

ENUMERATING MEANDRIC SYSTEMS WITH LARGE NUMBER OF LOOPS

MOTOHISA FUKUDA AND ION NECHITA

ABSTRACT. We investigate meandric systems with large number of loops using tools inspired by free probability. For any fixed integer r , we express the generating function of meandric systems on $2n$ points with $n - r$ loops in terms of a finite (the size depends on r) subclass of irreducible meandric systems, via the moment-cumulant formula from free probability theory. We show that the generating function, after an appropriate change of variable, is a rational function, and we bound its degree. Exact expressions for the generating functions are obtained for $r \leq 6$, as well as the asymptotic behavior of the meandric numbers for general r .

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1. INTRODUCTION

In this paper we consider the problem of enumerating meandric systems, which are a natural generalizations of meanders. This problem falls into the category of enumerating some non-crossing diagrams, and it has received a lot of interest from the mathematics and physics communities. An excellent reference providing an extensive overview of the problem and its various connections to various branches of mathematics is [DFGG97a].

Meandric systems of order n are defined as non-crossing closed loops which intersect a straight infinite line at $2n$ points. To configure all possible shapes, one can draw a horizontal line with $2n$ points and choose two non-crossing pair partitions on those $2n$ points for the upper and lower sides of the line so that connecting them gives a set of closed loops. In Figure 1, two non-crossing pairings $\{(1, 2), (3, 6), (4, 5)\}$ and $\{(1, 4), (2, 3), (5, 6)\}$ result in a meandric system with one loop, which is simply called a *meander*. For fixed n , we can define $M_n^{(k)}$ to be the number of meandric systems with $2n$ fixed points and k loops. Computing $M_n^{(1)}$, the number of meanders, is a notoriously difficult problem, while $M_n^{(n)}$ is just the *Catalan number* Cat_n , because such a meandric system is obtained when the same non-crossing pair partitions (or arches) are chosen for the upper and lower diagrams. The example in Figure 1 contributes to $M_3^{(1)}$. The problem of enumerating meanders and meandric systems is an important one, and has received a lot of attention in the last three decades [Slo99, Section 6].

One of main goals of research on meandric systems is to find explicit formulas for the numbers $M_n^{(k)}$ for any $k, n \in \mathbb{N}$. With the help of computers, these numbers (sequence A008828 in [OEI11])

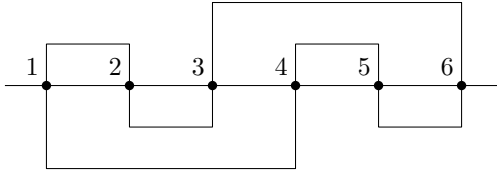


FIGURE 1. A meander by $\{(1, 2), (3, 6), (4, 5)\}$ and $\{(1, 4), (2, 3), (5, 6)\}$.

have been computed up to n of order 30, see e.g. [BS10]. In this paper, we focus on the formulas for $M_n^{(n-r)}$ for fixed $r \in \mathbb{N}$ and any $n \in \mathbb{N}$. Such formulas were obtained for $0 \leq r \leq 5$ in [DFGG97a], where the authors claimed to have proved them for $0 \leq r \leq 3$. In this work, we obtain the general formula for the generating functions of these numbers, and the exact values for $r \leq 6$. Moreover, we introduce a recipe for obtaining the generating function for any given r , which we implement on a computer algebra system, see [FN16]. Before moving on, let us be clear that we do not touch on what is probably considered the most important problem in the field, the enumeration of meanders, that is the numbers $M_n^{(1)}$ (sequence A005315 in [OEI11]).

Our approach consists in translating the problems of meandric systems into problems about non-crossing partitions and permutations, by using the bijection between non-crossing pairings of $2n$ points and the so-called geodesic permutations of n elements. These relations between meanders and permutations were investigated in [Sav09, Hal06], and precise and detailed discussions will be made in Section 3. A key concept in our analysis is the notion of *irreducible meandric systems*, which were introduced in [LZ93] (sequence A006664 in [OEI11]); the idea of counting combinatorial objects in terms of “irreducibles” dates back to Beissinger [Bei85]. However, we introduce the parameter r in the study of irreducible meandric systems to analyze the number of loops of a meandric system. Recently, Nica [Nic16] also analyzed irreducible meandric systems from the point of view of free probability, but with different goals and methods than ours. The focus in [Nic16] is on the set of irreducible meandric systems, independently of the parameter r , while r plays a key role in our investigations. We refer the reader to the comments at the end of Section 5 for a discussion of the similarities and differences between these two papers.

Inspired by the language of free probability, we show that the generating function of the sequence $(M_n^{(n-r)})_n$ for fixed r can be obtained from that for irreducible meandric systems, through natural transformations between moments and free cumulants (Theorem 5.1). Since irreducible meandric systems with fixed parameter r live on at most $n = 2r$ points, we can derive the generating function as described above. The following statement is our main result in this paper (see Theorem 5.6 for the precise statement), providing an alternative answer to the conjecture in [DFGG97a, Equation (2.4)] (see also Remark 5.7):

Theorem 1.1. *Let F_r be the generating function of the number of meanders on $2n$ points with $n - r$ loops*

$$F_r(t) = \sum_{n=r+1}^{\infty} M_n^{(n-r)} t^n.$$

Then, with the change of variables $t = w/(1+w)^2$, the functions F_r read

$$F_r(t) = \sum_{n=r+1}^{\infty} M_n^{(n-r)} \frac{w^n}{(1+w)^{2n}} = \frac{w^{r+1}(1+w)}{(1-w)^{2r-1}} \tilde{P}_r(w), \quad (1)$$

where $\tilde{P}_r(w)$ are polynomials of degree at most $3(r-1)$ (see Section 6 for the values of these polynomials up to $r = 6$).

The paper is organized as follows. In Section 2 we recall some basic properties of non-crossing partitions and permutations, which are used in Section 3 to make the connection to meandric systems. In Section 4 we introduce irreducible meandric systems, together with three parameters which are going to be used later for enumerating meandric systems. Section 5 contains the main result of the paper; we obtained the general form of the generating function of meandric systems with large number of loops, as well as their asymptotic behavior, after establishing their relations to irreducible meandric systems via the moment-cumulant formula. The first few exact values of the polynomials appearing in the generating functions are presented in Section 6.

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2. COMBINATORIAL ASPECTS OF NON-CROSSING PARTITIONS AND PERMUTATIONS

We gather in this section some well-known definitions and facts about non-crossing partitions and non-crossing permutations. Many of these facts are folklore, but one can follow [Bia97] or the excellent monograph [NS06].

For a permutation α in the symmetric group S_n , we denote by $\|\alpha\|$ its length, that is the minimal number m of transpositions τ_1, \dots, τ_m which multiply to α :

$$\|\alpha\| := \min\{m \geq 0 : \exists \tau_1, \dots, \tau_m \in \mathcal{S}_n \text{ transpositions s.t. } \alpha = \tau_1 \cdots \tau_m\}.$$

Sometimes the notation $|\cdot|$ is used for the length of permutations, but in our paper it stands for the cardinality of a set. The length function $\|\cdot\|$ induces a distance on \mathcal{S}_n by $d(\alpha, \beta) = \|\alpha^{-1}\beta\|$. Importantly, we have

$$\|\alpha\| + \#(\alpha) = n \tag{2}$$

where $\#(\alpha)$ is the number of cycles in α . Note also that both $\#(\cdot)$ and $\|\cdot\|$ are class functions (with respect to conjugation) and thus, e.g. $\|\alpha\| = \|\alpha^{-1}\|$ and $\|\alpha\beta\| = \|\beta\alpha\|$.

We recall next the concept of non-crossing partitions. A partition $A_1 \sqcup A_2 \sqcup \cdots \sqcup A_m = \{1, 2, \dots, n\}$ is called *non-crossing* if there do not exist $a, b \in A_i$ and $c, d \in A_j$ ($i \neq j$) such that $a < c < b < d$. An example of non-crossing partition $\{1, 4, 5\} \sqcup \{2, 3\}$ is found in Figure 2, and crossing $\{1, 3\} \sqcup \{2, 4, 5\}$ in Figure 3. The set of non-crossing partitions of $\{1, 2, \dots, n\}$ is denoted by $NC(n)$ or $NC(1, 2, \dots, n)$. We are sometime interested in $NC(n)$ restricted only to partitions with blocks of size 2, and we denote by $NC_2(n)$ this subset of $NC(n)$: $NC_2(n)$ is the set of non-crossing pairings and n must be an even number.

Non-crossing partitions of $\{1, 2, \dots, n\}$ are naturally identified to a subset of permutations in \mathcal{S}_n , called *geodesic (or non-crossing) permutations*, see [Bia97] or [NS06, Lecture 23]. The bijection corresponding to this identification associates to each block of a non-crossing partition a cycle in a permutation where the elements are ordered increasingly; the example in Figure 2 is identified to the permutation $(1, 4, 5)(2, 3) \in \mathcal{S}_5$. As it was shown by Biane in [Bia97], geodesic permutations are characterized by the fact that they saturate the triangle inequality; $\alpha \in \mathcal{S}_n$ is geodesic iff

$$\|\alpha\| + \|\alpha^{-1}\pi\| = \|\pi\| = n - 1,$$

where $\pi = (1, 2, \dots, n)$ is the full-cycle permutation; we say that α lies on the geodesic between $\text{id} = (1)(2) \cdots (n)$ and $\pi = (1, 2, \dots, n)$ in \mathcal{S}_n . We shall use the identification between non-crossing partitions and geodesic permutations. We write α^{-1} for a non-crossing partition $\alpha \in NC(n)$; this

should be understood as the permutation $\alpha^{-1} \in \mathcal{S}_n$ which lies on the geodesic between id and $\pi^{-1} = (n, n-1, \dots, 1)$.

