

Geometric and Information-Theoretic Properties of the Hoggar Lines

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We take a tour of a set of equiangular lines in eight-dimensional Hilbert space. This structure defines an informationally complete measurement, that is, a way to represent all quantum states of three-qubit systems as probability distributions. Investigating the shape of this representation of state space yields a pattern of connections among a remarkable spread of mathematical constructions. In particular, studying the Shannon entropy of probabilistic representations of quantum states leads to an intriguing link between the questions of real and of complex equiangular lines. Furthermore, we will find relations between quantum information theory and mathematical topics like octonionic integers and the 28 bitangents to a quartic curve.

I. INTRODUCTION

A set of *equiangular lines* is a collection of lines such that the angle made by each pair of lines is equal. These arrangements can be defined in real vector space \mathbb{R}^d or in complex vector space \mathbb{C}^d . The outstanding question is what the *maximum* size of such a set can be, as a function of the dimension d . This question is relevant to quantum physics, because the complex case corresponds to a particular type of *quantum measurement* with important properties [1–13]. One moral of our story will be that the real and the complex versions of the equiangular line question can intertwine in unexpected ways.

A symmetric, informationally complete, positive-operator valued measure—a SIC-POVM, or just a SIC—is a set of d^2 vectors $|\psi_j\rangle$ in a d -dimensional complex Hilbert space whose inner products satisfy

$$|\langle\psi_j|\psi_k\rangle|^2 = \frac{d\delta_{jk} + 1}{d + 1}. \quad (1)$$

It is often convenient to work with the rank-1 projection operators defined from these states,

$$\Pi_j = |\psi_j\rangle\langle\psi_j|. \quad (2)$$

When rescaled by the dimension, these operators sum to the identity:

$$\sum_j \frac{1}{d} \Pi_j = I. \quad (3)$$

Therefore, with this scaling, the operators Π_j can serve as the *effects* that comprise a general quantum measurement, or POVM. The index j labels the possible outcomes of an experiment that can, in principle, be carried out. It follows from Eq. (1) that such a measurement is *informationally complete* (IC). Given a probability distribution over the outcomes of an IC measurement, we can compute the probabilities for the outcomes of any other measurement. The *symmetric* IC POVMs make the calculations that interrelate different experiments take on a remarkably simple form [7, 9].

One can prove that no more than d^2 states in a d -dimensional Hilbert space can be equiangular. That is, the largest set of states for which

$$|\langle\psi_j|\psi_k\rangle|^2 = \alpha \quad (4)$$

whenever $j \neq k$ has size d^2 . In turn, for a maximal set the value of α is fixed by the dimension; it must be $1/(d + 1)$. So, a SIC is a maximal equiangular set in \mathbb{C}^d ; the question is whether they can be constructed for *all* values of the dimension. Despite a substantial number of exact solutions, as well as a longer list of high-precision numerical solutions [4, 10, 11], the problem remains open.

The real vector space analogue to Eq. (4) can be expressed in terms of the Euclidean inner product $\langle\cdot, \cdot\rangle$ in \mathbb{R}^n . An equiangular set of unit vectors \hat{v}_j satisfies

$$\langle\hat{v}_j, \hat{v}_k\rangle = \pm\alpha \quad (5)$$

for all $j \neq k$. Again, one can find an upper bound for the possible size of such a set. In Euclidean space \mathbb{R}^d , an equiangular set can contain at most

$$N = \binom{d + 1}{2} \quad (6)$$

lines. However, unlike the bound of d^2 in the complex case, it is known that this bound is *not* reached in all dimensions [14, 15]. For example, in \mathbb{R}^7 , one can construct $\binom{8}{2} = 28$ equiangular lines, but this is also the best that one can do in \mathbb{R}^8 and in \mathbb{R}^9 .

The general plan for this essay is as follows. In Sections II and III, we will see how SICs furnish a probabilistic representation of quantum state space, and we will introduce the particular SICs that will be our main focus of interest. Recent work by Szymusiak and Słomczyński has demonstrated the importance of these SIC solutions for understanding the *informational power* of quantum measurements [16]. We will touch on these calculations, in the context of extremizing the Shannon entropy of probabilistic representations for quantum states. These results connect complex geometry, finite group theory and information theory. Next, in Section IV, we will use the special properties of those SICs to simplify the equations

that indicate the shape of quantum state space. Section V will build on that development, showing how the way SIC vectors embed into Hilbert space leads to combinatorial design theory. Teasing out the structures that arise from an eight-dimensional SIC leads to an unforeseen connection between the real and the complex versions of the equiangular lines problem.

Section VI will study the pairing of two separate eight-dimensional SICs, a pattern of interlocking geometrical relationships that will lead, in Section VII, to another application of combinatorial design theory. The results will translate to probability and information theory in Section VIII, where we will see what they imply for the problem of distinguishing the consequences of different quantum-mechanical hypotheses. Investigating this further, we will arrive in Section IX at another connection between real and complex equiangular lines.

II. SIC REPRESENTATIONS OF QUANTUM STATES

In the textbook way of doing quantum theory, a quantum state for a system is a positive semidefinite operator ρ with unit trace. For a d -dimensional system (a *qudit*), ρ can be written as a $d \times d$ matrix of complex numbers. The set of all valid *density matrices* ρ is a convex set whose extreme points are the rank-1 projection operators. These extreme points are also known as *pure states*; states that are not pure are designated *mixed*.

Given a density matrix ρ and a POVM $\{E_i\}$, we find the probability of outcome i by using the *Born rule*:

$$\text{Prob}(i) = \text{tr}(\rho E_i). \quad (7)$$

If the POVM is informationally complete, we can reconstruct ρ entirely from these probabilities. In the case of a SIC [7], we can say that

$$\rho = \sum_i \left((d+1)p(i) - \frac{1}{d} \right) \Pi_i, \quad (8)$$

where

$$p(i) = \frac{1}{d} \text{tr}(\rho \Pi_i). \quad (9)$$

We will call the probability distribution $p(i)$ the *SIC representation* of the quantum state ρ .

Let us suppose we have a SIC solution for some dimension d . (In the next section, we will examine some examples in detail.) A state is pure if and only if its SIC representation satisfies the following two conditions. First, it must meet the quadratic constraint

$$\sum_j p(j)^2 = \frac{2}{d(d+1)}. \quad (10)$$

Second, it must satisfy the *QBic equation*,

$$\sum_{jkl} C_{jkl} p(j)p(k)p(l) = \frac{d+7}{(d+1)^3}, \quad (11)$$

where we have introduced the *triple products*,

$$C_{jkl} = \text{Re tr}(\Pi_j \Pi_k \Pi_l). \quad (12)$$

If two or more indices are equal, this reduces to

$$\text{tr}(\Pi_j \Pi_k) = \frac{d\delta_{jk} + 1}{d+1}. \quad (13)$$

The set of all valid states is the convex hull of the probability distributions that satisfy Eqs. (10) and (11).

The quadratic constraint (10) has a considerably simpler structure than the QBic equation, so we investigate the former first. One important consequence is an upper bound on the number of entries in $p(i)$ that can equal zero [17]. Normalization implies that

$$1 = \left(\sum_i p(i) \right)^2. \quad (14)$$

Writing n_0 for the number of zero-valued entries in $p(i)$, and applying the Cauchy–Schwarz inequality,

$$\left(\sum_i p(i) \right)^2 \leq (d^2 - n_0) \sum_{\{i|p(i) \neq 0\}} p(i)^2. \quad (15)$$

Consequently,

$$1 \leq (d^2 - n_0) \frac{2}{d(d+1)}. \quad (16)$$

Rearranging this, we find that

$$n_0 \leq \frac{d(d-1)}{2} = \binom{d}{2}. \quad (17)$$

When this bound was first derived, it was conjectured that one could improve upon it [17]. This bound can be reached in dimension 3. Note that when $d = 3$, the binomial coefficient $\binom{d}{2}$ reduces to d . It was conjectured that the true bound would turn out to be d in general. However, this is not the case [18]. In this paper, we will find examples in $d = 8$ where the number of zeros is $\binom{8}{2} = 28$. Therefore, the bound deduced from the Cauchy–Schwarz inequality is the best one possible in general.

