Stability of iterated polynomials and linear transformations preserving the strong q-log-convexity *

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Abstract

In this paper, we mainly study the stability of iterated polynomials and linear transformations preserving the strong q-log-convexity of polynomials

Let $[T_{n,k}]_{n,k\geq 0}$ be a triangular array of nonnegative numbers. We give two criterions for the linear transformation

$$y_n(q) = \sum_{k=0}^{n} T_{n,k} x_k(q)$$

preserving the strong q-log-convexity (resp. log-convexity). As applications, we derive that some linear transformations (for instance, the Stirling transformations of two kinds, the Jacobi-Stirling transformation of the second kind, the Legendre-Stirling transformation of the second kind, the central factorial transformations, the Catalan transformations of Aigner and Shaprio, the Motzkin transformation, the Bell transformation, and so on) preserve the strong q-log-convexity (resp. log-convexity) in a unified manner. In particular, we confirm a conjecture of Lin and Zeng and extend some results of Chen *et al.* and Zhu for strong q-log-convexity of polynomials.

The stability property of iterated polynomials implies the q-log-convexity. By applying the method of interlacing of zeros, we also present two criterions for the stability of the iterated Sturm sequences and q-log-convexity of polynomials. As consequences, we get the stabilities of iterated Eulerian polynomials of Types A and B, and their q-analogs. In addition, we also prove that the generating functions of alternating runs, the longest alternating subsequence and up-down runs of permutations form a q-log-convex sequence, respectively.

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1 Introduction

The main objective of this paper is twofold: one is to study the linear transformations preserving the strong q-log-convexity of the sequences of polynomials and another is to research one strong property of q-log-convexity called the stability of polynomials.

The Jacobi-Stirling numbers $\mathrm{JS}_n^k(z)$ of the second kind, which were introduced in [23], are the coefficients of the integral composite powers of the Jacobi differential operator

$$\ell_{\alpha,\beta}[y](t) = \frac{1}{(1-t)^{\alpha}(1+t)^{\beta}}\left(-(1-t)^{\alpha+1}(1+t)^{\beta+1}y'(t)\right)',$$

with fixed real parameters $\alpha, \beta \geq -1$. They also satisfy the following recurrence relation:

$$\begin{cases} \mathrm{JS}_0^0(z) = 1, \qquad \mathrm{JS}_n^k(z) = 0, \quad \text{if } k \notin \{1,\ldots,n\}, \\ \mathrm{JS}_n^k(z) = \mathrm{JS}_{n-1}^{k-1}(z) + k(k+z)\,\mathrm{JS}_{n-1}^k(z), \quad n,k \geq 1, \end{cases}$$

where $z = \alpha+\beta+1$. Actually, these numbers are a generalization of the Legendre-Stirling numbers of the second kind: it suffices to choose $\alpha = \beta = 0$. Recently, the Jacobi-Stirling numbers and Legendre-Stirling numbers have generated a significant amount of interest from some researchers in combinatorics, see Andrews *et al.* [4, 5, 6], Egge [21], Everitt *et al.* [22, 23], Gelineau and Zeng [26], Mongelli [39], Lin and Zeng [31] and Zhu [52] for details. In [31], Lin and Zeng proposed the next conjecture.

Conjecture 1.1. *[31] The Jacobi-Stirling transformation*

$$y_n = \sum_{k=0}^{n} JS_n^k(z)x_k$$

preserves the log-convexity for $z = 0, 1$.

Recall some notation and definitions. Let $\{a_n\}_{n\geq 0}$ be a sequence of nonnegative real numbers. It is called *log-convex* (resp. *log-concave*) if for all $k \geq 1$, $a_{k-1}a_{k+1} \geq a_k^2$ (resp. $a_{k-1}a_{k+1} \leq a_k^2$), which is equivalent to that $a_{n-1}a_{m+1} \geq a_n a_m$ (resp. $a_{n-1}a_{m+1} \leq a_n a_m$) for all $1 \leq n \leq m$. The log-concave sequences arise often in combinatorics, algebra, geometry, analysis, probability and statistics and have been extensively investigated, see survey articles Stanley [46] and Brenti [12] for details. On the other hand, the theory of log-convexity was recently developed by Liu and Wang [33] and Zhu [51].

For two polynomials with real coefficients $f(q)$ and $g(q)$, denote $f(q) \geq_q g(q)$ if the difference $f(q) - g(q)$ has only nonnegative coefficients. For a polynomial sequence $\{f_n(q)\}_{n\geq 0}$, it is called *q-log-concave* first suggested by Stanley if

$$f_n(q)^2 - f_{n+1}(q)f_{n-1}(q) \geq_q 0$$

for $n \geq 1$ and is called *strongly q-log-concave* introduced by Sagan if

$$f_{n+1}(q)f_{m-1}(q) - f_n(q)f_m(q) \geq_q 0$$

for any $m \geq n \geq 1$. Obviously, the strong q-log-concavity of polynomials implies the q-log-concavity. However, the converse does not hold. The q-log-concavity of polynomials have been extensively studied, see Butler [15], Leroux [30], and Sagan [41] for instance.

For the polynomial sequence $\{f_n(q)\}_{n\geq 0}$, it is called *q-log-convex* introduced by Liu and Wang if

$$f_{n+1}(q)f_{n-1}(q) - f_n(q)^2 \geq_q 0$$

for $n \geq 1$ and is called *strongly q-log-convex* defined by Chen *et al.* if

$$f_{n+1}(q)f_{m-1}(q) - f_n(q)f_m(q) \geq_q 0$$

for any $n \geq m \geq 1$. Clearly, strong q-log-convexity of polynomials implies the q-log-convexity. However, the converse does not hold.

The operator theory often is used to study the log-concavity or log-convexity. For example, the log-convexity and log-concavity are preserved under the binomial convolution respectively, see Davenport and Pólya [20] and Wang and Yeh [50]. Brändén [9] studied some linear transformations preserving the Pólya frequency property of sequences. Liu and Wang [33] also studied the linear transformation preserving the log-convexity. However, there are fewer results about the linear transformation preserving the strong q-log-convexity. This is our motivation of this paper.

Given a triangular array $[T_{n,k}]_{n,k\geq 0}$ of nonnegative real numbers and a sequence of polynomials $\{x_n(q)\}_{n\geq 0}$, define the polynomials

$$y_n(q) = \sum_{k=0}^{n} T_{n,k}x_k(q)$$

for $n \geq 0$. If we take $x_k(q) = q^k$, then it was demonstrated that the corresponding sequence $y_n(q)$ has the q-log-convexity or strong q-log-convexity for many famous triangles $[T_{n,k}]_{n,k\geq 0}$, including the Stirling triangle of the second kind, the Jacobi-Stirling triangle of the second kind, the Legendre-Stirling triangle of the second kind, the Eulerian triangles of Types A and B, the Narayana triangles of Types A and B, and so on, see Chen, *et al.*, [16, 17, 18], Liu and Wang [33], Liu and Zhu [34], and Zhu [51, 52] for instance. Thus it is natural to consider the strong q-log-convexity of the linear transformation $y_n(q)$ by that of $x_n(q)$. On the other hand, note that a log-convex sequence is one special case of the strongly q-log-convex sequence since the real number sequence $\{a_n\}_{n\geq 0}$ is log-convex if and only if $a_{n-1}a_{m+1} \geq a_n a_m$ for all $1 \leq n \leq m$. So it is easy to see that the linear transformation preserving the strong q-log-convexity also preserves the log-convexity.