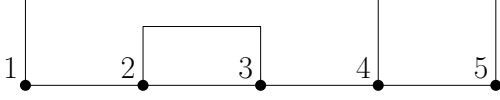


FIGURE 2. A non-crossing partition.

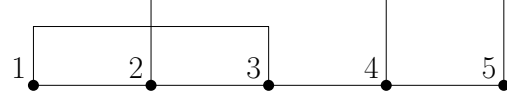


FIGURE 3. A crossing partition.

Next, we recall the notion of *Kreweras complement* for non-crossing partitions. The Kreweras complement of $\alpha \in NC(n)$ is another non-crossing partition, denoted $\alpha^{\text{Kr}} \in NC(n)$, defined in the following way [NS06, Definition 9.21]. First, expand the domain of partitions to $\{1, \bar{1}, 2, \bar{2}, \dots, n, \bar{n}\}$ and let then $\alpha^{\text{Kr}} \in NC(\bar{1}, \bar{2}, \dots, \bar{n}) \cong NC(n)$ be the largest non-crossing partition, with respect to the partial order defined in the next paragraph, such that $\alpha \sqcup \alpha^{\text{Kr}}$ is still a non-crossing partition on $\{1, \bar{1}, 2, \bar{2}, \dots, n, \bar{n}\}$. In the language of geodesic permutations, given a geodesic permutation $\text{id} - \alpha - \pi$, we define the geodesic permutation $\alpha^{\text{Kr}} \in \mathcal{S}_n$ as $\alpha^{\text{Kr}} = \alpha^{-1}\pi$ (see [NS06, Remark 23.24]). An example of Kreweras complement is found in Figure 4; we set $\alpha = (2, 6)(3, 4)$ and $\alpha^{\text{Kr}} = (1, 6)(2, 4, 5)$. Other trivial examples are $\text{id}^{\text{Kr}} = \pi$ and $\pi^{\text{Kr}} = \text{id}$.

The set $NC(n)$ is endowed with the partial order of *reversed refinement*: $\alpha \leq \beta$ if every block of α is contained in a block of β . On the level of geodesic permutations, the partial order $\alpha \leq \beta$ is equivalent to α being on the geodesic between id and β :

$$\|\alpha\| + \|\alpha^{-1}\beta\| = \|\beta\|.$$

Since $NC(n)$ is a lattice, for any $\alpha, \beta \in NC(n)$ we denote by $\alpha \wedge \beta$ the *meet* of α and β , that is the largest element $\gamma \in NC(n)$ such that $\gamma \leq \alpha, \beta$. Similarly, we write $\alpha \vee \beta$ for the uniquely defined *join* of α and β . Taking Kreweras complements, we have $(\alpha \wedge \beta)^{\text{Kr}} = \alpha^{\text{Kr}} \vee \beta^{\text{Kr}}$ and $(\alpha \vee \beta)^{\text{Kr}} = \alpha^{\text{Kr}} \wedge \beta^{\text{Kr}}$. The smallest element in $NC(n)$ is denoted by 0_n and corresponds to the partition made up of singletons, or to the identity permutation. The largest element of $NC(n)$ is denoted by 1_n and corresponds to the 1-block partition, or to the permutation $\pi \in \mathcal{S}_n$ defined previously.

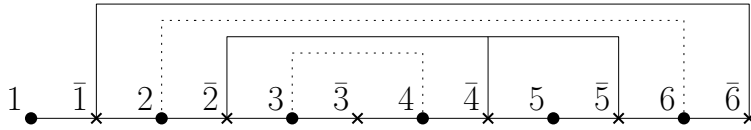


FIGURE 4. The Kreweras complement of $(2, 6)(3, 4)$.

Finally, let us discuss the well-known bijection between $NC(n)$ and $NC_2(2n)$, called *fattening*. For a given non-crossing partition $\alpha \in NC(n)$, we consider two points i_- and i_+ for both sides of each $i \in \{1, \dots, n\}$, left and right respectively, doubling the size on the index set. Associate now to α the following pairing: connect i_+ and j_- if $\alpha(i) = j$, where α is seen now as a permutation. It can be shown that the pair partition obtained is also non-crossing, see [NS06, Lecture 9]. We state a lemma on the fattening operation which is used later in the paper.

Lemma 2.1. *Take $\alpha \in NC(n)$ and denote its fattening by $\bar{\alpha} \in NC_2(2n)$.*

- (1) *Suppose (i_-, j_+) with $i \leq j$ is a pair in $\bar{\alpha}$. Then:*
 - (a) *if $i = j$ implies that i is a fixed point of the geodesic permutation α*
 - (b) *otherwise, $i < j$ and α has a cycle of the form (i, \dots, j) where the numbers in the bracket are in the increasing order.*
- (2) *Suppose (i_+, j_-) with $i < j$ is a pair in $\bar{\alpha}$. Then, α has a cycle of the form $(\dots i, j \dots)$ where the numbers in the bracket are in the increasing order.*

Proof. To show (1)-(a), notice that the paring (i_-, i_+) implies that $i \mapsto i$ by the definition of fattening. Similarly, the paring (i_-, j_+) implies that $j \mapsto i$, but since the permutation α is on the geodesic between id and $\pi = (1, 2, \dots, n)$ the claim is proved. The claim (2) also follows from the definition. \square

We end the combinatorial treatment of non-crossing partitions by giving the reader a taste of the connection between pairs of non-crossing partitions and meandric systems. These fact will be treated rigourously and in detail in Section 3. In Figure 5, one can find that the meander of Figure 1 is represented by drawing two geodesic permutations $\{(1), (2, 3)\}$ and $\{(1, 2), (3)\}$, one above and one below a horizontal line (the dotted lines show the fattening of the corresponding permutations). In Figure 6, the fattening operation is drawn with directions, which show how the original permutations act. Interestingly, the arrows from the upper and lower sides of the horizontal line are conflicting. However, if one inverts the directions of the lower side, one can have a loop with consistent directions. In fact, such a loop results from the following calculation

$$(1)(2, 3) \circ \{(1, 2)(3)\}^{-1} = (1, 3, 2)$$

which is equivalent to the fact that only one meander is generated from such pairs of permutations. Further details are found in Section 3 and one can understand more of this concept going through the example and the proof of Proposition 3.1.

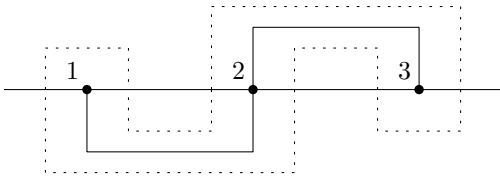


FIGURE 5. A meander obtained from two non-crossing partitions $\{(1), (2, 3)\}$ and $\{(1, 2), (3)\}$.

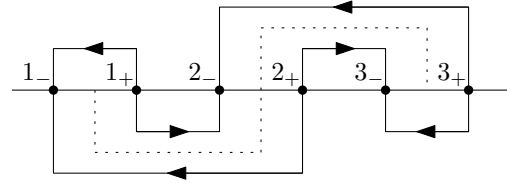


FIGURE 6. Fattening with directions.

We change now topics and discuss generating series associated to moments of probability measures and free cumulants. First, we define the moment generating function and R-transform:

$$M(z) = \sum_{n \geq 1} m_n z^n \quad \text{and} \quad R(z) = \sum_{n \geq 1} \kappa_n z^n$$

where m_n is the n -th moment and κ_n is the n -th free cumulant. Note that the free cumulants are defined by the following relation, called the *moment-cumulant formula* [NS06, Lecture 11]:

$$m_n = \sum_{\alpha \in NC(n)} \prod_{c \in \alpha} \kappa_{|c|} \quad (3)$$

Conversely, free cumulants can be expressed in terms of moments by using the Möbius function on the $NC(n)$ lattice. This implies that the two generating functions M and R are related by the following implicit equation (see [NS06, Remark 16.18]):

$$M(z) = R(z(1 + M(z))). \quad (4)$$

Historically, the notion of R-transform is introduced by Voiculescu in a slightly different form [Voi85, Voi86], and can be defined for each compactly supported measure μ via the moments: $m_n = \int x^n d\mu(x)$.

Next, we denote by \mathcal{F} the transformation mapping an arbitrary power series R to the unique power series M which consist of sequences in (3), or equivalently satisfy (4). In this case, we write

$$\mathcal{F} : R \mapsto M \quad (5)$$

The \mathcal{F} -transform is used in Section 5 where it plays a crucial role in the derivation of our main result. Its use in combinatorics predates its incarnation in free probability: in [Bei85], the author relates generating series for some classes of combinatorial objects to the generating series of *irreducible objects* of the same type. Remarkably, the lattices studied in [Bei85] are precisely the ones which appear in non-commutative probability theory: all partitions (in relation to classical, or tensor independence), non-crossing partitions (in relation to free independence) and “interval-block” partitions (in relation to Boolean independence).

3. FROM MEANDERS TO PAIRS OF NON-CROSSING PARTITIONS

The following result appears in several places in the literature [Hal06, Theorem 3.3], [Sav09, Theorem 5.7], [FS13, Section IV.C], or [Nic16, Section 3]. We state it here using our language, where we identify non-crossing partitions and geodesic permutations.

Proposition 3.1. *Meandric systems on $2n$ points with $n - r$ loops are in bijection with the set*

$$M_{n,r} := \{(\alpha, \beta) \in \text{NC}(n)^2 : \|\alpha^{-1}\beta\| = r\}. \quad (6)$$

We denote by $\mathcal{M}(\alpha, \beta)$ the meandric system associated to the pair (α, β) .

