \gamma_1 = -\frac{1}{(q-2)!}\left\{\frac{5}{3}\right\},$$

$$\gamma_2 = \frac{1}{2!(q-2)!}\left\{\frac{25}{9}(q-2)_2 + \frac{61}{6}(q-2) + 3\right\},$$

$$\gamma_3 = -\frac{1}{3!(q-2)!}\left\{\frac{125}{27}(q-2)_4 + \frac{305}{6}(q-2)_3 + \frac{681}{5}(q-2)_2 + 73(q-2) + 2\right\}.$$

The periodic parts are $-(-1)^n/4(q-3)!$ *for* γ_5 *and* $-(-1)^n/8(q-3)!$ *for* γ_6.

Unlike the case of bishops and semibishops, the period of $u_{\mathbb{Q}}(q;n)$ is not simple, although Kotěšovec [11, 6th ed., p. 31] makes the following remarkable conjecture.

Conjecture 8.2 (Kotěšovec)**.** The counting quasipolynomial for q queens has period $\operatorname{lcm}[F_q]$, the least common multiple of all positive integers up through the qth Fibonacci number F_q.

The observed periods up to $q = 7$ (see [11, 6th ed., pp. 19, 27–28] for $q = 7$) agree with this proposal, and the theory of Section 5.3 lends credence to its veracity.

Kotěšovec conjectures, yet more strongly, the exact form of the denominator of the generating function $\sum_{n\geq 0} u_{\mathbb{Q}}(q;n)x^n$: it is a product of specific cyclotomic polynomials raised to specific powers; see [11, 6th ed., p. 22]. The conjecture implies that, when written in standard Ehrhart form with denominator $(1-x^p)^{2q+1}$, the generating function has many cancellable factors. This, too, is not predicted by Ehrhart theory; but as it is too systematic and elegant to be accidental, it presents another tantalizing question. Kotěšovec's evidence, indeed, suggests that $u_{\mathbb{Q}}(q;n)$ has a recurrence relation of length far less than its period. A proof of these conjectures seems to require a new theoretical leap forward.

We summarize the known numerical results for queens in Table 8.1. Unlike in the case of bishops and semibishops, the period of $u_{\mathbb{Q}}(q;n)$ is not simple and we have no general formula in terms of q.

	Types	Period	Denom
$q = 1$	1	1	1
2	4	1	1
3	36	2	2
4	574*	6*	6
5	14206*	60*	—
6	501552†	840†	—
7	—	360360*	—

TABLE 8.1. The number of combinatorial configuration types of q nonattacking (unlabelled) queens in an $n \times n$ square board, with the period, and the denominator. Periods without denominators are unproved.
* is a number deduced from a formula in [11].
† is deduced from the formula of Karavaev; see [9, 11].

8.2. **The semiqueen.**

The *semiqueen* $\mathbb{Q}^{21}$ is the queen without one of its diagonal moves (think of it as having lost the left (or right) arm in battle).

Conjecture 5.7 gives a conjectural upper bound for the quasipolynomial period of $\operatorname{lcm}[F_{q/2}]$ when q is even and $\operatorname{lcm}[F_{(q+1)/2}-1]$ when q is odd. Since we do not expect all denominators in $[F_{q/2}]$ or $[F_{(q+1)/2}-1]$ to appear, we do not expect this bound to be tight for large q, although it agrees with Kotěšovec's formulas for $q \leq 6$ in [11, 6th ed., pp. 732–733].

8.3. **The frontal queen.**

The *frontal queen* is the partial queen $\mathbb{Q}^{12}$ that can advance and retreat but cannot move sideways. Our counting formulas for $q \leq 3$ are the same as Kotěšovec's heuristic ones [11, pp. 730–731].

Conjecture 5.9 implies a conjectural upper bound for the period p_q of the counting quasipolynomial for q frontal queens of $\operatorname{lcm}[2F_{q/2}-1]$ when q is even and $\operatorname{lcm}[F_{(q+3)/2}-1]$ when q is odd. Again, this should not be considered tight, because we do not expect that all denominators from 1 to the maximum will appear. However, Kotěšovec's heuristic approach did find that the period when $q=4$ is 6, which is our bound.

