Progress on Asymptotics of Klazar-type Set Partition Pattern Avoidance

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Abstract

We consider asymptotics of set partition pattern avoidance in the sense of Klazar. The main result of this paper extends work of Alweiss, and finds a classification for π such that the number of set partitions avoiding π grows more slowly than n^{cn}. Several conjectures are proposed, and the related question of parallel permutation pattern avoidance (which surprisingly seems to have not been addressed prior to now) is considered and solved in the case of the trivial permutation.

1 Introduction

Pattern avoidance has been a popular area of study in the last forty years, and a fundamental question that has been routinely asked is that of asymptotics. That is, for some pattern p, how does the avoidance function $A_n(p)$, equal to the number of patterns of size n avoiding p, grow? This has been especially well-studied in the most classical pattern avoidance area, that of permutations. The most famous result of this kind is the Stanley-Wilf Conjecture, proved by Marcus and Tardos in [7].

More recently, the study of pattern avoidance and the corresponding asymptotics has branched into other structures than permutations. The most well-studied type of pattern avoidance on set partitions is the RGF-type pattern avoidance, studied in great detail by for example Mansour [6], where many asymptotics of RGF-type avoidance are studied in detail.

Klazar [5] proposed a different, stronger notion of set partition pattern avoidance than RGF-type avoidance, proving several results about special cases involving the generating function of the avoidance sequence, and providing several conjectures when the partitions have what this paper will refer to as permutability 1 (Klazar refers to these as srps). The main theorem of this paper is a general asymptotic bound in this case.

2 Definitions and Preliminary Results

Definition. A *set partition* of n is a partition of $[n]$ into sets, where we ignore ordering of sets and ordering within the sets. We will write set partitions with slashes between the sets, as in $T_1/T_2/\cdots/T_m$ for some m. The *standard form* of a set partition is what is obtained from writing each T_i in increasing order, and then rearranging the sets so that $\min T_1 < \min T_2 < \cdots < \min T_m$. The T_i are called the *blocks* of the partition.

For example, 1635/24 and 24/1356 are not in standard form; the standard form for this partition is 1356/24. We now define a simple but useful statistic.

Definition. If $\pi = T_1/\cdots/T_m$ is a set partition of n, we define the *rank* of π, denoted $\text{rank}(\pi)$, to be $n - m$.

It is well known that the number of set partitions of n is the Bell number B_n, and the number of set partitions of n into m sets (or in other words, partitions of n of rank $n - m$) is the Stirling number of the second kind $S(n, m)$.

Definition. A set partition π of n *contains* a set partition π' of k in the Klazar sense (which we will use for the remainder of this paper) if there is a subset S of $[n]$ of cardinality k such that when π is restricted to the elements of S, the result is order-isomorphic to π'. Otherwise, we say π *avoids* π'.

For example, 136/5/27 contains 14/23 because when we restrict 136/5/27 to the set $\{2, 3, 6, 7\}$, the result is 36/27, which is order-isomorphic to 23/14, standardizing to 14/23. However, it avoids 1/234.

We can think of containment in the following way: if we have some $f : [m] \to [n]$ and a set partition of $[n]$, we can take the pullback under f to get a partition of $[m]$, where a and b are in the same partition if and only if $f(a)$ and $f(b)$ are. Then π contains π' if and only if π' is the pullback of π under some order-preserving injection.

Note that this Klazar notion of avoidance differs from the RGF notion of pattern avoidance in set partitions, studied in detail by Mansour [6], where switching the order of the sets during standardization is not allowed.

We will be concerned with the enumeration of the number of partitions of a given length that avoid a particular pattern.

Definition. If π is some set partition of k, let $B_n(\pi)$ be the number of set partitions of n that avoid π, and let $S(n, m, \pi)$ be the number of set partitions of n into m sets that avoid π.

This paper will primarily be devoted to finding log-asymptotics for $B_n(\pi)$ for particular π.

The notation is analogous to that for Bell and Stirling numbers.

Definition. A *layered partition* is a partition $T_1/\cdots/T_m$ such that $\max T_i < \min T_{i+1}$ for all $i \in [m-1]$. Equivalently, each set consists of an interval of consecutive integers.