Let $[T_{n,k}]_{n,k\geq 0}$ be an array of nonnegative numbers satisfying the recurrence relation:

$$T_{n,k} = (a_0 n + a_1 k^2 + a_2 k + a_3)T_{n-1,k} + (b_0 n + b_1 k^2 + b_2 k + b_3)T_{n-1,k-1} \tag{1.1}$$

with $T_{n,k} = 0$ unless $0 \leq k \leq n$ and $T_{0,0} = 1$. Note that all a_0, a_1, b_0, b_1 are nonnegative by the necessary condition for the nonnegativity of $[T_{n,k}]_{n,k\geq 0}$. For $a_2, b_2 \geq 0$ and $a_1 = b_1 = 0$, or $a_0 = b_0 = 0$ and $a_3, b_3 \geq 0$, then the criterions for the strong q-log-convexity of the row generating functions were gave by Chen *et al.* [18] and Zhu [52], respectively. An extensive result can be presented as follows.

Theorem 1.2. *Assume that the nonnegative triangle $[T_{n,k}]_{n,k\geq 0}$ satisfies the recurrence (1.1) with $a_1 + a_2 \geq 0$ and $b_1 + b_2 \geq 0$. If $\{x_n(q)\}_{n\geq 0}$ is strongly q-log-convex, then so is $y_n(q) = \sum_{k=0}^{n} T_{n,k}x_k(q)$. In particular, if $\{x_n\}_{n\geq 0}$ is log-convex, then so is $y_n = \sum_{k=0}^{n} T_{n,k}x_k$.*

In the following, we will also present another criterion for the linear transformation preserving the strong q-log-convexity.

Theorem 1.3. *Let $[T_{n,k}]_{n,k\geq 0}$ be an array of nonnegative numbers satisfying the recurrence relation:*

$$T_{n,k} = f_k\, T_{n-1,k-1} + g_k\, T_{n-1,k} + h_k\, T_{n-1,k+1} \tag{1.2}$$

with $T_{n,k} = 0$ unless $0 \leq k \leq n$ and $T_{0,0} = 1$. Assume that nonnegative sequences $\{f_k\}_{k\geq 0}$, $\{g_k\}_{k\geq 0}$ and $\{h_k\}_{k\geq 0}$ are increasing, respectively. If $g_k g_{k+1} \geq h_k f_{k+1}$, then the linear transformation

$$y_n(q) = \sum_{k=0}^{n} T_{n,k} x_k(q)$$

preserves the strong q-log-convexity, therefore preserves the log-convexity.

By Theorem 1.3, the next is immediate.

Proposition 1.4. *Let $[T_{n,k}]_{n,k\geq 0}$ satisfy the recurrence relation (1.2). Assume that $T_n(q) = \sum_{k=0}^{n} T_{n,k} q^k$ is the row generating functions. If nonnegative sequences $\{f_k\}_{k\geq 0}$, $\{g_k\}_{k\geq 0}$ and $\{h_k\}_{k\geq 0}$ are increasing respectively, and $g_k g_{k+1} \geq h_k f_{k+1}$ for $k \geq 0$, then generating functions $T_n(q)$ form a strongly q-log-convex sequence.*

Twenty five years ago Gian-Carlo Rota said "The one contribution of mine that I hope will be remembered has consisted in just pointing out that all sorts of problems of combinatorics can be viewed as problems of the locations of zeros of certain polynomials...", see the end of the introduction of [8]. In fact, polynomials with only real zeros play an important role in attacking log-concavity of sequences. One classical result is that if the polynomial $\sum_{i=0}^{n} a_i x^i$ with nonnegative coefficients has only real zeros, then the sequence $a_0, a_1, \ldots, a_n$ is log-concave. In addition, many log-concave sequences arising in combinatorics have the stronger property, see Liu and Wang [32] and Wang and Yeh [49] for instance. Using the algebraical method, Liu and Wang [33] found that many polynomials with real zeros have q-log-convexity. Thus at the end of their paper, they proposed the problem to research this relation between the q-log-convexity and real zeros. This is our another motivation.

One of the classical problems of the theory of equations is to find relations between the zeroes and coefficients of a polynomial. A real polynomial is *(Hurwitz) stable* if all of its zeros lie in the open left half of the complex plane. A well-known necessary condition for a real polynomial with positive leading coefficient to be stable is that all its coefficients are positive. Polynomial stability problems of various types arise in a number of problems in mathematics and engineering. We refer to [38, Chapter 9] for deep surveys on the stability theory. Clearly, the stability property of iterated polynomials implies the q-log-convexity. Thus it is natural to consider the following stronger problem.

Problem 1.5. *Given a sequence $\{f_n(q)\}_{n\geq 0}$ of polynomials with only real zeros, under which conditions can we obtain that $f_{n+1}(q)f_{n-1}(q) - f_n^2(q)$ is stable for $n \geq 1$?*

We say a polynomial is a *generalized stable* polynomial if all of its zeros excluding 0 lie in the open left half of the complex plane. The following result gives an answer to Problem 1.5.

Theorem 1.6. *Let $\{f_n(q)\}_{n\geq 0}$ be a sequence of polynomials with nonnegative coefficients, where $deg(f_n(q)) = deg(f_{n-1}(q)) + 1$ for $n \geq 1$. Assume that the sequence $\{f_n(q)\}_{n\geq 0}$ satisfies the recurrence relation*

$$f_n(q) = [a_1n + a_2 + (b_1n + b_2)q + (c_1n + c_2)q^2]f_{n-1}(q) + q(a_3 + b_3q + c_3q^2)f'_{n-1}(q),$$

where $a_1, b_1, c_1, a_1 + a_3, b_1 + b_3, c_1 + c_3$ are all nonnegative. If $\{f_n(q)\}_{n\geq 0}$ is a generalized Sturm sequence, then it is q-log-convex. Furthermore, assume that $\{a_1, b_1, c_1\} \neq \{0\}$. If $c_1 + c_3 = 0$, $a_1 + 2a_3 \geq 0$ and $b_1 \geq b_3$, then $f_{n+1}(q)f_{n-1}(q) - f_n^2(q)$ is a generalized stable polynomial for $n \geq 1$.

The generalized Sturm sequences arise often in combinatorics. In addition, the following result given by Liu and Wang [32] provides an approach to the generalized Sturm sequences.

Proposition 1.7. *Let $\{P_n(x)\}$ be a sequence of polynomials with nonnegative coefficients and $\deg P_n = \deg P_{n-1} + 1$. Suppose that*

$$P_n(x) = (a_nx + b_n)P_{n-1}(x) + x(c_nx + d_n)P'_{n-1}(x)$$

where $a_n, b_n \in \mathbb{R}$ and $c_n \leq 0, d_n \geq 0$. Then $\{P_n(x)\}$ forms a generalized Sturm sequence.

It is well-known that many classical combinatorial sequences of polynomials arising in certain triangular arrays, *e.g.*, Pascal triangle, Stirling triangle, Eulerian triangle and so on, satisfy the following recurrence relation (1.3). Let $\{T_{n,k}\}_{0\leq k\leq n}$ be an array of nonnegative numbers satisfying the recurrence relation

$$T_{n,k} = (a_1n + a_2k + a_3)T_{n-1,k} + (b_1n + b_2k + b_3)T_{n-1,k-1} \tag{1.3}$$

with $T_{n,k} = 0$ unless $0 \leq k \leq n$. Let its row generating function $T_n(q) = \sum_{k=0}^{n} T_{n,k}q^k$. By the recurrence relation (1.3), we have

$$T_n(q) = [a_1n + a_3 + (b_1n + b_2 + b_3)q]\, T_{n-1}(q) + (a_2 + b_2q)qT'_{n-1}(q).$$

By Proposition 1.7, we know that generating functions $T_n(q)$ form a generalized Sturm sequence. Thus, the next result follows from Theorem 1.6.