8.4. **The subqueen.**

The *subqueen* (Kotěšovec's "semi-rook + semi-bishop") is the partial queen $\mathbb{Q}^{11}$, with one horizontal or vertical and one diagonal move. It is the fourth and final piece whose counting function $u_{\mathbb{Q}^{11}}(q;n)$ is a polynomial for all q, if our Conjecture 4.3 is correct.

(There are two subqueens that differ in chirality: the right-handed subqueen has, say, a vertical move $(0,1)$ and a diagonal move $(1,1)$ to the right; the left-handed subqueen has the vertical move and a diagonal move $(-1,1)$ to the left. Mixing the two types is outside our competence since they will behave like two different pieces. In our work all subqueens are right-handed.)

Kotěšovec noticed that $(n)_q$ is a factor of $u_{\mathbb{Q}^{11}}(q;n)$ for q up to 8. For instance,

$$u_{\mathbb{Q}^{11}}(6;n) = \frac{(n)_6}{6!}\left(n^6 - 10n^5 + 45n^4 - \frac{1093}{9}n^3 + \frac{634}{3}n^2 - \frac{14033}{63}n + \frac{2278}{21}\right).$$

We would like to have an explanation for this.

Kotěšovec also presents a formula for the number of ways to place n nonattacking subqueens on an $n \times n$ board:

$$u_{\mathbb{Q}^{11}}(n;n) = \sum_{k=1}^{n} \binom{n+1}{k} \frac{k!}{2^k} S(n,k).$$

We hope that our method will be able to prove this.

9. The Nightrider

The basic move set for a nightrider $\mathbb{N}$ is $\{(1,2),(2,1),(1,-2),(2,-1)\}$. As always, $u_{\mathbb{N}}(1;n) = n^2$. It is easy to see that $u_{\mathbb{N}}(2;1)=0$, $u_{\mathbb{N}}(2;2)=6$, $u_{\mathbb{N}}(2;3)=28$, and not quite so easy to find $u_{\mathbb{N}}(2;4)=96$ by hand. Many more values of $u_{\mathbb{N}}(2;n)$ are in the OEIS (see Table 1.1). In Theorem II.3.1, all $(\hat{c},\hat{d}) = (1,2)$ and the period is 2, so $\bar{n} := (n \bmod 2) \in \{0,1\}$. Therefore,

$$u_{\mathbb{N}}(2;n) = \left\{\frac{n^4}{2} - \frac{5n^3}{6} + \frac{3n^2}{2} - \frac{11n}{12}\right\} + (-1)^n \frac{n}{4}.$$

This formula was found independently by Kotěšovec. His [11, 6th ed., p. 312] has an enormous formula for three nightriders (undoubtedly correct, though unproved) that is too complicated to reproduce here. A proof may be accessible using our techniques.

We summarize the known numerical results for nightriders in Table 9.1. We calculated the denominator for four nightriders by using Mathematica to find all vertices of the inside-out polytope and then the least common multiple of their denominators.

	Types	Period	Denom
$q = 1$	1	1	1
2	7	2	2
3	36*	60*	60
4	—	—	14559745200

TABLE 9.1. The number of combinatorial types of nonattacking placements of q (unlabelled) nightriders in an $n \times n$ square board; also the period and denominator.

* is a number derived from an unproved formula in [11].

From Theorem II.3.1 we get a generalization. Define a *partial nightrider* $\mathbb{N}^k$ to have any k $(1 \le k \le 4)$ of the nightrider's moves. (There are three different pieces $\mathbb{N}^2$; see Section 10.1. All have the same formula for two pieces.) Letting $\bar{n} := n \bmod 2 \in \{0, 1\}$,

$$\begin{aligned} u_{\mathbb{N}^k}(2; n) &= \frac{1}{2}n^4 - \frac{5k}{24}n^3 + \frac{k-1}{2}n^2 - \frac{k}{6}n - \frac{k\bar{n}}{8}n \\ &= \left\{ \frac{1}{2}n^4 - \frac{5k}{24}n^3 + \frac{k-1}{2}n^2 - \frac{11k}{48}n \right\} + (-1)^n \frac{3k}{48}n. \end{aligned} \tag{9.1}$$

A direct consequence of Theorem II.5.1 is that we know the second coefficient of the counting quasipolynomial of $\mathbb{N}^k$:

$$\gamma_1 = -\frac{5k}{24(q-2)!}. \tag{9.2}$$

This formula for $\mathbb{N}$ was conjectured by Kotěšovec. γ_1 is its own leading coefficient. Theorem II.5.1 gives the leading coefficient of every γ_i for all partial nightriders.