Before proving the result, let us describe how the bijection works on an example for $n = 5$. In Figure 7, two different geodesic permutations are represented by black lines in the upper and lower sides of the horizontal line: the permutation α is depicted on top, while β is depicted below the horizontal line. First, let us focus on the upper permutation, which is $\alpha = (1, 2, 3)(4, 5)$. The red lines are the fattening of the permutation which is the non-crossing pairing: $\bar{\alpha} = (1_-, 3_+)(1_+, 2_-)(2_+, 3_-)(4_-, 5_+)(4_+, 5_-)$. The red arrows show how this non-crossing pairing is related to the original permutation. Indeed, $1 \mapsto 2 \mapsto 3 \mapsto 1$ is represented by $1_+ \mapsto 2_-$, $2_+ \mapsto 3_-$ and $3_+ \mapsto 1_-$. Similarly, $4 \mapsto 5 \mapsto 4$ is indicated by $4_+ \mapsto 5_-$ and $5_+ \mapsto 4_-$. Next, the black lines in lower part represent $\beta = (1)(3)(2, 4, 5)$, and the fattening $\bar{\beta} = (1_-, 1_+)(2_-, 5_+)(2_+, 4_-)(3_-, 3_+)(4_+, 5_-)$ is drawn by blue lines. However, this time the arrows are reversed, i.e., they direct from $*_-$ to $*_+$ while $*_+$ to $*_-$ for the red lines, where $* \in \{1, \dots, 5\}$. This is because we want to consider $[(1)(3)(2, 4, 5)]^{-1}$ where $5 \mapsto 4 \mapsto 2 \mapsto 5$ is represented by $5_- \mapsto 4_+$, $4_- \mapsto 2_+$ and $2_- \mapsto 5_+$. Then, joining red and blue lines, we get the loop structure of the corresponding meanders and the number of loops in the meanders equals the number of loops of $(1, 2, 3)(4, 5) \circ (1)(3)(5, 4, 2) = (1, 2, 4, 3)(5)$, which is 2. In this example, we verified that the number of loops in the induced meandric system is $2 = 5 - 3$ where $\|\alpha\beta^{-1}\| = 3$.

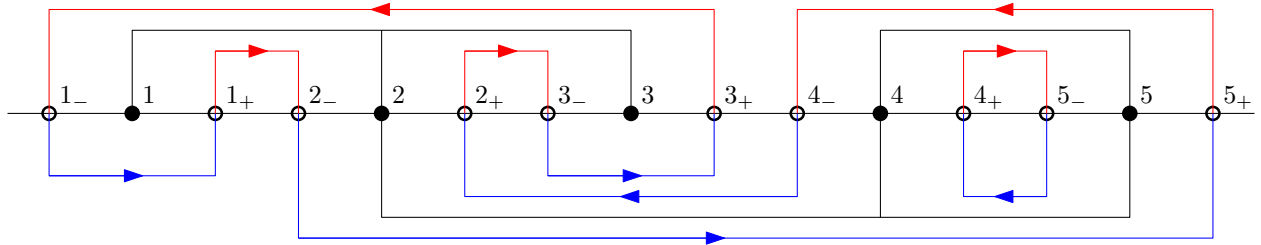


FIGURE 7. Meanders generated by $\alpha = (1, 2, 3)(4, 5)$ and $\beta = (1)(3)(2, 4, 5)$.

Sketch of proof of Proposition 3.1. We show the following bijection:

$$\text{NC}(n) \times \text{NC}(n) = \text{NC}_2(2n) \times \text{NC}_2(2n) = \{\text{meandric systems}\} \quad (7)$$

where the first identification is the fattening; $NC(n)$ on $\{i\}_{i=1}^n$ and $NC_2(2n)$ on $\{i_-, i_+\}_{i=1}^n$. Namely, for two permutations $\alpha, \beta \in S_n$, placing i_- on the left side of i and i_+ on the right, we identify

$$\begin{aligned} i &\mapsto \alpha(i) & j &\mapsto \beta^{-1}(j) \quad \text{respectively by} \\ i_+ &\mapsto \alpha(i)_- & j_- &\mapsto \beta^{-1}(j)_+. \end{aligned} \quad (8)$$

Here, the second line shows how the meandric system shapes as described in the above example. I.e., we can move along meanders to count the number of cycles:

$$1_- \rightarrow [\beta^{-1}(1)]_+ \rightarrow [\alpha\beta^{-1}(1)]_- \rightarrow [\beta^{-1}\alpha\beta^{-1}(1)]_+ \rightarrow \dots \rightarrow 1_- \quad (9)$$

As you can see, we visit $*_-$ and $*_+$ in turn ($* \in \{1, \dots, n\}$), and then $(\alpha\beta^{-1})^m(1_-) = 1_-$ for some $m \in \mathbb{N}$. This sequence corresponds to a cycle in the permutation $\alpha\beta^{-1}$ which is a loop in the meandric system. Then, we choose i such that i_- has not been visited, and identify another cycle in a similar way. We continue this process until we exhaust all $\{i_-\}_{i=1}^n$ to count the number of cycles in $\alpha\beta^{-1}$. To finish the proof, remember that $\#(\alpha\beta^{-1}) = n - \|\alpha\beta^{-1}\| = n - \|\alpha^{-1}\beta\|$. \square

To close this section we would like to overview how problems of meandric systems are related to other three interesting research topics, besides non-crossing partitions.

Firstly, the meandric numbers $M_n^{(1)}$ count also the number of configurations of a folding closed polymer in 2D, where a meander is regarded as a polymer. See Figure 8, which is compared with Figure 1. A folding open polymer can be thought of a *semi-meander*, but we do not treat this problem in this current paper. Interested readers are referred to [DFGG97a].

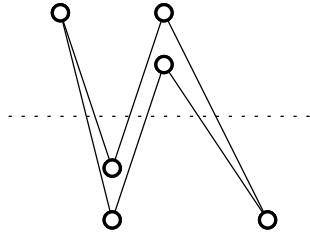


FIGURE 8. A folding closed polymer.

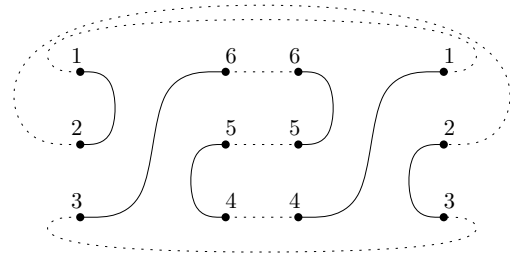


FIGURE 9. $\text{Tr}[(e_1e_2) \cdot (e_2e_1)^T]$ in the Temperley-Lieb Algebra.

Secondly, the scalar product in the Temperley-Lieb Algebra corresponds to a meandric system, which contributes to $M_n^{(k)}$. This algebra is generated by $\{1, e_1, \dots, e_{n-1}\}$ where

- (1) $e_i^2 = w \cdot e_i$ for $i = 1, \dots, n-1$.
- (2) $[e_i, e_j] = 0$ if $|i - j| \geq 2$.
- (3) $e_i \cdot e_{i\pm 1} \cdot e_i = e_i$ for $i = 1, \dots, n-1$.

Here, $w > 0$ is a scalar weight. The braid graphical representations of e_i and $e_1 \cdot e_2$ are found in Figure 10 and 11. As you can see in Figure 9, $e_1 \cdot e_2$ can be identified to the upper side of Figure 1 and $(e_2e_1)^T$ to the lower side so that the scalar product is the weight of loop, say w , powered to the number of loops of the meandric system. Interested readers are referred to [DFGG97b].

Thirdly, there is a mathematical relation to quantum information theory. While studying partial transpose of random quantum states [FS13], we came across *meander polynomials* :

$$M_n(x) := \sum_{k=1}^n M_n^{(k)} x^k \quad (10)$$

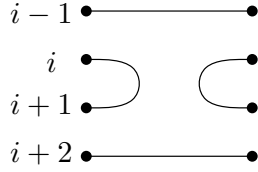


FIGURE 10. e_i in the braid representation.

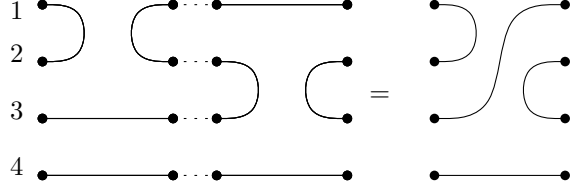


FIGURE 11. $e_1 \cdot e_2$ in the braid representation.

Based on this fact, a new random matrix model for the meander polynomials was found. That is, if you take a complex Gaussian random matrix $G \in M_{N^2, x}(\mathbb{C})$ with the mean 0 and the variance $1/N$ for each entry, we get it as the limiting moments:

$$M_n(x) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \text{Tr} \mathbb{E} ((GG^*)^\Gamma)^{2n} \quad (11)$$

where Γ is called *partial transpose*, with which we apply transpose only to one of spaces of the bipartite system $\mathbb{C}^n \otimes \mathbb{C}^n$. Note that GG^* will be a random quantum state (i.e. a positive semidefinite matrix with unit trace) with proper normalization.

4. IRREDUCIBLE MEANDRIC SYSTEMS

The concept of *irreducible meandric systems* was introduced by Lando and Zvonkin in [LZ93]. Informally, these are meandric systems on $2n$ points such that there is no interval $[a, b] \subseteq [1, 2n]$ with the property that the restriction of the meandric system to $[a, b]$ is another meandric system; see Figure 12 for examples. Let us explain here the elements appearing in Figure 12, since such diagrams will continue to be used from here on. The meandric systems are represented by the red, thin lines; here, the red curve(s) intersect the horizontal line $2n = 8$ times. The non-crossing partitions α and β generating the meander are represented with black lines, above and respectively below the horizontal line; in the left panel of Figure 12 we have $\alpha = (24)$ and $\beta = (13)$. Finally, the $n = 4$ points on which the geodesic permutations corresponding to α and β act are represented by blue dots.

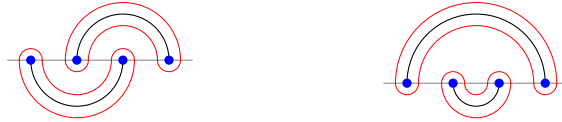


FIGURE 12. On the left, an irreducible meandric system. On the right a reducible meandric system: the diagram in the center gives a proper meandric system.