Since we have probability distributions, we can compute Shannon entropies. Of particular interest are the pure states which extremize the Shannon entropy of their SIC representations. It turns out (and the proof is not too long) that the pure states which *maximize* the Shannon entropy of their SIC representations are the SIC projectors $\{\Pi_i\}$ themselves.¹

¹ I first learned of this from unpublished notes by Huangjun Zhu, written in 2013.

What about *minimizing* the Shannon entropy? Imagine a probability distribution, not necessarily one corresponding to a quantum state. Under the constraint that $\sum_i p(i)^2$ is fixed,

$$\sum_i p(i)^2 = \frac{1}{N}, \quad (18)$$

then it can be shown [16] that the distributions of minimum entropy take the form

$$\left(\frac{1}{N}, \dots, \frac{1}{N}, 0, \dots, 0\right). \quad (19)$$

Exactly N entries are nonzero, and the others all vanish. If we take

$$N = \frac{d(d+1)}{2} = \binom{d+1}{2}, \quad (20)$$

then we see that a probability distribution with N non-vanishing, uniform entries is a pure state that minimizes the Shannon entropy—*provided* that it corresponds to a valid pure state. In other words, the minimizers we seek are those permutations of Eq. (19) that satisfy the QBic equation.

III. CONSTRUCTING SICs USING GROUPS

All known SICs have an additional kind of symmetry, above and beyond their definition: They are *group covariant*. Each SIC can be constructed by starting with a single vector, known as a *fiducial* vector, and acting upon it with the elements of some group. It is not known whether or not a SIC must be group covariant. Possibly, because group covariance simplifies the search procedure [4, 11], the fact that we only know of group-covariant SICs is merely an artifact. (However, we do have a proof that all SICs in $d = 3$ are group covariant [19].)

In all cases but one, namely the Hoggar SIC we will define below, the group that generates a SIC from a fiducial is an instance of a *Weyl–Heisenberg group*. We can define this group as follows. First, fix a value of d , and let $\omega_d = e^{2\pi i/d}$. Then, construct the shift and phase operators

$$X|j\rangle = |j+1\rangle, \quad Z|j\rangle = \omega_d^j |j\rangle, \quad (21)$$

where the shift is modulo d . The elements of the Weyl–Heisenberg group in dimension d are products of powers of X and Z , together with phase factors that depend on the dimension. For many purposes, those phase factors can be neglected.

In $d = 2$ —that is, for a system comprising a single qubit—a SIC is simply a tetrahedron, inscribed in the Bloch sphere [2]. (This configuration was described by Feynman, in a 1987 festschrift for Bohm [20].) Let r

and s be signs, and let σ_x , σ_y and σ_z denote the Pauli matrices. Then, the four pure states

$$\Pi_{r,s} = \frac{1}{2} \left(I + \frac{1}{\sqrt{3}} (r\sigma_x + s\sigma_y + rs\sigma_z) \right) \quad (22)$$

define a tetrahedron. Each point (x, y, z) lying within the unit ball (*Bloch ball*) defines a valid quantum state. The SIC representation of this state is the probability vector whose components are

$$p(r, s) = \frac{1}{4} + \frac{\sqrt{3}}{12} (sx + ry + srz). \quad (23)$$

Given the tetrahedron (22), we can define another, related to the first by inversion. Together, the two tetrahedra form a stellated octahedron, inscribed in the Bloch sphere. The SIC representations of the vertices of the second tetrahedron are the vector

$$\left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \quad (24)$$

and its permutations.

In what follows, we will make substantial use of two SICs. One of them is the *Hesse SIC* in $d = 3$, which we construct by applying the Weyl–Heisenberg group to the fiducial

$$|\psi_0^{(\text{Hesse})}\rangle = \frac{1}{\sqrt{2}}(0, 1, -1)^T. \quad (25)$$

The other lives in $d = 8$ and is designated the *Hoggar SIC*. (The construction was first devised by Hoggar [21, 22] by starting with 64 nonequiangular diagonals through the vertices of a quaternionic polytope, which become 64 equiangular lines when converted to complex space. Hoggar’s result was among the first discoveries of a maximal set of complex equiangular lines [23, pp. 731–33].) Actually, we have multiple choices of fiducial in this case, yielding distinct sets of $d^2 = 64$ states. However, all of these sets have the same symmetry group, and they are equivalent to one another up to unitary or antiunitary transformations. For brevity, then, we will refer to “the” Hoggar SIC [24].

A fiducial for the Hoggar SIC [25] can be written as follows:

$$|\psi_0\rangle \propto (-1 + 2i, 1, 1, 1, 1, 1, 1, 1)^T. \quad (26)$$

Upon this, we act with the elements of the group that is the tensor product of three copies of the $d = 2$ Weyl–Heisenberg group:

$$k = (k_0, k_1, \dots, k_5), \quad D_k = X^{k_0} Z^{k_1} \otimes X^{k_2} Z^{k_3} \otimes X^{k_4} Z^{k_5}. \quad (27)$$

Given a tetrahedral SIC, we can define a SIC representation of state space. Minimizing the Shannon entropy over pure states, as we discussed earlier, yields the four states of the counterpart tetrahedron. Performing the same procedure with the Hesse SIC, we find that the pure

states that minimize the Shannon entropy are twelve in number. They form a complete set of *Mutually Unbiased Bases* [18]. It is natural to ask what happens similarly for the Hoggar SIC; we will investigate this in a later section.

IV. SIMPLIFYING THE QBIC EQUATION

A quantum system for which $d = 3$ is known as a *qutrit*. In $d = 3$, we can simplify the QBic equation (11) considerably, using the Hesse SIC [18, 26, 27]. In the Hesse SIC representation of qutrit state space, the QBic equation can be reduced to

$$\sum_i p(i)^3 - 3 \sum_{(ijk) \in S} p(i)p(j)p(k) = 0, \quad (28)$$

where list S is a set of index triples (ijk) which can be constructed as the lines in a discrete affine plane of nine points [18, 26, 27]. This fact is quite handy when working with qutrit states, and it is a consequence of the triple products of the Hesse SIC states taking a simple form. In turn, the structure of the triple products simplifies because the Hesse SIC has the property that its symmetry group acts *doubly transitively*. This is a kind of symmetry beyond the definition of a SIC and beyond group covariance: Using unitary operators that map the Hesse SIC to itself, we can send any pair of states in the Hesse SIC to any other.

Zhu has proved [28] that the only SICs whose symmetry groups act doubly transitively are the tetrahedral SICs in $d = 2$, the Hesse SIC in $d = 3$ and the Hoggar SIC in $d = 8$. In $d = 2$, the QBic equation simplifies so far that it becomes redundant, and the quadratic constraint is sufficient to define the state space. As we have seen, the QBic equation also simplifies for the Hesse SIC, in a way that brings discrete geometry into the picture. It is reasonable to guess that something similar will happen in dimension $d = 8$.

When in dimension $d = 8$, using the Hoggar SIC, the number of distinct values the C_{jkl} take in this case is quite small: When the three indices are different, they can only be 0 or $\pm 1/27$.

Let S_+ denote the set of index triples (jkl) for which $C_{jkl} = 1/27$, and likewise, let S_- denote that set for which $C_{jkl} = -1/27$. We cull duplicates from these lists, so that, for example, if (jkl) belongs in S_+ , we do not also include its permutations (kjl) , (lkj) and so on. The sizes of these sets are

$$|S_+| = 16128 = 2^8 3^2 7, \quad |S_-| = 4032 = 2^6 3^2 7. \quad (29)$$

Simplifying the QBic equation (11) for the special case of the Hoggar SIC proceeds by fairly straightforward algebra. The only bit of moderate cleverness required is a

rearrangement by means of normalization:

$$\begin{aligned} \sum_j p(j)^2 \sum_{l \neq j} p(l) &= \sum_j p(j)^2 [1 - p(j)] \\ &= \sum_j p(j)^2 - \sum_j p(j)^3. \end{aligned} \quad (30)$$

The result of these manipulations is that a pure state must satisfy

$$\sum_j p(j)^3 + \frac{1}{3} \left[\sum_{S_+} p(j)p(k)p(l) - \sum_{S_-} p(j)p(k)p(l) \right] = \frac{11}{648}. \quad (31)$$

The remaining challenge is to characterize the sets S_+ and S_- .