Theorem 9.1. (I) *For a partial nightrider $\mathbb{N}^k$, the coefficient $q!\gamma_i$ of n^{2q-i} in $o_{\mathbb{N}^k}(q; n)$ is a polynomial in q, periodic in n, with leading term*

$$\left(-\frac{5k}{24} \right)^i \frac{q^{2i}}{i!}.$$

Specializing to the complete nightrider $\mathbb{N}$, computer algebra gives us a general formula for the third coefficient, γ_2. It and the periods of γ_3 and γ_4 are new. Both agree with Kotěšovec's data.

Theorem 9.2. *The third coefficient of the nightrider counting quasipolynomial is independent of n; it is*

$$\gamma_2 = \frac{1}{2!(q-2)!} \left\{ \left(\frac{5}{6}\right)^2 (q-2)_2 + \frac{1871}{720}(q-2) + 3 \right\}.$$

The next coefficient, γ_3, is periodic in n with period 2 and periodic part

$$(-1)^n \frac{1}{3!(q-2)!}\left\{\frac{3}{2}\right\}.$$

The coefficient γ_4 has period 2 and periodic part

$$-(-1)^n \frac{1}{4!(q-2)!}\left\{5(q-2)_2 + \frac{21}{2}(q-2)\right\}.$$

Proof. Just as in Theorem III.3.1, we calculate γ_2 by determining the contribution from all subspaces $\mathcal{U}$ defined by two move equations, each of the form $\mathcal{H}_{ij}^{d/c}$ for $d/c \in \{1/2, 2/1, -1/2, -2/1\}$ and $i, j \in [q]$. This is done in Lemma 9.3.

There is no contribution to γ_2 from subspaces of codimension 0 or 1.

The coefficient γ_3 may have contributions from subspaces of codimensions 1 to 3. Since the contribution of a subspace of codimension 3 comes from the leading coefficient, it is independent of n. We did not compute these leading coefficients. A subspace of codimension 2 contributes zero by Theorem II.4.2, or simply by observing the formula in Equation (II.2.5). Because by Equation (2.1) $\alpha(\mathcal{H}^{2/1}; n) = \alpha^{2/1}(n)$ provides a periodic contribution of $(-1)^n \frac{1}{8} n^{2q-3}\binom{q}{2}$ to $q!\gamma_3$, the periodic part of $(q-2)!\gamma_3$ is $(-1)^n \frac{1}{4} n^{2q-3}$, consisting of one contribution from each hyperplane in $\mathscr{A}_{\mathbb{N}}$.

The calculations for hyperplanes and subspaces of Type $\mathcal{U}^2_{3\mathrm{b}}$ imply that γ_4 is periodic with period 2. That is because a periodic contribution can come only from a subspace of codimension 1, 2, or 3. Equation (2.1) shows that hyperplanes make no contribution to γ_4. Subspaces of codimension 2 with periodic part all have period 2 with values that are the periodic coefficients of n^{2q-4} in the formulas for Types $\mathcal{U}^2_{3\mathrm{a}}$, $\mathcal{U}^2_{3\mathrm{b}}$, and $\mathcal{U}^2_{4*}$. Subspaces of codimension 3 contribute zero by Theorem II.4.2. Thus, the periodic part of $q!\gamma_4$ is the sum of $-(-1)^n \frac{1}{4}(q)_3$ from Type $\mathcal{U}^2_{3\mathrm{a}}$, $-(-1)^n \frac{3}{16}(q)_3$ from Type $\mathcal{U}^2_{3\mathrm{b}}$, and $-(-1)^n \frac{5}{24}(q)_4$ from Type $\mathcal{U}^2_{4*}$, giving a total periodic part of γ_4 of

$$-(-1)^n \frac{(q-2)(21+10(q-3))}{48} = -(-1)^n \frac{(q-2)(10q-9)}{48}. \qquad \square$$