In the language of non-crossing partitions, irreducible meandric systems have been shown [Nic16, Theorem 1.1 or Proposition 3.4] to be in bijection with the set

$$I_n := \{(\alpha, \beta) \in NC(n) : \alpha \wedge \beta = 0_n, \alpha \vee \beta = 1_n\}. \quad (12)$$

Interestingly, Lando and Zvonkin show that if $C(x)$ and, respectively, $I(x)$ are the generating series for the square Catalan numbers, and the number of irreducible meandric systems

$$C(x) = \sum_{n \geq 0} \text{Cat}_n^2 x^n, \quad I(x) = \sum_{n \geq 0} |I_n| x^n,$$

they satisfy the functional equation $C(x) = I(xC^2(x))$, allowing them to obtain the asymptotic growth rate of the number of irreducible meandric systems [LZ93]

$$\limsup_{n \rightarrow \infty} |I_n|^{1/n} = (\pi/(4 - \pi))^2.$$

We note that the problem of estimating the asymptotic growth of the sequence $(M_n^{(1)})_n$ (the number of meanders, i.e. meandric systems with one loop) is largely open. It is conjectured that

$$M_n^{(1)} \sim C\rho^n n^{-\kappa},$$

with $\kappa = (29 + \sqrt{145})/12$ [DFGG00, DFGJ00] and $\rho \approx 12.26287$, while it is known that $11.380 \leq \rho \leq 12.901$ [AP05]. We do not discuss this problem here, and we think that tackling it would require some new ideas (see [Nic16, Section 5] for some recent considerations). In this work, we do compute the asymptotic behavior of number of meandric systems on $2n$ points with $n - r$ loops (for fixed r), see Corollaries 5.8 and 6.2.

One of the main new insights of the current work is to further partition the set of irreducible meandric systems in terms of the lengths of the permutations α, β and in terms of the distance between α and β .

Definition 4.1. We call a pair of non-crossing partitions $(\alpha, \beta) \in NC(n)^2$ irreducible of type (n, r, a, b) if the following conditions are simultaneously satisfied:

- (1) $\alpha \wedge \beta = 0_n$
- (2) $\alpha \vee \beta = 1_n$
- (3) $\|\alpha^{-1}\beta\| = r$
- (4) $\|\alpha\| = a$
- (5) $\|\beta\| = b$.

The corresponding meandric system $\mathcal{M}(\alpha, \beta)$ is also called irreducible of type (r, a, b) . We write

$$I_{n,r,a,b} := \{(\alpha, \beta) \in NC(n)^2 : \alpha \wedge \beta = 0_n, \alpha \vee \beta = 1_n, \|\alpha^{-1}\beta\| = r, \|\alpha\| = a, \|\beta\| = b\}. \quad (13)$$

Let us consider some examples. At $n = 1$, we obtain the unique irreducible meandric system with $r = 0$: $\alpha = \beta = (1)$; the corresponding triple of parameters is $(0, 0, 0)$. At $n = 2$, we have the the following four possible parameter triples and the last two correspond to irreducible meandric systems (see Figure 13 for a graphical representation):

- $(0, 0, 0)$: $\alpha = \beta = (1)(2)$
- $(0, 1, 1)$: $\alpha = \beta = (12)$
- $(1, 0, 1)$: $\alpha = (1)(2)$, $\beta = (12)$
- $(1, 1, 0)$: $\alpha = (12)$, $\beta = (1)(2)$.

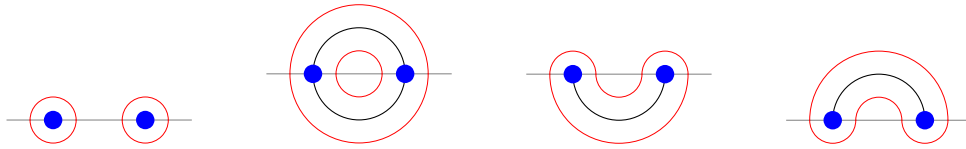


FIGURE 13. All meandric systems on $n = 2$ points: $[\alpha = \beta = (1)(2)]$, $[\alpha = \beta = (12)]$, $[\alpha = (1)(2), \beta = (12)]$, $[\alpha = (12), \beta = (1)(2)]$. Only the last two examples correspond to irreducible meandric systems.

One of the key facts that will be used in what follows is that the parameters n, r, a, b need to verify some restrictions in order for the set $I_{n,r,a,b}$ to be non-empty.

Definition 4.2. A quadruple of non-negative integers (n, r, a, b) is called compatible if it satisfies the following conditions:

- (1) $a, b \leq \max(2r - 2, 1)$
- (2) $|a - b| \leq r \leq a + b$
- (3) $a - b$ and $a + b$ have the same parity as r
- (4) If $r = |a - b|$, then $\min(a, b) = 0$ and $\max(a, b) = r$
- (5) $r + 1 \leq n \leq 2r + \mathbf{1}_{n=1}$.

In particular, a triple (r, a, b) is called compatible if it satisfies the first four conditions above for some fixed n .

As an example, in Figure 14, we have indicated by filled disks the possible values of a, b such that the quadruple $(n, r = 6, a, b)$ is compatible, disregarding the value of $n \leq 12$. It turns out however that the compatibility conditions are not sufficient to ensure that $I_{n,r,a,b} \neq \emptyset$: for all $n \leq 12$, the sets $I_{n,6,8,10}$, $I_{n,6,9,9}$, $I_{n,6,10,8}$, and $I_{n,6,10,10}$ are empty.

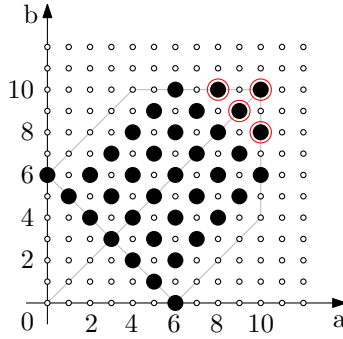


FIGURE 14. The set of pairs (a, b) such that the quadruple $(n, r = 6, a, b)$ is compatible is depicted by large, filled disks. It turns out that some of these quadruples (the ones marked with a red circle) correspond to empty sets $I_{n,r,a,b}$: this implies that the compatibility conditions from Definition 4.2 are not sufficient for $I_{n,r,a,b} \neq \emptyset$.

We prove now the main result of this section.

Proposition 4.3. *If a quadruple (n, r, a, b) is not compatible, then the corresponding set $I_{n,r,a,b}$ is empty.*

Proof. The claim (2) comes from the triangle inequality in the metric space \mathcal{S}_n with the distance $d(\alpha, \beta) = \|\alpha^{-1}\beta\|$. The claim (3) is clear because of well-definedness of parity; if one writes a permutation as products of transpositions, the parity is same for all possible products. To show (4), suppose $a \leq b$ without loss of generality. Then, $r = b - a$ corresponds to a tight case for the triangle inequality, and we have the geodesic: $\text{id} - \alpha - \beta$, so that $\alpha \wedge \beta = 0_n$ implies that $\alpha = 0_n$ and hence the claim. Next, we prove the upper-bound in (5). The case $n = 1$ corresponds to the case $(n, r, a, b) = (1, 0, 0, 0)$. Suppose now $n \geq 2$ and we show that $n/2$ bounds from above the second term of the right hand side of the following identity:

$$n = r + (\text{the number of loops in the meanders}).$$

To this end, note that for irreducible meandric systems every loop intersects the horizontal line at least four times. Indeed, consider a meandric system of two non-crossing pairings $\bar{\alpha}, \bar{\beta} \in NC_2(2n)$ of $\{1_-, 1_+, \dots, n_-, n_+\}$. Suppose for a contradiction that there is a loop which intersects the horizontal line at only two points, for example, $\{i_-, j_+\}$ with $i \leq j$ or $\{i_+, j_-\}$ with $i < j$. By Lemma 2.1, $i = j$ implies that $\alpha \vee \beta \neq 1_n$, and $i \neq j$ that $\alpha \wedge \beta \neq 0_n$. This contradiction proves that the number of loops in an irreducible meandric system is bounded by $\frac{2n}{4}$, because it is impossible to draw a loop which intersects with the horizontal line an odd number of times. The lower bound in (5) follows from the fact that the diameter of \mathcal{S}_n is $n - 1$. Finally, we show (1) for a (the proof

for b being similar). When $n \leq 2$, the claim is true based on the above observation. For $n \geq 3$ it is easy to see from (5) that

$$a \leq n - 1 \leq 2r - 1$$

Suppose $a = n - 1$ and this implies that $\alpha = 1_n$ and hence $\beta = 0_n$ for irreducible meandric systems. Then, we have $n = r + 1 \leq 2r - 1$ so that $a \leq 2r - 2$. This completes the proof. \square

The bound on n in the result above is interesting: if we are interested in irreducible meandric systems with a fixed parameter r , we only need to investigate non-crossing partitions of sizes at most $2r$; this fact will be useful in the proof of our main result and also in the numerical procedures used to generate irreducible meandric systems [FN16]. In Figure 15, we have represented all irreducible meandric systems with $r = 2$; note that the maximal size of non-crossing partitions appearing in the list is $n = 4$.

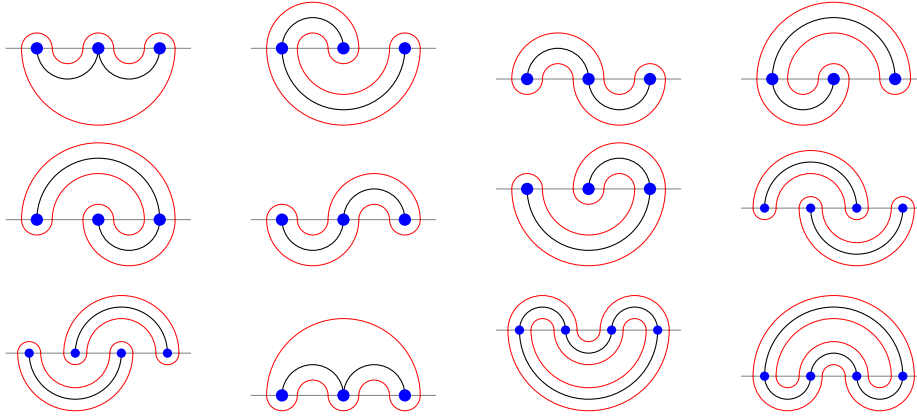


FIGURE 15. All irreducible meandric systems of type $(2, *, *)$. From top to bottom, left to right, the first meander is of type $(2, 0, 2)$, the next 8 are of type $(2; 1, 1)$, the next is of type $(2, 2, 0)$, and the last two are of type $(2, 2, 2)$.

