Convolution identities for Tetranacci numbers

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Abstract

We give convolution identities without binomial coefficients for Tetranacci numbers and convolution identities with binomial coefficients for Tetranacci and Tetranacci-type numbers.

1 Introduction

Convolution identities for various types of numbers (or polynomials) have been studied, with or without binomial coefficients, including Bernoulli, Euler, Genocchi, Catalan, Cauchy, Stirling, Fibonacci and Tribonacci numbers ([1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]).Tetranacci sequence has been studied in [12, 13, 14].

Tetranacci numbers T_n are defined by the recurrence relation

$$T_n = T_{n-1}+T_{n-2}+T_{n-3}+T_{n-4} \quad (n \ge 4) \quad \text{with} \quad T_0 = 0,\ T_1 = T_2 = 1,\ T_3 = 2 \tag{1}$$

and their sequence is given by

$$\{T_n\}_{n\ge 0} = 0, 1, 1, 2, 4, 8, 15, 29, 56, 108, 208, \dots$$

([15, A000078]).

The generating function without factorials is given by

$$T(x) := \frac{x}{1-x-x^2-x^3-x^4} = \sum_{n=0}^{\infty} T_n x^n \tag{2}$$

because of the recurrence relation (1).

On the other hand, the generating function with binomial coefficients is given by

$$t(x) := c_1 e^{\alpha x} + c_2 e^{\beta x} + c_3 e^{\gamma x} + c_4 e^{\delta x} = \sum_{n=0}^{\infty} T_n \frac{x^n}{n!}\,, \tag{3}$$

where α, β, γ and δ are the roots of $x^4 - x^3 - x^2 - x - 1 = 0$ and

$$\begin{aligned}
c_1 &:= \frac{2 - (\beta + \gamma + \delta) + (\beta\gamma + \gamma\delta + \delta\beta)}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} \\
&= \frac{1}{-\alpha^3 + 6\alpha - 1}\,, \\
c_2 &:= \frac{2 - (\alpha + \gamma + \delta) + (\alpha\gamma + \gamma\delta + \delta\alpha)}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} \\
&= \frac{1}{-\beta^3 + 6\beta - 1}\,, \\
c_3 &:= \frac{2 - (\alpha + \beta + \delta) + (\alpha\beta + \beta\delta + \delta\alpha)}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} \\
&= \frac{1}{-\gamma^3 + 6\gamma - 1}\,, \\
c_4 &:= \frac{2 - (\alpha + \beta + \gamma) + (\alpha\beta + \beta\gamma + \gamma\alpha)}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)} \\
&= \frac{1}{-\delta^3 + 6\delta - 1}\,.
\end{aligned}$$

Notice that

$$\begin{aligned}
c_1 + c_2 + c_3 + c_4 &= 0\,, \\
c_1\alpha + c_2\beta + c_3\gamma + c_4\delta &= 1\,, \\
c_1\alpha^2 + c_2\beta^2 + c_3\gamma^2 + c_4\delta^2 &= 1\,, \\
c_1\alpha^3 + c_2\beta^3 + c_3\gamma^3 + c_4\delta^3 &= 2,
\end{aligned}$$

because t_n has a Binet-type formula:

$$T_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n + c_4\gamma^n \quad (n \ge 0)\,.$$

In this paper, we give convolution identities without binomial coefficients for Tetranacci numbers and convolution identities with binomial coefficients for Tetranacci and Tetranacci-type numbers.

2 Convolution identities without binomial coefficients

By (2), we have

$$T'(x) = \frac{1 + x^2 + 2x^3 + 3x^4}{(1 - x - x^2 - x^3 - x^4)^2}\,.$$

Hence,

$$(1 + x^2 + 2x^3 + 3x^4)T(x)^2 = x^2 T'(x)\,. \tag{4}$$

The left-hand side of (4) is

$$
\begin{aligned}
&(1+x^2+2x^3+3x^4)\sum_{n=0}^\infty\sum_{k=0}^n T_k T_{n-k}x^n\\
&=\sum_{n=0}^\infty\sum_{k=0}^n T_k T_{n-k}x^n+\sum_{n=2}^\infty\sum_{k=0}^{n-2} T_k T_{n-k-2}x^n\\
&\quad+2\sum_{n=3}^\infty\sum_{k=0}^{n-3} T_k T_{n-k-3}x^n+3\sum_{n=4}^\infty\sum_{k=0}^{n-4} T_k T_{n-k-4}x^n\\
&=\sum_{n=4}^\infty\sum_{k=0}^{n-4} t_k(T_{n-k}+T_{n-k-2}+2T_{n-k-3}+3T_{n-k-4})x^n\\
&\quad+\sum_{n=4}^\infty(T_{n-1}+T_{n-2}+3T_{n-3})x^n+x^2+2x^3\,.
\end{aligned}
$$

The right-hand side of (4) is

$$
x^2\sum_{n=0}^\infty(n+1)T_{n+1}x^n=\sum_{n=2}^\infty(n-1)T_{n-1}x^n\,.
$$

Therefore, we get the following result.

Theorem 1. *For $n\ge 4$, we have*

$$
\sum_{k=0}^{n-4} T_k(T_{n-k}+T_{n-k-2}+2T_{n-k-3}+3T_{n-k-4})=(n-2)T_{n-1}-T_{n-2}-3T_{n-3}\,.
$$

The identity (4) can be written as

$$
T(x)^2=\frac{x^2}{1+x^2+2x^3+3x^4}T'(x)\,. \tag{5}
$$

Since

$$
\begin{aligned}
\frac{1}{1+x^2+2x^3+3x^4}&=\sum_{l=0}^\infty(-1)^l x^{2l}(1+2x+3x^2)^l\\
&=\sum_{l=0}^\infty(-1)^l x^{2l}\sum_{\substack{i+j+k=l\\ i,j,k\ge 0}}\binom{l}{i,j,k}1^i(2x)^j(3x^2)^k\\
&=\sum_{m=0}^\infty\sum_{\substack{j,k=0\\ 3j+2k\le m\\ j+4k\le m}}(-1)^{\frac{m-j-2k}{2}}\frac{1+(-1)^{m-j-2k}}{2}\\
&\qquad\times\binom{\frac{m-j-2k}{2}}{\frac{m-j-2k}{2}-j-k,j,k}2^j 3^k x^m,
\end{aligned}
$$

and

$$T'(x)=\sum_{n=0}^\infty (n+1)T_{n+1}x^n\,,$$

the right-hand side of (5) is

$$\begin{aligned} x^2 A\sum_{l=0}^\infty (l+1)T_{l+1}x^l &= x^2\sum_{n=0}^\infty\sum_{l=0}^n B(l+1)T_{l+1}x^n\\ &=\sum_{n=2}^\infty\sum_{l=0}^{n-2} C(l+1)T_{l+1}x^n\,, \end{aligned}$$

where

$$A=\sum_{m=0}^\infty \sum_{\substack{j,k=0\\3j+2k\le m\\j+4k\le m}} (-1)^{\frac{m-j-2k}{2}}\frac{1+(-1)^{m-j-2k}}{2}\binom{\frac{m-j-2k}{2}}{\frac{m-j-2k}{2}-j-k,j,k}2^j3^kx^m,$$

$$B=\sum_{\substack{j,k=0\\3j+2k\le n-l\\j+4k\le n-l}} (-1)^{\frac{n-l-j-2k}{2}}\frac{1+(-1)^{n-l-j-2k}}{2}\binom{\frac{n-l-j-2k}{2}}{\frac{n-l-3j-4k}{2},j,k}2^j3^k,$$

$$C=\sum_{\substack{j,k=0\\3j+2k\le n-2-l\\j+4k\le n-2-l}} (-1)^{\frac{n-2-l-j-2k}{2}}\frac{1+(-1)^{n-2-l-j-2k}}{2}\binom{\frac{n-2-l-j-2k}{2}}{\frac{n-2-l-3j-4k}{2},j,k}2^j3^k.$$

Since the left-hand side of (5) is

$$\sum_{n=0}^\infty\sum_{k=0}^n T_kT_{n-k}x^n\,,$$

comparing the coefficients on both sides, we obtain the following result without binomial coefficient.

Theorem 2. *For $n\ge 2$,*

$$\sum_{k=0}^n T_kT_{n-k}=\sum_{l=0}^{n-2}(l+1)T_{l+1}D,$$

where

$$D=\sum_{\substack{j,k=0\\3j+2k\le n-2-l\\j+4k\le n-2-l}} (-1)^{\frac{n-2-l-j-2k}{2}}\frac{1+(-1)^{n-2-l-j-2k}}{2}\binom{\frac{n-2-l-j-2k}{2}}{\frac{n-2-l-3j-4k}{2},j,k}2^j3^k.$$

3 Some preliminary lemmas

For convenience, we shall introduce modified Tetranacci numbers $T_n^{(s_0,s_1,s_2,s_3)}$, satisfying the recurrence relation

$$T_n^{(s_0,s_1,s_2,s_3)}=T_{n-1}^{(s_0,s_1,s_2,s_3)}+T_{n-2}^{(s_0,s_1,s_2,s_3)}+T_{n-3}^{(s_0,s_1,s_2,s_3)}+T_{n-4}^{(s_0,s_1,s_2,s_3)}\quad (n\ge 4)$$

with given initial values $T_0^{(s_0,s_1,s_2,s_3)} = s_0$, $T_1^{(s_0,s_1,s_2,s_3)} = s_1$, $T_2^{(s_0,s_1,s_2,s_3)} = s_2$,and $T_3^{(s_0,s_1,s_2,s_3)} = s_3$. Hence, $T_n = T_n^{(0,1,1,2)}$ are ordinary Tetranacci numbers.

First, we shall prove the following four lemmata.

Lemma 1. *We have*

$$c_1^2 e^{\alpha x} + c_2^2 e^{\beta x} + c_3^2 e^{\gamma x} + c_4^2 e^{\delta x} = \frac{1}{563}\sum_{n=0}^\infty T_n^{(40,64,215,344)}\frac{x^n}{n!}\,.$$

Proof. For Tetranacci-type numbers s_n, satisfying the recurrence relation $s_n = s_{n-1} + s_{n-2} + s_{n-3} + s_{n-4}$ $(n \ge 4)$ with given initial values s_0, s_1, s_2 and s_3, we have

$$d_1 e^{\alpha x} + d_2 e^{\beta x} + d_3 e^{\gamma x} + d_4 e^{\delta x} = \sum_{n=0}^\infty s_n\frac{x^n}{n!}\,. \tag{6}$$

Since d_1, d_2, d_3 and d_4 satisfy the system of the equations

$$\begin{aligned} d_1 + d_2 + d_3 + d_4 &= s_0\,,\\ d_1\alpha + d_2\beta + d_3\gamma + d_4\gamma &= s_1\,,\\ d_1\alpha^2 + d_2\beta^2 + d_3\gamma^2 + d_4\gamma^2 &= s_2\,,\\ d_1\alpha^3 + d_2\beta^3 + d_3\gamma^3 + d_4\gamma^3 &= s_3\,, \end{aligned}$$

we have

$$d_1 = \frac{\begin{vmatrix} s_0 & 1 & 1 & 1\\ s_1 & \beta & \gamma & \delta\\ s_2 & \beta^2 & \gamma^2 & \delta^2\\ s_3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1\\ \alpha & \beta & \gamma & \delta\\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2\\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix}} = \frac{s_0\beta\gamma\delta + s_2(\beta+\gamma+\delta) - s_3 - s_1(\beta\gamma+\beta\delta+\gamma\delta)}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)}\,,$$

$$d_2 = \frac{\begin{vmatrix} 1 & s_0 & 1 & 1\\ \alpha & s_1 & \gamma & \delta\\ \alpha^2 & s_2 & \gamma^2 & \delta^2\\ \alpha^3 & s_3 & \gamma^3 & \delta^3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1\\ \alpha & \beta & \gamma & \delta\\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2\\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix}} = \frac{s_0\gamma\delta\alpha + s_2(\gamma+\delta+\alpha) - s_3 - s_1(\gamma\delta+\gamma\alpha+\delta\alpha)}{(\gamma-\beta)(\delta-\beta)(\alpha-\beta)}\,,$$

$$d_3 = \frac{\begin{vmatrix} 1 & 1 & s_0 & 1 \\ \alpha & \beta & s_1 & \delta \\ \alpha^2 & \beta^2 & s_2 & \delta^2 \\ \alpha^3 & \beta^3 & s_3 & \delta^3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix}} = \frac{s_0\delta\alpha\beta + s_2(\delta+\alpha+\beta) - s_3 - s_1(\delta\alpha+\delta\beta+\alpha\beta)}{(\delta-\gamma)(\alpha-\gamma)(\beta-\gamma)},$$

$$d_4 = \frac{\begin{vmatrix} 1 & 1 & 1 & s_0 \\ \alpha & \beta & \gamma & s_1 \\ \alpha^2 & \beta^2 & \gamma^2 & s_2 \\ \alpha^3 & \beta^3 & \gamma^3 & s_3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix}} = \frac{s_0\alpha\beta\gamma + s_2(\alpha+\beta+\gamma) - s_3 - s_1(\alpha\beta+\alpha\gamma+\beta\gamma)}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}.$$