For example, 12/3456/789 is layered while 13/2456/789 is not.

Alweiss [1] found the correct log-asymptotic for $B_n(\pi)$ in the case where π is layered. This represented the first evaluation of the correct log-asymptotic that works in exponentially many cases in n.

An important notion in this paper will be relating set-partition pattern avoidance to permutation pattern avoidance. To this end, we define the following notion.

Definition. Let $\sigma_1, \ldots, \sigma_k$ be permutations $[n] \to [n]$. We define the *set partition correspondent to* $(\sigma_1, \ldots, \sigma_k)$ to be the partition $T_1/\cdots/T_n$ of $(k+1)n$ such that $B_i = \{i, n + \sigma_1(i), 2n + \sigma_2(i), \ldots, kn + \sigma_k(i)\}$. It is easy to see that this is indeed a set partition, and we will write it $[\sigma_1, \ldots, \sigma_k]$.

Notice that a set partition of $(k+1)n$ is correspondent to some $(\sigma_1, \ldots, \sigma_k)$ if and only if every set in the partition contains exactly one element from each of $\{1, \ldots, n\}$, $\{n + 1, \ldots, 2n\}$,$\ldots$,$\{kn + 1, \ldots, (k+1)n\}$. Klazar [5] studied the avoidance generating functions of partitions of the form $[\sigma]$ (which he referred to as srps).

Now, we define what we will call *parallel pattern avoidance* for k-tuples of permutations $(\sigma_1, \ldots, \sigma_k)$.

Definition. If $\sigma_1, \ldots, \sigma_k \in S_n$ and $\sigma_1', \ldots, \sigma_k'$ are permutations of S_m, $(\sigma_1, \ldots, \sigma_k)$ contains (respectively avoids) $(\sigma_1', \ldots, \sigma_k')$ if there exists (respectively does not exist) indices $c_1 < \cdots < c_m$ such that $\sigma_i(c_1)\sigma_i(c_2)\cdots\sigma_i(c_m)$ is order-isomorphic to σ_i' for all i.

We will occasionally say 'contains/avoids in parallel' to refer to this notion in particular.

For $k = 1$, parallel pattern avoidance is equivalent to the classical case of permutation pattern containment/avoidance. This idea of parallel avoidance in k-tuples of permutations also reduces to several other interesting concepts in special cases; for example, (σ_1, σ_2) avoids $(12, 21)$ if and only if $\sigma_1^{-1} \leq \sigma_2^{-1}$ in the Weak Bruhat Order, which has been previously studied; for example, see [4] and $A007767$ in [8].

We now relate this to our topic of partition pattern avoidance.

Proposition 2.1. *Let $\sigma_1, \ldots, \sigma_m$ be partitions of n and $\sigma_1', \ldots, \sigma_m'$ be partitions of m. The following two statements are equivalent:*

- *The k-tuple of permutations $(\sigma_1, \ldots, \sigma_k)$ contains the m-tuple of permutations $\sigma_1', \ldots, \sigma_k'$.*
- *The set partition $[\sigma_1, \ldots, \sigma_k]$ contains the set partition $[\sigma_1', \ldots, \sigma_k']$.*

Proof. If $(\sigma_1, \ldots, \sigma_k)$ contains $(\sigma_1', \ldots, \sigma_k')$, we have indices $c_1 < \cdots < c_m$ with $\sigma_i(c_1)\cdots\sigma_i(c_m)$ order-isomorphic to σ_i. $[\sigma_1, \ldots, \sigma_k]$ has blocks $T_1, \ldots, T_n$ given by $T_i = \{i, n + \sigma_1(i), 2n + \sigma_2(i), \ldots, kn + \sigma_k(i)\}$. Restricting this to simply the elements in $T_{c_1}, \ldots, T_{c_m}$, we have blocks given by $\{c_i, n + \sigma_1(c_i), \ldots, kn + \sigma_k(c_i)\}$. We show that this is order-isomorphic to $[\sigma_1', \ldots, \sigma_k']$. Since $c_i = \min T_{c_i}$, and the c_i are increasing, the block T_{c_i} must correspond the i^{th} block of $[\sigma_1', \ldots, \sigma_k']$, which is $\{i, m + \sigma_1'(i), \ldots, km + \sigma_k'(i)\}$. Thus we must show that $j_1 n + \sigma_{j_1}(c_{i_1}) < j_2 n + \sigma_{j_2}(c_{i_2})$ if and only if $j_1 m + \sigma_{j_1}(i_1) < j_2 m + \sigma_{j_2}(i_2)$. But since