V. TRIPLE PRODUCTS AND COMBINATORIAL DESIGNS

Group covariance tells us that any C_{jkl} can be written as C_{0mn} for some m and n . This implies a d^2 -fold degeneracy among the triple products. In our case, we know that the sizes of S_+ and S_- must be multiples of 64. And, in fact,

$$|S_+| = 64 \cdot 4 \cdot 3^2 7, \quad |S_-| = 64 \cdot 3^2 7. \quad (32)$$

In forming the triple product C_{0mn} , we have

$$\binom{63}{2} = \frac{63 \cdot 62}{2} = 1953 \quad (33)$$

ways of choosing the subscripts m and n . We find that

$$\frac{|S_-|}{64} = \frac{1}{31} \binom{63}{2}, \quad \frac{|S_+|}{64} = \frac{4}{31} \binom{63}{2}. \quad (34)$$

It is now time to go into the group theory of SIC structures in more detail. We define the *multipartite Weyl–Heisenberg group* in a prime-power dimension p^n to be the tensor product of n copies of the Weyl–Heisenberg group in dimension p . The *Clifford group* in dimension p^n is the group of unitaries that stabilize the multipartite Weyl–Heisenberg group. The order of the Clifford group [28] is

$$p^{n^2+2n} \prod_{j=1}^n (p^{2j} - 1). \quad (35)$$

Therefore, in dimension $8 = 2^3$, the Clifford group has order

$$\begin{aligned} 2^{3^2+2 \cdot 3} \prod_{j=1}^3 (2^{2j} - 1) &= 2^{15} \cdot 3 \cdot 15 \cdot 63 = 2^{15} \cdot 3^4 \cdot 5 \cdot 7 \\ &= 2^9 \cdot 3^2 \cdot (|S_+| + |S_-|). \end{aligned} \quad (36)$$

The symmetry group of the Hoggar SIC is a subgroup of the Clifford group with order

$$64 \cdot 6048 = 2^{11} \cdot 3^3 \cdot 7 = 24|S_+| = 96|S_-|. \quad (37)$$

The factor of $64 = 2^6$ comes from the triple-qubit Pauli group. Take any vector from the Hoggar SIC, and consider those unitaries in the symmetry group that leave that vector fixed while permuting the others. These form the stabilizer subgroup of that vector. For any vector in the Hoggar SIC, the stabilizer subgroup is isomorphic to the projective special unitary group $PSU(3,3)$, which has 6048 elements. This explains the other factor in Eq. (37).²

Let N_k^+ be the number of triples in the set S_+ that contain the value k , and likewise for N_k^- and S_- . One finds that

$$N_k^- = 189, N_k^+ = 756 \quad \forall k. \quad (38)$$

These values factorize as

$$N_k^- = 3^3 \cdot 7, N_k^+ = 2^2 \cdot 3^3 \cdot 7. \quad (39)$$

Furthermore, if we let N_{kl}^\pm denote the number of triples in S_+ (respectively, S_-) that contain the pair (k,l) , we obtain

$$N_{kl}^- = 6, N_{kl}^+ = 24, \quad \forall k, l. \quad (40)$$

This leads us into *combinatorial design theory*. A *balanced incomplete block design* (BIBD) is a collection of v points and b blocks, such that there are k points within each block, and r blocks contain any given point. Consistency requires that

$$bk = vr. \quad (41)$$

The final parameter, λ , specifies the number of blocks containing any two specific points. This constant must satisfy

$$\lambda(v-1) = r(k-1). \quad (42)$$

In a *symmetric design*, $b = v$, and so $r = k$. Any two blocks meet in the same number of points, and that number is λ . Ryser's theorem [33] establishes that this is an if-and-only-if relationship.

² If one constructs the Hoggar SIC as Zhu does, then its fiducial vector's stabilizer group is generated by his unitary operators U_7 and U_{12} . Construct the new unitaries $U_a = U_{12}U_7$ and $U_b = U_{12}^2$. These satisfy the relations for the generators of $PSU(3,3)$ as presented in the *Atlas of Finite Group Representations* [29]. Also, the conjugacy classes in Zhu's Table 10.1 can be matched with those for $PSU(3,3)$ computed, for example, using the GAP software [30]. The group $PSU(3,3)$, as well as the stabilizer groups for the other doubly-transitive SICs, can all be constructed from the *octavian integers* [13, 31]. As Baez notes, "Often you can classify some sort of gizmo, and you get a beautiful systematic list, but also some number of exceptions. Nine times out of 10 those exceptions are related to the octonions" [32].

The set S_- contains 4032 "blocks," where each block is made of three points drawn from a set of 64 possibilities. We found earlier that each point occurs in 189 different blocks, and that each pair of points occurs in 6 different blocks. Therefore, S_- is a BIBD with

$$v = 64, b_- = |S_-| = 4032, k = 3, r_- = 189, \lambda_- = 6. \quad (43)$$

Likewise, S_+ is a BIBD with

$$v = 64, b_+ = |S_+| = 16128, k = 3, r_+ = 756, \lambda_+ = 24. \quad (44)$$

Referring back to Eq. (37), we have that

$$b_\pm = |S_\pm| = \frac{6048v}{4\lambda_\mp} = \frac{6048v}{4} \frac{\lambda_\pm}{\lambda_- \lambda_+} = \frac{6048v}{576} \lambda_\pm = 672\lambda_\pm. \quad (45)$$

Zhu proves that the Hoggar SIC is "doubly transitive," *i.e.*, for any distinct pair of vectors, there is a symmetry operation that takes it to any other distinct pair [28]. This has implications for the structure coefficient matrices C_i , defined by

$$(C_i)_{jk} = C_{ijk}. \quad (46)$$

Group covariance means that

$$C_{ijk} = C_{0j'k'} \quad (47)$$

for some j' and k' . So, the entries in all the matrices $\{C_i\}$ are elements of the matrix C_0 . The additional requirement that the action of the symmetry group is doubly transitive means that if we want to understand the triple products C_{ijk} , we only need to look at C_{01k} , because any triple of distinct indices (ijk) can be mapped to some $(01k')$, leaving the triple product invariant.

We expect to see some values occur in sets of six, or multiples of six. Why? Because the triple product function is completely symmetric:

$$C_{ijk} = C_{jki} = C_{kij} = C_{jik} = C_{kji} = C_{ikj}. \quad (48)$$

By applying unitaries in the symmetry group, we can turn the first pair of indices into ij across the board:

$$C_{ijk} = C_{ij\sigma_1(i)} = C_{ij\sigma_2(j)} = C_{ij\sigma_3(k)} = C_{ij\sigma_4(i)} = C_{ij\sigma_5(j)}. \quad (49)$$

Here, the $\{\sigma_1, \dots, \sigma_5\}$ are permutations of the set of indices $\{0, \dots, 63\}$. They are defined by relations of the form

$$\sigma_1(j) = i, \sigma_1(k) = j. \quad (50)$$

Unless these permutations happen to align in such a way that, for example, $\sigma_4(i) = \sigma_5(j)$, we will have six elements in the j^{th} row of the matrix C_i , all equal.

Explicit computation bears this idea out. We need the values of C_{01k} , where the subscripts "0" and "1" refer to the first and second projectors in the ordering defined by Eq. (27). Note that two entries will be the trivial value,

$1/(d+1)$. We can display the results by arranging them in a $4 \times 4 \times 4$ cube. Define the sequence

$$\sigma = \{I, \sigma_z, \sigma_x, \sigma_x \sigma_z\}. \quad (51)$$

Then, interpreting the index k as an ordered tuple (k_0, k_1, \dots, k_5) , we have

$$C_{01k} = \text{Re tr}(\Pi_0 \Pi_1 D_k \Pi_0 D_k^\dagger), \quad (52)$$

where

$$D_k = \sigma_{k_1+2k_0} \otimes \sigma_{k_3+2k_2} \otimes \sigma_{k_5+2k_4}. \quad (53)$$

We can therefore display C_{01k} for all k in a three-dimensional cube, which is portrayed in Figure 1.

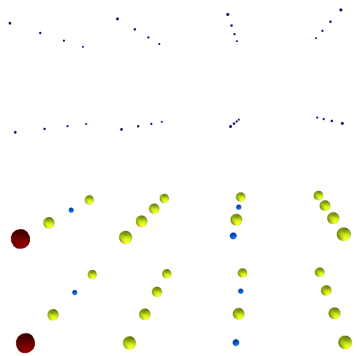


FIG. 1: Visual representation of C_{01k} for the Hoggar SIC. Small dots indicate $C_{01k} = 0$. Large spheres (red) indicate the trivial value, $C_{01k} = 1/9$. Intermediate spheres (yellow) indicate $C_{01k} = 1/27$, and the six slightly smaller spheres (blue) stand for $C_{01k} = -1/27$.