When $s_0 = 40$, $s_1 = 64$, $s_2 = 215$ and $s_3 = 344$, by $\alpha+\beta+\gamma+\delta = 1$, $\beta\gamma+\beta\delta+\gamma\delta = -1-(\alpha\beta+\alpha\gamma+\alpha\delta) = \alpha^2-\alpha-1$, $\alpha\beta\gamma\delta = 1$ and $\alpha^4 = \alpha^3+\alpha^2+\alpha+1$, we have

$$d_1 = \frac{40\beta\gamma\delta + 215(\beta+\gamma+\delta) - 344 - 64(\beta\gamma+\beta\delta+\gamma\delta)}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)} = 563c_1^2.$$

. Similarly, we have $d_2 = 563c_2^2$, $d_3 = 563c_3^2$ and $d_4 = 563c_4^2$. □

Lemma 2. *We have*

$$\sum_{n=0}^{\infty} t_n \frac{x^n}{n!} = c_1c_2e^{(\alpha+\beta)x} + c_1c_3e^{(\alpha+\gamma)x} + c_1c_4e^{(\alpha+\delta)x} + c_2c_3e^{(\beta+\gamma)x} + c_2c_4e^{(\beta+\delta)x} + c_3c_4e^{(\gamma+\delta)x},$$

where

$$t_n = \frac{1}{2}\left(\sum_{k=0}^{n}\binom{n}{k}T_kT_{n-k} - \frac{2^n}{563}T_n^{(40,64,215,344)}\right).$$

Proof. Since

$$\begin{aligned}&(c_1e^{\alpha x} + c_2e^{\beta x} + c_3e^{\gamma x} + c_4e^{\delta x})^2\\ &= c_1^2e^{\alpha x} + c_2^2e^{\beta x} + c_3^2e^{\gamma x} + c_4^2e^{\delta x} + 2(c_1c_2e^{(\alpha+\beta)x} + c_1c_3e^{(\alpha+\gamma)x} + c_1c_4e^{(\alpha+\delta)x}\\ &\quad + c_2c_3e^{(\beta+\gamma)x} + c_2c_4e^{(\beta+\delta)x} + c_3c_4e^{(\gamma+\delta)x}),\end{aligned}$$

we can obtain the following identity:

$$\left(\sum_{n=0}^\infty T_n\frac{x^n}{n!}\right)^2=\sum_{n=0}^\infty\sum_{k=0}^n\binom{n}{k}T_kT_{n-k}\frac{x^n}{n!}$$
$$=\frac{1}{563}\sum_{n=0}^\infty T_n^{(40,64,215,344)}\frac{(2x)^n}{n!}+2\sum_{n=0}^\infty t_n\frac{x^n}{n!}.$$

Comparing the coefficients on both sides, we get the desired result. □

Lemma 3. *We have*

$$c_2c_3c_4e^{\alpha x}+c_3c_4c_1e^{\beta x}+c_4c_1c_2e^{\gamma x}+c_1c_2c_3e^{\delta x}=-\frac{1}{563}\sum_{n=0}^\infty T_n^{(-5,2,13,32)}\frac{x^n}{n!}\,.$$

Proof. In the proof of Lemma 1, we put $s_0=-5$, $s_1=2$, $s_2=13$ and $s_3=32$, instead. We have

$$d_1=\frac{-5\beta\gamma\delta+13(\beta+\gamma+\delta)-32-2(\beta\gamma+\beta\delta+\gamma\delta)}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)}=-563c_2c_3c_4.$$

. Similarly, we have $d_2=-563c_3c_4c_1$, $\quad d_3=-563c_4c_1c_2$ and $d_4=-563c_1c_2c_3$. □

Lemma 4. *We have*

$$c_1c_2c_3c_4=-\frac{1}{563}\,.$$

Proof. By $\alpha+\beta+\gamma+\delta=1$, $\beta\gamma+\beta\delta+\gamma\delta=-1-(\alpha\beta+\alpha\gamma+\alpha\delta)=\alpha^2-\alpha-1$, $\alpha\beta\gamma\delta=1$ and $\alpha^4=\alpha^3+\alpha^2+\alpha+1$, we have

$$\begin{aligned}&c_1c_2c_3c_4\\&=\frac{\alpha^2}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\frac{\beta^2}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\\&\quad\times\frac{\gamma^2}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}\frac{\delta^2}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\\&=\frac{\alpha^2\beta^2\gamma^2\delta^2}{(\alpha-\beta)^2(\alpha-\gamma)^2(\alpha-\delta)^2(\beta-\gamma)^2(\gamma-\beta)^2(\beta-\delta)^2}\\&=\frac{1}{(4\alpha^3-3\alpha^2-2\alpha-1)^2(39\alpha^3-58\alpha^2-23\alpha-23)}\\&=-\frac{1}{563}.\end{aligned}$$

□

4 Convolution identities for three and four Tetranacci numbers

Before giving more convolution identities,we shall give some elementary algebraic identities in symmetric form.It is not so difficult to determine the relations among coefficients.

Lemma 5. *The following equality holds:*

$$\begin{aligned}
&(a+b+c+d)^3\\
&= A(a^3+b^3+c^3+d^3)+B(abc+abd+acd+bcd)\\
&\quad +C(a^2+b^2+c^2+d^2)(a+b+c+d)\\
&\quad +D(ab+ac+ad+bc+bd+cd)(a+b+c+d),
\end{aligned}$$

where $A=D-2, \quad B=-3D+6, \quad C=-D+3.$

Lemma 6. *The following equality holds:*

$$\begin{aligned}
&(a+b+c+d)^4\\
&= A(a^4+b^4+c^4+d^4)+Babcd+C(a^3+b^3+c^3+d^4)(a+b+c+d)\\
&\quad +D(a^2+b^2+c^2+d^2)^2+E(a^2+b^2+c^2+d^2)(ab+ac+ad+bc+bd+cd)\\
&\quad +F(ab+ac+ad+bc+bd+cd)^2+G(a^2+b^2+c^2+d^2)(a+b+c+d)^2\\
&\quad +H(ab+ac+ad+bc+bd+cd)(a+b+c)^2\\
&\quad +I(abc(a+b+c)+abd(a+b+d)+bcd(b+c+d)+acd(a+c+d))\\
&\quad +J(abc+abd+bcd+acd)(a+b+c+d),
\end{aligned}$$

where $A=-D+E+G+H-3, \quad B=12D+12G-4J-12,$
$C=-E-2G-H+4, \quad F=-2D-2G-2H+6, \quad I=4D-E+2G-H-J.$

Lemma 7. *The following equality holds:*

$$\begin{aligned}
&(a+b+c+d)^5\\
&= A(a^5+b^5+c^5+d^5)\\
&\quad +B(abc(ab+bc+ca)+abd(ab+bd+ad)+acd(ac+ad+cd)+bcd(bc+bd+cd))\\
&\quad +C(abc(a^2+b^2+c^2)+abd(b^2+c^2+d^2)\\
&\qquad +acd(a^2+c^2+d^2)+bcd(b^2+c^2+d^2))\\
&\quad +D(abc(a+b+c)^2+abd(a+b+d)^2+acd(a+c+d)^2+bcd(b+c+d)^2)\\
&\quad +E(a^4+b^4+c^4+d^4)(a+b+c+d)+F(a+b+c+d)abcd\\
&\quad +G(a+b+c+d)\\
&\qquad \times(abc(a+b+c)+abd(a+b+d)+bcd(b+c+d)+acd(a+c+d))\\
&\quad +H(a^3+b^3+c^3+d^3)(a^2+b^2+c^2+d^2)
\end{aligned}$$

$$
\begin{aligned}
&+ I(a^3+b^3+c^3+d^3)(ab+ac+ad+bc+bd+cd) \\
&+ J(abc+abd+acd+bcd)(a^2+b^2+c^2+d^2) \\
&+ K(abc+abd+acd+bcd)(ab+ac+ad+bc+bd+cd) \\
&+ L(a^3+b^3+c^3+d^3)(a+b+c+d)^2 \\
&+ M(abc+abd+acd+bcd)(a+b+c+d)^2 \\
&+ N(a^2+b^2+c^2+d^2)^2(a+b+c+d) \\
&+ P(ab+ac+ad+bc+bd+cd)^2(a+b+c+d) \\
&+ Q(a^2+b^2+c^2+d^2)(ab+ac+ad+bc+bd+cd)(a+b+c+d) \\
&+ R(a^2+b^2+c^2+d^2)(a+b+c+d)^3 \\
&+ S(ab+ac+ad+bc+bd+cd)(a+b+c+d)^3,
\end{aligned}
$$

where

$$
\begin{aligned}
&A = I+2L+2N+P+2Q+6R+4S-14, \\
&B = -2D-2G-K-2M-2N-5P-2Q-6R-12S+30, \\
&C = -D-G-I-J-2L-M-2P-3Q-6R-7S+20, \\
&E = -I-2L-N-Q-3R-S+5, \\
&F = -3G-J-3K-7M-12P-3Q-6R-27S+60, \\
&H = -L-2N-P-Q-4R-3S+10.
\end{aligned}
$$

Now, let us consider the sum of three products with trinomial coefficients.

Lemma 8. *We have*

$$
c_1^3 e^{\alpha x} + c_2^3 e^{\beta x} + c_3^3 e^{\gamma x} + c_4^3 e^{\delta x} = \frac{1}{563}\sum_{n=0}^{\infty} T_n^{(15,27,48,107)}\frac{x^n}{n!}\,.
$$

Proof. In the proof of Lemma 1, we put $s_0 = 15$, $s_1 = 27$, $s_2 = 48$ and $s_3 = 107$, instead. We can obtain that

$$
d_1 = \frac{15\beta\gamma\delta + 48(\beta+\gamma+\delta) - 107 - 27(\beta\gamma+\beta\delta+\gamma\delta)}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)} = 563c_1^3.
$$

. Similarly, we have $d_2 = 563c_2^3$, $d_3 = 563c_3^3$ and $d_4 = 563c_4^3$. □

By using Lemmata 1, 2, 3, 5 and 8, we get the following result.