Proposition 1.8. *Let $\{T_{n,k}\}_{n,k\geq 0}$ be the nonnegative array as above (1.3) and the row generating function $T_n(q) = \sum_{k=0}^{n} T_{n,k}q^k$. If $a_2 \geq 0 \geq b_2$, then $\{T_{n+1}(q)T_{n-1}(q) - T_n^2(q)\}_{n\geq 1}$ is a sequence of the generalized stable polynomials.*

The remainder of this paper is structured as follows. In Section 2, we will present the proofs of Theorems 1.2 and 1.3. In Section 3, we give the proof of Theorem 1.6. In Section 4, we apply Theorems 1.2 and 1.3 to some famous triangular arrays in a unified manner, including Stirling triangles of two kinds, the Jacobi-Stirling triangle of the second kind, the Legendre-Stirling triangle of the second kind, the central factorial numbers triangle, the Catalan triangles of Aigner and Shaprio, the Motzkin triangle, the Bell triangle, and so on. In particular, we solve the Conjecture 1.1. Finally, we also apply Proposition 1.8 to Eulerian polynomials of Types A and B, and their q-analogs. Using Theorem 1.6, we also obtain the q-log-convexities of the generating functions of alternating runs, the longest alternating subsequence and up-down runs of permutations, respectively.

2 Proof of Theorems 1.2 and 1.3

Recall that a matrix $M = (m_{ij})_{i,j\geq 0}$ of nonnegative numbers is said to be *r-order totally positive* (TP_r for short) if its all minors of order at most r are nonnegative. It is called *totally positive* (TP for short) if its all minors of order are nonnegative. Total positivity of matrices has been extensively studied and is very useful, see Karlin [27] for more details.

The following result for log-concavity is a classical result, which can be applied to many combinatorial triangular arrays.

Lemma 2.1. *[29] Assume that a nonnegative triangular array $[A(n,k)]_{n,k\geq 0}$ satisfies the recurrence*

$$A(n,k) = f(n,k)A(n-1,k-1) + g(n,k)A(n-1,k-1).$$

If $2f(n,k) \geq f(n,k+1)+f(n,k-1)$ and $2g(n,k) \geq g(n,k+1)+g(n,k-1)$ for $n > k > 1$, then

$$A^2(n,k) \geq A(n,k+1)A(n,k-1)$$

for $n \geq k \geq 0$.

The next result for TP_2 is very important in our proof.

Lemma 2.2. *Let $[T_{n,k}]_{n,k\geq 0}$ be an array of nonnegative numbers satisfying the recurrence*

$$T_{n,k} = [a_0 n + f(k)]T_{n-1,k} + [b_0 n + g(k)]T_{n-1,k-1} \tag{2.1}$$

with $T_{n,k} = 0$ unless $0 \leq k \leq n$. If both $f(k)$ and $g(k)$ are nonnegative and increasing in k, then $[T_{n,k}]_{n,k\geq 0}$ is TP_2.

Proof. For $j \geq i$, by recurrence (2.1), we have

$$\begin{aligned}
& T_{n+1,j}T_{n,i} - T_{n+1,i}T_{n,j} \\
= & \left[(a_0 n + f(j))T_{n,j} + (b_0 n + g(j))T_{n,j-1}\right]T_{n,i} - \left[(a_0 n + f(i))T_{n,i} + (b_0 n + g(i))T_{n,i-1}\right]T_{n,j} \\
\geq & [f(j) - f(i)]T_{n,j}T_{n,i} + [b_0 n + g(i)][T_{n,j-1}T_{n,i} - T_{n,i}T_{n,j}] \\
\geq & 0
\end{aligned}$$

by Lemma 2.1 and the monotonicity of $f(k)$ and $g(k)$. Thus we have

$$T_{n+1,j}T_{n,i} - T_{n+1,i}T_{n,j} \geq 0$$

for $j \geq i$. It follows that

$$\begin{aligned}
T_{n+1,j}T_{n,i} &\geq T_{n+1,i}T_{n,j}, \\
T_{n+2,j}T_{n+1,i} &\geq T_{n+2,i}T_{n+1,j}, \\
T_{n+3,j}T_{n+2,i} &\geq T_{n+3,i}T_{n+2,j}, \\
&\vdots \\
T_{m,j}T_{m-1,i} &\geq T_{m,i}T_{m-1,j},
\end{aligned}$$

which imply that

$$T_{m,j}T_{n,i} \geq T_{m,i}T_{n,j}$$

for any $m \geq n$ and $j \geq i$, that is to say that $[T_{n,k}]_{n,k\geq 0}$ is TP_2. This completes the proof. ∎

The next lemma plays an important role in our proof.

Lemma 2.3. *[48] Given four sequences* $\{a_i\}_{i=0}^n$, $\{b_i\}_{i=0}^n$, $\{c_i\}_{i=0}^n$ *and* $\{d_i\}_{i=0}^n$, *then we have*

$$\sum_{i=0}^n a_ic_i \sum_{i=0}^n b_id_i - \sum_{i=0}^n a_id_i \sum_{i=0}^n b_ic_i = \sum_{0\le i<j\le n} (a_ib_j - a_jb_i)(c_id_j - c_jd_i).$$

Proof of Theorem 1.2: In the following proof, we simply write x_k for $x_k(q)$.

In order to prove the strong q-log-convexity of $\{y_n(q)\}_{n\ge0}$, it suffices to show for $n \ge m$ that

$$y_{n+1}(q)y_{m-1}(q) - y_n(q)y_m(q) \ge_q 0.$$

Assume that $f(k) = a_1k^2 + a_2k$ and $g(k) = b_1k^2 + b_2k$. Since all $a_1, b_1, a_1 + a_2, b_1 + b_2$ are nonnegative, both $f(k)$ and $g(k)$ are nonnegative and increasing in k. So it follows from Lemma 2.2 that the matrix $[T_{n,k}]_{n,k\ge0}$ is TP$_2$. In addition, for $j \ge i$, we also have

$$g(j+1)x_ix_{j+1} - g(i+1)x_{i+1}x_j \ge_q g(i+1)[x_ix_{j+1} - x_{i+1}x_j] \ge_q 0 \tag{2.2}$$

since $\{x_n(q)\}_{n\ge0}$ is strongly q-log-convex. Thus if we view $T_{m-1,k}$, $T_{n,k}$, x_k and $f(k)x_k$ as a_k, b_k, c_k and d_k in Lemma 2.3, respectively, then we obtain that

$$\begin{aligned}
& \sum_{k=0}^n f(k)T_{n,k}x_k \sum_{k=0}^{m-1} T_{m-1,k}x_k - \sum_{k=0}^n T_{n,k}x_k \sum_{k=0}^{m-1} f(k)T_{m-1,k}x_k \\
= & \sum_{k=0}^n f(k)T_{n,k}x_k \sum_{k=0}^{n} T_{m-1,k}x_k - \sum_{k=0}^n T_{n,k}x_k \sum_{k=0}^{n} f(k)T_{m-1,k}x_k \\
= & \sum_{0\le i<j\le n} [f(j) - f(i)](T_{m-1,i}T_{n,j} - T_{m-1,j}T_{n,i})x_ix_j \\
\ge_q & \ 0
\end{aligned} \tag{2.3}$$

since $f(k)$ is increasing and the matrix $[T_{n,k}]_{n,k\ge0}$ is TP$_2$. Similarly by Lemma 2.3, it follows from (2.2), 2-order positivity of $[T_{n,k}]_{n,k\ge0}$ and the strong q-log-convexity of $\{x_n(q)\}_{n\ge0}$ that we have

$$\begin{aligned}
& \sum_{k=0}^n g(k+1)T_{n,k}x_{k+1} \sum_{k=0}^{m-1} T_{m-1,k}x_k - \sum_{k=0}^n T_{n,k}x_k \sum_{k=0}^{m-1} g(k+1)T_{m-1,k}x_{k+1} \\
= & \sum_{k=0}^n g(k+1)T_{n,k}x_{k+1} \sum_{k=0}^{n} T_{m-1,k}x_k - \sum_{k=0}^n T_{n,k}x_k \sum_{k=0}^{n} g(k+1)T_{m-1,k}x_{k+1} \\
= & \sum_{0\le i<j\le n} [T_{m-1,i}T_{n,j} - T_{m-1,j}T_{n,i}]\,[g(j+1)x_ix_{j+1} - g(i+1)x_{i+1}x_j] \\
\ge_q & \ 0
\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
& \sum_{k=0}^n [b_0n + b_3]T_{n,k}x_{k+1} \sum_{k=0}^{m-1} T_{m-1,k}x_k - \sum_{k=0}^n T_{n,k}x_k \sum_{k=0}^{m-1} (b_0m + b_3)T_{m-1,k}x_{k+1} \\
= & \sum_{0\le i<j\le n} (b_0m + b_3)\,[T_{m-1,i}T_{n,j} - T_{m-1,j}T_{n,i}]\,[x_ix_{j+1} - x_{i+1}x_j] \\
\ge_q & \ 0.
\end{aligned} \tag{2.5}$$