$1 \leq \sigma_a(b) \leq n$ and $1 \leq \sigma'_a(b) \leq m$ for all a, b, the first statement is equivalent to $j_1 < j_2$ or $j_1 = j_2 = j$ and $\sigma_j(c_{i_1}) < \sigma_j(c_{i_2})$, and the second is equivalent to $j_1 < j_2$ or $j_1 = j_2 = j$ and $\sigma'_j(i_1) < \sigma'_j(i_2)$. These are equivalent by the definition of pattern containment for k-tuples of permutations.

Now suppose $[\sigma_1, \ldots, \sigma_k]$ contains $[\sigma'_1, \ldots, \sigma'_k]$. Since all blocks of both partitions have size $k+1$, the blocks of the latter partition must correspond exactly to m block of the former, say blocks $T_{c_1}, \ldots, T_{c_m}$ with $c_1 < \cdots < c_m$. Now following the exact same argument in reverse, we see that $(\sigma_1, \ldots, \sigma_k)$ contains $(\sigma'_1, \ldots, \sigma'_k)$ (at indices $c_1, \ldots, c_m$), as we showed the ordering information is exactly equivalent in both cases. $\square$

Now, we discuss the concepts of layered and permutation-correspondent partitions to form two useful statistics.

3 Thickness and Permutability

Definition. Let π be a partition. The *thickness* of π, which we will call $\mathrm{th}(\pi)$, is the maximum rank of a layered partition contained in π.

Definition. The *permutability* of π, which we will call $\mathrm{pm}(\pi)$, is the minimum k such that there exists a k-tuple of permutations $(\sigma_1, \ldots, \sigma_k)$ such that the correspondent partition $[\sigma_1, \ldots, \sigma_k]$ contains π.

Note that as one would expect, $[\sigma_1, \ldots, \sigma_k]$ has permutability k, as it has a block of size $k+1$, which is not contained in $[\sigma'_1, \ldots, \sigma'_{k-1}$ for any choice of the σ'_i. Similarly, a layered partition of rank k has thickness k, as taking elements away from a partition can only maintain or decrease the rank.

Proposition 3.1. *For all partitions π, $pm(\pi) \geq th(\pi)$, and if π is permutation-correspondent then equality holds.*

Proof. Suppose there are permutations $\sigma_1, \ldots, \sigma_k$ such that $[\sigma_1, \ldots, \sigma_k]$ contains π. We must show that all layered partitions π' contained in π have rank at most k. Since π' is contained in π which is in turn contained in $[\sigma_1, \ldots, \sigma_k]$, it suffices to show that any layered partition contained in $[\sigma_1, \ldots, \sigma_k]$ has rank at most k–and to show equality in the desired case, we must just exhibit a layered partition of rank exactly k.

Suppose $\sigma_i \in S_n$ for all i. Take a layered partition π' of $[m]$ of rank ℓ. In a layered partition, the number of pairs $(i, i+1)$ that are in different sets is equal to the number of blocks minus one, because in $T_1/\cdots/T_m$ the i that satisfy this are $\max T_j$, $1 \leq j \leq m-1$. Since the total number of such pairs is the number of elements minus one, the number of pairs $(i, i+1)$ that are in the same set is the rank of the partition, ℓ.

Sort these ℓ pairs in increasing order, so (a_1, a_1+1), (a_2, a_2+1),..., $(a_\ell, a_\ell+1)$, with $a_{i+1} \geq a_i + 1$. Since π' is contained in $[\sigma_1, \ldots, \sigma_k]$, there is some order-preserving function $f : [m] \to [(k+1)n]$ such that the pullback of the partition $[\sigma_1, \ldots, \sigma_n]$ via f is π'. In particular, $f(a_i) < f(a_i+1) \leq f(a_{i+1})$ for all i and $f(a_i)$ and $f(a_i+1)$ are in the same set.