The pairing of values follows from the facts that $C_{01k} = C_{10k}$ by symmetry and

$$D_1^2 = (I \otimes I \otimes Z)^2 = I \otimes I \otimes I. \quad (54)$$

This makes the triple product insensitive to a Z factor on one qubit. However, if the displacement operator includes a factor of X on that qubit, then the triple product C_{01k} vanishes. Inspection reveals that among the non-vanishing values, $C_{01k} = -1/27$ when the displacement operator D_k includes only factors of X , apart from the third qubit, which is insensitive to Z .

Define the complex triple products

$$T_{jkl} = \langle \psi_j | \psi_k \rangle \langle \psi_k | \psi_l \rangle \langle \psi_l | \psi_j \rangle = \text{tr}(\Pi_j \Pi_k \Pi_l). \quad (55)$$

Up until now, we have taken the real part of this quantity. We can instead scale by the magnitude to obtain a phase [5]:

$$\tilde{T}_{jkl} = \frac{T_{jkl}}{|T_{jkl}|} = e^{i\theta_{jkl}}. \quad (56)$$

It follows from the definition of T_{jkl} that, in general,

$$e^{i\theta_{mjk}} e^{i\theta_{mkl}} e^{i\theta_{mlj}} = e^{i\theta_{jkl}}. \quad (57)$$

For the Hoggar SIC, θ_{jkl} takes the values $0, \pi$ and $\pm\pi/2$.

Note that the definition of a SIC implies that

$$\langle \psi_j | \psi_k \rangle = \frac{1}{\sqrt{d+1}} e^{i\theta_{jk}} \quad (58)$$

for some angles θ_{jk} . This two-index object is related to the three-index object θ_{jkl} by

$$e^{i\theta_{jkl}} = e^{i\theta_{jk}} e^{i\theta_{kl}} e^{i\theta_{lj}}. \quad (59)$$

With this relation, we can understand more about the triple products C_{jkl} using the following sneaky trick. The operators X and Z are Hermitian, but XZ is not. We can fix this by defining

$$Y = iXZ, \quad (60)$$

which is a Hermitian operator (and equal to the familiar Pauli matrix σ_y). A tensor-product operator like $X \otimes Z \otimes XZ$ will not be Hermitian, but $X \otimes Z \otimes Y$ will be. So, by introducing appropriate phase factors, we can fix up the Weyl–Heisenberg displacement operators D_k so that they are Hermitian matrices. The phase with which we modify D_k includes a factor of i for every instance of Y in the tensor product:

$$\hat{D}_k = (-e^{i\pi/d})^{\#(Y)} D_k. \quad (61)$$

These operators serve just as well for generating a SIC.

But notice: Our displacement operators are now Hermitian matrices, that is, *quantum observables*, and their expectation values are real. Consequently, for any \hat{D}_k ,

$$\langle \psi_0 | \hat{D}_k | \psi_0 \rangle \in \mathbb{R}. \quad (62)$$

In turn, this implies that

$$e^{i\theta_{0k}} = \pm 1. \quad (63)$$

Denote by S_0 the set of all triples (jkl) for which C_{jkl} vanishes. For these triples, it must be the case that T_{jkl} is pure imaginary. Let us focus on the case $j = 0$, with $k \neq 0$ and $l \neq 0$. Here, the only place a factor of i can enter is the middle:

$$e^{i\theta_{0kl}} = e^{i\theta_{0k}} e^{i\theta_{kl}} e^{i\theta_{l0}}. \quad (64)$$

The middle factor is the phase of the inner product

$$\langle \psi_k | \psi_l \rangle = \langle \psi_0 | \hat{D}_k^\dagger \hat{D}_l | \psi_0 \rangle. \quad (65)$$

This can yield an imaginary part for some values of k and l , thanks to the phase factors we introduced to obtain Hermiticity. Write $\{\cdot, \cdot\}$ for the symplectic form

$$\{a, b\} = a_1 b_0 - b_1 a_0. \quad (66)$$

Then the phase we obtain is

$$(-i)^{\{(k_0, k_1), (l_0, l_1)\} + \{(k_2, k_3), (l_2, l_3)\} + \{(k_4, k_5), (l_4, l_5)\}}. \quad (67)$$

If we fix the index k , say to

$$(k_0, k_1, k_2, k_3, k_4, k_5) = (0, 0, 0, 0, 0, 1), \quad (68)$$

then the phase contribution will be an imaginary number for exactly 32 of the 64 possible choices of the index l . These are the values for which $C_{0ll} = 0$.

If we define the matrix

$$\Omega = I_{3 \times 3} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (69)$$

then we can write the exponent in Eq. (67) as

$$f(k, l) = k \Omega l^T, \quad (70)$$

where we are interpreting k and l as row vectors of six elements each. The matrix Ω is invertible and antisymmetric, so $f(k, l)$ is a *symplectic bilinear form*.

Let us consider again the three-index angle tensor θ_{jkl} . We know that

$$e^{i\theta_{jkl}} = \pm i, \text{ for } (jkl) \in S_0. \quad (71)$$

If (mjk) , (mkl) and (mlj) are three triples in S_0 , then

$$e^{i\theta_{jkl}} = \pm i. \quad (72)$$

That is, (jkl) must then be a member of S_0 , too. On the other hand, if (mjk) , (mkl) and (mlj) are all *outside* of S_0 , then $e^{i\theta_{jkl}}$ is the product of three real numbers, and so it must be real itself. Therefore, if (mjk) , (mkl) and (mlj) are in the complement of S_0 , then so is (jkl) .

This means that S_0 qualifies as a *two-graph*. Much studied in discrete mathematics, a two-graph can be defined [34] as a set T of triples such that

$$(pqr), (pqs), (prs) \in T \Rightarrow (qrs) \in T, \quad (73)$$

and likewise for the complement of T .

One application of two-graphs is generating sets of equiangular lines in *real* vector spaces. Pick a point in \mathbb{R}^d , and draw a set of lines through it, such that any two meet at an angle whose cosine is $\pm\alpha$ (with $\alpha \neq 0$). For some triples of those intersecting lines, the product of the cosines will be negative, and for others, it will be positive. The triples for which the product is negative constitute a two-graph. Going in the other direction, any two-graph can be formulated in this way.

Notice what has happened here: We started with a set of *complex* equiangular lines, the Hoggar SIC, and in considering the additional symmetries that set enjoys above and beyond its definition, we have arrived at *real* equiangular lines.

This will happen again.

Two-graphs have been taxonomied to an extent, with the aid of the classification theorem for finite simple groups. Those two-graphs with doubly transitive automorphism groups were classified by Taylor [35]. Our set S_0 is Taylor's example B.xi, the two-graph on 64 vertices whose automorphism group contains $PSU(3, 3)$.

Knowing the automorphism group of this two-graph, we can find the stabilizer of any pair of vertices. This will be the subgroup whose action leaves that pair fixed. For example, automorphisms in the stabilizer subgroup of the pair $(0, 1)$ will send the triple $(01k)$ to the triple $(01k')$. Taylor [35] observes that the stabilizer of two points in a triple has orbits of length 6, 24 and 32 on the remaining points. Combining this with Zhu's observation [24] that two triples in the Hoggar SIC can be mapped to each other by a symmetry operation if and only if they have the same triple product, and we see a combinatorial origin of the patterns we observed in Figure 1.

Given a two-graph T , one can construct a regular graph G that embodies its structure in the following manner [35]. Copy over the list of vertices from T to G . Then, select a vertex v of the two-graph T , and draw the edges of G so that u and w are neighbors whenever $(uvw) \in T$. Let A be the *Seidel adjacency matrix* of the graph G . This matrix is constructed so that $A_{uw} = -1$ if u and w are adjacent, $A_{uw} = 1$ if they are not, and $A_{uu} = 0$ on the diagonal. Suppose that the smallest eigenvalue of A is λ , and this eigenvalue occurs with multiplicity m . Then, $M = I - (1/\lambda)A$ is a symmetric, positive definite matrix, and the rank of M will be the number $|A|$ of vertices in the graph minus the multiplicity m . Consequently, M can be taken as the Gram matrix for a set of vectors

$$\{v_1, v_2, \dots, v_{|A|}\}, \quad (74)$$

with each vector living in $\mathbb{R}^{|A|-m}$.