5. COUNTING MEANDRIC SYSTEMS USING IRREDUCIBLE MEANDRIC SYSTEMS

We list three important sets of meandric systems, i.e. pair of non-crossing partitions in $NC(n)^2$:

$$I_{n,r,a,b} := \{(\alpha, \beta) : \alpha \wedge \beta = 0, \alpha \vee \beta = 1, \|\alpha^{-1}\beta\| = r, \|\alpha\| = a, \|\beta\| = b\} \quad (14)$$

$$K_{n,r,a,b} := \{(\alpha, \beta) : \alpha \vee \beta = 1, \|\alpha^{-1}\beta\| = r, \|\alpha\| = n - 1 - a, \|\beta\| = n - 1 - b\} \quad (15)$$

$$M_{n,r,a,b} := \{(\alpha, \beta) : \|\alpha^{-1}\beta\| = r, \|\alpha^{-1}(\alpha \vee \beta)\| = a, \|\beta^{-1}(\alpha \vee \beta)\| = b\} \quad (16)$$

The three sets above count pairs of non-crossing partitions (or, equivalently, geodesic permutations), according to the r, a, b statistics, having some particular geometric properties in the lattice $NC(n)$. The first one was already introduced in (13) and its formal generating series for irreducible meandric systems, together with their statistics r, a, b , is

$$\begin{aligned} I(X, Y, A, B) &= \sum_{(n,r,a,b) \text{ compatible}} |I_{n,r,a,b}| X^n Y^r A^a B^b \\ &= \sum_{r,a,b \geq 0} e_{r,a,b}(X) \cdot Y^r A^a B^b \quad \text{where} \quad e_{r,a,b}(X) = \sum_{n \geq 1} |I_{n,r,a,b}| X^n \\ &= \sum_{n \geq 1} e_n(Y, A, B) \cdot X^n \quad \text{where} \quad e_n(Y, A, B) = \sum_{r,a,b \geq 0} |I_{n,r,a,b}| Y^r A^a B^b \end{aligned} \quad (17)$$

This series starts as follows

$$I(X, Y, A, B) = X + X^2YA + X^2YB + X^3Y^2A^2 + 6X^3Y^2AB + X^3Y^2B^2 + 2X^4Y^2AB + 2X^4Y^2A^2B^2 + o(Y^2).$$

Note that the coefficients of Y^2 (corresponding to $r = 2$) are associated to the irreducible meandric systems from Figure 15.

In a similar fashion, we introduce a refinement of the set $M_{n,r}$ from (6), to take into account the statistics a, b , and we denote by M its formal generating series

$$\begin{aligned} M(X, Y, A, B) &= \sum_{(n,r,a,b) \text{ compatible}} |M_{n,r,a,b}| X^n Y^r A^a B^b \\ &= \sum_{r,a,b \geq 0} g_{r,a,b}(X) \cdot Y^r A^a B^b \quad \text{where} \quad g_{r,a,b}(X) = \sum_{n \geq 1} |M_{n,r,a,b}| X^n \\ &= \sum_{n \geq 1} g_n(Y, A, B) \cdot X^n \quad \text{where} \quad g_n(Y, A, B) = \sum_{r,a,b \geq 0} |M_{n,r,a,b}| Y^r A^a B^b \end{aligned} \quad (18)$$

where $M_{n,r,a,b}$ is defined in (16). Note that for all $r \geq 0$, the generating function F_r of meandric systems on $2n$ points with $n - r$ loops is given by

$$F_r(X) := \sum_{n \geq 1} M_n^{(n-r)} X^n = [Y^r] M(X, Y, 1, 1).$$

For this reason, our final goal is to collect information on generating function $M(X, Y, A, B)$.

Finally we introduce the formal generating series for $K_{n,r,a,b}$ defined in (15):

$$\begin{aligned} K(X, Y, A, B) &= \sum_{(n,r,a,b) \text{ compatible}} |K_{n,r,a,b}| X^n Y^r A^a B^b \\ &= \sum_{r,a,b \geq 0} f_{r,a,b}(X) \cdot Y^r A^a B^b \quad \text{where} \quad f_{r,a,b}(X) = \sum_{n \geq 1} |K_{n,r,a,b}| X^n \\ &= \sum_{n \geq 1} f_n(Y, A, B) \cdot X^n \quad \text{where} \quad f_n(Y, A, B) = \sum_{r,a,b \geq 0} |K_{n,r,a,b}| Y^r A^a B^b \end{aligned} \quad (19)$$

which is an intermediate definition bridging $I(X, Y, A, B)$ and $M(X, Y, A, B)$.

We prove now the main result of this paper, connecting the two formal power series I and M .

Theorem 5.1. *The formal generating series for I , K and M from (17), (19) and (18), respectively, are related by the following relation:*

$$I \xrightarrow{\mathcal{F}} K \xrightarrow{\mathcal{F}} M \quad (20)$$

or equivalently,

$$\{e_n(Y, A, B)\}_{n \geq 1} \xrightarrow{\mathcal{F}} \{f_n(Y, A, B)\}_{n \geq 1} \xrightarrow{\mathcal{F}} \{g_n(Y, A, B)\}_{n \geq 1} \quad (21)$$

for all $r, a, b \geq 0$. Remember that the transform \mathcal{F} is defined in (5).

Proof. First, we have

$$\begin{aligned}
 g_n(Y, A, B) &= \sum_{\alpha, \beta \in NC(n)} Y^{\|\alpha^{-1}\beta\|} A^{\|\alpha^{-1}(\alpha \vee \beta)\|} B^{\|\beta^{-1}(\alpha \vee \beta)\|} \\
 &= \sum_{\sigma \in NC(n)} \sum_{\substack{\alpha, \beta \in NC(n) \\ \alpha \vee \beta = \sigma}} Y^{\|\alpha^{-1}\beta\|} A^{\|\alpha^{-1}\sigma\|} B^{\|\beta^{-1}\sigma\|} \\
 &= \sum_{\sigma \in NC(n)} \prod_{c \in \sigma} \sum_{\substack{\alpha, \beta \in NC(|c|) \\ \alpha \vee \beta = 1_{|c|}}} Y^{\|\alpha^{-1}\beta\|} A^{\|\alpha^{-1}1_{|c|}\|} B^{\|\beta^{-1}1_{|c|}\|} = \sum_{\sigma \in NC(n)} \prod_{c \in \sigma} f_{|c|}(Y, A, B)
 \end{aligned}$$

where $c \in \sigma$ is a cycle in σ and $|c|$ is its length. The main idea here is that, for fixed σ , the functions $\|\alpha^{-1}\beta\|$, $\|\alpha^{-1}\sigma\|$ and $\|\beta^{-1}\sigma\|$ are multiplicative with respect to the cycles of σ if $\alpha \vee \beta = \sigma$. We have proved that $\mathcal{F} : K \mapsto M$.

Second, we can apply a similar calculation to show $\mathcal{F} : I \mapsto K$, but this time we take the Kreweras complement:

$$\begin{aligned}
 f_n(Y, A, B) &= \sum_{\substack{\alpha, \beta \in NC(n) \\ \alpha \vee \beta = 1_n}} Y^{\|\alpha^{-1}\beta\|} A^{\|\alpha^{-1}1_n\|} B^{\|\beta^{-1}1_n\|} = \sum_{\substack{\alpha, \beta \in NC(n) \\ \alpha \wedge \beta = 0_n}} Y^{\|\alpha^{-1}\beta\|} A^{\|\alpha\|} B^{\|\beta\|} \\
 &= \sum_{\sigma \in NC(n)} \sum_{\substack{\alpha, \beta \in NC(n) \\ \alpha \wedge \beta = 0_n \\ \alpha \vee \beta = \sigma}} Y^{\|\alpha^{-1}\beta\|} A^{\|\alpha\|} B^{\|\beta\|} \\
 &= \sum_{\sigma \in NC(n)} \prod_{c \in \sigma} \sum_{\substack{\alpha, \beta \in NC(|c|) \\ \alpha \wedge \beta = 0_{|c|} \\ \alpha \vee \beta = 1_{|c|}}} Y^{\|\alpha^{-1}\beta\|} A^{\|\alpha\|} B^{\|\beta\|} = \sum_{\sigma \in NC(n)} \prod_{c \in \sigma} e_{|c|}(Y, A, B)
 \end{aligned}$$

This completes the proof. □

Corollary 5.2. *We have*

$$K(X, Y, A, B) = I(X(1 + K(X, Y, A, B)), Y, A, B) \quad (22)$$

$$M(X, Y, A, B) = K(X(1 + M(X, Y, A, B)), Y, A, B) \quad (23)$$

Before starting evaluating the functions $f_{r,a,b}$ and $g_{r,a,b}$ we introduce some notation: for a polynomial $P(Y, A, B)$,

- $[Y^r A^a B^b]P$ is the coefficient of $Y^r A^a B^b$ in $P(Y, A, B)$.
- $D[r, a, b]P = \frac{\partial^{r+a+b}}{\partial Y^r \partial A^a \partial B^b} P(Y, A, B)$
- $D_0[r, a, b]P = \frac{\partial^{r+a+b}}{\partial Y^r \partial A^a \partial B^b} \Big|_{Y=A=B=0} P(Y, A, B)$

This means that

$$D_0[r, a, b]P = r!a!b! \cdot [Y^r A^a B^b]P. \quad (24)$$

Proposition 5.3. *The coefficients of the formal power series $K(X, Y, A, B)$ are as follows:*

- For $r = 0$ we have

$$f_{0,0,0}(X) = \frac{X}{1-X} \quad (25)$$

and $f_{0,a,b}(X) = 0$ for $(a, b) \neq (0, 0)$.