In $[\sigma_1, \ldots, \sigma_k]$, it is easy to see that if $a < b$ are in the same set, then there is a multiple of n in the interval $[a, b)$, by the definition of permutation-correspondent partitions. Thus the $[f(a_i), f(a_i+1))$ are non-intersecting half-open intervals each containing a multiple of n. Thus in $[f(a_1), f(a_\ell+1)) \subset [1, (k+1)n)$, there are at least ℓ multiple of n. But $[1, (k+1)n)$ has k multiples of n, so $\ell \leq k$, as desired.

For equality, it suffices to realize that $[\sigma_1, \ldots, \sigma_k]$ has blocks of size $k+1$, and a single block of size $k+1$ is a layered partition of rank k. □

A useful characterization of permutability is the following:

Proposition 3.2. *For a fixed k and a set partition π of m, the following are equivalent.*

- *$pm(\pi) \leq k$.*
- *There exists $0 = a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq a_{k+1} = m$ such that the $k+1$ intervals $(0, a_1]$, $(a_1, a_2]$, …, $(a_{k-1}, a_k]$, and $(a_k, m]$ each contain at most one element from each block of π. (In other words, $[m]$ can be divided into $k+1$ intervals, each of which contains at most one element from each block.)*

Proof. Suppose the first bullet holds. Then there is $n \in \mathbb{N}$ and $\sigma_1, \ldots, \sigma_k \in S_n$ such that π is contained in $[\sigma_1, \ldots, \sigma_n]$. Then we have an order-preserving $f : [m] \to [(k+1)n]$ with π being the pullback of $[\sigma_1, \ldots, \sigma_k]$ under f. Letting $a_i = \max\{i : f(i) \leq in\}$ (for $0 \leq i \leq k$, and $a_{k+1} = m$), we have that the image of $(a_i, a_{i+1}]$ under f contains only elements in $(in+1, (i+1)n]$. But this interval contains exactly one element from each block of $[\sigma_1, \ldots, \sigma_k]$, so since f is injective, we still get at most one element from each block of π in $(a_i, a_{i+1}]$.

Now suppose the second bullet holds. Suppose π has n blocks. Since $(a_i, a_{i+1}]$ contains at most one element from each block, $a_{i+1} - a_i \leq n$ for all $i \in \{0, 1, \ldots, k\}$. Thus there is an order-preserving injection $f : [m] \to [(k+1)n]$ that sends a_i to in for $i \in [k+1]$ (simply choose values for the images of the elements in $(a_i, a_{i+1}]$ in $(in, (i+1)n]$ so that they are in the correct order). This induces a partition on the image of f in $[(k+1)n]$. We will include the remaining elements of $[(k+1)n]$ in this image as follows: Since in each interval $(in, (i+1)n]$ there is at most one element from each block, and there are n blocks and n elements in that interval in total, we can assign the remaining elements to the blocks so that there is exactly one element from each set in that block. Then this resulting partition, call it π', will pull back to π under f, so π is contained in π'. Let $T_1, \ldots, T_n$ be the blocks of π'. T_i contains exactly one element of $(jn, (j+1)n]$ for all $i \in [n]$, $j \in \{0, \ldots, k\}$, call it c_{ij}. The set $\{c_{ij} : i \in [n]\}$ must be the interval $(jn, (j+1)n]$, so if $\sigma_j : [n] \to [n]$ is defined via $\sigma_j(i) = c_{ij} - jn$, then σ_j is a permutation. Since T_i consists of the elements $c_{ij} = \sigma_j(i) + jn$ for $1 \leq j \leq k$ and the element i (corresponding to when $j = 0$), we see that $\pi' = [\sigma_1, \ldots, \sigma_k]$, as desired. □

From this we can see that we do not always have $\mathrm{pm}(\pi) = \mathrm{th}(\pi)$; for example, $\mathrm{th}(1267/345) = 3$ but by the proposition above we can see that $\mathrm{pm}(1267/345) = 4$; the coarsest division into intervals that satisfy the conditions of the proposition is $[1], [2, 3], [4], [5, 6], [7]$.