Theorem 3. *For $n \ge 0$,*

$$\sum_{\substack{k_1+k_2+k_3=n\\k_1,k_2,k_3\ge 0}} \binom{n}{k_1,k_2,k_3} T_{k_1}T_{k_2}T_{k_3}$$
$$= \frac{A}{563} 3^n T_n^{(15,27,48,107)} - \frac{B}{563}\sum_{k=0}^{n}\binom{n}{k} T_k^{(-5,2,13,32)}(-1)^k$$
$$+ \frac{C}{563}\sum_{k=0}^{n}\binom{n}{k} 2^{n-k} T_{n-k}^{(40,64,215,344)} T_k + D\sum_{k=0}^{n}\binom{n}{k} T_k t_{n-k}\,.$$

where $A = D-2,\ B = -3D+6,\ C = -D+3,$

$$t_n = \frac{1}{2}\left(\sum_{k=0}^{n}\binom{n}{k} T_k T_{n-k} - \frac{2^n}{563} T_n^{(40,64,215,344)}\right).$$

Remark. If we take $D = 0$, we have for $n \ge 0$,

$$\sum_{\substack{k_1+k_2+k_3=n\\k_1,k_2,k_3\ge 0}} \binom{n}{k_1,k_2,k_3} T_{k_1}T_{k_2}T_{k_3}$$
$$= -\frac{2}{563} 3^n T_n^{(15,27,48,107)} - \frac{6}{563}\sum_{k=0}^{n}\binom{n}{k} T_k^{(-5,2,13,32)}(-1)^k$$
$$+ \frac{3}{563}\sum_{k=0}^{n}\binom{n}{k} 2^{n-k} T_{n-k}^{(40,64,215,344)} T_k\,.$$

Proof. First, by Lemmata1, 2, 3, 5 and 8, we have

$$(c_1e^{\alpha x} + c_2e^{\beta x} + c_3e^{\gamma x} + c_4e^{\delta x})^3$$
$$= A(c_1^3e^{3\alpha x} + c_2^3e^{3\beta x} + c_3^3e^{3\gamma x} + c_4^3e^{3\delta x})$$
$$+ B(c_1c_2c_3e^{(\alpha+\beta+\gamma)x} + c_2c_3c_4e^{(\beta+\gamma+\delta)x} + c_1c_2c_4e^{(\alpha+\beta+\delta)x} + c_1c_3c_4e^{(\alpha+\gamma+\delta)x})$$
$$+ C(c_1^2e^{2\alpha x} + c_2^2e^{2\beta x} + c_3^2e^{2\gamma x} + c_4^2e^{2\delta x})(c_1e^{\alpha x} + c_2e^{\beta x} + c_3e^{\gamma x} + c_4e^{\delta x})$$
$$+ D(c_1c_2e^{(\alpha+\beta)x} + c_1c_3e^{(\alpha+\gamma)x} + c_1c_4e^{(\alpha+\delta)x} + c_2c_3e^{(\beta+\gamma)x} + c_2c_4e^{(\beta+\delta)x} + c_3c_4e^{(\gamma+\delta)x})$$
$$\times (c_1e^{\alpha x} + c_2e^{\beta x} + c_3e^{\gamma x} + c_4e^{\delta x})$$
$$= \frac{A}{563}\sum_{n=0}^{\infty} T_n^{(15,27,48,107)}\frac{(3x)^n}{n!} - \frac{B}{563}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} T_k^{(-5,2,13,32)}(-1)^k\frac{x^n}{n!}$$
$$+ \frac{C}{563}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} 2^{n-k}T_{n-k}^{(40,64,215,344)}T_k\frac{x^n}{n!} + D\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} T_k t_{n-k}\frac{x^n}{n!}\,.$$

On the other hand,

$$\left(\sum_{n=0}^{\infty} T_n\frac{x^n}{n!}\right)^3 = \sum_{\substack{k_1+k_2+k_3=n\\k_1,k_2,k_3\ge 0}} \binom{n}{k_1,k_2,k_3} T_{k_1}T_{k_2}T_{k_3}\frac{x^n}{n!}\,.$$

Comparing the coefficients on both sides, we get the desired result. □

Next, we shall consider the sum of the products of four tetranacci numbers. We need the following supplementary result. The proof is similar to that of Lemma 8 and omitted.

Lemma 9. *We have*

$$c_1^4 e^{\alpha x} + c_2^4 e^{\beta x} + c_3^4 e^{\gamma x} + c_4^4 e^{\delta x} = \frac{1}{563^2} \sum_{n=0}^{\infty} T_n^{(3052,4658,8804,16451)} \frac{x^n}{n!} \,.$$

By using Lemmata1, 2, 3, 6, 8, and 9,letting $I = 0$ in Lemma 6, comparing the coefficients on both sides, we can get the following theorem.

Theorem 4. *For $n \ge 0$,*

$$\begin{aligned}
&\sum_{\substack{k_1+k_2+k_3+k_4=n\\ k_1,k_2,k_3,k_4\ge 0}} \binom{n}{k_1,k_2,k_3,k_4} T_{k_1}T_{k_2}T_{k_3}T_{k_4}\\
&= \frac{A}{563^2} 4^n T_n^{(3052,4658,8804,16451)} - \frac{B}{563} + \frac{C}{563} \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} T_{n-k}^{(15,27,48,107)} T_k\\
&\quad + \frac{D}{563^2} \sum_{k=0}^{n} \binom{n}{k} 2^n T_{n-k}^{(40,64,215,344)} T_k^{(40,64,215,344)}\\
&\quad + \frac{E}{563} \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} T_{n-k}^{(40,64,215,344)} t_k + F \sum_{k=0}^{n} \binom{n}{k} t_{n-k} t_k\\
&\quad + \frac{G}{563} \sum_{\substack{k_1+k_2+k_3=n\\ k_1,k_2,k_3\ge 0}} \binom{n}{k_1,k_2,k_3} T_{k_1}^{(40,64,215,344)} 2^{k_1} T_{k_2} T_{k_3}\\
&\quad + H \sum_{\substack{k_1+k_2+k_3=n\\ k_1,k_2,k_3\ge 0}} \binom{n}{k_1,k_2,k_3} t_{k_1} T_{k_2} T_{k_3}\\
&\quad - \frac{J}{563} \sum_{\substack{k_1+k_2+k_3=n\\ k_1,k_2,k_3\ge 0}} \binom{n}{k_1,k_2,k_3} T_{k_1}^{(-5,2,13,32)} (-1)^{k_1} T_{k_2},
\end{aligned}$$

where $A = -D + E + G + H - 3$, $B = -4D + 4E + 4G + 4H - 12$, $C = -E - 2G - H + 4$,$F = -2D - 2G - 2H + 6$, $J = 4D - E + 2G - H$,

$$t_n = \frac{1}{2} \left(\sum_{k=0}^{n} \binom{n}{k} T_k T_{n-k} - \frac{2^n}{563} T_n^{(40,64,215,344)} \right).$$

Remark. If $D = E = G = H = 0$, then by $A = -3$, $B = -12$, $C = 4$, $F = 6$

and $J=0$, we have for $n\geq 0$,

$$\sum_{\substack{k_1+k_2+k_3+k_4=n\\ k_1,k_2,k_3,k_4\geq 0}}\binom{n}{k_1,k_2,k_3,k_4}T_{k_1}T_{k_2}T_{k_3}T_{k_4}$$
$$=-\frac{3}{563^2}4^nT_n^{(3052,4658,8804,16451)}+\frac{12}{563}+\frac{4}{563}\sum_{k=0}^{n}\binom{n}{k}3^{n-k}T_{n-k}^{(15,27,48,107)}T_k$$
$$+6\sum_{k=0}^{n}\binom{n}{k}t_{n-k}t_k\,,$$

Let

$$\sum_{n=0}^{\infty}t_n^1\frac{x^n}{n!}$$
$$=c_1c_2c_3e^{(\alpha+\beta+\gamma)x}(c_1e^{\alpha x}+c_2e^{\beta x}+c_3e^{\gamma x})+c_2c_3c_4e^{(\beta+\gamma+\delta)x}(c_2e^{\beta x}+c_3e^{\gamma x}+c_4e^{\delta x})$$
$$+c_1c_2c_4e^{(\alpha+\beta+\delta)x}(c_1e^{\alpha x}+c_2e^{\beta x}+c_4e^{\delta x})+c_1c_3c_4e^{(\alpha+\gamma+\delta)x}(c_1e^{\alpha x}+c_3e^{\gamma x}+c_4e^{\delta x}).$$

By using Lemmata1, 2, 3, 6, 8, and 9, comparing the coefficients on both sides, we can get the following theorem.

Theorem 5. *For $n\geq 0$,$I\neq 0$*

$$It_n^1=\sum_{\substack{k_1+k_2+k_3+k_4=n\\ k_1,k_2,k_3,k_4\geq 0}}\binom{n}{k_1,k_2,k_3,k_4}T_{k_1}T_{k_2}T_{k_3}T_{k_4}$$
$$-\frac{A}{563^2}4^nT_n^{(3052,4658,8804,16451)}+\frac{B}{563}-\frac{C}{563}\sum_{k=0}^{n}\binom{n}{k}3^{n-k}T_{n-k}^{(15,27,48,107)}T_k$$
$$-\frac{D}{563^2}\sum_{k=0}^{n}\binom{n}{k}2^nT_{n-k}^{(40,64,215,344)}T_k^{(40,64,215,344)}$$
$$-\frac{E}{563}\sum_{k=0}^{n}\binom{n}{k}2^{n-k}T_{n-k}^{(40,64,215,344)}t_k-F\sum_{k=0}^{n}\binom{n}{k}t_{n-k}t_k$$
$$-\frac{G}{563}\sum_{\substack{k_1+k_2+k_3=n\\ k_1,k_2,k_3\geq 0}}\binom{n}{k_1,k_2,k_3}T_{k_1}^{(40,64,215,344)}2^{k_1}T_{k_2}T_{k_3}$$
$$-H\sum_{\substack{k_1+k_2+k_3=n\\ k_1,k_2,k_3\geq 0}}\binom{n}{k_1,k_2,k_3}t_{k_1}T_{k_2}T_{k_3}$$
$$+\frac{J}{563}\sum_{\substack{k_1+k_2+k_3=n\\ k_1,k_2,k_3\geq 0}}\binom{n}{k_1,k_2,k_3}T_{k_1}^{(-5,2,13,32)}(-1)^{k_1}T_{k_2},$$

where $A=-D+E+G+H-3,\quad B=12D+12G-4J-12,$

$C = -E - 2G - H + 4, \quad F = -2D - 2G - 2H + 6, \quad I = 4D - E + 2G - H - J,$

$$t_n = \frac{1}{2}\left(\sum_{k=0}^{n}\binom{n}{k}T_k T_{n-k} - \frac{2^n}{563}T_n^{(40,64,215,344)}\right).$$

Remark. If $D = E = G = H = 0$, $J = -1$, then by $A = -3$, $B = -8$, $C = 4$, $F = 6$ and $I = 1$, we have for $n \ge 0$,

$$\begin{aligned} t_n^1 = &\sum_{\substack{k_1+k_2+k_3+k_4=n \\ k_1,k_2,k_3,k_4\ge 0}}\binom{n}{k_1,k_2,k_3,k_4}T_{k_1}T_{k_2}T_{k_3}T_{k_4} \\ &+ \frac{3}{563^2}4^n T_n^{(3052,4658,8804,16451)} - \frac{8}{563} - \frac{4}{563}\sum_{k=0}^{n}\binom{n}{k}3^{n-k}T_{n-k}^{(15,27,48,107)}T_k \\ &- 6\sum_{k=0}^{n}\binom{n}{k}t_{n-k}t_k - \frac{1}{563}\sum_{\substack{k_1+k_2+k_3=n \\ k_1,k_2,k_3\ge 0}}\binom{n}{k_1,k_2,k_3}T_{k_1}^{(-5,2,13,32)}(-1)^{k_1}T_{k_2}. \end{aligned}$$

5 Convolution identities for five Tetranacci numbers

We shall consider the sum of the products of five tetranacci numbers. We need the following supplementary result. The proof is similar to that of Lemma 8 and omitted.

Lemma 10.

$$c_1^5 e^{\alpha x} + c_2^5 e^{\beta x} + c_3^5 e^{\gamma x} + c_4^5 e^{\delta x} = \frac{1}{563^2}\sum_{n=0}^{\infty}T_n^{(500,1423,2598,4986)}\frac{x^n}{n!}.$$

By using Lemmata1, 2, 3, 7, 8, 9 and 10, comparing the coefficients on both sides, we can get the following theorems.