Lemma 9.3. *The total contribution to $o_{\mathbb{N}}(q;n)$ of all subspaces with codimension 2 is*

$$\begin{aligned}
&\left\{\left[\frac{3}{2}(q)_2 + \frac{1}{4}(q)_3 + \frac{1511}{1440}(q)_3 + \frac{25}{72}(q)_4\right] n^{2q-2} + \left[\frac{7}{12}(q)_3 + \frac{227}{144}(q)_3 + \frac{55}{72}(q)_4\right] n^{2q-4}\right. \\
&\qquad \left. + \left[\frac{1}{8}(q)_3 + \frac{65}{144}(q)_4\right] n^{2q-6}\right\} \\
&- (-1)^n \left\{\left[\frac{1}{4}(q)_3 + \frac{3}{16}(q)_3 + \frac{5}{24}(q)_4\right] n^{2q-4} + \left[\frac{1}{8}(q)_3 + \frac{11}{48}(q)_4\right] n^{2q-6}\right\} \\
&+ \left[\left(\frac{527}{1728} - \frac{1}{8}\zeta_{12}^{3n} + \frac{2}{27}\zeta_{12}^{4n} - \frac{13}{64}\zeta_{12}^{6n} + \frac{2}{27}\zeta_{12}^{8n} - \frac{1}{8}\zeta_{12}^{9n}\right.\right. \\
&\qquad + \frac{599}{1600} - \frac{4}{25}\zeta_{20}^{4n} + \frac{1}{8}\zeta_{20}^{5n} - \frac{4}{25}\zeta_{20}^{8n} + \frac{1}{64}\zeta_{20}^{10n} - \frac{4}{25}\zeta_{20}^{12n} + \frac{1}{8}\zeta_{20}^{15n} - \frac{4}{25}\zeta_{20}^{16n} \\
&\qquad \left.\left. + \frac{51}{256} - \frac{19}{256}\zeta_4^{n} - \frac{13}{256}\zeta_4^{2n} - \frac{19}{256}\zeta_4^{3n}\right) 2(q)_3\right] n^{2q-6}.
\end{aligned}$$

The striking symmetry in the coefficients of powers of each ζ_r in the last n^{2q-6}-term is due to the coefficients' being real numbers.

Proof. There are four types of subspace, of which only Type $\mathcal{U}^2_{3\mathrm{b}}$ involves calculations that are substantially different from those in Lemma III.3.4.

From Section 2, for a subspace determined by equations involving κ pieces, $\alpha(\mathcal{U};n)$ is a quasipolynomial of degree $2\kappa - \operatorname{codim}\mathcal{U}$.

Type $\mathcal{U}^2_2$: The subspace $\mathcal{U}$ is defined by two move equations involving the same two pieces. The contribution to $o_{\mathbb{N}}(q;n)$ is $n^{2q-4}o_{\mathbb{N}}(2;n) = \frac{3}{2}(q)_2 n^{2q-2}$. Since $\mathcal{U}$ lies in four hyperplanes, the Möbius function is $\mu(\hat{0},\mathcal{U}) = 3$ and the contribution to $q!\gamma_2$ is $\frac{3}{2}(q)_2$.

Type $\mathcal{U}^2_{3\mathrm{a}}$: The subspace $\mathcal{U}$ is defined by two move equations of the same slope involving three pieces. There is one subspace of this type for each of the four slopes. The number of points in each subspace is $\beta^{2/1}(n)$ from Equation (2.1). There are $(q)_3/3!$ ways to choose three nightriders, and the Möbius function is 2. Thus we multiply $\beta^{2/1}(n)$ by $\frac{8}{3!}(q)_3 n^{2q-6}$ to find that the contribution to $o_{\mathbb{N}}(q;n)$ is

$$(q)_3\left\{\frac{1}{4}n^{2q-2} + \frac{7}{12}n^{2q-4} + \frac{1}{8}n^{2q-6} - (-1)^n\left[\frac{1}{4}n^{2q-4} + \frac{1}{8}n^{2q-6}\right]\right\},$$

so that to $q!\gamma_2$ is $\frac{1}{4}(q)_3$ and that to $q!\gamma_4$ is $\left[\frac{7}{12} - (-1)^n\frac{1}{4}\right](q)_3$.