We also have the following corollary.

Corollary 3.3. *If π is layered, then $th(\pi) = pm(\pi)$.*

Proof. We already have that the second quantity is at least the first by Proposition 3.1. To prove the reverse inequality, supposing π is a layered partition of $[m]$ of rank k, it suffices by Proposition 3.2 to show that we can divide $[m]$ into $k+1$ intervals, each containing at most one element of each block of π. We do this by breaking $[m]$ at each point where the two consecutive elements are in the same block. By the layeredness of π, this satisfies the desired condition, and we break it at k points by the definition of rank and layeredness of π (since two consecutive elements are not in the same interval if and only if we are going from the maximal element of one block up to the minimal element of another, which occurs a number of times equal to the number of blocks minus one). Therefore, we end up with $k+1$ intervals, as desired. $\square$

4 Current and Prior Results

The following result is due to Alweiss. [1]

Theorem 4.1. *Let π be a set partition. Then there exists a constant $c_1(\pi) > 0$ such that*

$$B_n(\pi) \geq c_1(\pi)^n n^{n\left(1-\frac{1}{th(\pi)}\right)}$$

for all n.

Furthermore, if π is layered (and so $th(\pi)$ is simply the rank of π in this case), then there exists another constant $c_2(\pi)$ such that

$$B_n(\pi) \leq c_2(\pi)^n n^{n\left(1-\frac{1}{th(\pi)}\right)}$$

for all n.

Alweiss also conjectured that $1 - \frac{1}{\mathrm{th}(\pi)}$ in the exponent is optimal in all cases, but this is false, as the following result shows.

Theorem 4.2. *Let π be a set partition. Then there exists a constant $c_1(\pi) > 0$ such that*

$$B_n(\pi) \geq c_1(\pi)^n n^{n\left(1-\frac{1}{pm(\pi)}\right)}$$

for all n.

As the earlier example of 1267/345 shows, it is possible for $\mathrm{pm}(\pi) > \mathrm{th}(\pi)$ to occur, so this disproves Alweiss's conjecture. However, this lower bound and previous results suggest a similar conjecture may be true.

Conjecture 1. *Let π be a set partition. Then the following bounds hold.*

- *(Weak Form)* $\displaystyle\lim_{n\to\infty} \frac{\log B_n(\pi)}{n \log n} = 1 - \frac{1}{pm(\pi)}$.

- *(Strong Form) There exists a constant $c_2(\pi)$ such that $B_n(\pi) \leq c_2(\pi)^n n^{n\left(1-\frac{1}{pm(\pi)}\right)}$ for all n.*

The naming is accurate as the second bullet and Theorem 4.2 together imply the first bullet. When $\pm(\pi) = 1$, the strong form of this conjecture was proposed by Klazar as Problem 1 in [5].

The largest result of this paper is the following.

Theorem 4.3. *The weak form of Conjecture 1 holds in the case where $pm(\pi) = 1$. That is, if $pm(\pi) = 1$, then*

$$\lim_{n\to\infty} \frac{\log B_n(\pi)}{n \log n} = 0.$$

This result, together with Theorem 4.2, shows that $\mathrm{pm}(\pi) = 1$ is a necessary and sufficient condition for $B_n(\pi)$ to grow slower than n^{cn} for all $c > 0$.

5 Proof of Theorem 4.2

We will now prove Theorem 4.2.

Let π be a set partition with $\mathrm{pm}(\pi) = k$. Assume $k > 1$, as the case $k = 1$ is trivial. By the interval criterion for permutability, removing blocks containing one element from π does not change its permutability (as it preserves intervals containing exactly one element from each set). Thus if π' is π with all one-element blocks removed, any partition avoiding π' must avoid π since π contains π', so $B_n(\pi) \geq B_n(\pi')$ and $\mathrm{pm}(\pi') = k$. So it suffices to show the problem for π'; that is, we can assume without loss of generality that π has no blocks of size 1. This means that we can add any blocks of size 1 to a partition of $[n-i]$ avoiding π to get a partition of $[n]$ avoiding π. If we only range over partitions of $[n-i]$ with no blocks of size 1, the resulting partitions will all be distinct. Let $B'_n([\sigma])$ be the number of partitions of $[n]$ avoiding $[\sigma]$ with no blocks of size 1. Then since we can perform the process of adding single blocks in $\binom{n}{i}$ ways, we have $B_n(\sigma) \geq \binom{n}{i} B'_{n-i}(\sigma)$