5.1

Let $B = C = D = 0$, we can obtain the following theorem.

Theorem 6. *For* $n \ge 0$,

$$\begin{aligned} &\sum_{\substack{k_1+\cdots+k_5=n \\ k_1,\dots,k_5\ge 0}}\binom{n}{k_1,\dots,k_5}T_{k_1}\cdots T_{k_5} \\ &= \frac{A}{563^2}5^n T_n^{(500,1423,2598,4986)} + \frac{E}{563^2}\sum_{k=0}^{n}\binom{n}{k}4^{n-k}T_{n-k}^{(3052,4658,8804,16451)}T_k \\ &\quad - \frac{F}{563}\sum_{k=0}^{n}\binom{n}{k}T_k + G\sum_{k=0}^{n}\binom{n}{k}t_k^1 T_{n-k} + \frac{H}{563^2}\sum_{k=0}^{n}\binom{n}{k}3^{n-k}2^k T_{n-k}^{(15,27,48,107)}T_k^{(40,64,215,344)} \end{aligned}$$

$$
\begin{aligned}
&+\frac{I}{563}\sum_{k=0}^{n}\binom{n}{k}3^kT_k^{(15,27,48,107)}t_{n-k}\\
&-\frac{J}{563^2}\sum_{\substack{k_1+k_2+k_3=n\\k_1,k_2,k_3\geq 0}}\binom{n}{k_1,k_2,k_3}(-1)^{k_1}2^{k_2}T_{k_1}^{(-5,2,13,32)}T_{k_2}^{(40,64,215,344)}\\
&-\frac{K}{563}\sum_{\substack{k_1+k_2+k_3=n\\k_1,k_2,k_3\geq 0}}\binom{n}{k_1,k_2,k_3}(-1)^{k_1}T_{k_1}^{(-5,2,13,32)}t_{k_2}\\
&+\frac{L}{563}\sum_{\substack{k_1+k_2+k_3=n\\k_1,k_2,k_3\geq 0}}\binom{n}{k_1,k_2,k_3}3^{k_1}T_{k_1}^{(15,27,48,107)}T_{k_2}T_{k_3}\\
&-\frac{M}{563}\sum_{\substack{k_1+k_2+k_3+k_4=n\\k_1,k_2,k_3,k_4\geq 0}}\binom{n}{k_1,k_2,k_3,k_4}(-1)^{k_1}T_{k_1}^{(-5,2,13,32)}T_{k_2}T_{k_3}\\
&+\frac{N}{563^2}\sum_{\substack{k_1+k_2+k_3=n\\k_1,k_2,k_3\geq 0}}\binom{n}{k_1,k_2,k_3}2^{k_1}T_{k_1}^{(40,64,215,344)}2^{k_2}T_{k_2}^{(40,64,215,344)}T_{k_3}\\
&+P\sum_{\substack{k_1+k_2+k_3=n\\k_1,k_2,k_3\geq 0}}\binom{n}{k_1,k_2,k_3}t_{k_1}t_{k_2}T_{k_3}\\
&+\frac{Q}{563}\sum_{\substack{k_1+k_2+k_3=n\\k_1,k_2,k_3\geq 0}}\binom{n}{k_1,k_2,k_3}2^{k_1}T_{k_1}^{(40,64,215,344)}t_{k_2}T_{k_3}\\
&+\frac{R}{563}\sum_{\substack{k_1+k_2+k_3+k_4=n\\k_1,k_2,k_3,k_4\geq 0}}\binom{n}{k_1,k_2,k_3,k_4}2^{k_1}T_{k_1}^{(40,64,215,344)}T_{k_2}T_{k_3}T_{k_4}\\
&+S\sum_{\substack{k_1+k_2+k_3+k_4=n\\k_1,k_2,k_3,k_4\geq 0}}\binom{n}{k_1,k_2,k_3,k_4}t_{k_1}T_{k_2}T_{k_3}T_{k_4},
\end{aligned}
$$

where

$$
\begin{aligned}
A&=I+2L+2N+P+2Q+6R+4S-14,\\
E&=-I-2L-N-Q-3R-S+5,\\
F&=4G+I+2L+6N+5P+6Q+18R+16S-50,\\
H&=-L-2N-P-Q-4R-3S+10,\\
J&=-G-I-2L-M-2P-3Q-6R-7S+20,\\
K&=-2G-2M-2N-5P-2Q-6R-12S+30,
\end{aligned}
$$

t_n *and* t_n^1 *are same as those in theorem 2 and theorem 5, respectively.*

Remark. If $G=I=L=M=N=P=Q=R=S=0$, then by $A=-14$,

$E = 5$, $F = -50$, $H = 10$, $J = 20$ and $K = 30$, we have for $n \geq 0$,

$$\begin{aligned}
&\sum_{\substack{k_1+\cdots+k_5=n\\ k_1,\dots,k_5\geq 0}} \binom{n}{k_1,\dots,k_5} T_{k_1}\cdots T_{k_5}\\
&= -\frac{14}{563^2}5^n T_n^{(500,1423,2598,4986)} + \frac{5}{563^2}\sum_{k=0}^{n}\binom{n}{k}4^{n-k}T_{n-k}^{(3052,4658,8804,16451)}T_k\\
&\quad + \frac{50}{563}\sum_{k=0}^{n}\binom{n}{k}T_k + \frac{10}{563^2}\sum_{k=0}^{n}\binom{n}{k}3^{n-k}2^k T_{n-k}^{(15,27,48,107)}T_k^{(40,64,215,344)}\\
&\quad - \frac{20}{563^2}\sum_{\substack{k_1+k_2+k_3=n\\ k_1,k_2,k_3\geq 0}}\binom{n}{k_1,k_2,k_3}(-1)^{k_1}2^{k_2}T_{k_1}^{(-5,2,13,32)}T_{k_2}^{(40,64,215,344)}\\
&\quad - \frac{30}{563}\sum_{\substack{k_1+k_2+k_3=n\\ k_1,k_2,k_3\geq 0}}\binom{n}{k_1,k_2,k_3}(-1)^{k_1}T_{k_1}^{(-5,2,13,32)}t_{k_2}\,.
\end{aligned}$$

5.2

Let $B \neq 0$, $C = D = 0$, we can obtain the following theorem.
Let

$$\begin{aligned}
&\sum_{n=0}^{\infty} t_n^2 \frac{x^n}{n!}\\
&= c_1c_2c_3e^{(\alpha+\beta+\gamma)x}(c_1c_2e^{(\alpha+\beta)x} + c_2c_3e^{(\beta+\gamma)x} + c_3c_1e^{(\gamma+\alpha)x}) + \cdots\\
&\quad + c_2c_3c_4e^{(\beta+\gamma+\delta)x}(c_2c_3e^{(\beta+\gamma)x} + c_3c_4e^{(\gamma+\delta)x} + c_4c_2e^{(\delta+\beta)x}).
\end{aligned}$$

Theorem 7. *For $n \geq 0$,*

$$\begin{aligned}
Bt_n^2 &= \sum_{\substack{k_1+\cdots+k_5=n\\ k_1,\dots,k_5\geq 0}} \binom{n}{k_1,\dots,k_5} T_{k_1}\cdots T_{k_5} - \frac{A}{563^2}5^n T_n^{(500,1423,2598,4986)} - \cdots\\
&\quad - S\sum_{\substack{k_1+k_2+k_3+k_4=n\\ k_1,k_2,k_3,k_4\geq 0}}\binom{n}{k_1,k_2,k_3,k_4} t_{k_1}T_{k_2}T_{k_3}T_{k_4},
\end{aligned}$$

where

$$\begin{aligned}
&A = -G - J - M + 2N - P - Q - 3S + 6,\\
&B = -2G - K - 2M - 2N - 5P - 2Q - 6R - 12S + 30,\\
&E = G + J + M - N + 2P + 2Q + 3R + 6S - 15,\\
&F = -3G - J - 3K - 7M - 12P - 3Q - 6R - 27S + 60,\\
&H = -L - 2N - P - Q - 4R - 3S + 10,\\
&I = -G - J - 2L - M - 2P - 3Q - 6R - 7S + 20\,,
\end{aligned}$$

t_n and t_n^1 are same as those in theorem 2 and theorem 5, respectively.

Remark. If $G = J = K = L = M = N = P = Q = R = S = 0$, then by $A = 6$, $B = 30$, $E = -15$, $F = 60$, $H = 10$ and $I = 20$, we have for $n \geq 0$,

$$\begin{aligned}30t_n^2 = &\sum_{\substack{k_1+\cdots+k_5=n\\k_1,\dots,k_5\geq 0}} \binom{n}{k_1,\dots,k_5} T_{k_1}\cdots T_{k_5} - \frac{6}{563^2}5^n T_n^{(500,1423,2598,4986)}\\&+\frac{15}{563^2}\sum_{k=0}^{n}\binom{n}{k}4^{n-k}T_{n-k}^{(3052,4658,8804,16451)}T_k + \frac{60}{563}\sum_{k=0}^{n}\binom{n}{k}T_k\\&-\frac{10}{563^2}\sum_{k=0}^{n}\binom{n}{k}3^{n-k}2^k T_{n-k}^{(15,27,48,107)}T_k^{(40,64,215,344)} - \frac{20}{563}\sum_{k=0}^{n}\binom{n}{k}3^k T_k^{(15,27,48,107)}t_{n-k}\,.\end{aligned}$$

5.3

Let $C \neq 0$, $B = D = 0$, we can obtain the following theorem.
Let

$$\begin{aligned}&\sum_{n=0}^{\infty} t_n^3 \frac{x^n}{n!}\\&= c_1c_2c_3e^{(\alpha+\beta+\gamma)x}(c_1^2e^{2\alpha x}+c_2^2e^{2\beta x}+c_3^2e^{2\gamma x})+\cdots\\&\quad + c_2c_3c_4e^{(\beta+\gamma+\delta)x}(c_2^2e^{2\beta x}+c_3^2e^{2\gamma x}+c_4^2e^{2\delta x}).\end{aligned}$$

Theorem 8. *For $n \geq 0$,*

$$\begin{aligned}Ct_n^3 = &\sum_{\substack{k_1+\cdots+k_5=n\\k_1,\dots,k_5\geq 0}} \binom{n}{k_1,\dots,k_5} T_{k_1}\cdots T_{k_5} - \frac{A}{563^2}5^n T_n^{(500,1423,2598,4986)} - \cdots\\&- S\sum_{\substack{k_1+k_2+k_3+k_4=n\\k_1,k_2,k_3,k_4\geq 0}} \binom{n}{k_1,k_2,k_3,k_4} t_{k_1}T_{k_2}T_{k_3}T_{k_4},\end{aligned}$$

where

$$\begin{aligned}&A = I + 2L + 2N + P + 2Q + 6R + 4S - 14,\\&C = -G - I - J - 2L - M - 2P - 3Q - 6R - 7S + 20,\\&E = -I - 2L - N - Q - 3R - S + 5,\\&F = 3G - J - M + 6N + 3P + 3Q + 12R + 9S - 30,\\&H = -L - 2N - P - Q - 4R - 3S + 10,\\&K = -2G - 2M - 2N - 5P - 2Q - 6R - 12S + 30,\end{aligned}$$

t_n and t_n^1 are same as those in theorem 2 and theorem 5, respectively.