In our case, the matrix A has only two eigenvalues: 7, with multiplicity 36; and -9 , with multiplicity 28. This means that M is the Gram matrix for a set of *equiangular lines* (as it should be, since we derived G from a two-graph).

From the triple-product structure of the Hoggar SIC, we have arrived at a set of 64 equiangular lines in \mathbb{R}^{36} .

The numbers 28 and 36 will recur in the next developments.

VI. THE TWIN OF THE HOGGAR SIC

Now, we investigate the eight-dimensional analogue of what happens when we minimize the Shannon entropy for qubit pure states.

The "twin Hoggar SIC" can be constructed by applying the triple-Pauli displacement operators to the fiducial vector

$$|\tilde{\psi}_0\rangle \propto (-1 - 2i, 1, 1, 1, 1, 1, 1, 1)^T. \quad (75)$$

This is related to our original fiducial vector, Eq. (26), by complex conjugation.

In the SIC representation defined by the original Hoggar lines, the vectors comprising the "twin Hoggar SIC" have $(8 - 1)8/2 = 28$ elements equal to zero, and the other $(8 + 1)8/2 = 36$ elements equal to $1/36$ [16]. Consequently, the Hoggar lines provide a counterexample to

```

1110111011100001111011101110000111101110111000010001000100011110
110111011101001011011101110100101101110111010010001000100101101
1011101110110100101110111011010010111011101101000100010001001011
0111011101110000111011101110000111011101110001000100010000111

```

TABLE I: Four SIC representations of states from the twin Hoggar lines, written as bit sequences.

the conjecture that the best upper bound on the number of zero-valued entries in dimension d is just d . The bound $d(d-1)/2$ deduced from the Cauchy-Schwarz inequality [17] is, actually, tight. Furthermore, the states of the twin Hoggar SIC minimize the Shannon entropy of their SIC representations, as we discussed above. One can, in fact, find the the twin Hoggar SIC-set by testing all the states of the form (19) to see which ones satisfy the QBic equation.

For any vector p in the twin Hoggar SIC set,

$$\sum_j p(j)^3 = 36 \left(\frac{1}{36} \right)^2 = \frac{1}{1296}. \quad (76)$$

Eq. (31) then becomes

$$\frac{1}{1296} + \frac{1}{3} \left[\sum_{S_+} p(j)p(k)p(l) - \sum_{S_-} p(j)p(k)p(l) \right] = \frac{11}{648}. \quad (77)$$

The bracketed sum must therefore equal

$$\left[\sum_{S_+} p(j)p(k)p(l) - \sum_{S_-} p(j)p(k)p(l) \right] = \frac{7}{144}. \quad (78)$$

Furthermore, any product $p(j)p(k)p(l)$ that does not evaluate to zero must equal

$$p(j)p(k)p(l) = \left(\frac{1}{36} \right)^3 = \frac{1}{46,656}. \quad (79)$$

From this, we can calculate the net number of contributions that the sums over S_+ and S_- must make, if the state is to be valid:

$$\frac{\frac{7}{144}}{\frac{1}{46,656}} = 2,268 = 2^2 3^{47} = \frac{3(|S_+| - |S_-|)}{2^4}. \quad (80)$$

If π_i and π_j are two projectors in the twin set, then

$$\text{tr}(\pi_i \pi_j) = d(d+1) \sum_k p_i(k)p_j(k) - 1 = \frac{1}{d+1}. \quad (81)$$

Therefore,

$$\sum_k p_i(k)p_j(k) = \frac{d+2}{d(d+1)^2} = \frac{5}{324}. \quad (82)$$

Now, each element in p_i is either 0 or $1/36$, and likewise for p_j . Let n denote the number of overlapping nonzero entries in these two vectors. We know that

$$n \left(\frac{1}{36} \right)^2 = \frac{5}{324}, \quad (83)$$

and so

$$n = 20. \quad (84)$$

This result will be important for understanding the twin Hoggar SIC using combinatorial design theory.

VII. COMBINATORIAL DESIGNS FROM THE TWIN HOGGAR SIC

We have a set of $d^2 = 64$ “blocks,” each one of which essentially is a binary string of length 64. And each block contains exactly 36 of the nonzero entries that a length-64 block could in principle contain. We can think of this as there being 64 “points,” and each block contains 36 of them. Table I gives examples of four such blocks.

If we fix $v = b = 64$ and $k = 36$, then

$$\lambda \cdot 63 = 36 \cdot 35 \Rightarrow \lambda = 20. \quad (85)$$

This is just what we found before when we calculated the number of overlapping 1s in any pair of vectors in the twin Hoggar set. Therefore, the twin Hoggar SIC defines a symmetric design. Specifically, it is a “2-(64,36,20) design.”

If we apply a NOT to each of our bit-strings, then we arrive at a new design. Generally, the *complement* of a design is found by replacing each block with its complement: The points that were included in a block are now excluded, and vice versa. The new design has parameters

$$v' = v, b' = b, k' = v - k, r' = b - r, \lambda' = \lambda + b - 2r. \quad (86)$$

The complement to our Hoggar design therefore satisfies

$$v' = b' = 64, k' = r' = 28, \lambda' = 12. \quad (87)$$

Therefore, we can designate it a “2-(64,28,12) design.”

The existence of a symmetric design with parameters

$$(v, k, \lambda) = (4u^2, 2u^2 - u, u^2 - u) \quad (88)$$

is known to be equivalent to the existence of a regular Hadamard matrix possessing dimensions $4u \times 4u$. Setting $u = 4$, we find that the complement of the Hoggar design meets the Hadamard criterion. The incidence matrix of the design can be transformed into a regular Hadamard matrix by simple substitutions.

The complement of the Hoggar design is equivalent to an *orthogonality graph* for the Hoggar SIC and its twin. In an orthogonality graph, vertices stand for states, and

vertices are linked by an edge if the corresponding states are orthogonal. If a point V_i lies within block B_j , then the i^{th} vector in the Hoggar SIC is orthogonal to the j^{th} vector in the twin SIC. This can be visualized as a bipartite graph containing two sets of 64 vertices apiece, where each vertex in the first set is linked to 28 vertices in the second set.

We can generate the Hoggar design in another way by the following procedure. Start with this Hadamard matrix:

$$H_2 = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}. \quad (89)$$

Construct the tensor product of three copies of H_2 :

$$H_6 = H_2 \otimes H_2 \otimes H_2. \quad (90)$$

Then, use this to create an incidence matrix by replacing all the entries that equal -1 with 0:

$$M = \frac{H_6 + 1}{2}. \quad (91)$$

The resulting 64×64 array is the incidence matrix of the Hoggar design, containing all the same rows as the (appropriately renormalized) SIC representations of the twin set. This ties us firmly into the literature on combinatorial designs: The Hoggar design is a *symplectic design on 64 points*.³ A symplectic design [36–38], denoted $\mathcal{S}^\epsilon(2m)$ with m a positive integer and $\epsilon = \pm 1$, is a BIBD with

$$b = v = 2^{2m}, \quad k = 2^{2m-1} + \epsilon 2^{m-1}, \quad \lambda = 2^{2m-2} + \epsilon 2^{m-1}. \quad (92)$$

The object that we found by way of SIC-POVMs is exactly $\mathcal{S}^1(2m)$ for $m = 3$. Symplectic designs for larger m can be constructed by taking the tensor product of m copies of the Hadamard matrix H_2 .

That is how to construct the symplectic designs $\mathcal{S}^\pm(6)$, as combinatorial geometries. Does the matrix H_2 have a meaning in quantum physics? In fact, it does. In qubit state space, a SIC is a tetrahedron inscribed within the Bloch sphere. Finding the minimum-entropy pure states, as we did for the Hoggar SIC, they turn out to form a second tetrahedron, dual to the first. Together, the two SICs constitute a stellated octahedron in the Bloch-sphere representation. Each projector in the new SIC is orthogonal to exactly one of the four projectors in the original SIC. Let $J_{4 \times 4}$ be the 4×4 matrix whose entries are all 1. Then, up to normalization, the

SIC representations of the four new projectors can be written as the rows of the matrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = J_{4 \times 4} - I_{4 \times 4}. \quad (93)$$

This is clearly just the Hadamard matrix H_2 , shifted and rescaled. So, the structure of orthogonalities between the Hoggar SIC and its twin is, essentially, the tensor product of three copies of the analogous structure for a qubit SIC.