Type $\mathcal{U}^2_{3\mathrm{b}}$: The subspace $\mathcal{U}$ is defined by two move equations of different slopes involving three pieces, say $\mathcal{U} = \mathcal{H}^{d/c}_{12} \cap \mathcal{H}^{d'/c'}_{23}$. It suffices to find the contributions when $d/c = 1/2$ and $d'/c' \in \{2/1, -2/1, -1/2\}$; the other three combinations of slopes are symmetric to these three, generating a multiplicative factor of 2. We write $\mathcal{U} = \mathcal{U}^{d'/c'}$ when we need to mention the slope.

The Möbius function is $\mu(\hat{0},\mathcal{U}) = 1$.

We choose $\mathbb{N}_2$ in q ways, $\mathbb{N}_1$ in $q-1$ ways, and $\mathbb{N}_3$ in $q-2$ ways.

For each value of d'/c' we calculated the denominators of all vertex coordinates of $[0,1]^{2q} \cap \mathcal{U}$ using Mathematica. That gave us the denominator $D([0,1]^{2q} \cap \mathcal{U})$ and hence an upper bound on the period of $\alpha(\mathcal{U};n)$ in each case. Using Mathematica again we found quasipolynomial formulas for the number of placements of the three nightriders, $\alpha(\mathcal{U};n)$. These formulas were calculated by varying the position of $\mathbb{N}_2$ in the $n \times n$ grid as n varied in a residue class modulo $D([0,1]^{2q} \cap \mathcal{U})$. The calculations were carried out for $n = 1, \ldots, 100$, which covers at least five periods in every case. By Theorem 2.1 and the fact that $\alpha(\mathcal{U};n)$ has degree $4 = 2 \cdot 3 - \operatorname{codim}\mathcal{U}$, there are $2p+1$ coefficients to determine in $\alpha(\mathcal{U};n)$; as the periods are bounded by 20 in every case, that is enough data to infer them all with redundancy. We found that the period always equals the denominator.

Case $d'/c' = 2/1$. The vertex denominators here are $2,3,4$ so $D([0,1]^{2q} \cap \mathcal{U}) = \operatorname{lcm}(2,3,4) = 12$. The number of placements is

$$\alpha(\mathcal{U}^{2/1};n) = \begin{cases} \frac{53}{288}n^4 + \frac{7}{36}n^2 & \text{for } n \equiv 0 \bmod 12, \\ \frac{53}{288}n^4 + \frac{7}{36}n^2 - \frac{2}{9} & \text{for } n \equiv \pm 4 \bmod 12, \\ \frac{53}{288}n^4 + \frac{7}{36}n^2 + \frac{1}{2} & \text{for } n \equiv 6 \bmod 12, \\ \frac{53}{288}n^4 + \frac{7}{36}n^2 + \frac{5}{18} & \text{for } n \equiv \pm 2 \bmod 12, \\ \frac{53}{288}n^4 + \frac{55}{144}n^2 + \frac{21}{32} & \text{for } n \equiv 3 \bmod 6, \\ \frac{53}{288}n^4 + \frac{55}{144}n^2 + \frac{125}{288} & \text{for } n \equiv \pm 1 \bmod 6. \end{cases}$$

Note that the coefficient of n^2, which becomes a contribution to γ_4, has period 2.

Case $d'/c' = -2/1$. The vertex coordinate denominators here are 2, 4, and 5 so $D([0,1]^{2q} \cap \mathcal{U}) = 20$. The number of placements is

$$\alpha(\mathcal{U}^{-2/1};n) = \begin{cases} \frac{27}{160}n^4 + \frac{1}{4}n^2 & \text{for } n \equiv 0 \bmod 20, \\ \frac{27}{160}n^4 + \frac{1}{4}n^2 + \frac{4}{5} & \text{for } n \equiv \pm 4, \pm 8 \bmod 20, \\ \frac{27}{160}n^4 + \frac{1}{4}n^2 - \frac{1}{2} & \text{for } n \equiv 10 \bmod 20, \\ \frac{27}{160}n^4 + \frac{1}{4}n^2 + \frac{3}{10} & \text{for } n \equiv \pm 2, \pm 6 \bmod 20, \\ \frac{27}{160}n^4 + \frac{1}{4}n^2 - \frac{9}{32} & \text{for } n \equiv \pm 5 \bmod 20, \\ \frac{27}{160}n^4 + \frac{1}{4}n^2 + \frac{83}{160} & \text{for odd } n \not\equiv \pm 5 \bmod 20. \end{cases}$$