Thus, for $n \geq m$, by the recurrence relation (1.1), we have

$$
\begin{aligned}
& y_{n+1}(q)y_{m-1}(q) - y_n(q)y_m(q) \\
= \; & \left[\sum_{k=0}^{n}(a_0n + f(k) + a_3 + a_0)T_{n,k}x_k + \sum_{k=1}^{n+1}(b_0n + g(k) + b_3 + b_0)T_{n,k-1}x_k\right]\sum_{k=0}^{m-1}T_{m-1,k}x_k \\
& -\sum_{k=0}^{n}T_{n,k}x_k\left[\sum_{k=0}^{m-1}(a_0m + f(k) + a_3)T_{m-1,k}x_k + \sum_{k=1}^{m}(b_0m + g(k) + b_3)T_{m-1,k-1}x_k\right] \\
= \; & \left[\sum_{k=0}^{n}[a_0n + f(k) + a_0]T_{n,k}x_k\sum_{k=0}^{m-1}T_{m-1,k}x_k - \sum_{k=0}^{n}T_{n,k}x_k\sum_{k=0}^{m-1}[a_0m + f(k)]T_{m-1,k}x_k\right] + \\
& \left[\sum_{k=1}^{n+1}(b_0n + g(k) + b_3 + b_0)T_{n,k-1}x_k\sum_{k=0}^{m-1}T_{m-1,k}x_k - \sum_{k=0}^{n}T_{n,k}x_k\sum_{k=1}^{m}(b_0m + g(k) + b_3)T_{m-1,k-1}x_k\right] \\
= \; & \left[\sum_{k=0}^{n}f(k)T_{n,k}x_k\sum_{k=0}^{m-1}T_{m-1,k}x_k - \sum_{k=0}^{n}T_{n,k}x_k\sum_{k=0}^{m-1}f(k)T_{m-1,k}x_k\right] \\
& +a_0(n-m+1)\sum_{k=0}^{n}T_{n,k}x_k\sum_{k=0}^{m-1}T_{m-1,k}x_k + b_0(n-m+1)\sum_{k=0}^{n}T_{n,k}x_{k+1}\sum_{k=0}^{m-1}T_{m-1,k}x_k \\
& +\left[\sum_{k=0}^{n}(b_0m + b_3)T_{n,k}x_{k+1}\sum_{k=0}^{m-1}T_{m-1,k}x_k - \sum_{k=0}^{n}T_{n,k}x_k\sum_{k=0}^{m-1}(b_0m + b_3)T_{m-1,k}x_{k+1}\right] \\
& +\left[\sum_{k=0}^{n}g(k+1)T_{n,k}x_{k+1}\sum_{k=0}^{m-1}T_{m-1,k}x_k - \sum_{k=0}^{n}T_{n,k}x_k\sum_{k=0}^{m-1}g(k+1)T_{m-1,k}x_{k+1}\right] \\
= \; & \sum_{0\leq i<j\leq n}[f(j) - f(i)](T_{m-1,i}T_{n,j} - T_{m-1,j}T_{n,i})x_ix_j + a_0(n-m+1)\sum_{k=0}^{n}T_{n,k}x_k\sum_{k=0}^{m-1}T_{m-1,k}x_k \\
& +b_0(n-m+1)\sum_{k=0}^{n}T_{n,k}x_{k+1}\sum_{k=0}^{m-1}T_{m-1,k}x_k \\
& +\sum_{0\leq i<j\leq n}(b_0m + b_3)\left[T_{m-1,i}T_{n,j} - T_{m-1,j}T_{n,i}\right]\left[x_ix_{j+1} - x_{i+1}x_j\right] \\
& +\sum_{0\leq i<j\leq n}\left[T_{m-1,i}T_{n,j} - T_{m-1,j}T_{n,i}\right]\left[g(j+1)x_ix_{j+1} - g(i+1)x_{i+1}x_j\right] \\
\geq_q \; & 0
\end{aligned}
$$

by (2.6), (2.4) and (2.5). This completes the proof. □

In what follows we will prove Theorem 1.3. The following lemma has been proved in [51, Theorem 1].

Lemma 2.4. *Let $[T_{n,k}]_{n,k\geq 0}$ be an array of nonnegative numbers satisfying the recurrence relation:*

$$T_{n,k} = f_k\,T_{n-1,k-1} + g_k\,T_{n-1,k} + h_k\,T_{n-1,k+1}$$

with $T_{n,k} = 0$ unless $0 \leq k \leq n$ and $T_{0,0} = 1$. Assume that nonnegative sequences $\{f_k\}_{k\geq 0}$, $\{g_k\}_{k\geq 0}$ and $\{h_k\}_{k\geq 0}$ satisfy $g_kg_{k+1} \geq h_kf_{k+1}$, then the lower triangle matrix $[T_{n,k}]_{n,k\geq 0}$ is TP_2.

Proof of Theorem 1.3: In the following proof, we simply write x_k for $x_k(q)$.

In order to prove the strong q-log-convexity of $\{y_n(q)\}_{n\geq 0}$, it suffices to show for $n \geq m$ that