Now suppose n is a multiple of k, $n = km$. Then if $\sigma_1, \ldots, \sigma_{k-1} \in S_m$ are permutations, then $[\sigma_1, \ldots, \sigma_{k-1}]$ will be a partition of $[m(k-1+1)] = [n]$, and by the definition of permutability, it must avoid π. Since these all correspond to different partitions, and all blocks have size $k > 1$, we can count them to see that

$$B'_n(\pi) \geq m!^{k-1} = \left(\frac{n}{k}\right)!^{k-1}.$$

By Stirling Approximation, there is $c > 0$ such that $\left(\frac{n}{k}\right)! > c^{\frac{n}{k}} \left(\frac{n}{k}\right)^{\frac{n}{k}} = \left(\frac{c}{k}\right)^{\frac{n}{k}} n^{\frac{n}{k}}$. Substituting this in,

$$B'_n(\pi) \geq \left(\frac{c}{k}\right)^{\frac{(k-1)n}{k}} n^{\frac{(k-1)n}{k}} = c_0^n n^{n\left(1-\frac{1}{k}\right)},$$

where $c_0 = \left(\frac{c}{k}\right)^{\frac{k-1}{k}}$.

Now, let $n = km + i$, $0 \le i \le k-1$. Since we are dealing with asymptotics we may assume that $n > k$. We have that since $n - i$ is a multiple of k, assuming $c_0 < 1$ without loss of generality for ease of manipulation,

$$\begin{aligned}
B_n(\pi) &\ge \binom{n}{i} B'_{n-i}(\pi) \\
&\ge \binom{n}{i} c_0^{n-i}(n-i)^{(n-i)\left(1-\frac{1}{k}\right)} \\
&\ge \frac{(n-i)^i}{i!} c_0^{n-i}(n-i)^{(n-i)\left(1-\frac{1}{k}\right)} \\
&= \frac{c_0^n}{c_0^i i!}(n-i)^{(n-i)\left(1-\frac{1}{k}\right)+i} \\
&= \frac{c_0^n}{c_0^i i!}(n-i)^{n\left(1-\frac{1}{k}\right)+\frac{i}{k}} \\
&\ge \frac{c_0^n}{c_0^i i!}(n-i)^{n\left(1-\frac{1}{k}\right)} \\
&= \frac{c_0^n}{c_0^i i!}\left(1-\frac{i}{n}\right)^{n\left(1-\frac{1}{k}\right)} n^{n\left(1-\frac{1}{k}\right)} \\
&> \frac{c_0^n}{k!}\left(1-\frac{k}{n}\right)^{n\left(1-\frac{1}{k}\right)} n^{n\left(1-\frac{1}{k}\right)}.
\end{aligned}$$

Since $\left(1-\frac{k}{n}\right)^n$ is positive for $n \in [k+1, \infty]$ and limits to $e^{-k} \ne 0$ as $n \to \infty$, it must have a minimum, call it d, on $n \in [k+1, \infty]$. Substituting this in and noting $d < 1$,

$$\begin{aligned}
B_n(\pi) &> \frac{c_0^n}{k!}\left(1-\frac{k}{n}\right)^{n\left(1-\frac{1}{k}\right)} n^{n\left(1-\frac{1}{k}\right)} \\
&= \frac{c_0^n}{k!}\left(\left(1-\frac{k}{n}\right)^n\right)^{\left(1-\frac{1}{k}\right)} n^{n\left(1-\frac{1}{k}\right)} \\
&\ge \frac{c_0^n}{k!} d^{\left(1-\frac{1}{k}\right)} n^{n\left(1-\frac{1}{k}\right)} \\
&\ge \frac{c_0^n d}{k!} n^{n\left(1-\frac{1}{k}\right)} \\
&> \left(\frac{c_0 d}{k!}\right)^n n^{n\left(1-\frac{1}{k}\right)} \\
&= c_1^n n^{n\left(1-\frac{1}{k}\right)},
\end{aligned}$$

where $c_1 = \frac{c_0 d}{k!}$. This concludes the proof of the theorem.