Remark. If $G = I = J = L = M = N = P = Q = R = S = 0$, then by

$A=-14$, $C=20$, $E=5$, $F=-30$, $H=10$ and $K=30$, we have for $n\geq 0$,

$$\begin{aligned}20t_n^3 &= \sum_{\substack{k_1+\cdots+k_5=n\\k_1,\ldots,k_5\geq 0}}\binom{n}{k_1,\ldots,k_5}T_{k_1}\cdots T_{k_5}+\frac{14}{563^2}5^nT_n^{(500,1423,2598,4986)}\\&\quad-\frac{5}{563^2}\sum_{k=0}^n\binom{n}{k}4^{n-k}T_{n-k}^{(3052,4658,8804,16451)}T_k+\frac{30}{563}\sum_{k=0}^n\binom{n}{k}T_k\\&\quad-\frac{10}{563^2}\sum_{k=0}^n\binom{n}{k}3^{n-k}2^kT_{n-k}^{(15,27,48,107)}T_k^{(40,64,215,344)}\\&\quad+\frac{30}{563}\sum_{\substack{k_1+k_2+k_3=n\\k_1,k_2,k_3\geq 0}}\binom{n}{k_1,k_2,k_3}(-1)^{k_1}T_{k_1}^{(-5,2,13,32)}t_{k_2}\,.\end{aligned}$$

5.4

Let $D\neq 0$, $B=C=0$, we can obtain the following theorem.
Let

$$\begin{aligned}&\sum_{n=0}^{\infty}t_n^4\frac{x^n}{n!}\\&=c_1c_2c_3e^{(\alpha+\beta+\gamma)x}(c_1e^{\alpha x}+c_2e^{\beta x}+c_3e^{\gamma x})^2+\cdots\\&\quad+c_2c_3c_4e^{(\beta+\gamma+\delta)x}(c_2e^{\beta x}+c_3e^{\gamma x}+c_4e^{\delta x})^2.\end{aligned}$$

Theorem 9. *For $n\geq 0$,*

$$\begin{aligned}Dt_n^4 &= \sum_{\substack{k_1+\cdots+k_5=n\\k_1,\ldots,k_5\geq 0}}\binom{n}{k_1,\ldots,k_5}T_{k_1}\cdots T_{k_5}-\frac{A}{563^2}5^nT_n^{(500,1423,2598,4986)}-\cdots\\&\quad-S\sum_{\substack{k_1+k_2+k_3+k_4=n\\k_1,k_2,k_3,k_4\geq 0}}\binom{n}{k_1,k_2,k_3,k_4}t_{k_1}T_{k_2}T_{k_3}T_{k_4},\end{aligned}$$

where

$$\begin{aligned}&A=I+2L+2N+P+2Q+6R+4S-14,\\&D=-G-I-J-2L-M-2P-3Q-6R-7S+20,\\&E=-I-2L-N-Q-3R-S+5,\\&F=-3G-6I-7J-7M+6N-12L-9P-15Q-24R-33S+90,\\&H=-L-2N-P-Q-4R-3S+10,\\&K=2I+2J-2N+4L-P+4Q+6R+2S-10,\end{aligned}$$

t_n and t_n^1 are same as those in theorem 2 and theorem 5, respectively.

Remark. If $G = I = J = L = M = N = P = Q = R = S = 0$, then by $A = -14$, $D = 20$, $E = 5$, $F = 90$, $H = 10$ and $K = -10$, we have for $n \ge 0$,

$$
\begin{aligned}
20t_n^4 &= \sum_{\substack{k_1+\cdots+k_5=n\\ k_1,\dots,k_5\ge 0}} \binom{n}{k_1,\dots,k_5} T_{k_1}\cdots T_{k_5} + \frac{14}{563^2} 5^n T_n^{(500,1423,2598,4986)}\\
&\quad - \frac{5}{563^2}\sum_{k=0}^n \binom{n}{k} 4^{n-k} T_{n-k}^{(3052,4658,8804,16451)} T_k + \frac{90}{563}\sum_{k=0}^n \binom{n}{k} T_k\\
&\quad - \frac{10}{563^2}\sum_{k=0}^n \binom{n}{k} 3^{n-k} 2^k T_{n-k}^{(15,27,48,107)} T_k^{(40,64,215,344)}\\
&\quad - \frac{10}{563} \sum_{\substack{k_1+k_2+k_3=n\\ k_1,k_2,k_3\ge 0}} \binom{n}{k_1,k_2,k_3} (-1)^{k_1} T_{k_1}^{(-5,2,13,32)} t_{k_2}\,.
\end{aligned}
$$

6 More general results

We shall consider the general case of Lemmata 1, 8 and 9. Similarly to the proof of Lemma 1, for tetranacci-type numbers $s_{1,n}^{(n)}$, satisfying the recurrence relation $s_{1,k}^{(n)} = s_{1,k-1}^{(n)} + s_{1,k-2}^{(n)} + s_{1,k-3}^{(n)} + s_{1,k-4}^{(n)}$ ($k \ge 4$) with given initial values $s_{1,0}^{(n)}$, $s_{1,1}^{(n)}$, $s_{1,2}^{(n)}$ and $s_{1,3}^{(n)}$, we have the form

$$
d_1^{(n)} e^{\alpha x} + d_2^{(n)} e^{\beta x} + d_3^{(n)} e^{\gamma x} + d_4^{(n)} e^{\delta x} = \sum_{k=0}^\infty s_{1,k}^{(n)} \frac{x^k}{k!}\,.
$$

Theorem 10. *For $n \ge 1$, we have*

$$
c_1^n e^{\alpha x} + c_2^n e^{\beta x} + c_3^n e^{\gamma x} + c_4^n e^{\delta x} = \frac{1}{A_1^{(n)}} \sum_{k=0}^\infty T_{1,k}^{(s_{1,0}^{(n)}, s_{1,1}^{(n)}, s_{1,2}^{(n)}, s_{1,3}^{(n)})} \frac{x^k}{k!}\,,
$$

where $s_{1,0}^{(n)}$, $s_{1,1}^{(n)}$, $s_{1,2}^{(n)}$, $s_{1,3}^{(n)}$ and $A_1^{(n)}$ satisfy the recurrence relations:

$$
\begin{aligned}
&s_{1,0}^{(n)} = \pm\mathrm{lcm}(b_1,b_2,b_3), \quad s_{1,1}^{(n)} = M s_{1,0}^{(n)}, \quad s_{1,2}^{(n)} = N s_{1,0}^{(n)}, \quad s_{1,3}^{(n)} = P s_{1,0}^{(n)},\\
&A_1^{(n)} = \frac{A_1^{(n-1)}}{s_{1,2}^{(n-1)}} (4 s_{1,3}^{(n)} - 3 s_{1,2}^{(n)} - 2 s_{1,1}^{(n)} - s_{1,0}^{(n)}),
\end{aligned}
$$

b_1, b_2, b_3, M, N and P are determined in the proof.

Proof. By $d_1^{(n)} = A_1^{(n)} c_1^n$, $d_1^{(n-1)} = A_1^{(n-1)} c_1^{n-1}$,

$$\begin{aligned} d_1^{(n)} &= \frac{s_{1,0}^{(n)}\beta\gamma\delta + s_{1,2}^{(n)}(\beta+\gamma+\delta) - s_{1,3}^{(n)} - s_{1,1}^{(n)}(\beta\gamma+\beta\delta+\gamma\delta)}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)}, \\ c_1 &= \frac{2-(\beta+\gamma+\delta)+(\beta\gamma+\gamma\delta+\delta\beta)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} \\ &= \frac{\alpha^2}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} \\ &= \frac{1}{-\alpha^3+6\alpha-1}, \end{aligned}$$

we can obtain the following recurrence relation:

$$\begin{aligned} &A = -3s_{1,1}^{(n-1)} - s_{1,2}^{(n-1)} + s_{1,3}^{(n-1)}, \quad B = 5s_{1,1}^{(n-1)} + 5s_{1,2}^{(n-1)} - 5s_{1,3}^{(n-1)}, \\ &C = 5s_{1,1}^{(n-1)} - 2s_{1,2}^{(n-1)} + 2s_{1,3}^{(n-1)}, \quad D = 5s_{1,1}^{(n-1)} - s_{1,2}^{(n-1)} + s_{1,3}^{(n-1)}, \\ &E = s_{1,0}^{(n-1)} - s_{1,1}^{(n-1)}, \quad F = -5s_{1,0}^{(n-1)}, \quad G = 2s_{1,0}^{(n-1)} + 6s_{1,1}^{(n-1)}, \end{aligned}$$

$$\begin{aligned} &H = s_{1,0}^{(n-1)} - s_{1,1}^{(n-1)}, \quad I = 4s_{1,1}^{(n-1)} + 6s_{1,2}^{(n-1)}, \quad J = -3s_{1,1}^{(n-1)} - 5s_{1,2}^{(n-1)}, \\ &K = -2s_{1,1}^{(n-1)} + 2s_{1,2}^{(n-1)}, \quad L = -s_{1,1}^{(n-1)} + s_{1,2}^{(n-1)}, \\ &M = \frac{(LA-DI)(FA-BE)-(HA-DE)(JA-BI)}{(GA-CE)(JA-BI)-(KA-CI)(FA-BE)}, \\ &N = \frac{M(KA-CI)+(LA-DI)}{BI-JA}, \quad P = -\frac{1}{A}(BN+CM+D), \\ &M = \frac{a_1}{b_1}, \quad N = \frac{a_2}{b_2}, \quad P = \frac{a_3}{b_3}, \quad \text{with} \quad \gcd(a_i,b_i)=1, \\ &s_{1,0}^{(n)} = \pm\mathrm{lcm}(b_1,b_2,b_3), \quad s_{1,1}^{(n)} = Ms_{1,0}^{(n)}, \quad s_{1,2}^{(n)} = Ns_{1,0}^{(n)}, \quad s_{1,3}^{(n)} = Ps_{1,0}^{(n)}, \\ &A_1^{(n)} = \frac{A_1^{(n-1)}}{s_{1,2}^{(n-1)}}(4s_{1,3}^{(n)} - 3s_{1,2}^{(n)} - 2s_{1,1}^{(n)} - s_{1,0}^{(n)}). \end{aligned}$$

We choose the symbol of $s_{1,0}^{(n)}$ such that for some $k_0, \forall k \geq k_0$, $T_{1,k}^{(s_{1,0}^{(n)},s_{1,1}^{(n)},s_{1,2}^{(n)},s_{1,3}^{(n)})}$ is positive. ◻

Next we shall consider the general case of Lemma 3. Similarly to the proof of Lemma 3, for tetranacci-type numbers $s_{1,k}^{(n)}$, satisfying the recurrence relation $s_{1,k}^{(n)} = s_{1,k-1}^{(n)} + s_{1,k-2}^{(n)} + s_{1,k-3}^{(n)} + s_{1,k-4}^{(n)}$ $(k \geq 4)$ with given initial values $s_{1,0}^{(n)}$, $s_{1,1}^{(n)}$, $s_{1,2}^{(n)}$ and $s_{1,3}^{(n)}$, we have the form

$$r_1^{(n)}e^{\alpha x} + r_2^{(n)}e^{\beta x} + r_3^{(n)}e^{\gamma x} + r_4^{(n)}e^{\delta x} = \sum_{k=0}^{\infty} s_{1,k}^{(n)}\frac{x^k}{k!},$$

where $r_1^{(n)}$, $r_2^{(n)}$, $r_3^{(n)}$ and $r_4^{(n)}$ are determined by solving the system of the equations.