$$y_{n+1}(q)y_{m-1}(q) - y_n(q)y_m(q) \geq_q 0.$$

Thus, by the recurrence relation (1.2), we have

$$\begin{aligned}
& y_{n+1}(q)y_{m-1}(q) - y_n(q)y_m(q) \\
= & \left[\sum_{k=0}^{n+1} f_k T_{n,k-1}x_k + \sum_{k=0}^{n+1} g_k T_{n,k}x_k + \sum_{k=0}^{n+1} h_k T_{n,k+1}x_k\right]\sum_{k=0}^{m-1} T_{m-1,k}x_k \\
& -\sum_{k=0}^{n} T_{n,k}x_k \left[\sum_{k=0}^{m} f_k T_{m-1,k-1}x_k + \sum_{k=0}^{m} g_k T_{m-1,k}x_k + \sum_{k=0}^{m} h_k T_{m-1,k+1}x_k\right] \\
= & \left[\sum_{k=1}^{n+1} f_k T_{n,k-1}x_k + \sum_{k=0}^{n} g_k T_{n,k}x_k + \sum_{k=0}^{n-1} h_k T_{n,k+1}x_k\right]\sum_{k=0}^{m-1} T_{m-1,k}x_k \\
& -\sum_{k=0}^{n} T_{n,k}x_k \left[\sum_{k=1}^{m} f_k T_{m-1,k-1}x_k + \sum_{k=0}^{m-1} g_k T_{m-1,k}x_k + \sum_{k=0}^{m-2} h_k T_{m-1,k+1}x_k\right] \\
= & \left[\sum_{k=1}^{n+1} f_k T_{n,k-1}x_k \sum_{k=0}^{m-1} T_{m-1,k}x_k - \sum_{k=0}^{n} T_{n,k}x_k \sum_{k=1}^{m} f_k T_{m-1,k-1}x_k\right] + \\
& \left[\sum_{k=0}^{n} g_k T_{n,k}x_k \sum_{k=0}^{m-1} T_{m-1,k}x_k - \sum_{k=0}^{n} T_{n,k}x_k \sum_{k=0}^{m-1} g_k T_{m-1,k}x_k\right] + \\
& \left[\sum_{k=0}^{n-1} h_k T_{n,k+1}x_k \sum_{k=0}^{m-1} T_{m-1,k}x_k - \sum_{k=0}^{n} T_{n,k}x_k \sum_{k=0}^{m-2} h_k T_{m-1,k+1}x_k\right] \\
= & \left[\sum_{k=0}^{n} f_{k+1} T_{n,k}x_{k+1} \sum_{k=0}^{m-1} T_{m-1,k}x_k - \sum_{k=0}^{n} T_{n,k}x_k \sum_{k=0}^{m-1} f_{k+1} T_{m-1,k}x_{k+1}\right] + \\
& \left[\sum_{k=0}^{n} g_k T_{n,k}x_k \sum_{k=0}^{m-1} T_{m-1,k}x_k - \sum_{k=0}^{n} T_{n,k}x_k \sum_{k=0}^{m-1} g_k T_{m-1,k}x_k\right] + \\
& \left[\sum_{k=1}^{n} h_{k-1} T_{n,k}x_{k-1} \sum_{k=0}^{m-1} T_{m-1,k}x_k - \sum_{k=0}^{n} T_{n,k}x_k \sum_{k=1}^{m-1} h_{k-1} T_{m-1,k}x_{k-1}\right] \\
= & \left[\sum_{k=0}^{n} f_{k+1} T_{n,k}x_{k+1} \sum_{k=0}^{n} T_{m-1,k}x_k - \sum_{k=0}^{n} T_{n,k}x_k \sum_{k=0}^{n} f_{k+1} T_{m-1,k}x_{k+1}\right] + \\
& \left[\sum_{k=0}^{n} g_k T_{n,k}x_k \sum_{k=0}^{n} T_{m-1,k}x_k - \sum_{k=0}^{n} T_{n,k}x_k \sum_{k=0}^{n} g_k T_{m-1,k}x_k\right] + \\
& \left[\sum_{k=1}^{n} h_{k-1} T_{n,k}x_{k-1} \sum_{k=1}^{n} T_{m-1,k}x_k - \sum_{k=1}^{n} T_{n,k}x_k \sum_{k=1}^{n} h_{k-1} T_{m-1,k}x_{k-1}\right] + \\
& \left[\sum_{k=1}^{n} h_{k-1}\left(T_{n,k}T_{m-1,0} - T_{n,0}T_{m-1,k}\right)x_{k-1}x_0\right].
\end{aligned}$$

By Lemma 2.4, it is clear that $\sum_{k=1}^{n} h_{k-1}\left(T_{n,k}T_{m-1,0}-T_{n,0}T_{m-1,k}\right)x_{k-1}x_0 \geq_q 0$.

On the other hand, similar to the proof of Theorem 1.2, by Lemma 2.3, we have the following:

$$\begin{aligned}
& \left[\sum_{k=0}^{n} f_{k+1}T_{n,k}x_{k+1}\sum_{k=0}^{n} T_{m-1,k}x_k-\sum_{k=0}^{n} T_{n,k}x_k\sum_{k=0}^{n} f_{k+1}T_{m-1,k}x_{k+1}\right]\\
= & \sum_{0\leq i<j\leq n}\left[T_{m-1,i}T_{n,j}-T_{m-1,j}T_{n,i}\right]\left[f_{j+1}x_ix_{j+1}-f_{i+1}x_{i+1}x_j\right]\\
\geq_q & \ 0,\\
& \left[\sum_{k=0}^{n} g_kT_{n,k}x_k\sum_{k=0}^{n} T_{m-1,k}x_k-\sum_{k=0}^{n} T_{n,k}x_k\sum_{k=0}^{n} g_kT_{m-1,k}x_k\right]\\
= & \sum_{0\leq i<j\leq n}[g_j-g_i](T_{m-1,i}T_{n,j}-T_{m-1,j}T_{n,i})x_ix_j\\
\geq_q & \ 0
\end{aligned}$$

and

$$\begin{aligned}
& \left[\sum_{k=1}^{n} h_{k-1}T_{n,k}x_{k-1}\sum_{k=1}^{n} T_{m-1,k}x_k-\sum_{k=1}^{n} T_{n,k}x_k\sum_{k=1}^{n} h_{k-1}T_{m-1,k}x_{k-1}\right]\\
= & \sum_{1\leq i<j\leq n}\left[T_{m-1,i}T_{n,j}-T_{m-1,j}T_{n,i}\right]\left[h_{j-1}x_ix_{j+1}-h_{i-1}x_{i+1}x_j\right]\\
\geq_q & \ 0.
\end{aligned}$$

Hence for any $n \geq m$, we have

$$y_{n+1}(q)y_{m-1}(q)-y_n(q)y_m(q) \geq_q 0.$$

This completes the proof. $\square$

3 Proof of Theorem 1.6

Following Wagner [47], a real polynomial is said to be *standard* if either it is identically zero or its leading coefficient is positive. Suppose that $f, g \in \mathrm{RZ}$. Let $\{r_i\}$ and $\{s_j\}$ be all zeros of f and g in nondecreasing order respectively. We say that g *interlaces* f if $\deg f = \deg g + 1 = n$ and

$$r_n \leq s_{n-1} \leq \cdots \leq s_2 \leq r_2 \leq s_1 \leq r_1. \tag{3.1}$$

By $g \preceq f$ we denote "g interlaces f". For notational convenience, let $a \preceq bx + c$ for any real constants a, b, c and $f \preceq 0, 0 \preceq f$ for all real polynomial f with only real zeros.

Let $\{P_n(x)\}_{n\geq 0}$ be a sequence of standard polynomials. Recall that $\{P_n(x)\}$ is a *Sturm sequence* if $\deg P_n = n$ and $P_{n-1}(r)P_{n+1}(r) < 0$ whenever $P_n(r) = 0$ and $n \geq 1$. We say that $\{P_n(x)\}$ is a *generalized Sturm sequence* if $P_n \in \mathrm{RZ}$ and $P_0 \preceq P_1 \preceq \cdots \preceq P_{n-1} \preceq P_n \preceq \cdots$. For example, if P is a standard polynomial with only real zeros and $\deg P = n$, then $P^{(n)}, P^{(n-1)}, \ldots, P', P$ form a generalized Sturm sequence by Rolle's theorem.

In order to simplify our proof, we need the following lemma.

Lemma 3.1. [24, Lemma 1.20] *Let both $f(x)$ and $g(x)$ be standard real polynomials with only real zeros. Assume that $\deg(f(x)) = n$ and all real zeros of $f(x)$ are $s_1, \ldots, s_n$. If $\deg(g) = n-1$ and we write*

$$g(x) = \sum_{i=1}^{n} \frac{c_i f(x)}{x - s_i},$$

then $g(x)$ interlaces $f(x)$ if and only if all c_i are positive.