An automorphism of a symmetric design is a permutation of the points that preserves the block structure, sending blocks to blocks. The symplectic designs admit *2-transitive automorphism groups*. That is, the automorphism group of a symplectic design $\mathcal{S}^\epsilon(2m)$ contains permutations that map any pair of points to any other pair of points. Furthermore, the automorphism group of a design is 2-transitive for points if and only if it is so for blocks as well. Therefore, the automorphism group of a symplectic design includes transformations that can map any pair of blocks to any other pair of blocks.

The *symmetric difference* of two sets is defined to be the set of those elements contained in their union but not their intersection. For example,

$$\begin{aligned} \{\text{John, Paul, George}\} \ominus \{\text{George, Ringo}\} \\ = \{\text{John, Paul, Ringo}\}. \end{aligned} \quad (94)$$

If the symmetric difference of any *three* blocks in a design is either a block or the complement of a block, then that design is said to have the *symmetric difference property*. If a design enjoys the symmetric difference property, then that design or its complement meets the following condition [39] on its parameters:

$$v = 2^{2m}, \quad k = 2^{2m-1} - 2^{m-1}, \quad \lambda = 2^{2m-2} - 2^{m-1}. \quad (95)$$

The complement of the Hoggar design satisfies these conditions with $m = 3$.

The 64-point designs with the symmetric difference property can be completely classified [39]. There exist four inequivalent such designs, distinguished by their automorphism groups [37]. The symplectic design, which we found by way of the Hoggar SIC, is the most symmetric: It is the only one of the four whose automorphism group is 2-transitive.

Let \mathbb{F}_2 denote the finite field of order two, and let $\text{Sp}(2m, \mathbb{F})$ denote the group of $2m \times 2m$ symplectic matrices over the field \mathbb{F} . Then, the automorphism group of the symplectic design $\mathcal{S}^1(6)$ is isomorphic to

$$G = (\mathbb{Z}_2)^6 \times \text{Sp}(6, \mathbb{F}_2). \quad (96)$$

The stabilizer of any point is $\text{Sp}(6, \mathbb{F}_2)$.

The original SIC and the twin SIC have the same symmetry group. Let Π_i be a projector in the original set and π_i a projector in the twin set. Suppose that g is

³ While these notes were in preparation, Szymusiak and Słomczyński updated an earlier arXiv paper of theirs with an independent derivation of this point [16].

an element of the symmetry group that takes π_j to π_k . Then

$$\text{tr}(\Pi_i \pi_k) = \text{tr}(\Pi_i g \pi_j g^\dagger) = \text{tr}(g^\dagger \Pi_i g \pi_j). \quad (97)$$

So, the SIC representation of π_j is just the SIC representation of π_k , with the entries permuted. Any element of the Hoggar SIC's symmetry group corresponds to a permutation that preserves the combinatorial design structure. However, the converse is not true: Not all elements in the automorphism group G can be implemented by unitaries that belong to the Hoggar SIC's symmetry group. This is a restatement of the fact that the symmetry group of the Hoggar SIC is a proper subgroup of the triple-qubit Clifford group.

VIII. POST-PEIERLS COMPATIBILITY

A quantum state can be thought of as a hypothesis for how a quantum system will behave when experimented upon. When are two such hypotheses different in a meaningful way? One way of quantifying this is the idea of *compatibility* between quantum states. Two quantum states ρ and ρ' are *post-Peierls incompatible* if a measurement exists that meets the following condition [40]. Let the measurement outcomes be labeled by j , so that the operators $\{E_j\}$ form a POVM,

$$\sum_j E_j = I. \quad (98)$$

The probabilities for the outcomes are computed using the Born rule:

$$q(j) = \text{tr}(\rho E_j), \quad q'(j) = \text{tr}(\rho' E_j). \quad (99)$$

If one can devise a measurement $\{E_j\}$ such that for *any* outcome j , at least one of $q(j)$ or $q'(j)$ is zero, then the states ρ and ρ' are post-Peierls (PP) incompatible. This can naturally be generalized to the question of compatibility among three or more states.

Is it possible for quantum states to be PP incompatible with respect to a SIC measurement? Yes, but not if we only consider two states at a time. For example, these are three valid states for the Hesse SIC representation in dimension $d = 3$.

$$\begin{aligned} & (0, 0, 0, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}); \\ & (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, 0, 0, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}); \\ & (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, 0, 0). \end{aligned} \quad (100)$$

Note that there is exactly one zero in each column. In other words, for each outcome of the Hesse SIC, exactly one of these three states assigns that outcome a probability of zero.

This is a situation where the relationship among three entities is not clearly apparent from the relationships

within each pair. In such a case, it can be helpful to portray the configuration diagrammatically [41–43]. We do so in Figure 2. Each circle in Figure 2 stands for one of the three states given in Eq. (100). The numbers contained within a circle are the labels of the outcomes that are consistent with that state. Note that these outcomes are only written in the areas where two circles overlap. No outcome belongs to a single state alone, and no outcome belongs to all three.

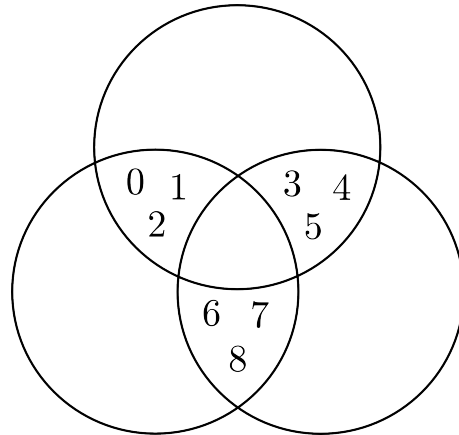


FIG. 2: Pictorial representation of the hypotheses defined in Eq. (100). Each circle corresponds to a quantum state. The numbers indicate the outcomes that are consistent with that state, *i.e.*, the outcomes for which that state implies nonzero probability.

Suppose we have three pure states in dimension $d = 3$. We denote them by $|\psi\rangle$, $|\psi'\rangle$ and $|\psi''\rangle$. These can be considered as three different hypotheses that an agent Alice is willing to entertain about a quantum system. If they are PP incompatible, then there exists some measurement that Alice can perform such that for any outcome of that measurement, at least one of the three hypotheses deems that outcome impossible. If we specialize to *von Neumann* measurements, then we can give a criterion for “PP-ODOP” compatibility (One-Dimensional, Orthogonal Projectors). A necessary and sufficient condition [18, 40] for three pure states in $d = 3$ to be PP-ODOP incompatible is for the following inequalities to be satisfied. First,

$$|\langle\psi|\psi'\rangle|^2 + |\langle\psi'|\psi''\rangle|^2 + |\langle\psi''|\psi\rangle|^2 < 1, \quad (101)$$

and second,

$$\begin{aligned} & (|\langle\psi|\psi'\rangle|^2 + |\langle\psi'|\psi''\rangle|^2 + |\langle\psi''|\psi\rangle|^2 - 1)^2 \\ & \geq 4 |\langle\psi|\psi'\rangle|^2 |\langle\psi'|\psi''\rangle|^2 |\langle\psi''|\psi\rangle|^2. \end{aligned} \quad (102)$$

Consider what happens if the three states are drawn from a SIC set. No set of three vectors can span more than three dimensions, so even though our states naturally live in a higher-dimensional Hilbert space, we press forward and use the three-dimensional criterion. In that

case,

$$|\langle\psi|\psi'\rangle|^2 = |\langle\psi'|\psi''\rangle|^2 = |\langle\psi''|\psi\rangle|^2 = \frac{1}{d+1}. \quad (103)$$

The first inequality becomes

$$\frac{3}{d+1} < 1, \quad (104)$$

and the second inequality becomes

$$\left(\frac{3}{d+1} - 1\right)^2 \geq \frac{4}{(d+1)^3}. \quad (105)$$

We can simplify the latter expression to

$$(d-2)^2 \geq \frac{4}{d+1}. \quad (106)$$

Both inequalities are satisfied simultaneously for $d \geq 3$.