Theorem 11.

$$c_2^n c_3^n c_4^n e^{\alpha x} + c_1^n c_3^n c_4^n e^{\beta x} + c_1^n c_2^n c_4^n e^{\gamma x} + c_1^n c_2^n c_3^n e^{\delta x} = \frac{1}{A_2^{(n)}} \sum_{k=0}^{\infty} T_{2,k}^{(s_{2,0}^{(n)}, s_{2,1}^{(n)}, s_{2,2}^{(n)}, s_{2,3}^{(n)})} \frac{x^k}{k!},$$

where $s_{2,0}^{(n)}$, $s_{2,1}^{(n)}$, $s_{2,2}^{(n)}$, $s_{2,3}^{(n)}$ *and* $A_2^{(n)}$ *satisfy the recurrence relations:*

$$s_{2,0}^{(n)} = \pm \mathrm{lcm}(b_1, b_2, b_3), \quad s_{2,1}^{(n)} = M s_{2,0}^{(n)}, \quad s_{2,2}^{(n)} = N s_{2,0}^{(n)}, \quad s_{2,3}^{(n)} = P s_{2,0}^{(n)},$$
$$A_2^{(n)} = \frac{A_2^{(n-1)}}{2s_{2,2}^{(n-1)} - s_{2,3}^{(n-1)}} (-16 s_{2,3}^{(n)} + 103 s_{2,2}^{(n)} - 157 s_{2,1}^{(n)} - 10 s_{2,0}^{(n)}),$$

b_1,b_2, b_3, M, N *and* P *are determined in the proof.*

Proof. By $r_1^{(n)} = A_2^{(n)} c_2^n c_3^n c_4^n$, we can obtain the following recurrence relation:

$A = -16 s_{2,0}^{(n-1)} - 16 s_{2,1}^{(n-1)} + 158 s_{2,2}^{(n-1)} - 71 s_{2,3}^{(n-1)}$,
$B = 103 s_{2,0}^{(n-1)} + 103 s_{2,1}^{(n-1)} - 243 s_{2,2}^{(n-1)} + 70 s_{2,3}^{(n-1)}$,
$C = -157 s_{2,0}^{(n-1)} - 157 s_{2,1}^{(n-1)} - 209 s_{2,2}^{(n-1)} + 183 s_{2,3}^{(n-1)}$,
$D = -10 s_{2,0}^{(n-1)} - 10 s_{2,1}^{(n-1)} - 42 s_{2,2}^{(n-1)} + 26 s_{2,3}^{(n-1)}$,
$E = 32 s_{2,0}^{(n-1)} + 16 s_{2,1}^{(n-1)} - 330 s_{2,2}^{(n-1)} + 157 s_{2,3}^{(n-1)}$,
$F = -206 s_{2,0}^{(n-1)} - 103 s_{2,1}^{(n-1)} + 365 s_{2,2}^{(n-1)} - 131 s_{2,3}^{(n-1)}$,
$G = 314 s_{2,0}^{(n-1)} + 157 s_{2,1}^{(n-1)} + 351 s_{2,2}^{(n-1)} - 254 s_{2,3}^{(n-1)}$,
$H = 20 s_{2,0}^{(n-1)} + 10 s_{2,1}^{(n-1)} + 216 s_{2,2}^{(n-1)} - 113 s_{2,3}^{(n-1)}$,
$I = 32 s_{2,1}^{(n-1)} - 36 s_{2,2}^{(n-1)} + 10 s_{2,3}^{(n-1)}$, $J = -206 s_{2,1}^{(n-1)} + 91 s_{2,2}^{(n-1)} + 6 s_{2,3}^{(n-1)}$,
$K = 314 s_{2,1}^{(n-1)} + 69 s_{2,2}^{(n-1)} - 113 s_{2,3}^{(n-1)}$, $L = 20 s_{2,1}^{(n-1)} - 304 s_{2,2}^{(n-1)} + 147 s_{2,3}^{(n-1)}$,

$$M = \frac{(LA - DI)(FA - BE) - (HA - DE)(JA - BI)}{(GA - CE)(JA - BI) - (KA - CI)(FA - BE)},$$
$$N = \frac{M(GA - CE) + (HA - DE)}{BE - FA}, \quad P = -\frac{1}{A}(BN + CM + D),$$
$$M = \frac{a_1}{b_1}, \quad N = \frac{a_2}{b_2}, \quad P = \frac{a_3}{b_3}, \quad \gcd(a_i, b_i) = 1,$$
$$s_{2,0}^{(n)} = \pm \mathrm{lcm}(b_1, b_2, b_3), \quad s_{2,1}^{(n)} = M s_{2,0}^{(n)}, \quad s_{2,2}^{(n)} = N s_{2,0}^{(n)}, \quad s_{2,3}^{(n)} = P s_{2,0}^{(n)},$$
$$A_2^{(n)} = \frac{A_2^{(n-1)}}{2s_{2,2}^{(n-1)} - s_{2,3}^{(n-1)}} (-16 s_{2,3}^{(n)} + 103 s_{2,2}^{(n)} - 157 s_{2,1}^{(n)} - 10 s_{2,0}^{(n)}).$$

We choose the symbol of $s_{2,0}^{(n)}$ such that for some k_0,$\forall k \geq k_0$, $T_{2,k}^{(s_{2,0}^{(n)}, s_{2,1}^{(n)}, s_{2,2}^{(n)}, s_{2,3}^{(n)})}$ is positive. □

As application, we compute some values of $s_{2,0}^{(n)},\ s_{2,1}^{(n)},\ s_{2,2}^{(n)},\ A_2^{(n)}$ for some n. For $n = 2$, we have

$$A = -170, \quad B = -1228, \quad C = 3610, \quad D = 316, \quad E = 606, \quad F = 1377,$$
$$G = -4821, \quad H = -888, \quad I = -84, \quad J = 963, \quad K = -2091, \quad L = 792,$$
$$M = -\frac{34}{15}, \quad N = -6, \quad P = -\frac{44}{15},$$
$$s_{2,0}^2 = -15, \quad s_{2,1}^2 = 34, \quad s_{2,2}^2 = 90, \quad s_{2,3}^2 = 44, \quad A_2^2 = 563^2,$$
$$c_2^2c_3^2c_4^2e^{\alpha x} + c_1^2c_3^2c_4^2e^{\beta x} + c_1^2c_2^2c_4^2e^{\gamma x} + c_1^2c_2^2c_3^2e^{\delta x} = \frac{1}{563^2}\sum_{k=0}^{\infty} T_{2,k}^{(-15,34,90,44)}\frac{x^k}{k!}.$$

For $n = 3$, we have

$$A = 10792, \quad B = -16833, \quad C = 13741, \quad D = -2826, \quad E = -22728, \quad F = 26674,$$
$$G = 21042, \quad H = 14508, \quad I = -1712, \quad J = 1450, \quad K = 11914, \quad L = -20212,$$
$$M = \frac{353}{175}, \quad N = -\frac{21}{25}, \quad P = \frac{38}{25},$$
$$s_{2,0}^3 = 175, \quad s_{2,1}^3 = 353, \quad s_{2,2}^3 = -147, \quad s_{2,3}^3 = 266, \quad A_2^3 = -563^3,$$
$$c_2^3c_3^3c_4^3e^{\alpha x} + c_1^3c_3^3c_4^3e^{\beta x} + c_1^3c_2^3c_4^3e^{\gamma x} + c_1^3c_2^3c_3^3e^{\delta x} = -\frac{1}{563^3}\sum_{k=0}^{\infty} T_{2,k}^{(175,353,-147,266)}\frac{x^k}{k!}.$$

We can obtain more convolution identities for any fixed n, but we only some of the results.The proof of next eight theorems are similar to the proofs of theorem (lemma) 2, 3, 4, 5, 6, 7, 8 and 9, and omitted.

Let

$$c_1^n c_2^n e^{(\alpha+\beta)x} + \cdots + c_3^n c_4^n e^{(\gamma+\delta)x} = \sum_{k=0}^{\infty} t_{1,k}^{(n)}\frac{x^k}{k!},$$

then by previous algebraic identities ,we can obtain the following theorems.

Theorem 12. *For $m \geq 0$, $n \geq 1$,*

$$c_1^n c_2^n e^{(\alpha+\beta)x} + \cdots + c_3^n c_4^n e^{(\gamma+\delta)x} = \sum_{k=0}^{\infty} t_{1,m}^{(n)}\frac{x^m}{m!},$$

where

$$t_{1,m}^{(n)} = \frac{1}{2}\left(\frac{1}{(A_1^{(n)})^2}\sum_{k=0}^{m}\binom{m}{k} T_{1,k}^{(s_{1,0}^{n},s_{1,1}^{(n)},s_{1,2}^{(n)},s_{1,3}^{(n)})} T_{1,m-k}^{(s_{1,0}^{(n)},s_{1,1}^{(n)},s_{1,2}^{(n)},s_{1,3}^{(n)})} - \frac{2^m}{A_1^{(2n)}} T_{1,m}^{(s_{1,0}^{(n)},s_{1,1}^{(n)},s_{1,2}^{(n)},s_{1,3}^{(n)})}\right).$$

Theorem 13. *For $m \geq 0$, $n \geq 1$,*

$$\frac{1}{(A_1^{(n)})^3} \sum_{\substack{k_1+k_2+k_3=m \\ k_1,k_2,k_3 \geq 0}} \binom{m}{k_1,k_2,k_3} T_{1,k_1}^{(s_{1,0}^{(n)},s_{1,1}^{(n)},s_{1,2}^{(n)},s_{1,3}^{(n)})} T_{1,k_2}^{(s_{1,0}^{(n)},s_{1,1}^{(n)},s_{1,2}^{(n)},s_{1,3}^{(n)})} T_{1,k_3}^{(s_{1,0}^{(n)},s_{1,1}^{(n)},s_{1,2}^{(n)},s_{1,3}^{(n)})}$$

$$= \frac{A}{A_1^{(3n)}} 3^m T_{1,m}^{(s_{1,0}^{(3n)},s_{1,1}^{(3n)},s_{1,2}^{(3n)},s_{1,3}^{(3n)})} + \frac{B}{A_2^{(n)}} \sum_{k=0}^{m} \binom{m}{k} T_{2,k}^{(s_{2,0}^{(n)},s_{2,1}^{(n)},s_{2,2}^{(n)},s_{2,3}^{(n)})} (-1)^k$$

$$+ \frac{C}{A_1^{(n)} A_1^{(2n)}} \sum_{k=0}^{m} \binom{m}{k} T_{1,k}^{(s_{1,0}^{(n)},s_{1,1}^{(n)},s_{1,2}^{(n)},s_{1,3}^{(n)})} 2^{m-k} T_{1,m-k}^{(s_{1,0}^{(2n)},s_{1,1}^{(2n)},s_{1,2}^{(2n)},s_{1,3}^{(2n)})}$$

$$+ \frac{D}{A_1^{(n)}} \sum_{k=0}^{m} \binom{m}{k} T_{1,k}^{(s_{1,0}^{(n)},s_{1,1}^{(n)},s_{1,2}^{(n)},s_{1,3}^{(n)})} t_{1,m-k}^{(n)},$$

where

$$A = D - 2, \quad B = -3D + 6, \quad C = -D + 3,$$

$t_{1,m}^{(n)}$ *is determined in theorem 12.*

Theorem 14. *For $m \geq 0$, $n \geq 1$,*

$$\frac{1}{(A_1^{(n)})^4} \sum_{\substack{k_1+k_2+k_3+k_4=m \\ k_1,k_2,k_3,k_4 \geq 0}} \binom{m}{k_1,k_2,k_3,k_4} T_{1,k_1}^{(s_{1,0}^{(n)},s_{1,1}^{(n)},s_{1,2}^{(n)},s_{1,3}^{(n)})} \cdots T_{1,k_4}^{(s_{1,0}^{(n)},s_{1,1}^{(n)},s_{1,2}^{(n)},s_{1,3}^{(n)})}$$

$$= \frac{A}{A_1^{(4n)}} 4^m T_{1,m}^{(s_{1,0}^{(4n)},\cdots,s_{1,3}^{(4n)})} + B(\frac{-1}{563})^n + \frac{C}{A_1^{(3n)} A_1^{(n)}} \sum_{k=0}^{m} \binom{m}{k} 3^{m-k} T_{1,m-k}^{(s_{1,0}^{(3n)},\cdots,s_{1,3}^{(3n)})} T_{1,k}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}$$

$$+ \frac{D}{(A_1^{(2n)})^2} \sum_{k=0}^{m} \binom{m}{k} 2^m T_{1,k}^{(s_{1,0}^{(2n)},\cdots,s_{1,3}^{(2n)})} T_{1,m-k}^{(s_{1,0}^{(2n)},\cdots,s_{1,3}^{(2n)})}$$

$$+ \frac{E}{A_1^{(2n)}} \sum_{k=0}^{m} \binom{m}{k} 2^{m-k} T_{1,m-k}^{(s_{1,0}^{(2n)},\cdots,s_{1,3}^{(2n)})} t_{1,m-k}^{(n)} + F \sum_{k=0}^{m} \binom{m}{k} t_{1,k}^{(n)} t_{1,m-k}^{(n)}$$

$$+ \frac{G}{A_1^{(2n)} (A_1^{(n)})^2} \sum_{\substack{k_1+k_2+k_3=m \\ k_1,k_2,k_3 \geq 0}} \binom{m}{k_1,k_2,k_3} T_{1,k_1}^{(s_{1,0}^{(2n)},\cdots,s_{1,3}^{(2n)})} 2^{k_1} T_{1,k_2}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})} T_{1,k_3}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}$$

$$+ \frac{H}{(A_1^{(n)})^2} \sum_{\substack{k_1+k_2+k_3=m \\ k_1,k_2,k_3 \geq 0}} \binom{m}{k_1,k_2,k_3} t_{1,k_1}^{(n)} T_{1,k_2}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})} T_{1,k_3}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}$$

$$+ \frac{J}{A_2^{(n)} A_1^{(n)}} \sum_{\substack{k_1+k_2+k_3=m \\ k_1,k_2,k_3 \geq 0}} \binom{m}{k_1,k_2,k_3} (-1)^{k_1} T_{2,k_1}^{(s_{2,0}^{(n)},\cdots,s_{2,3}^{(n)})} T_{1,k_2}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})},$$

where $A = -D + E + G + H - 3, \quad B = -4D + 4E + 4G + 4H - 12,$
$C = -E - 2G - H + 4, \quad F = -2D - 2G - 2H + 6, \quad J = 4D - E + 2G - H,$
$t_{1,m}^{(n)}$ *is determined in theorem 12.*