Case $d'/c' = -1/2$. The vertex denominators here are 2 and 4 so $D([0,1]^{2q} \cap \mathcal{U}) = 4$. The number of placements is

$$\alpha(\mathcal{U}^{-1/2};n) = \begin{cases} \frac{11}{64}n^4 + \frac{1}{4}n^2 & \text{for } n \equiv 0 \bmod 4, \\ \frac{11}{64}n^4 + \frac{1}{4}n^2 + \frac{1}{4} & \text{for } n \equiv 2 \bmod 4, \\ \frac{11}{64}n^4 + \frac{1}{4}n^2 + \frac{19}{64} & \text{for } n \text{ odd}. \end{cases}$$

The contribution to γ_2 is therefore $2(q)_3\big(\frac{53}{288} + \frac{27}{160} + \frac{11}{64}\big) = \frac{1511}{1440}(q)_3$. That to γ_4 is $\big[\frac{227}{144} - (-1)^n\frac{3}{16}\big](q)_3$. Last, the contribution to γ_6 has period $60 = \operatorname{lcm}(12,20,4)$ and is $2(q)_3$ times

$$\begin{aligned} &\big(\tfrac{527}{1728} - \tfrac{1}{8}\zeta_{12}^{3n} + \tfrac{2}{27}\zeta_{12}^{4n} - \tfrac{13}{64}\zeta_{12}^{6n} + \tfrac{2}{27}\zeta_{12}^{8n} - \tfrac{1}{8}\zeta_{12}^{9n}\big) \\ &+ \big(\tfrac{599}{1600} - \tfrac{4}{25}\zeta_{20}^{4n} + \tfrac{1}{8}\zeta_{20}^{5n} - \tfrac{4}{25}\zeta_{20}^{8n} + \tfrac{1}{64}\zeta_{20}^{10n} - \tfrac{4}{25}\zeta_{20}^{12n} + \tfrac{1}{8}\zeta_{20}^{15n} - \tfrac{4}{25}\zeta_{20}^{16n}\big) \\ &+ \big(\tfrac{51}{256} - \tfrac{19}{256}\zeta_4^{n} - \tfrac{13}{256}\zeta_4^{2n} - \tfrac{19}{256}\zeta_4^{3n}\big). \end{aligned}$$

Type $\mathcal{U}^2_{4*}{:}\mathcal{U}^1_2\mathcal{U}^1_2$: The subspace $\mathcal{U}$ is defined by two move equations involving four distinct pieces. For every pair of hyperplanes, the number of attacking configurations is $\big(\alpha^{2/1}(n)\big)^2$, whose value is given in Equation (2.1). We must also multiply by the number of ways in which we can choose this pair of hyperplanes, which is $4\frac{(q)_4}{8} + 6\frac{(q)_4}{4} = 2(q)_4$. The Möbius function is 1. We conclude that the contribution to $o_{\mathbb{N}}(q;n)$ is

$$(q)_4\left\{\frac{25}{72}n^{2q-2} + \frac{55}{72}n^{2q-4} + \frac{65}{144}n^{2q-6} - (-1)^n\left[\frac{5}{24}n^{2q-4} + \frac{11}{48}n^{2q-6}\right]\right\},$$

that to γ_2 is $\frac{25}{72}(q)_4$, and that to γ_4 is $\left(\frac{55}{72} - (-1)^n \frac{5}{24}\right)(q)_4$.

Curiously, not only is the quasipolynomial for every subspace as a whole an even function, so is each constituent; equivalently, opposite constituents $\alpha_i(\mathcal{U}; n)$ and $\alpha_{-i}(\mathcal{U}; n)$ are equal, for every i. We do not know why. □

Type $\mathcal{U}^2_{3b}$ contributes period 60 to γ_6, as one can see from Lemma 9.3. We therefore expect γ_6 to have period that is a multiple of 60; however, we are far from proving this.