Proof of Theorem 1.6:
Since

$$f_n(q) = [a_1 n + a_2 + (b_1 n + b_2)q + (c_1 n + c_2)q^2] f_{n-1}(q) + q(a_3 + b_3 q + c_3 q^2) f'_{n-1}(q),$$

it follows that

$$\begin{aligned}
& f_{n+1}(q) f_{n-1}(q) - f_n^2(q) \\
= & \{[a_1 n + a_2 + a_1 + (b_1 n + b_2 + b_1)q + (c_1 n + c_2 + c_1)q^2] f_n(q) + q(a_3 + b_3 q + c_3 q^2) f'_n(q)\} \\
& f_{n-1}(q) - \{[a_1 n + a_2 + (b_1 n + b_2)q + (c_1 n + c_2)q^2] f_{n-1}(q) + q(a_3 + b_3 q + c_3 q^2) f'_{n-1}(q)\} f_n(q) \\
= & (a_1 + b_1 q + c_1 q^2) f_n(q) f_{n-1}(q) + q(a_3 + b_3 q + c_3 q^2) \left[f'_n(q) f_{n-1}(q) - f_n(q) f'_{n-1}(q) \right] \\
= & f_n^2(q) \left[(a_1 + b_1 q + c_1 q^2) \frac{f_{n-1}(q)}{f_n(q)} - q(a_3 + b_3 q + c_3 q^2) \left(\frac{f_{n-1}(q)}{f_n(q)} \right)' \right]. \qquad (3.2)
\end{aligned}$$

Note that $\{f_n(q)\}_{n \geq 0}$ is a generalized Sturm sequence. Thus, if we assume that the all non-positive zeros of $f_n(q)$ are ordered as $r_1 \geq r_2 \geq \ldots \geq r_n$, then $f_{n-1}(q) = f_n(q) \sum_{i=1}^{n} \frac{s_i}{q - r_i}$ by Lemma 3.1, where $s_i > 0$ for $1 \leq i \leq n$. Hence,

$$\begin{aligned}
& f_{n+1}(q) f_{n-1}(q) - f_n^2(q) \\
= & f_n^2(q) \left[(a_1 + b_1 q + c_1 q^2) \frac{f_{n-1}(q)}{f_n(q)} - q(a_3 + b_3 q + c_3 q^2) \left(\frac{f_{n-1}(q)}{f_n(q)} \right)' \right] \\
= & f_n^2(q) \left[(a_1 + b_1 q + c_1 q^2) \sum_{i=1}^{n} \frac{s_i}{q - r_i} + q(a_3 + b_3 q + c_3 q^2) \sum_{i=1}^{n} \frac{s_i}{(q - r_i)^2} \right] \\
= & f_n^2(q) \sum_{i=1}^{n} \frac{s_i \left[(a_1 + b_1 q + c_1 q^2)(q - r_i) + q(a_3 + b_3 q + c_3 q^2) \right]}{(q - r_i)^2} \\
= & \sum_{i=1}^{n} s_i \left[(c_1 + c_3) q^3 + (b_1 + b_3 - c_1 r_i) q^2 + (a_1 + a_3 - b_1 r_i) q - a_1 r_i \right] \left(\frac{f_n(q)}{q - r_i} \right)^2,
\end{aligned}$$

which is a polynomial with nonnegative coefficients since

$$(c_1 + c_3) q^3 + (b_1 + b_3 - c_1 r_i) q^2 + (a_1 + a_3 - b_1 r_i) q - a_1 r_i, \frac{f_n(q)}{q - r_i}$$

are all polynomials with nonnegative coefficients for $1 \leq i \leq n$. Thus $\{f_n(q)\}_{n \geq 0}$ is q-log-convex.

In the following, we proceed to demonstrate the second part that

$$f_{n+1}(q) f_{n-1}(q) - f_n^2(q)$$

is a generalized stable polynomial for each $n \geq 1$. Note that

$$\begin{aligned} & f_{n+1}(q)f_{n-1}(q) - f_n^2(q) \\ = \quad & f_n^2(q)\sum_{i=1}^{n} \frac{s_i\left[(a_1+b_1q+c_1q^2)(q-r_i)+q(a_3+b_3q+c_3q^2)\right]}{(q-r_i)^2}. \end{aligned}$$

Thus we only need to show that

$$\sum_{i=1}^{n} \frac{s_i\left[(b_1+b_3-c_1r_i)q^2+(a_1+a_3-b_1r_i)q-a_1r_i\right]}{(q-r_i)^2} \tag{3.3}$$

has no zeros in the right half plane since $c_1+c_3=0$. Let $q=x+yi$, where i is the imaginary number unit. Then, for $x \geq 0$ and $r \leq 0$, it follows from $a_1+2a_3 \geq 0$ and $b_1 \geq b_3$ that we have

$$\begin{aligned} & [(x-r)^2-y^2]^2 Re\left(\frac{(b_1+b_3-c_1r)q^2+(a_1+a_3-b_1r)q-a_1r}{(q-r)^2}\right) \\ = \quad & [(x-r)^2-y^2]^2 Re\left(\frac{(b_1+b_3-c_1r)(x+yi)^2+(a_1+a_3-b_1r)(x+yi)-a_1r}{(x-r+yi)^2}\right) \\ = \quad & (B-c_1r)(x^2-y^2)^2+(B-c_1r)x^2(r^2-2xr)+(xA-xb_1r-a_1r)(x-r)^2+ \\ & y^2\left\{4x^2B+xA-(2B+b_1)xr-(a_1+2a_3)r+(b_1-b_3)r^2-xr[(2x-r)^2-2xr]\right\} \\ \geq \quad & 0, \end{aligned}$$

where $A=a_1+a_3$ and $B=b_1+b_3$. Thus,

$$Re\left(\sum_{k=1}^{n} \frac{c_k\left[(b_1+b_2)q^2+(a_1+a_2-b_1r_k)q-a_1r_k\right]}{(q-r_k)^2}\right) \geq 0$$

with the equality if and only if $q=0$ since not all of a_1, b_1, c_1 is equal to 0. This completes the proof. □

4 Applications

In this section, we give some applications of the main results.

4.1 Stirling transformations of two kinds

The Bell polynomial is the generating function $B_n(q)=\sum_{k=0}^{n} S_{n,k}q^k$ of the Stirling numbers of the second kind, where the Stirling numbers of the second kind satisfy the recurrence

$$S_{n+1,k}=kS_{n,k}+S_{n,k-1}.$$

Let $c_{n,k}$ be the signless Stirling number of the first kind, i.e., the number of permutations of $[n]$ which contain exactly k permutation cycles. Similarly, singless Stirling numbers of the first kind $c_{n,k}$ satisfy the recurrence

$$c_{n,k}=(n-1)c_{n-1,k}+c_{n-1,k-1}.$$

The q-log-convexity and strong q-log-convexity of $\{B_n(q)\}_{n\geq 0}$ have been proved, see Liu and Wang [33], Chen *et al.* [18] and Zhu [51, 52] for instance. Liu and Wang [33] also proved that both the Stirling transformation of two kinds preserve the log-convexity, respectively. By Theorem 1.2, we can extend above results to the strong q-log-convexity as follows.

Proposition 4.1. *The linear transformation $y_n(q) = \sum_{k=0}^{n} S_{n,k} x_k(q)$ preserves the strong q-log-convexity.*

Proposition 4.2. *The linear transformation $y_n(q) = \sum_{k=0}^{n} c_{n,k} x_k(q)$ preserves the strong q-log-convexity.*

4.2 Jacobi-Stirling transformation of the second kind

Note that the Jacobi-Stirling numbers $\mathrm{JS}_n^k(z)$ of the second kind satisfy the following recurrence relation:

$$\begin{cases} \mathrm{JS}_0^0(z) = 1, \qquad \mathrm{JS}_n^k(z) = 0, \quad \text{if } k \notin \{1,\ldots,n\}, \\ \mathrm{JS}_n^k(z) = \mathrm{JS}_{n-1}^{k-1}(z) + k(k+z)\,\mathrm{JS}_{n-1}^k(z), \quad n,k \geq 1, \end{cases} \tag{4.1}$$

where $z = \alpha + \beta + 1$. Lin and Zeng [31] and Zhu [52] independently proved the strong q-log-convexity of the row generating functions of Jacobi-Stirling numbers. By Theorem 1.2, we have the following generalization, which in particular confirms Conjecture 1.1.

Proposition 4.3. *The Jacobi-Stirling transformation*

$$z_n(q) = \sum_{k=0}^{n} JS_n^k(z) x_k(q)$$

preserves the strong q-log-convexity for $z \geq 0$. In particular, the Jacobi-Stirling transformation

$$z_n = \sum_{k=0}^{n} JS_n^k(z) x_k$$

preserves the log-convexity for $z \geq 0$.