- More generally, when (r, a, b) is compatible, we have

$$f_{r,a,b}(X) = \frac{X^{r+1}Q_{r,a,b}(X)}{(1-X)^{2r+1}}, \quad (26)$$

where $Q_{r,a,b}(X)$ is a polynomial of degree at most $r - \mathbf{1}_{r \geq 1}$, with integer coefficients.

Proof. Notice that $K_{n,0,a,b}$ is empty unless $(a, b) = (0, 0)$, so that $|K_{n,0,0,0}| = 1$ for all $n \geq 1$. Hence

$$f_{0,0,0}(X) = X + X^2 + X^3 + \dots$$

which proves the first statement.

For the induction step, we introduce the order relation $<$ on compatible triples naturally; $(r_1, a_1, b_1) < (r_2, a_2, b_2)$ if $r_1 \leq r_2$, $a_1 \leq a_2$, $b_1 \leq b_2$ and $(r_1, a_1, b_1) \neq (r_2, a_2, b_2)$. We always assume that those triples are non-negative: $(r, a, b) \geq (0, 0, 0)$. Let us assume now the conclusion holds for all triples $(r, a, b) < (r_0, a_0, b_0)$; we only think of the case $r_0 \geq 1$ because of the compatibility condition. We write $f = f_{r_0, a_0, b_0}$ and $K = K(X, Y, A, B)$ below. Since

$$\sum_{n \geq 1} |I_{n,0,0,0}| (X(1+K))^n Y^r A^a B^b = X + XK$$

we have by using Corollary 5.2

$$\begin{aligned} f(X) &= [Y^{r_0} A^{a_0} B^{b_0}] \left((X + XK) + \sum_{(r,a,b) \neq (0,0,0)} \sum_{n \geq 1} |I_{n,r,a,b}| \left((X(1+K))^n Y^r A^a B^b \right) \right) \\ &= Xf(X) + \sum_{\substack{(0,0,0) \\ < (r,a,b) \leq \\ (r_0, a_0, b_0)}} \sum_{n=r+1}^{2r} |I_{n,r,a,b}| X^n [Y^{r_0-r} A^{a_0-a} B^{b_0-b}] (1+K)^n, \end{aligned} \quad (27)$$

where we have used the fact that $I_{n,r,a,b}$ is empty, unless $r+1 \leq n \leq 2r$, see Proposition 4.3. Using the recurrence hypothesis we have

$$\begin{aligned} [Y^{r_0-r} A^{a_0-a} B^{b_0-b}] (1+K)^n &= \sum_{\substack{\sum_{i=1}^n (r_i, a_i, b_i) \\ = (r_0-r, a_0-a, b_0-b)}} \prod_{i=1}^n \frac{X^{s(r_i)} Q_{r_i, a_i, b_i}(X)}{(1-X)^{2r_i+1}} \\ &= \frac{X^{r_0-r}}{(1-X)^{2(r_0-r)+n}} \cdot \hat{Q}_{r,a,b}(X) \end{aligned} \quad (28)$$

because $(r_i, a_i, b_i) < (r_0, a_0, b_0)$ holds by the condition $(r, a, b) \neq (0, 0, 0)$. Here, $s(r) = r + \mathbf{1}_{r \geq 1}$, where the indicator function comes from the fact that $[Y^0 A^0 B^0](1+K) = 1/(1-X)$. Note that

$$\hat{Q}_{r,a,b} = \sum_{\substack{\sum_{i=1}^n (r_i, a_i, b_i) \\ = (r_0-r, a_0-a, b_0-b)}} \prod_{i=1}^n X^{\mathbf{1}_{r_i \geq 1}} \cdot Q_{r_i, a_i, b_i} \quad (29)$$

is a polynomial and moreover

$$\deg(\hat{Q}_{r,a,b}) \leq \sum_{i=1}^n [\mathbf{1}_{r_i \geq 1} + r_i - \mathbf{1}_{r_i \geq 1}] = r_0 - r \quad (30)$$

The compatibility condition may give 0 in (28), but we do not treat such cases separately, because it does not make any difference. Hence, putting (27) and (28) together we have

$$(1 - X)^{2r_0+1} \cdot f(X) = \sum_{\substack{(0,0,0) \\ \langle r,a,b \rangle \leq \\ (r_0,a_0,b_0)}} \sum_{n=r+1}^{2r} |I_{n,r,a,b}| X^n \cdot X^{r_0-r} (1 - X)^{2r-n} \cdot \hat{Q}_{r,a,b} \quad (31)$$

$$= X^{r_0+1} \sum_{\substack{(0,0,0) \\ \langle r,a,b \rangle \leq \\ (r_0,a_0,b_0)}} \sum_{n=r+1}^{2r} |I_{n,r,a,b}| X^{n-r-1} (1 - X)^{2r-n} \cdot \hat{Q}_{r,a,b} \quad (32)$$

To finish the proof, notice that the powers of X and $1 - X$ are both non-negative for $r+1 \leq n \leq 2n$, and moreover

$$\deg \left(X^{n-r-1} (1 - X)^{2r-n} \cdot \hat{Q}_{r,a,b} \right) \leq (n - r - 1) + (2r - n) + (r_0 - r) = r_0 - 1 \quad (33)$$

Note also that the polynomial $\hat{Q}_{r,a,b}$ has integer coefficients, so the same must hold for Q_{r_0,a_0,b_0} . This completes the proof. \square

We prove now a similar result for $M(X, Y, A, B)$. To this end, we need the following two results from classical multivariate analysis: the *generalized Leibniz product rule*:

$$\frac{\partial^n}{\partial x_1 \partial x_2 \cdots \partial x_n} (uv) = \sum_{\emptyset \subseteq S \subseteq [n]} \frac{\partial^{|S|} u}{\prod_{i \in S} \partial x_i} \cdot \frac{\partial^{|S^c|} v}{\prod_{i \in S^c} \partial x_i},$$

and the *generalized Faà di Bruno chain rule*:

$$\frac{\partial^n}{\partial x_1 \partial x_2 \cdots \partial x_n} (u \circ v) = \sum_{\pi \in \Pi_n} u^{(\#\pi)}(v) \cdot \prod_{\lambda \in \pi} \frac{\partial^{|\lambda|} v}{\prod_{i \in \lambda} \partial x_i},$$

where, in the first formula, the sum runs over all subsets S of $[n] = \{1, 2, \dots, n\}$ and, in the second formula, the sum runs over all (possibly crossing) partitions π of $[n]$. As before, we denote by $\#\pi$ the number of blocks of the partition π , and we write $\lambda \in \pi$ for a block λ of π .

In the following statement we show that, after a change of variables, the generating series M has a simple form. The change of variables is motivated by the $Y = A = B = 0$ series: $W \mapsto W/(1+W)^2$ is the functional inverse of the function $g_{0,0,0}$ from (35).

Proposition 5.4. *The coefficients of the formal power series $M(X, Y, A, B)$, after the change of variable $\tilde{M}(W, Y, A, B) = M(W/(1+W)^2, Y, A, B)$, are as follows: by writing $\tilde{g}_{r,a,b}(W) = g_{r,a,b}(W/(1+W)^2)$,*

- For $r = 0$, we have $\tilde{g}_{0,0,0}(W) = W$ and $\tilde{g}_{0,a,b}(W) = 0$ for $(a, b) \neq (0, 0)$.
- For $r \geq 1$ and any a, b such that (r, a, b) is compatible,

$$\tilde{g}_{r,a,b}(W) = \frac{W^{r+1}(1+W)P_{r,a,b}(W)}{(1-W)^{2r-1}}, \quad (34)$$

for some polynomial $P_{r,a,b}$ of degree at most $3r - 3$.

Proof. Notice that the set $M_{n,0,a,b}$ is empty unless $(a, b) = (0, 0)$, so that $|M_{n,0,0,0}| = \text{Cat}_n$ for all $n \geq 1$. This means that $1 + g_{0,0,0}(X)$ is the generating function of Catalan numbers so that

$$g_{0,0,0}(X) = \frac{1 - \sqrt{1 - 4X}}{2X} - 1 \quad (35)$$

By replacing X by $W/(1+W)^2$ we can prove the first statement.