Let

$$\sum_{k=0}^{\infty} t_{2,k}^{(n)} \frac{x^k}{k!} = c_1^n c_2^n c_3^n e^{(\alpha+\beta+\gamma)x}(c_1^n e^{\alpha x} + c_2^n e^{\beta x} + c_3^n e^{\gamma x}) + \cdots$$
$$+ c_2^n c_3^n c_4^n e^{(\alpha+\gamma+\delta)x}(c_2^n e^{\alpha x} + c_3^n e^{\gamma x} + c_4^n e^{\delta x}).$$

Theorem 15. *For $m \geq 0$, $n \geq 1$,$I \neq 0$,*

$$It_{2,m}^{(n)} = \frac{1}{(A_1^{(n)})^4} \sum_{\substack{k_1+k_2+k_3+k_4=m \\ k_1,k_2,k_3,k_4 \geq 0}} \binom{m}{k_1,k_2,k_3,k_4} T_{1,k_1}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})} \cdots T_{1,k_4}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}$$
$$- \frac{A}{A_1^{(4n)}} 4^m T_{1,m}^{(s_{1,0}^{(4n)},\cdots,s_{1,3}^{(4n)})} - \cdots$$
$$- \frac{J}{A_2^{(n)} A_1^{(n)}} \sum_{\substack{k_1+k_2+k_3=m \\ k_1,k_2,k_3 \geq 0}} \binom{m}{k_1,k_2,k_3} (-1)^{k_1} T_{2,k_1}^{(s_{2,0}^{(n)},\cdots,s_{2,3}^{(n)})} T_{1,k_2}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})},$$

where $A = -D + E + G + H - 3$,$B = 12D + 12G - 4J - 12$,
$C = -E - 2G - H + 4$, $F = -2D - 2G - 2H + 6$, $I = 4D - E + 2G - H - J$,
$t_{1,m}^{(n)}$ is determined in theorem 12.

Lemma 7 will be discussed in four cases.
Case 1: $B = C = D = 0$.

Theorem 16. *For $m \geq 0$, $n \geq 1$,*

$$\frac{1}{(A_1^{(n)})^5} \sum_{\substack{k_1+\cdots+k_5=m \\ k_1,\cdots,k_5 \geq 0}} \binom{m}{k_1,\cdots,k_5} T_{1,k_1}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})} \cdots T_{1,k_5}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}$$
$$= \frac{A}{A_1^{(5n)}} 5^m T_{1,m}^{(s_{1,0}^{(5n)},\cdots,s_{1,3}^{(5n)})} + \frac{E}{A_1^{(4n)} A_1^n} \sum_{k=0}^{m} \binom{m}{k} 4^{m-k} T_{1,m-k}^{(s_{1,0}^{(4n)},\cdots,s_{1,3}^{(4n)})} T_{1,k}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}$$
$$+ F(\frac{-1}{563})^n \sum_{k=0}^{m} \binom{m}{k} T_{1,k}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})} + \frac{G}{A_1^{(n)}} \sum_{k=0}^{m} \binom{m}{k} t_{2,k}^{(n)} T_{1,m-k}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}$$
$$+ \frac{H}{A_1^{(3n)} A_1^{(2n)}} \sum_{k=0}^{m} \binom{m}{k} 3^{m-k} 2^k T_{1,m-k}^{(s_{1,0}^{(3n)},\cdots,s_{1,3}^{(3n)})} T_{1,k}^{(s_{1,0}^{(2n)},\cdots,s_{1,3}^{(2n)})}$$
$$+ \frac{I}{A_1^{(3n)}} \sum_{k=0}^{m} \binom{m}{k} 3^k T_{1,k}^{(s_{1,0}^{(3n)},\cdots,s_{1,3}^{(3n)})} t_{1,m-k}^{(n)}$$
$$+ \frac{J}{A_2^{(n)} A_1^{(2n)}} \sum_{\substack{k_1+k_2+k_3=m \\ k_1,k_2,k_3 \geq 0}} \binom{m}{k_1,k_2,k_3} (-1)^{k_1} T_{2,k_1}^{(s_{2,0}^{(n)},\cdots,s_{2,3}^{(n)})} 2^{k_2} T_{1,k_2}^{(s_{1,0}^{(2n)},\cdots,s_{1,3}^{(2n)})}$$

$$+\frac{K}{A_2^{(n)}}\sum_{\substack{k_1+k_2+k_3=m\\k_1,k_2,k_3\geq 0}}\binom{m}{k_1,k_2,k_3}(-1)^{k_1}T_{2,k_1}^{(s_{2,0}^{(n)},\cdots,s_{2,3}^{(n)})}t_{1,k_2}^{(n)}$$

$$+\frac{L}{A_1^{(3n)}(A_1^{(n)})^2}\sum_{\substack{k_1+k_2+k_3=m\\k_1,k_2,k_3\geq 0}}\binom{m}{k_1,k_2,k_3}3^{k_1}T_{1,k_1}^{(s_{1,0}^{(3n)},\cdots,s_{1,3}^{(3n)})}T_{1,k_2}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}T_{1,k_3}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}$$

$$+\frac{M}{A_2^{(n)}(A_1^{(n)})^2}\sum_{\substack{k_1+k_2+k_3+k_4=m\\k_1,k_2,k_3,k_4\geq 0}}\binom{m}{k_1,k_2,k_3,k_4}(-1)^{k_1}T_{2,k_1}^{(s_{2,0}^{(n)},\cdots,s_{2,3}^{(n)})}T_{1,k_2}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}T_{1,k_3}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}$$

$$+\frac{N}{(A_1^{(2n)})^2A_1^n}\sum_{\substack{k_1+k_2+k_3=m\\k_1,k_2,k_3\geq 0}}\binom{m}{k_1,k_2,k_3}2^{k_1}T_{1,k_1}^{(s_{1,0}^{(2n)},\cdots,s_{1,3}^{(2n)})}2^{k_2}T_{1,k_2}^{(s_{1,0}^{(2n)},\cdots,s_{1,3}^{(2n)})}T_{1,k_3}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}$$

$$+\frac{P}{A_1^{(n)}}\sum_{\substack{k_1+k_2+k_3=m\\k_1,k_2,k_3\geq 0}}\binom{m}{k_1,k_2,k_3}t_{1,k_1}^{(n)}t_{1,k_2}^{(n)}T_{1,k_3}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}$$

$$+\frac{Q}{A_1^{(2n)}A_1^{(n)}}\sum_{\substack{k_1+k_2+k_3=m\\k_1,k_2,k_3\geq 0}}\binom{m}{k_1,k_2,k_3}2^{k_1}T_{1,k_1}^{(s_{1,0}^{(2n)},\cdots,s_{1,3}^{(2n)})}t_{1,k_2}^{(n)}T_{1,k_3}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}$$

$$+\frac{R}{A_1^{(2n)}(A_1^{(n)})^3}\sum_{\substack{k_1+k_2+k_3+k_4=m\\k_1,k_2,k_3,k_4\geq 0}}\binom{m}{k_1,k_2,k_3,k_4}2^{k_1}T_{1,k_1}^{(s_{1,0}^{(2n)},\cdots,s_{1,3}^{(2n)})}T_{1,k_2}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}T_{1,k_3}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}T_{1,k_4}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}$$

$$+\frac{S}{(A_1^{(n)})^3}\sum_{\substack{k_1+k_2+k_3+k_4=m\\k_1,k_2,k_3,k_4\geq 0}}\binom{m}{k_1,k_2,k_3,k_4}t_{1,k_1}^{(n)}T_{1,k_2}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}T_{1,k_3}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}T_{1,k_4}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})},$$

where

$$\begin{aligned}
A&=I+2L+2N+P+2Q+6R+4S-14,\\
E&=-I-2L-N-Q-3R-S+5,\\
F&=4G+I+2L+6N+5P+6Q+18R+16S-50,\\
H&=-L-2N-P-Q-4R-3S+10,\\
J&=-G-I-2L-M-2P-3Q-6R-7S+20,\\
K&=-2G-2M-2N-5P-2Q-6R-12S+30,
\end{aligned}$$

$t_{1,m}^{(n)}$ *and* $t_{2,m}^{(n)}$ *are determined in theorem 12 and 15, respectively.*

Case 2: $B\neq 0$, $C=D=0$. Let

$$\begin{aligned}
&\sum_{k=0}^{\infty}t_{3,k}^{(n)}\frac{x^k}{k!}\\
&=c_1^nc_2^nc_3^ne^{(\alpha+\beta+\gamma)x}(c_1^nc_2^ne^{(\alpha+\beta)x}+c_2^nc_3^ne^{(\beta+\gamma)x}+c_3^nc_1^ne^{(\gamma+\alpha)x})+\cdots\\
&\quad+c_2^nc_3^nc_4^ne^{(\beta+\gamma+\delta)x}(c_2^nc_3^ne^{(\beta+\gamma)x}+c_3^nc_4^ne^{(\gamma+\delta)x}+c_4^nc_2^ne^{(\delta+\beta)x}).
\end{aligned}$$

Theorem 17. *For $m \geq 0$, $n \geq 1$,*

$$\begin{aligned}
Bt_{3,m}^{(n)} &= \frac{1}{(A_1^{(n)})^5} \sum_{\substack{k_1+\cdots+k_5=m \\ k_1,\cdots,k_5 \geq 0}} \binom{m}{k_1,\cdots,k_5} T_{1,k_1}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})} \cdots T_{1,k_5}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})} \\
&- \frac{A}{A_1^{(5n)}} 5^m T_{1,m}^{(s_{1,0}^{(5n)},\cdots,s_{1,3}^{(5n)})} - \cdots \\
&- \frac{S}{(A_1^{(n)})^3} \sum_{\substack{k_1+k_2+k_3+k_4=m \\ k_1,k_2,k_3,k_4 \geq 0}} \binom{m}{k_1,k_2,k_3,k_4} t_{1,k_1}^{(n)} T_{1,k_2}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})} T_{1,k_3}^{(s_{1,0}^{n},\cdots,s_{1,3}^{(n)})} T_{1,k_4}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})},
\end{aligned}$$

where

$$\begin{aligned}
A &= -G - J - M + 2N - P - Q - 3S + 6, \\
B &= -2G - K - 2M - 2N - 5P - 2Q - 6R - 12S + 30, \\
E &= G + J + M - N + 2P + 2Q + 3R + 6S - 15, \\
F &= -3G - J - 3K - 7M - 12P - 3Q - 6R - 27S + 60, \\
H &= -L - 2N - P - Q - 4R - 3S + 10, \\
I &= -G - J - 2L - M - 2P - 3Q - 6R - 7S + 20,
\end{aligned}$$