Remark 4.4. If $z = 1$, then $\mathrm{JS}_n^k(1)$ are the Legendre-Stirling numbers of the second kind.

4.3 Central factorial transformations

The *central factorial numbers* of the second kind $T(n,k)$ are defined in Riordan's book [40, p. 213-217] by

$$x^n = \sum_{k=0}^{n} T(n,k)\, x \prod_{i=1}^{k-1} \left(x + \frac{k}{2} - i \right). \tag{4.2}$$

Therefore, if we denote the central factorial numbers of even indices by $U(n,k) = T(2n, 2k)$, then

$$U(n,k) = U(n-1,k-1) + k^2 U(n-1,k).$$

For the central factorial numbers of odd indices, set $V(n,k) = 4^{n-k}T(2n+1, 2k+1)$. By the definition, we have the following recurrence relation:

$$V(n,k) = V(n-1, k-1) + (2k+1)^2 V(n-1, k).$$

Zhu [52] proved that the row generating functions of $U(n,k)$ and $V(n,k)$ form a strongly q-log-convex sequence, respectively. These can be extended to the following in view of Theorem 1.2.

Proposition 4.5. *The linear transformation $y_n(q) = \sum_{k=0}^{n} U(n,k)x_k(q)$ preserves the strong q-log-convexity. In particular, $y_n = \sum_{k=0}^{n} U(n,k)x_k$ preserves the log-convexity.*

Proposition 4.6. *The linear transformation $y_n(q) = \sum_{k=0}^{n} V(n,k)x_k(q)$ preserves the strong q-log-convexity. In particular, $y_n = \sum_{k=0}^{n} V(n,k)x_k$ preserves the log-convexity.*

4.4 Ramanujan transformation

Let $r_{n,k}$ be the number of rooted labeled trees on n vertices with k improper edges. Shor [43] proved that $r_{n,k}$ satisfies the following recurrence relation:

$$r_{n,k} = (n-1)r_{n-1,k} + (n+k-2)r_{n-1,k-1}$$

where $r_{1,0} = 1$, $n \geq 1$, $k \leq n-1$, and $r_{n,k} = 0$ otherwise. It was proved by that the row generating functions of $[r_{n,k}]_{n,k\geq 0}$ are the famous Ramanujan polynomials $r_n(y)$, which are defined by the recurrence relation

$$r_1(y) = 1,\ r_{n+1} = n(1+y)r_n(y) + y^2 r'(y).$$

The first values of the polynomials $r_n(y)$ are

$$r_2(y) = 1 + y,\ r_3(y) = 2 + 4y + 3y^2,\ r_4(y) = 6 + 18y + 25y^2 + 15y^3.$$

Chen *et al.* [18] proved that $r_n(y)$ forms a strongly q-log-convex sequence, which can be extended to the following by Theorem 1.2.

Proposition 4.7. *If $\{x_n(q)\}_{n\geq 0}$ is strongly q-log-convex, then the linear transformation $y_n(q) = \sum_{k=0}^{n} r_{n,k}x_k(q)$ is q-log-convexity. In particular, $y_n = \sum_{k=0}^{n} r_{n,k}x_k$ preserves the log-convexity.*

4.5 Associated Lah transformation

The associated Lah numbers defined by

$$L_m(n,k) = (n!/k!) \sum_{i=1}^{k} (-1)^{k-i} \binom{k}{i} \binom{n+mi-1}{n}$$

satisfy the recurrence

$$L_m(n,k) = (mk+n-1)L_m(n-1,k) + mL_m(n-1,k-1).$$

Let $L_n(q) = \sum_{k=0}^{n} L_m(n,k)q^k$, which has only real zeros, see Wang and Yeh [49] for instance. By virtue of Theorem 1.2 and Proposition 1.8, we have the following two results, respectively.

Proposition 4.8. *The linear transformation $z_n(q) = \sum_{k=0}^{n} L_m(n,k)x_k(q)$ preserves the strong q-log-convexity. In particular, $z_n = \sum_{k=0}^{n} L_m(n,k)x_k$ preserves the log-convexity.*

Proposition 4.9. *$L_{n+1}(q)L_{n-1}(q) - L_n^2(q)$ is a stable polynomial for each $n \geq 1$.*

4.6 Catalan transformations of two kinds

Aigner [2] defined the Catalan triangle

$$C = [C_{n,k}]_{n,k\geq 0} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 2 & 3 & 1 & & \\ 5 & 9 & 5 & 1 & \\ \vdots & & & & \ddots \end{bmatrix},$$

where $C_{n+1,k} = C_{n,k-1} + 2\,C_{n,k} + C_{n,k+1}$ and $C_{n+1,0} = C_{n,0} + C_{n,1}$. The numbers in the 0th column are the Catalan numbers C_n. Shaprio [42] also introduced another Catalan triangle of

$$C' = [C'_{n,k}]_{n,k\geq 0} = \begin{bmatrix} 1 & & & & \\ 2 & 1 & & & \\ 5 & 4 & 1 & & \\ 14 & 14 & 6 & 1 & \\ \vdots & & & & \ddots \end{bmatrix},$$

where $C'_{n+1,k} = C'_{n,k-1} + 2\,C'_{n,k} + C'_{n,k+1}$ for $k \geq 0$. The numbers in the 0th column are also the Catalan numbers C_n. By Theorem 1.3, the next is immediate.

Proposition 4.10. *The Catalan transformations of two kinds $y_n(q) = \sum_{k=0}^{n} C_{n,k}x_k(q)$ and $y_n(q) = \sum_{k=0}^{n} C'_{n,k}x_k(q)$ preserve the strong q-log-convexity, respectively. In particular, they preserve the log-convexity, respectively.*

4.7 The Motzkin transformation

The Motzkin triangle [1, 2] is

$$M = [M_{n,k}]_{n,k\geq 0} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 2 & 2 & 1 & & \\ 4 & 5 & 3 & 1 & \\ \vdots & & & & \ddots \end{bmatrix},$$

where $M_{n+1,k} = M_{n,k-1} + M_{n,k} + M_{n,k+1}$ and $M_{n+1,0} = M_{n,0} + M_{n,1}$. The numbers in the 0th column are the Motzkin numbers M_n. Using Theorem 1.3, we immediately have the following.

Proposition 4.11. *The Motzkin transformation $z_n(q) = \sum_{k=0}^{n} M_{n,k}x_k(q)$ preserves the strong q-log-convexity. In particular, it preserves the log-convexity.*

4.8 The Bell transformation

The Bell triangle [3] is

$$B = [B_{n,k}]_{n,k\geq 0} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 2 & 3 & 1 & & \\ 5 & 10 & 6 & 1 & \\ \vdots & & & & \ddots \end{bmatrix},$$

where $B_{n+1,k} = B_{n,k-1} + (1+k)\,B_{n,k} + (1+k)\,B_{n,k+1}$ and $B_{n+1,0} = B_{n,0} + B_{n,1}$. The numbers in the 0th column are the Bell numbers. The next is immediate from Theorem 1.3.

Proposition 4.12. *The Bell transformation $y_n(q) = \sum_{k=0}^{n} B_{n,k}x_k(q)$ preserves the strong q-log-convexity. In particular, it preserves the log-convexity.*

4.9 Eulerian polynomials of Types A and B

Let $\pi = a_1a_2\cdots a_n$ be a permutation of $[n]$. An element $i \in [n-1]$ is called a descent of π if $a_i > a_{i+1}$. The Eulerian number $A_{n,k}$ is defined as the number of permutations of $[n]$ having $k-1$ descents, which satisfies the recurrence

$$A_{n,k} = kA_{n-1,k} + (n-k+1)A_{n-1,k-1}.$$

Let $A_n(q) = \sum_{k=0}^{n} A_{n,k}q^k$ be the classical Eulerian polynomial.