We shall prove the general case by recurrence, as we did in Proposition 5.3. To do so, fix (r_0, a_0, b_0) with $r_0 \geq 1$, and let us assume that the expression (34) holds for all (compatible) triples (r, a, b) with $(r, a, b) < (r_0, a_0, b_0)$. Then, we can write, separating the case $(r, a, b) = (0, 0, 0)$,

$$K(X, Y, A, B) = f_{0,0,0}(X) + \sum_{(r,a,b) \neq (0,0,0)} f_{r,a,b}(X) \cdot Y^r A^a B^b,$$

so that, via Corollary 5.2, we get

$$\tilde{M} = \underbrace{f_{0,0,0} \left(\frac{W}{(1+W)^2} (1 + \tilde{M}) \right)}_{\clubsuit} + \sum_{(r,a,b) \neq (0,0,0)} \underbrace{f_{r,a,b} \left[\frac{W}{(1+W)^2} (1 + \tilde{M}) \right]}_{\spadesuit(r,a,b)} Y^r A^a B^b. \quad (36)$$

Here and below we write $\tilde{M} = \tilde{M}(W, Y, A, B)$. In order to find the coefficient $\tilde{g}_{r_0, a_0, b_0}(W) = [Y^{r_0} A^{a_0} B^{b_0}] \tilde{M}$, we shall take the derivative of (36), and then set $Y = A = B = 0$:

$$D_0[r_0, a_0, b_0] \tilde{M} = (r_0!)(a_0!)(b_0!) \cdot \tilde{g}_{r_0, a_0, b_0}(W) \quad (37)$$

First, let us deal with the first term of the RHS of (36): \clubsuit . Since $f_{0,0,0}(X) = X/(1-X)$ via Proposition 5.3, the p -th derivative is given by

$$\forall p \geq 1, \quad f_{0,0,0}^{(p)}(X) = \frac{p!}{(1-X)^{p+1}}. \quad (38)$$

Using the Faà di Bruno formula, the derivative of \clubsuit reads,

$$D[r_0, a_0, b_0] \clubsuit = \sum_{\pi \in \Pi_{r_0+a_0+b_0}} f_{000}^{(\#\pi)} \left[\frac{W}{(1+W)^2} (1 + \tilde{M}) \right] \cdot \prod_{\lambda \in \pi} D[\lambda_1, \lambda_2, \lambda_3] \frac{W}{(1+W)^2} (1 + \tilde{M}) \quad (39)$$

$$= \sum_{\pi \in \Pi_{r_0+a_0+b_0}} f_{000}^{(\#\pi)} \left[\frac{W}{(1+W)^2} (1 + \tilde{M}) \right] \cdot \left(\frac{W}{(1+W)^2} \right)^{\#\pi} \cdot \prod_{\lambda \in \pi} D[\lambda_1, \lambda_2, \lambda_3] \tilde{M} \quad (40)$$

Here, for a block $\lambda \in \pi$, we let λ_1 (resp. λ_2 and λ_3) be the number of indices $i \in \lambda$ such that $x_i = Y$ (resp. $x_i = A$ and $x_i = B$) in the sense that we regard π as a partition of

$$(x_i) = \underbrace{(Y, \dots, Y)}_{r_0}, \underbrace{(A, \dots, A)}_{a_0}, \underbrace{(B, \dots, B)}_{b_0} \quad (41)$$

To calculate further we use the induction hypothesis. For $(\lambda_1, \lambda_2, \lambda_3) < (r_0, a_0, b_0)$ we have

$$D_0[\lambda_1, \lambda_2, \lambda_3] \tilde{M} = \lambda_1! \lambda_2! \lambda_3! \cdot \frac{W^{\lambda_1+1} (1+W) P_{\lambda_1, \lambda_2, \lambda_3}(W)}{(1-W)^{2\lambda_1-1}} \quad (42)$$

Note that this vanishes based on the first statement of the current proposition (the compatibility condition), but this fact will not be used. Also, the polynomial $P_{\lambda_1, \lambda_2, \lambda_3}$ is of degree at most $3\lambda_1 - 3$. Therefore, for $\pi \neq 1_{r_0+a_0+b_0}$ by the induction hypothesis we have

$$\prod_{\lambda \in \pi} D_0[\lambda_1, \lambda_2, \lambda_3] \tilde{M} = \frac{W^{r_0+\#\pi} (1+W)^{\#\pi}}{(1-W)^{2r_0-\#\pi}} \cdot \hat{P}_\pi(W) \quad (43)$$

If $\#\pi > r_0$ this quantity vanishes because one of the blocks of π must give 0 in (42), but this fact does not make any difference in the current paper. More importantly,

$$\deg(\hat{P}_\pi) \leq 3r_0 - 3\#\pi \quad (44)$$

On the other hand, for $\pi = 1_{r_0+a_0+b_0}$ we have $(r_0!)(a_0!)(b_0!) \cdot \tilde{g}_{r_0, a_0, b_0}(W)$ instead.

Now we are ready to analyze $D_0[r_0, a_0, b_0] \clubsuit$. Since $\tilde{M}(W, 0, 0, 0) = W$ from the first claim, by using (38) we have

$$f_{000}^{(\#\pi)} \left[\frac{W}{(1+W)^2} (1 + \tilde{M}) \right] = (\#\pi)! \cdot (1+W)^{\#\pi+1} \quad (45)$$

Here, we used the following equality:

$$\frac{W}{(1+W)^2} (1 + \tilde{M}(W, 0, 0, 0)) = \frac{W}{1+W} \quad (46)$$

Then,

$$\begin{aligned} D_0[r_0, a_0, b_0] \clubsuit &= (r_0!)(a_0!)(b_0!)W \cdot \tilde{g}_{r_0, a_0, b_0}(W) + \sum_{\substack{\pi \in \Pi_{r_0+a_0+b_0} \\ \pi \neq 1_{r_0+a_0+b_0}}} (\#\pi)! \cdot \frac{W^{r_0+2\#\pi}(1+W)}{(1-W)^{2r_0-\#\pi}} \cdot \hat{P}_\pi(W) \\ &= (r_0!)(a_0!)(b_0!)W \cdot \tilde{g}_{r_0, a_0, b_0}(W) + \frac{W^{r_0+1}(1+W)}{(1-W)^{2_0-2}} \cdot \hat{P}(W) \end{aligned} \quad (47)$$

where $\#\pi \geq 2$ and

$$\hat{P}(W) = \sum_{\substack{\pi \in \Pi_{r_0+a_0+b_0} \\ \pi \neq 1_{r_0+a_0+b_0}}} (\#\pi)! \cdot W^{2\#\pi-1} (1-W)^{\#\pi-2} \cdot \hat{P}_\pi(W) \quad (48)$$

This is a polynomial because the powers in the above formula are all non-negative, and moreover,

$$\deg(\hat{P}) \leq (2\#\pi - 1) + (\#\pi - 2) + (3r_0 - 3\#\pi) = 3r_0 - 3. \quad (49)$$

Let us now focus on the second term in (36): $\spadesuit(r, a, b)$. For fixed (r, a, b) and (r_0, a_0, b_0)

$$\begin{aligned} D_0[r_0, a_0, b_0] \spadesuit(r, a, b) &= \sum_{r'=0}^{r_0} \sum_{a'=0}^{a_0} \sum_{b'=0}^{b_0} c_{r'a'b'} \cdot \left(D_0[r', a', b'] f_{rab} \left[\frac{W}{(1+W)^2} (1 + \tilde{M}) \right] \right) \\ &\quad \cdot \left(D_0[r_0 - r', a_0 - a', b_0 - b'] (Y^r, A^a, B^b) \right) \\ &= c_{r_0-r, a_0-r, b_0-b} \cdot D_0[r_0 - r, a_0 - a, b_0 - b] f_{r,a,b} \left[\frac{W}{(1+W)^2} (1 + \tilde{M}) \right] \end{aligned}$$

where $c_{r',a',b'}$ is some combinatorial factor. Here, we used the generalized Leibniz product rule.

To continue our calculation, let us first compute the derivative (in X) of the function $f_{r,a,b}$ for $r \geq 1$ in (26). For an arbitrary order of derivation $p \geq 1$, we have

$$f_{r,a,b}^{(p)}(X) = \sum_{p=s+t+u} X^{r+1-s} (1-X)^{-2r-1-t} Q_{r,a,b,s,t,u}(X),$$

where $Q_{r,a,b,s,t,u}$ is a polynomial of degree at most $r-1-u$, which also incorporates the combinatorial factors obtained from the derivation of the powers of X and $1-X$. Note that $f_{rab}^{(p)}$ vanishes unless $s \leq r+1$ and $u \leq r-1$.

Then, by using generalized Faà di Bruno chain rule together with (46) and (43) (replacing r_0 by $r_0 - r$) we have, with $l = r_0 + a_0 + b_0 - (r + a + b)$,

$$\begin{aligned}
& D_0[r_0 - r, a_0 - a, b_0 - b] f_{r,a,b} \left[\frac{W}{(1+W)^2} (1 + \tilde{M}) \right] \\
&= \sum_{\pi \in \Pi_l} f_{r,a,b}^{(\#\pi)} \left[\frac{W}{(1+W)^2} (1 + \tilde{M}) \right] \cdot \prod_{\lambda \in \pi} D[\lambda_1, \lambda_2, \lambda_3] \frac{W}{(1+W)^2} (1 + \tilde{M}) \\
&= \sum_{\pi \in \Pi_l} \sum_{\#\pi = s+t+u} \left(\frac{W}{1+W} \right)^{r+1-s} \left(\frac{1}{1+W} \right)^{-2r-1-t} \\
&\quad \cdot Q_{r,a,b,s,t,u} \left(\frac{W}{1+W} \right) \cdot \left(\frac{W}{(1+W)^2} \right)^{\#\pi} \cdot \frac{W^{r_0-r+\#\pi} (1+W)^{\#\pi}}{(1-W)^{2r_0-2r-\#\pi}} \hat{P}_\pi(W) \\
&= \frac{W^{r_0+1} (1+W)}{(1-W)^{2r_0-2}} \cdot \tilde{P}_{r,a,b}(W)
\end{aligned} \tag{50}$$

where

$$\tilde{P}_{r,a,b}(W) = \sum_{\pi \in \Pi_l} \sum_{\#\pi = s+t+u} W^{2\#\pi-s} (1-W)^{\#\pi+2r-2} \cdot R_{r,a,b,s,t,u}(W) \cdot \hat{P}_\pi(W) \tag{51}$$

with

$$R_{r,a,b,s,t,u}(W) = (1+W)^{r-1-u} \cdot Q_{r,a,b,s,t,u} \left(\frac{W}{1+W} \right). \tag{52}$$

Note that $R_{r,a,b,s,t,u}$ is a polynomial of degree at most $r - 1 - u$. Hence $\tilde{P}_{r,a,b}$ is also a polynomial such that

$$\begin{aligned}
\deg(\tilde{P}_{r,a,b}) &\leq (2\#\pi - s) + (\#\pi + 2r - 2) + (3r_0 - 3r - 3\#\pi) + (r - 1 - u) \\
&= 3r_0 - 3 - s - u \leq 3r_0 - 3.
\end{aligned} \tag{53}$$

Therefore

$$\sum_{(r,a,b) \neq (0,0,0)} D_0[r_0, a_0, b_0] \spadesuit(r, a, b) = \frac{W^{r_0+1} (1+W)}{(1-W)^{2r_0-2}} \cdot \tilde{P}(W) \tag{54}$$

with the degree of \tilde{P} at most $3r_0 - 3$.