This computational method can be applied to larger numbers of any piece, limited only by human effort and computing power. It should be feasible to deduce, at the least, nightrider formulas for γ_3, γ_4, and $u_{\mathbb{N}}(3; n)$.

10. Conclusions, Conjectures, Extensions

Work on nonattacking chess placements raises many questions, some of which have general interest.

10.1. **Simplified riders.**

We cannot reach satisfactorily strong conclusions about the queen and nightrider in part because their periods grow too rapidly as q increases, which we now understand by way of the twisted Fibonacci spirals in Section 5.3. It would be desirable to study simplified analogs, hoping not only for hints to solve those pieces but to find general patterns in the period and coefficients. As having four move directions is complicated, we propose handicapping the pieces by eliminating some of their moves.

As we saw in Part III, partial queens $\mathbb{Q}^{hk}$ are approachable because the queen's moves are individually simple. We suggest further study of the following variants, some of which have been investigated by Kotěšovec.

(a) A generalization of the subqueen is a rider with two moves, $(1, 0)$ and (c, d). The denominator of this piece was determined in Proposition 4.4. This piece, and especially its period, would facilitate analysis of the effect of non-unit slopes.

The nightrider's main complication comes from the non-unit slopes. We propose as worthy subjects the partial nightriders with only two moves:

(b) The *lateral nightrider*, which moves along slope $\pm 1/2$ (or equivalently $\pm 2/1$). We conjecture a period of 4 for $q \geq 3$. We verified this for $q = 3, 4$.

(c) The *inclined nightrider*, which moves along slopes $1/2$ and 2. We propose a period of $3 \cdot 2^{q-1}$ for $q \geq 3$. We only know this for $q = 3$ but we have evidence for $q = 4$ from analyzing trajectories (with help from Arvind Mahankali) and it is certainly correct if Conjecture 4.3 is true.

(d) The *orthonightrider*, whose directions have the orthogonal slopes $1/2$ and -2. We propose the period is $5 \cdot 2^{q-1}$ for $q \geq 3$. This is correct for $q = 3$ and evidence suggests it for higher q.

These should exhibit some of the complexity of the nightrider without being so opaque. It appears that the lateral nightrider should behave more nicely than the other two. For one, from our analysis of subspaces of Type $\mathcal{U}^2_{3b}$ in Theorem 9.2, it is expected to have a smaller period. Furthermore, the denominators generated by configurations similar to those in Figure 4.1 behave much more nicely than the others.

(e) A simple three-move rider would have moves $(1, 0), (0, 1), (c, d)$. This should be investigated.

(f) The partial nightrider $\mathbb{N}^3$. We discussed it briefly in Section 9, finding the counting formula and the period 2 for $q = 2$. The period for three pieces appears to be 60. These periods are the same as for the complete nightrider but we expect $\mathbb{N}^3$ to have a smaller period than $\mathbb{N}$ when $q \geq 4$.

10.2. **Counting nonattacking combinatorial types.**

It would be valuable to produce a conjectural expression for the number of combinatorial types of nonattacking configuration for the queen, a partial queen with three moves, or any other piece with more than two moves (one or two moves being easy; see Proposition I.5.6).

10.3. **A surprise in Ehrhart theory.**

We found that periods equal denominators. That is not a general truth about Ehrhart quasipolynomials. Is there always equality in nonattacking rider problems, and if so, is there an interesting reason?

10.4. **Points and lines.**

In Section 8 we saw that denominators arising from vertex configurations of queens can be determined by understanding configurations of points and lines. Indeed, the Fibonacci numbers seem to arise from optimal configurations of points and lines—in some imprecise notion of optimality. Understanding which denominators appear is a fundamental problem.

10.5. **Connection with billiards.**

The configurations in Section 4.2 may be related to the theory of billiards. If there are only two moves, with a linear transformation one can ensure that the angle of incidence equals the angle of reflection and make it equal to any desired angle less than a right angle. The square board becomes a parallelogram, but a theory of two-move riders on polygonal boards would be able to handle that.