$t_{1,m}^{(n)}$ *and* $t_{2,m}^{(n)}$ *are determined in theorem 12 and 15, respectively.*

Case 3: $C \neq 0$, $B = D = 0$. Let

$$\begin{aligned}
&\sum_{k=0}^{\infty} t_{4,k}^{(n)} \frac{x^k}{k!} \\
&= c_1^n c_2^n c_3^n e^{(\alpha+\beta+\gamma)x} (c_1^{2n} e^{2\alpha x} + c_2^{2n} e^{2\beta x} + c_3^{2n} e^{2\gamma x}) + \cdots \\
&\quad + c_2^n c_3^n c_4^n e^{(\beta+\gamma+\delta)x} (c_2^{2n} e^{2\beta x} + c_3^{2n} e^{2\gamma x} + c_4^{2n} e^{2\delta x}).
\end{aligned}$$

Theorem 18. *For $m \geq 0$, $n \geq 1$,*

$$\begin{aligned}
Ct_{4,m}^{(n)} &= \frac{1}{(A_1^{(n)})^5} \sum_{\substack{k_1+\cdots+k_5=m \\ k_1,\cdots,k_5 \geq 0}} \binom{m}{k_1,\cdots,k_5} T_{1,k_1}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})} \cdots T_{1,k_5}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})} \\
&- \frac{A}{A_1^{(5n)}} 5^m T_{1,m}^{(s_{1,0}^{(5n)},\cdots,s_{1,3}^{(5n)})} - \cdots \\
&- \frac{S}{(A_1^{(n)})^3} \sum_{\substack{k_1+k_2+k_3+k_4=m \\ k_1,k_2,k_3,k_4 \geq 0}} \binom{m}{k_1,k_2,k_3,k_4} t_{1,k_1}^{(n)} T_{1,k_2}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})} T_{1,k_3}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})} T_{1,k_4}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})},
\end{aligned}$$

where

$$
\begin{aligned}
A &= I + 2L + 2N + P + 2Q + 6R + 4S - 14,\\
C &= -G - I - J - 2L - M - 2P - 3Q - 6R - 7S + 20,\\
E &= -I - 2L - N - Q - 3R - S + 5,\\
F &= 3G - J - M + 6N + 3P + 3Q + 12R + 9S - 30,\\
H &= -L - 2N - P - Q - 4R - 3S + 10,\\
K &= -2G - 2M - 2N - 5P - 2Q - 6R - 12S + 30,
\end{aligned}
$$

$t^{(n)}_{1,m}$ *and* $t^{(n)}_{2,m}$ *are determined in theorem 12 and 15, respectively.*

Case 4: $D \neq 0$, $B = C = 0$. Let

$$
\begin{aligned}
&\sum_{k=0}^{\infty} t^{(n)}_{5,k}\frac{x^k}{k!}\\
&= c_1^n c_2^n c_3^n e^{(\alpha+\beta+\gamma)x}(c_1^n e^{\alpha x} + c_2^n e^{\beta x} + c_3^n e^{\gamma x})^2 + \cdots\\
&\quad + c_2^n c_3^n c_4^n e^{(\beta+\gamma+\delta)x}(c_2^n e^{\beta x} + c_3^n e^{\gamma x} + c_4^n e^{\delta x})^2.
\end{aligned}
$$

Theorem 19. *For* $m \geq 0$, $n \geq 1$,

$$
\begin{aligned}
Dt^{(n)}_{5,m} &= \frac{1}{(A_1^{(n)})^5} \sum_{\substack{k_1+\cdots+k_5=m\\ k_1,\cdots,k_5\geq 0}} \binom{m}{k_1,\cdots,k_5} T_{1,k_1}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})} \cdots T_{1,k_5}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})}\\
&- \frac{A}{A_1^{(5n)}} 5^m T_{1,m}^{(s_{1,0}^{(5n)},\cdots,s_{1,3}^{(5n)})} - \cdots\\
&- \frac{S}{(A_1^{(n)})^3} \sum_{\substack{k_1+k_2+k_3+k_4=m\\ k_1,k_2,k_3,k_4\geq 0}} \binom{m}{k_1,k_2,k_3,k_4} t^{(n)}_{1,k_1} T_{1,k_2}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})} T_{1,k_3}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})} T_{1,k_4}^{(s_{1,0}^{(n)},\cdots,s_{1,3}^{(n)})},
\end{aligned}
$$

where

$$
\begin{aligned}
A &= I + 2L + 2N + P + 2Q + 6R + 4S - 14,\\
D &= -G - I - J - 2L - M - 2P - 3Q - 6R - 7S + 20,\\
E &= -I - 2L - N - Q - 3R - S + 5,\\
F &= -3G - 6I - 7J - 7M + 6N - 12L - 9P - 15Q - 24R - 33S + 90,\\
H &= -L - 2N - P - Q - 4R - 3S + 10,\\
K &= 2I + 2J - 2N + 4L - P + 4Q + 6R + 2S - 10,
\end{aligned}
$$

$t^{(n)}_{1,m}$ *and* $t^{(n)}_{2,m}$ *are determined in theorem 12 and 15, respectively.*

7 Some more interesting general expressions

We shall give some more interesting general expressions.

Lemma 11. *For $n \geq 1$, we have*

$$\begin{aligned}&(c_2c_3+c_3c_4+c_4c_2)e^{\alpha x}+(c_3c_4+c_4c_1+c_1c_3)e^{\beta x}\\&+(c_1c_2+c_1c_4+c_4c_2)e^{\gamma x}+(c_1c_2+c_2c_3+c_1c_3)e^{\delta x}\\=&\frac{1}{563}\sum_{k=0}^{\infty}T_k^{(146,416,581,1080)}\frac{x^k}{k!}.\end{aligned}$$

Theorem 20.

$$\begin{aligned}&(c_2c_3+c_3c_4+c_4c_2)^ne^{\alpha x}+(c_3c_4+c_4c_1+c_1c_3)^ne^{\beta x}\\&+(c_1c_2+c_1c_4+c_4c_2)^ne^{\gamma x}+(c_1c_2+c_2c_3+c_1c_3)^ne^{\delta x}\\=&\frac{1}{A_3^n}\sum_{k=0}^{\infty}T_{3,k}^{(s_{3,0}^n,s_{3,1}^n,s_{3,2}^n,s_{3,3}^n)}\frac{x^k}{k!},\end{aligned}$$

where $s_{3,0}^{(n)}$, $s_{3,1}^{(n)}$, $s_{3,2}^{(n)}$, $s_{3,3}^{(n)}$ and $A_3^{(n)}$ satisfy the recurrence relations:

$$\begin{aligned}&s_{3,0}^{(n)}=\pm\mathrm{lcm}(b_1,b_2,b_3),\quad s_{3,1}^{(n)}=Ms_{3,0}^{(n)},\quad s_{3,2}^{(n)}=Ns_{3,0}^{(n)},\quad s_{3,3}^{(n)}=Ps_{3,0}^{(n)},\\&A_3^{(n)}=A_3^{(n-1)}\frac{(-16s_{3,3}^{(n)}+103s_{3,2}^{(n)}-157s_{3,1}^{(n)}-10s_{3,0}^{(n)})}{-8s_{3,3}^{(n)}+10s_{3,2}^{(n)}+7s_{3,1}^{(n)}-6s_{3,0}^{(n)}}.\end{aligned}$$

b_1,b_2, b_3, M, N and P are determined in the proof.

Proof. Similarly to the proof of Theorem 12, we consider the form

$$h_1^{(n)}e^{\alpha x}+h_2^{(n)}e^{\beta x}+h_3^{(n)}e^{\gamma x}+h_4^{(n)}e^{\delta x}=\sum_{k=0}^{\infty}s_{3,k}^{(n)}\frac{x^k}{k!}.$$

By $h_1^{(n)}=A_3^{(n)}(c_2c_3+c_3c_4+c_4c_2)^n$, we can obtain the following recurrence relation:

$$\begin{aligned}A&=-650s_{3,0}^{(n-1)}+385s_{3,1}^{(n-1)}+854s_{3,2}^{(n-1)}-664s_{3,3}^{(n-1)},\\B&=1862s_{3,0}^{(n-1)}+231s_{3,1}^{(n-1)}-1627s_{3,2}^{(n-1)}+1178s_{3,3}^{(n-1)},\\C&=-1100s_{3,0}^{(n-1)}-2380s_{3,1}^{(n-1)}-417s_{3,2}^{(n-1)}+522s_{3,3}^{(n-1)},\\D&=16s_{3,0}^{(n-1)}-252s_{3,1}^{(n-1)}-170s_{3,2}^{(n-1)}+148s_{3,3}^{(n-1)},\\E&=1198s_{3,0}^{(n-1)}-1083s_{3,1}^{(n-1)}-1906s_{3,2}^{(n-1)}+1368s_{3,3}^{(n-1)},\\F&=-2434s_{3,0}^{(n-1)}+814s_{3,1}^{(n-1)}+3473s_{3,2}^{(n-1)}-1769s_{3,3}^{(n-1)},\end{aligned}$$

$$
\begin{aligned}
G &= 988s_{3,0}^{(n-1)} + 1935s_{3,1}^{(n-1)} - 757s_{3,2}^{(n-1)} - 933s_{3,3}^{(n-1)},\\
H &= -518s_{3,0}^{(n-1)} + 801s_{3,1}^{(n-1)} + 920s_{3,2}^{(n-1)} - 834s_{3,3}^{(n-1)},\\
I &= 268s_{3,0}^{(n-1)} + 186s_{3,1}^{(n-1)} - 132s_{3,2}^{(n-1)} - 16s_{3,3}^{(n-1)},\\
J &= -1303s_{3,0}^{(n-1)} - 1690s_{3,1}^{(n-1)} + 146s_{3,2}^{(n-1)} + 666s_{3,3}^{(n-1)},\\
K &= -3980s_{3,0}^{(n-1)} - 3273s_{3,1}^{(n-1)} + 1638s_{3,2}^{n-1} + 620s_{3,3}^{(n-1)},\\
L &= 1012s_{3,0}^{(n-1)} - 869s_{3,1}^{(n-1)} - 1490s_{3,2}^{(n-1)} + 1116s_{3,3}^{(n-1)},\\
M &= \frac{(LA-DI)(FA-BE)-(HA-DE)(JA-BI)}{(GA-CE)(JA-BI)-(KA-CI)(FA-BE)},\\
N &= \frac{M(GA-CE)+(HA-DE)}{BE-FA}, \quad P = -\frac{1}{A}(BN+CM+D),
\end{aligned}
$$

$$
\begin{aligned}
&M = \frac{a_1}{b_1}, \quad N = \frac{a_2}{b_2}, \quad P = \frac{a_3}{b_3}, \quad \gcd(a_i, b_i) = 1,\\
&s_{3,0}^{(n)} = \pm \mathrm{lcm}(b_1, b_2, b_3), \quad s_{3,1}^{(n)} = Ms_{3,0}^{(n)}, \quad s_{3,2}^{(n)} = Ns_{3,0}^{(n)}, \quad s_{3,3}^{(n)} = Ps_{3,0}^{(n)},\\
&A_3^{(n)} = A_3^{(n-1)} \frac{(-16s_{3,3}^{(n)} + 103s_{3,2}^{(n)} - 157s_{3,1}^{(n)} - 10s_{3,0}^{(n)})}{-8s_{3,3}^{(n)} + 10s_{3,2}^{(n)} + 7s_{3,1}^{(n)} - 6s_{3,0}^{(n)}}.
\end{aligned}
$$

We choose the symbol of $s_{3,0}^{(n)}$ such that for some k_0,$\forall k \geq k_0$, $T_{3,k}^{(s_{3,0}^{(n)}, s_{3,1}^{(n)}, s_{3,2}^{(n)}, s_{3,3}^{(n)})}$ is positive. □

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