Putting (37), (47) and (54) together, (36) gives the following:

$$(r_0!)(a_0!)(b_0!) \cdot (1-W) \cdot \tilde{g}_{r_0, a_0, b_0}(W) = \frac{W^{r_0+1} (1+W)}{(1-W)^{2r_0-2}} \cdot [\hat{P}(W) + \tilde{P}(W)] \tag{55}$$

This completes the proof. \square

Remark 5.5. *We believe that the polynomials $P_{r,a,b}$ have integer coefficients, and in fact this is the case for $r \leq 6$ (see Section 6). Perhaps one could try and show this by a more careful analysis of the combinatorial factors appearing in the proof.*

From the previous proposition, our main theorem states as:

Theorem 5.6. *For any fixed $r \geq 1$ there exists a polynomial \tilde{P}_r of degree at most $3r - 3$ such that the generating function of the number of meanders on $2n$ points with $n - r$ loops*

$$F_r(t) = \sum_{n=r+1}^{\infty} M_n^{(n-r)} t^n,$$

with the change of variables $t = w/(1+w)^2$, reads

$$F_r(t) = \sum_{n=r}^{\infty} M_n^{(n-r)} \frac{w^n}{(1+w)^{2n}} = \frac{w^{r+1}(1+w)}{(1-w)^{2r-1}} \tilde{P}_r(w). \quad (56)$$

Proof. The claim follows directly from Proposition 5.4, by setting $A = B = 1$ and writing

$$\tilde{P}_r = \sum_{a,b: (r,a,b) \text{ compatible}} P_{r,a,b}.$$

□

Remark 5.7. One can obtain $M_n^{(n-r)}$, the number of meandric systems on $2n$ points with $n-r$ loops, from the series F_r , after a change of variables and a series expansion. It would be interesting to relate the explicit form of F_r from (56) to the conjecture from [DFGG97a, Equation (2.4)].

As a corollary of the formula (56) for the generating series, we obtain the asymptotic behavior of the meandric numbers $M_n^{(n-r)}$.

Corollary 5.8. For any fixed $r \geq 1$, assuming that $\tilde{P}_r(1) \neq 0$, the number of meandric systems on $2n$ points having $n-r$ loops has the following asymptotic behavior as $n \rightarrow \infty$:

$$M_n^{(n-r)} \sim \frac{\tilde{P}_r(1)}{2^{2r-2} \Gamma((2r-1)/2)} 4^n n^{(2r-3)/2}. \quad (57)$$

Proof. The generating function F_r from (56) is analytic on $\mathbb{C} \setminus [1/4, \infty)$, hence the exponential growth of the meandric numbers is 4^n , see [FS09, Theorem IV.7]. For the more precise statement, we use the transfer results from [FS09, Section VI]. Note that the behavior of $w(x)$ at $x \rightarrow 1/4$ and of $F_r(w)$ at $w \rightarrow 1$ are given respectively by

$$\begin{aligned} w(x) &\sim 1 - 4\sqrt{1/4 - x} \\ F_r(w) &\sim 2\tilde{P}_r(1)(1-w)^{1-2r}. \end{aligned}$$

Hence, in a ‘‘Camambert region’’ with the opening at $x = 1/4$, we have the following equivalent when $x \rightarrow 1/4$

$$F_r(x) \sim \frac{2\tilde{P}_r(1)}{4^{2r-1}} (1/4 - x)^{-(2r-1)/2}.$$

By [FS09, Theorem VI.1], it follows that

$$M_n^{(n-r)} \sim \frac{2\tilde{P}_r(1)}{4^{2r-1}} \frac{4^{(2r-1)/2}}{\Gamma((2r-1)/2)} 4^n n^{(2r-3)/2},$$

which is the announced result. □

We end this section with some comments relating our approach to the meander generating series with previous results. Lando and Zvonkin were the first ones to study irreducible meandric systems [LZ93]. They show that the following formal equality holds:

$$B(x) = N(xB^2(x)), \quad (58)$$

where $B(x)$ is the generating series for the square Catalan numbers $B(x) = \sum_{n=0}^{\infty} \text{Cat}_n^2 x^n$ and N is the generating series for irreducible meandric systems. In our notation, $N(X) = 1 + I(X)$ and $B(X) = 1 + M(X)$. Then, our formula gives theirs by setting $Y = A = B = 1$ where we just split the reduction in two steps. Indeed, the series I, K, M are related by the relations

$$K(X) = I(X(1 + K(X))) \quad (59)$$

$$M(X) = K(X(1 + M(X))). \quad (60)$$

Plugging (59) into (60), we get

$$\begin{aligned} M(X) &= K(X(1 + M(X))) \\ &= I[X(1 + M(X)) \cdot (1 + K(X(1 + M(X))))] \\ &= I[X(1 + M(X))^2], \end{aligned}$$

which is precisely (58). Similarly, the computations in [Nic16, Remark 4.5] can be shown to be equivalent to the special case $Y = A = B = 1$ of Theorem 5.1 in a similar straightforward fashion. Although the derivations in [LZ93, Nic16] seem simpler, since they only require one implicit functional equation to be solved, the quadratic term appearing in (58) makes this equation more complicated to deal with.

6. EXACT FORMULAS FOR SMALL VALUES OF r

We gather in this section the formulas for the generating functions of the numbers of meandric systems on n points with $n - r$ loops, for small values of r ($r \leq 6$). Let us emphasize that our method could be used, in principle, to obtain the generating functions for all (fixed, but arbitrarily large) values of r ; we are limited by the following computational tasks:

- (1) computing the number of irreducible meandric systems of type (r, a, b)
- (2) performing the formal power series inversion in (59), (60).

We have implemented the above computational steps and automated the computation of the generating functions. First, a C program computes all the irreducible meandric systems of size p , storing the results for later use. Then, a symbolic MATHEMATICA function computes automatically the generating function (for given r), using the irreducible meandric system data. All the software is available at [FN16]. We think that our crude computer implementation could be optimized to reach larger values of r .

Finally, let us once more make the observation that our method is not well suited to tackle the (most important) problem of enumerating connected meanders. Connected meanders correspond to taking $r = n - 1$, while in our method r is a fixed parameter which is not allowed to grow with n .

Proposition 6.1. *The polynomials \tilde{P}_r appearing in the generating function (56) for meandric systems on $2n$ points having $n - r$ loops are as follows*

$$\begin{aligned} \tilde{P}_1(w) &= 2 \\ \tilde{P}_2(w) &= 4w^3 - 12w^2 + 4w + 8 \\ \tilde{P}_3(w) &= 18w^6 - 92w^5 + 134w^4 + 8w^3 - 146w^2 + 52w + 42 \\ \tilde{P}_4(w) &= 112w^9 - 770w^8 + 1864w^7 - 1344w^6 - 1656w^5 + 3052w^4 - 520w^3 - 1440w^2 + 520w + 262 \\ \tilde{P}_5(w) &= 820w^{12} - 7052w^{11} + 23264w^{10} - 31788w^9 - 3108w^8 + 60568w^7 - 54912w^6 - 16808w^5 \\ &\quad + 48012w^4 - 11660w^3 - 13664w^2 + 4948w + 1828 \\ \tilde{P}_6(w) &= 6632w^{15} - 68322w^{14} + 283820w^{13} - 558256w^{12} + 311016w^{11} + 798210w^{10} - 1587476w^9 \\ &\quad + 556540w^8 + 1213592w^7 - 1278814w^6 - 76668w^5 + 652408w^4 - 181480w^3 - 129026w^2 \\ &\quad + 46692w + 13820. \end{aligned}$$

Note that all the polynomials above have even integer coefficients. Although we have not proved this fact (see Remark 5.5), the factor 2 appearing in front of each coefficient of \tilde{P} has a simple interpretation: for every $r \geq 1$, for each pair (α, β) contributing to the series M , there is the pair $(\beta, \alpha) \neq (\alpha, \beta)$ which also contributes.

Plugging these values above into Corollary 5.8, we obtain the exact asymptotic behavior of meandric numbers for $r \leq 6$.

Corollary 6.2. *The first 6 series of meandric numbers $M_n^{(n-r)}$ have the following asymptotic behavior as $n \rightarrow \infty$*

$$\begin{aligned} M_n^{(n-1)} &\sim \frac{2}{\sqrt{\pi}} 4^n n^{-1/2} \\ M_n^{(n-2)} &\sim \frac{2}{\sqrt{\pi}} 4^n n^{1/2} \\ M_n^{(n-3)} &\sim \frac{4}{3\sqrt{\pi}} 4^n n^{3/2} \\ M_n^{(n-4)} &\sim \frac{2}{3\sqrt{\pi}} 4^n n^{5/2} \\ M_n^{(n-5)} &\sim \frac{4}{15\sqrt{\pi}} 4^n n^{7/2} \\ M_n^{(n-6)} &\sim \frac{4}{45\sqrt{\pi}} 4^n n^{9/2}. \end{aligned}$$

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MF: YAMAGATA UNIVERSITY, 1-4-12 KOJIRAKAWA, YAMAGATA, 990-8560 JAPAN
E-mail address: `fukuda@sci.kj.yamagata-u.ac.jp`

IN: ZENTRUM MATHEMATIK, M5, TECHNISCHE UNIVERSITÄT MÜNCHEN, BOLTZMANNSTRASSE 3, 85748 GARCHING, GERMANY AND CNRS, LABORATOIRE DE PHYSIQUE THÉORIQUE, IRSAMC, UNIVERSITÉ DE TOULOUSE, UPS, F-31062 TOULOUSE, FRANCE
E-mail address: `nechita@irsamc.ups-tlse.fr`