For Coxeter groups of type B_n, let $E_{n,k}$ be the Eulerian numbers of type B_n counting the elements of B_n with k B-descents. Then the Eulerian numbers of type B_n satisfy the recurrence

$$E_{n,k} = (2k+1)E_{n-1,k} + (2n-2k+1)E_{n-1,k-1}. \tag{4.3}$$

Assume that $E_n(q) = \sum_{k=0}^{n} E_{n,k}q^k$ be the Eulerian polynomial of type B_n. It is known that $A_n(q)$ and $E_n(q)$ forms a strong q-log-convex sequence, respectively, see [52, 34]. By Theorem 1.6, the following is immediate.

Proposition 4.13. *Both $A_{n+1}(q)A_{n-1}(q) - A_n^2(q)$ and $E_{n+1}(q)E_{n-1}(q) - E_n^2(q)$ are generalized stable polynomials in q for $n \geq 1$.*

4.10 Q-Eulerian polynomials

Given a finite Coxeter group W, define the Eulerian polynomials of W by

$$P(W,x) = \sum_{\pi\in W} x^{d_W(\pi)},$$

where $d_W(\pi)$ is the number of W-descents of π. We refer the reader to Björner and Brenti [7] for relevant definitions.

For Coxeter groups of type A_n, it is known that $P(A_n, x) = A_n(x)/x$, where $A_n(x)$ is the classical Eulerian polynomial. Foata and Schützenberger [25] introduced a q-analog of the classical Eulerian polynomials, defined by

$$A_n(x; q) = \sum_{\pi \in \mathcal{S}_n} x^{\mathrm{exc}\,(\pi)+1} q^{c(\pi)},$$

where $\mathrm{exc}\,(\pi)$ and $c(\pi)$ denote the numbers of excedances and cycles in π respectively. It is clear that $A_n(x; 1) = A_n(x)$ is precisely the classical Eulerian polynomial. Brenti showed that q-Eulerian polynomials satisfy the recurrence relation

$$A_n(x; q) = (nx + q - 1)A_{n-1}(x; q) + x(1 - x)\frac{\partial}{\partial x} A_{n-1}(x; q),$$

with $A_0(x; q) = x$, see [14, Proposition 7.2].

For Coxeter groups of type B_n, Brenti [13] defined a q-analogues of $P(B_n, x)$, which reduces to $A_n(x)$ when $q = 0$ and to $P(B_n, q)$ when $q = 1$, by

$$B_n(x; q) = \sum_{\pi \in B_n} q^{N(\pi)} x^{d_B(\pi)}$$

where $N(\pi) = |\{i \in [n] : \pi(i) < 0\}|$. He showed that $\{B_n(x; q)\}$ satisfies the recurrence relation

$$B_n(x; q) = \{1 + [(1 + q)n - 1]x\}B_{n-1}(x; q) + (1 + q)x(1 - x)\frac{\partial}{\partial x} B_{n-1}(x; q),$$

with $B_0(x; q) = 1$, see [13, Theorem 3.4 (i)]. Thus the following result follows from Proposition 1.8.

Proposition 4.14. *Both $A_{n+1}(x; q)A_{n-1}(x; q) - A_n^2(x; q)$ and $B_{n+1}(x; q)B_{n-1}(x; q) - B_n^2(x; q)$ are generalized stable polynomials in x for any fixed $q \geq 0$.*

4.11 Alternating runs

Let $\mathcal{S}_n$ denote the symmetric group of all permutations of $[n]$, where $[n] = \{1, 2, \ldots, n\}$. Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathcal{S}_n$. We say that π changes direction at position i if either $\pi(i - 1) < \pi(i) > \pi(i + 1)$, or $\pi(i - 1) > \pi(i) < \pi(i + 1)$. We say that π has k alternating runs if there are $k - 1$ indices i such that π changes direction at these positions. Let $R(n, k)$ denote the number of permutations in S_n having k alternating runs. Then

$$R(n, k) = kR(n - 1, k) + 2R(n - 1, k - 1) + (n - k)R(n - 1, k - 2) \tag{4.4}$$

for $n, k \geq 1$, where $R(1, 0) = 1$ and $R(1, k) = 0$ for $k \geq 1$, see Bóna [10] for a combinatorial proof. Let $R_n(x) = \sum_{k=1}^{n-1} R(n, k)x^k$. Then the recurrence (4.4) induces

$$R_{n+2}(x) = x(nx + 2)R_{n+1}(x) + x(1 - x^2)R'_{n+1}(x)$$

with $R_1(x) = 1$ and $R_2(x) = 2x$. The polynomial $R_n(x)$ is closely related to the classical Eulerian polynomial $A_n(x)$:

$$R_n(x) = \left(\frac{1 + x}{2}\right)^{n-1} (1 + w)^{n+1} A_n\left(\frac{1 - w}{1 + w}\right), w = \sqrt{\frac{1 - x}{1 + x}},$$

see Knuth [28]. The polynomials $R_n(x)$ have only real non-positive zeros and $R_n(x) \preceq R_{n+1}(x)$, see Ma and Wang [37]. Thus it follows from Theorem 1.6 that the next result is immediate.

Proposition 4.15. *The generating functions of alternating runs $R_n(q)$ form a q-log-convex sequence.*

4.12 The longest alternating subsequence and up-down runs of permutations

Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathcal{S}_n$. An *alternating subsequence* of π is a subsequence $\pi(i_1)\cdots\pi(i_k)$ satisfying

$$\pi(i_1) > \pi(i_2) < \pi(i_3) > \cdots \pi(i_k).$$

Denote by $\mathrm{as}\,(\pi)$ the length of the longest alternating subsequence of π. Let

$$a_k(n) = \#\{\pi \in \mathcal{S}_n : \mathrm{as}\,(\pi) = k\},$$

and let $t_n(x) = \sum_{k=1}^{n} a_k(n)x^k$. Define

$$T(x,z) = \sum_{n\geq 0} t_n(x)\frac{z^n}{n!}.$$

Remarkably, Stanley [45, Theorem 2.3] obtained the following closed-form formula:

$$T(x,z) = (1-x)\frac{1+\rho+2xe^{\rho z}+(1-\rho)e^{2\rho z}}{1+\rho-x^2+(1-\rho-x^2)e^{2\rho z}},$$

where $\rho = \sqrt{1-x^2}$.

For $n \geq 2$, Bóna [10, Section 1.3.2] obtained the following identity:

$$t_n(x) = \frac{1}{2}(1+x)R_n(x).$$

Ma [35] also proved that the polynomials $t_n(x)$ satisfy the recurrence relation

$$t_{n+1}(x) = x(nx+1)t_n(x) + x\left(1-x^2\right)t_n'(x),$$

with initial conditions $t_0(x) = 1$ and $t_1(x) = x$.

On the other hand, $a_k(n)$ is also the number of permutations in $\mathcal{S}_n$ with k up-down runs. The *up-down runs* of a permutation π are the alternating runs of π endowed with a 0 in the front, see [44, A186370]. The up-down runs of a permutation are closely related to interior peaks and left peaks. Based on the interior peaks and left peaks, Ma [36] defined polynomials $M_n(x)$, which satisfy the recurrence relation

$$M_{n+1}(x) = (1+nx^2)M_n(x) + x(1-x^2)M_n'(x),$$

with initial conditions $M_1(x) = 1+x$ and $M_2(x) = 1+2x+x^2$, see [36, Section 2]. In addition, $M_n(x)$ has only real non-positive zeros and $M_n(x) \preceq M_{n+1}(x)$, see Ma [36]. So, we have the following result by Theorem 1.6.

Proposition 4.16. *Both $\{t_n(q)\}_{n\geq 0}$ and $\{M_n(q)\}_{n\geq 0}$ are q-log-convex sequences.*

5 Remarks

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