The three-dimensional criterion tells us that there is *some* von Neumann measurement with respect to which the three states drawn from the SIC set are incompatible. Yet the SIC is itself a measurement, and with respect to *that* measurement, the three states are entirely compatible. In the representation that the SIC itself defines, the states $|\psi\rangle$, $|\psi'\rangle$ and $|\psi''\rangle$ all have the form

$$e_k(i) = \frac{1}{d(d+1)} + \frac{1}{d+1}\delta_{ik} \quad (107)$$

for some values of k , and these vectors contain no zeros at all. We have here a rather cute situation. Classically, an “informationally complete measurement” would be something like an experiment that discovers the exact values of all the positions and momenta of the particles comprising a system. We tend to think of any other measurement as a coarse-graining of that one, a measurement that throws away some of the information that is, in principle, available. And throwing away information makes classical configurations *harder* to distinguish from one another. If we can rule out a hypothesis using a clumsy, imprecise measurement, then *surely* we could do so using a *maximally informative* one! How could a measurement that is less exhaustive be *better* at ruling out a hypothesis?

This is indicative of the way in which quantum physics runs counter to classical intuition. An informationally complete quantum measurement is *not* the determination of the values of all hidden variables, or the narrowing of a Liouville density to a delta function. A vector in a SIC representation is not a probability distribution over a putative hidden-variable configuration space. And we do not calculate the probabilities for outcomes of other experiments merely by blurring over IC ones.

The double-transitivity of the Hoggar SIC simplifies the structure of the triple products, as we saw above. It does the same for considerations of PP compatibility [40], as well.

Any two SIC vectors are PP compatible. However, a set of three SIC vectors when taken together can be PP incompatible. In dimension 3, the measurements that reveal PP incompatibility for the Hesse SIC are a collection of vectors originally known for other reasons: They comprise four Mutually Unbiased Bases (MUB) [18]. What about with the Hoggar SIC?

Use one set of Hoggar lines to define a SIC representation of state space, and translate the twin Hoggar lines into this representation. Any two projectors in the twin Hoggar set will be pairwise PP compatible. Direct computation shows that any set of three distinct projectors will also be compatible, in the sense that the Hoggar SIC measurement itself will not reveal any incompatibility. However, a set of four lines from the twin Hoggar set can be PP-POVM incompatible, with that incompatibility revealed by the original Hoggar SIC-POVM itself. We will refer to this as “PP-H incompatibility.” For example, in Table I we gave the SIC representations of four lines from the twin Hoggar set.

As we noted earlier, a “1” in a bitstring means that the entry in that place is the nonzero value appropriate for the dimension, which here is 1/36. A “0” means that the vector is zero in that slot.

There is at least one 0 in each column, meaning that for every possible outcome of the Hoggar SIC-POVM, one of these four state assignments deems that outcome impossible. However, if we leave out any of the four rows, this is no longer true.

We illustrate this in Figure 3. Each ellipse stands for a state vector, that is, for a row in Table I. The central region, where all four ellipses overlap, contains no outcomes.

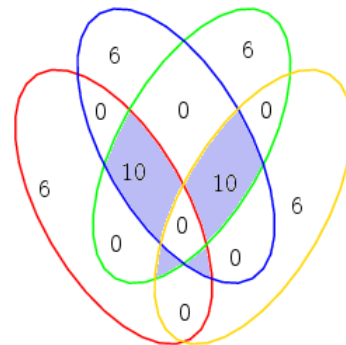


FIG. 3: Venn diagram for the set of four states from the twin Hoggar SIC given in Table I. Each ellipse stands for a quantum state. Labels indicate the number of outcomes of the Hoggar SIC for which that state implies nonzero probability. The shaded regions, where exactly three of the four ellipses overlap, contain 10 outcomes. Each ellipse contains three such regions, as well as a region all to itself. In total, each ellipse contains a value of 36. The central region, where all four ellipses intersect, contains 0. (Figure based on [44].)

There are lots of other examples; we do, after all, have 64 vectors to choose from. However, we should be able to simply the problem, and understand what’s going on

by considering only a subset of all those combinations. Why?

If we apply the same permutation to the four rows shown above, the columns still line up, meaning that there is still at least one zero in each column. Consequently, the transformed states will also be PP-H incompatible.

Because we can take any distinct pair (π_j, π_k) to the pair (π_0, π_1) , then we should be able to understand the PP-H compatibility properties of all quadruples by working out what happens with $(\pi_0, \pi_1, \pi_m, \pi_n)$.

We now apply our knowledge of combinatorial design theory. Let B_i denote the bitstring representation of the state π_i in the twin Hoggar SIC. These 64 sequences, which we can think of as the rows in a square matrix, form a symmetric design, as we showed earlier, and this design has the symmetric difference property. In terms of bitstrings, the symmetric difference $B_i \ominus B_j$ is equivalent to an XOR operation:

$$(B_i \ominus B_j)(n) = B_i(n) \text{ XOR } B_j(n). \quad (108)$$

This is readily verified, and implies the convenient fact that the symmetric difference is associative:

$$(B_i \ominus B_j) \ominus B_k = B_i \ominus (B_j \ominus B_k). \quad (109)$$

In Table II, we show the values resulting from applying XOR symmetrically to three bits, and the complementary values.

a	b	c	$a \text{ XOR } b \text{ XOR } c$	$\text{NOT}(a \text{ XOR } b \text{ XOR } c)$
0	0	0	0	1
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	0

TABLE II: The XOR of three bits, and its complement.

Because the Hoggar design has the symmetric difference property, the symmetric difference of any three blocks is either a block or the complement of a block. Suppose that the symmetric difference of B_i , B_j and B_k is the complement of B_l . Then B_l is the complement of $B_i \ominus B_j \ominus B_k$. We can find each element of B_l by locating the proper row in Table II. It follows that for all $n \in \{0, \dots, 63\}$, the set

$$\{B_i(n), B_j(n), B_k(n), B_l(n)\} \quad (110)$$

contains either 1, 2 or 3 zeroes. That is, these elements are never all zero, nor are they all ever one. Consequently, a measurement of the Hoggar SIC-POVM reveals PP-H incompatibility among the four states $\{\pi_i, \pi_j, \pi_k, \pi_l\}$ in the twin Hoggar SIC.

What else can we say about the symmetric differences of the blocks $\{B_i\}$? Each $B_i \ominus B_j$ for a distinct pair $i \neq j$ is a list of positions where exactly one of $B_i(n)$ and $B_j(n)$ equals one. By direct computation, we find that each such list is 32 items long. We can pick a pair of distinct blocks in 2,016 different ways. However, not all choices yield different lists of positions. In fact, only 126 lists occur. This is a consequence of a result noticed by Kantor [36]: The symmetric differences in the symplectic designs $\mathcal{S}(2m)$ correspond to the *hyperplanes* in the $2m$ -dimensional discrete affine space on the finite field of order 2, denoted \mathbb{F}_2 . In the case $m = 3$, there are 126 such hyperplanes, each containing $2^5 = 32$ points. Each hyperplane is the symmetric difference of 16 different choices of block pairs.

From Kantor's work, we can also extract a criterion for when a set of three blocks $\{B_i, B_j, B_k\}$ will be part of a PP-H-incompatible quadruple. As we deduced, this occurs when the symmetric difference of the three blocks is the complement of a block. The quantity

$$|(B_i \ominus B_j) \cap B_k| \quad (111)$$

equals either 16 or 20, depending on the choice of blocks. When it equals 16, the symmetric difference of the three blocks is itself a block. On the other hand, when it equals 20, then the symmetric difference is the complement of a block, and we have the incompatibility we seek. This can be interpreted in terms of another affine space on the finite field \mathbb{F}_2 . In this space, the points are the 64 bitstrings of the twin Hoggar SIC. For a fixed B_i and B_j with $j \neq i$, the set of all B_k such that

$$|(B_i \ominus B_j) \cap B_k| = 16 \quad (112)$$

defines a hyperplane in this affine space. Points that lie outside this hyperplane correspond to bitstrings which, together with B_i and B_j , can form part of a PP-H-incompatible quartet.