Dictionary of Notation

(c, d) – coordinates of move vector (pp. 6)
$(\hat{c}, \hat{d})$ – $(\min, \max)$ of c, d (p. 8)
d/c – slope of line or move (p. 6)
F_q – Fibonacci numbers (p. 16)
h – # of horizontal, vertical moves of partial queen (p. 9)
k – # of diagonal moves of partial queen (p. 9)
$m = (c, d)$ – basic move (p. 6)
$m^\perp = (d, -c)$ – orthogonal vector to move m (p. 6)
n – size of square board (p. 2)
$\bar{n} := n \bmod \hat{d}$ (p. 8)
$[n]^2$ – square board (p. 5)
$o_{\mathbb{P}}(q; n)$ – # of nonattacking labelled configurations (p. 7)
p – period of quasipolynomial (p. 3)
q – # of pieces on a board (p. 2)
$s(n, k)$ – Stirling number of the first kind (p. 24)
$S(n, k)$ – Stirling number of the second kind (p. 25)
$u_{\mathbb{P}}(q; n)$ – # of nonattacking unlabelled configurations (p. 2)
$u^W_{\mathbb{B}}(n; i), u^B_{\mathbb{B}}(n; i)$ – Arshon's bishops numbers (p. 27)
$z = (x, y)$, $z_i = (x_i, y_i)$ – piece position (p. 6)

$\mathbf{z} = (z_1, \ldots, z_q)$ – configuration (p. 6)

$\alpha^{d/c}(n)$ – # of 2-piece attacks on slope d/c (p. 8)
$\alpha(\mathcal{U}; n)$ – # of attacking configurations in essential part of subspace $\mathcal{U}$ (p. 7)
$\beta^{d/c}(n)$ – # of 3-piece attacks on slope d/c (p. 8)
γ_i – coefficient of n^{2q-i} in $u_{\mathbb{P}}$ (p. 7)
$\zeta_r = e^{2\pi i/r}$ – primitive r-th root of unity (p. 29)
κ – # of of pieces in equations of $\mathcal{U}$
μ – Möbius function of intersection lattice (p. 6)
φ – golden ratio $(1+\sqrt{5})/2$ (p. 16)

D – denominator of inside-out polytope (p. 4)
$\Delta(\mathbf{z})$ – denominator of vertex $\mathbf{z}$ (p. 9)

$\mathbf{M}$ – set of basic moves (p. 6)

$\mathscr{A}_{\mathbb{P}}$ – move arrangement of piece $\mathbb{P}$ (p. 6)
$\mathcal{B}$ – closed board: usually the square $[0,1]^2$ (p. 6)
$\mathcal{E}$ – edge line of the board (p. 8)
$\mathcal{H}_{ij}^{d/c}$ – hyperplane for move (c, d) (p. 6)
$\mathscr{L}(\mathscr{A}_{\mathbb{P}})$ – intersection lattice (p. 6)
$\mathcal{P}, \mathcal{P}^\circ$ – closed, open polytope (p. 6)
$[0,1]^{2q}, (0,1)^{2q}$ – closed, open hypercube (p. 6)
$(\mathcal{P}, \mathscr{A}_{\mathbb{P}}), ([0,1]^{2q}, \mathscr{A}_{\mathbb{P}})$ – inside-out polytope (p. 6)
$(\mathcal{P}^\circ, \mathscr{A}_{\mathbb{P}}), ((0,1)^{2q}, \mathscr{A}_{\mathbb{P}})$ – open inside-out polytope (p. 6)
$\mathcal{T}$ – triangular board $0 \leq x \leq y \leq 1$ (p. 28
$\mathcal{U}$ – subspace in intersection lattice (p. 6)
$\widetilde{\mathcal{U}}$ – essential part of subspace $\mathcal{U}$ (p. 6)
$\mathcal{X}_{ij}$ – hyperplane of equal x coordinates (p. 15)
$\mathcal{Y}_{ij}$ – hyperplane of equal y coordinates (p. 15)

$\mathbb{R}$ – real numbers
$\mathbb{R}^{2q}$ – configuration space (p. 4)
$\mathbb{Z}$ – integers

$\mathbb{B}$ – bishop (p. 25)
$\mathbb{N}$ – nightrider (p. 31)
$\mathbb{P}$ – piece (p. 6)
$\mathbb{Q}$ – queen (p. 29)
$\mathbb{Q}^{hk}$ – partial queen (p. 9)
$\mathbb{R}$ – rook (p. 24)

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