This construction also tells us about the triple products, in a way that relates back to our symplectic bilinear form, Eq. (70). Consider the quartet formed by B_i , B_j , B_k and their symmetric difference. If this quartet is PP-H incompatible, then

$$\text{Re tr}(\Pi_i \Pi_j \Pi_k) = \text{Re tr}(\pi_i \pi_j \pi_k) = 0. \quad (113)$$

In dimension 3, the triple products of the Hesse SIC depend on whether or not three points are collinear [18]. Now, we see that in dimension 8, triple products depend upon whether three points lie in the same hyperplane.

In fact, the implication works both ways: If $(ijk) \in S_0$, then B_i , B_j and B_k can be extended to form a PP-H-incompatible quartet.

IX. DEEPER INTO THE BITSTRINGS

The bit $B_j(n)$ will be 0 if the inner product

$$\text{tr}(\Pi_n \pi_j) = \text{tr}(D_n \Pi_0 D_n^\dagger D_j \pi_0 D_j^\dagger) \quad (114)$$

vanishes. Here, the displacement operators D_n and D_j are built from tensor products of the Pauli matrices. Note that we can use the cyclic property of the trace to reduce the problem to investigating inner products of the form

$$\text{tr}(\Pi_0 D_m \pi_0 D_m^\dagger). \quad (115)$$

The product $\Pi_0 \pi_0$ is a symmetric matrix. If we want the trace to vanish, we should try introducing an asymmetry somehow.

Of the four Pauli matrices, three (counting the identity) are symmetric. Only Y , which is proportional to the product XZ , is antisymmetric. We therefore make the educated guess that the inner product will vanish if the displacement operator D_m involves an *odd number of factors* of the Pauli matrix Y . This happens in 28 out of the 64 possible displacement operators D_m , which is the number we're looking for. Why 28? If we want one factor of Y , we have three places to put it, and we have $3^2 = 9$ choices for the other two factors. This gives us 27 possible operators. Then, the operator YYY is also antisymmetric, making a total of 28.

It is straightforward to check that these zeros fall in the correct places to reproduce the first row of Table I.

The displacement operator D_m will be antisymmetric if a certain sum has odd parity:

$$m_0 m_1 + m_2 m_3 + m_4 m_5 = 1 \pmod{2}. \quad (116)$$

This construction for picking 28 configurations out of 64 also arises in the study of *bitangents* to *quartic curves* [45]. Take the plane \mathbb{R}^2 , and define a curve on the plane by a fourth-degree equation in two variables. Such a curve can have as many as 28 bitangent lines, *i.e.*, lines that are tangent to the curve at exactly two places. By extending to the complex projective plane, one can always find a full set of 28 bitangents. Each one is labeled by a set of binary coordinates satisfying Eq. (116).

Rather unexpectedly, then, the study of SICs has made contact with the theory of algebraic curves!

Consider the elements of the index k that indicate the powers to which we raise X when constructing D_k , that is, the ordered triple (k_0, k_2, k_4) . This triple can take eight different values, seven of them nonzero. Likewise, we have seven nonzero possibilities for (k_1, k_3, k_5) . Let us group the possibilities for these two ordered triples according to when the dot product has even parity:

$$k_0 k_1 + k_2 k_3 + k_4 k_5 = 0 \pmod{2}. \quad (117)$$

For each choice of (k_1, k_3, k_5) , there are three choices for (k_0, k_2, k_4) that satisfy Eq. (117).

(k_0, k_2, k_4)	(k_1, k_3, k_5)	
010, 011, 001	100	
001, 101, 100	010	
010, 110, 100	001	
001, 111, 110	110	(118)
010, 111, 101	101	
011, 111, 100	011	
110, 101, 011	111	

This configuration has a name. The choices for (k_0, k_2, k_4) label the points of the *Fano plane*, and (k_1, k_3, k_5) label the lines. The Fano plane has seven points and seven lines. Each point lies on three lines, and each line contains three points. A line and a point of the Fano plane are incident if and only if their coordinates satisfy Eq. (117).

In the Fano plane, there are 28 ways to select a point and a line *not* incident with it: For each point, four of the seven lines do not go through that point, and we have seven ways to choose a point. In discrete geometry, a *flag* is the combination of a line and a point lying on that line, and an *anti-flag* is a line with a point lying off that line. So, there are 28 anti-flags in the Fano plane, and for each of them, the dot product of the point and line labels has *odd* parity. That is, for each anti-flag, the label of the point and the label of the line satisfy Eq. (116).

Look back at Table I. Each occurrence of the bit 0 is an anti-flag in a Fano plane! We use the powers to which we raise X to pick a point, and the powers to which we raise Z to pick a line (or vice versa). If the point lies off the line, we write a 0. All other bits in the sequence, we set to 1.

We have not yet exhausted the numerology of the integer 28. The bitangents to a quartic curve can also be identified [46, 47] with *pairs of opposing vertices* in the *Gosset polytope* 3_{21} , an object living in \mathbb{R}^7 that is related to the Lie algebra E_7 . We can construct this polytope in the following way [48]. Start with the two vectors

$$(3, 3, -1, -1, -1, -1, -1) \text{ and } (-3, -3, 1, 1, 1, 1, 1), \quad (119)$$

which both live in \mathbb{R}^8 . Permute the entries of these vectors in all possible ways. This creates 56 vectors in \mathbb{R}^8 . All of them are orthogonal to the vector

$$(1, 1, 1, 1, 1, 1, 1), \quad (120)$$

so they actually all fit into \mathbb{R}^7 . These are the vertices of the Gosset polytope. Each pair of opposite vectors defines a line through the origin, yielding 28 lines... which turn out to be equiangular.

We have here another unforeseen relation between the complex and the real versions of the equiangular lines question. Starting with one maximal set of complex equiangular lines, we construct another. The fact of a vector in one set being orthogonal to a vector in the other corresponds to a *real* line in a maximal equiangular set thereof.

X. CONCLUDING REMARKS

SICs are a confluence of multiple topics in mathematics. Weyl–Heisenberg SIC solutions in dimensions larger than 3 turn out to have deep number-theoretic properties, connecting quantum information theory to Hilbert’s twelfth problem [11]. The other known SIC solutions, which we have termed the sporadic SICs, relate by way

of group theory to sphere packing and the octonions [13]. By asking a physicist’s question—“Given this constraint, which states maximize and minimize the entropy?”—we launched ourselves into symplectic designs, two-graphs, bitangents to quartic curves and Gosset polytopes. Prolonged exposure to the SIC problem makes one suspect that the interface between physics and mathematics does not have the shape that one first expected.

For each of the SICs with doubly transitive symmetry groups, the pure states that minimize the Shannon entropy of the SIC representation are related to equiangular real lines. In dimension 2, they form a SIC [16], which is a tetrahedron in the Bloch ball, and that yields four real lines. In dimension 3, they form 12 MUB states [18]. Picking one state from each MUB, we obtain four equiangular lines in nine-dimensional real space. (There are 81 ways to do this.) And in dimension 8, the procedure yields the twin Hoggar SIC, which is equivalent to 64 equiangular lines in \mathbb{C}^8 . Furthermore, when we consider the relation between the original SIC and its twin, we find a set of 28 lines, which are a maximal set for 7- or 8-dimensional real vector space. And, as we remarked before, the triple-product structure of the Hoggar SIC leads to a two-graph on 64 vertices, which is itself equivalent

to a set of equiangular lines in \mathbb{R}^{36} .

That the solutions to the real and complex versions of the equiangular lines problem should be related in this way is rather surprising.

To draw this essay to a close, we should note that the Hoggar SIC provides a rather clean and elementary introduction to several mathematical structures that have been employed in the study of three-qubit quantum systems [49]. For example, we encountered the group $PSU(3, 3)$: It was (up to isomorphism) simply the group of transformations that permute the vectors in the Hoggar SIC while leaving the fiducial untouched. This group has also appeared [50] in studies of Bell–Kochen–Specker phenomena, that is, of the nonclassical meshing together of probability assignments [7, 9, 18, 51, 52]. Likewise, the sorting of tensor products of Pauli operators into symmetric and antisymmetric matrices has been invoked in other problems [53]. We remarked upon the appearance of a polytope related to an exceptional Lie algebra; this, too, is a type of structure pertinent to Bell–Kochen–Specker phenomena in three-qubit systems [54, 55]. All this suggests that more ideas might yet be grown from the Hoggar SIC.

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