6 Proof of Theorem 4.3

We now turn to the proof of Theorem 4.3; that is, the case of $\mathrm{pm}(\pi) = 1$.

A key idea which will reappear later in this paper is that results on set partition pattern avoidance depend on (via permutation-correspondent partitions) results in the concept defined earlier of parallel permutation pattern avoidance, because as proven earlier $[\sigma_1, \ldots, \sigma_k]$ avoids $[\sigma'_1, \ldots, \sigma'_k]$ if and only if $(\sigma_1, \ldots, \sigma'_k)$ avoids $(\sigma'_1, \ldots, \sigma'_k)$ in parallel avoidance. As we are dealing with the case $k = 1$, this corresponds to the classical case of asymptotics in permutation pattern avoidance. To this end, we will utilise the Stanley-Wilf Conjecture, famously proved by Marcus and Tardos in 2003. [7]

Theorem 6.1 (Marcus, Tardos)**.** *For any permutation $\sigma \in S_k$, let $S_n(\sigma)$ be the set of permutations in S_n avoiding σ. Then for all σ there exists a constant $c(\sigma)$ such that*

$$|S_n(\sigma)| \leq c^n.$$

We will now turn to the main proof. We proceed via recursion. For now we will assume n is a power of 2. Notice that we can define a set partition of $[n]$ via three pieces of data, two of which lead to recursion:

1. The induced partition on $\{1, \ldots, \frac{n}{2}\}$.
2. The induced partition on $\{\frac{n}{2} + 1, \ldots, n\}$.
3. The information of which sets in the two induced partitions above correspond to each other; that is, are the same set in the full partition.

As an example, for the partition 1246/35/78, we have the induced partition 124/3, the induced partition 5/6/78, and the matching data that 3 corresponds to 5 and 124 corresponds to 6.

Let π be a set partition with $\mathrm{pm}(\pi) = 1$. π is contained in some partition $[\sigma]$ for some permutation σ by the definition of permutability. Since $[\sigma]$ contains π, $B_n([\sigma]) \geq B_n(\pi)$, and since $\mathrm{pm}([\sigma]) = 1$, it suffices to prove Theorem 4.3 for $[\sigma]$. Thus we may assume without loss of generality that we are dealing with a partition of the form $\pi = [\sigma]$.

Take any set partition π' of $[n]$, say with m blocks, avoiding $[\sigma]$. Then the restriction of π' to each of $\{1, \ldots, \frac{n}{2}\}$ and $\{\frac{n}{2} + 1, \ldots, n\}$ must avoid σ. Suppose these restrictions have a and b blocks, respectively. Then in the third piece of data above, we must match exactly $a + b - m$ blocks from the first restriction to $a + b - m$ blocks from the second restriction. There are $\binom{a}{a+b-m}\binom{b}{a+b-m}$ ways to choose the blocks that we will match to each other. Now it remains to choose the ordering of the matching.

Suppose we match blocks $T_1, \ldots, T_{a+b-m}$ from the restriction of π' to $\{1, \ldots, \frac{n}{2}\}$ to blocks $R_1, \ldots, R_{a+b-m}$ from the restriction of π' to $\{\frac{n}{2} + 1, \ldots, n\}$. As usual, we suppose that the T_i and R_i are sorted in increasing order of smallest element. Our matching of these will take the form of a permutation $\sigma' \in S_{a+b-m}$, where T_i is matched to $R_{\sigma'(i)}$.

Now, restricting π' to the smallest elements of $T_1, \ldots, T_{a+b-m}, R_1, \ldots, R_{a+b-m}$, we see that π' contains the partition with blocks $\{i, a + b - m + \sigma(i)\}$ for all $i \in [a + b - m]$–which is just the partition $[\sigma']$. Thus $[\sigma']$ may not contain $[\sigma]$, so by Proposition 2.1, σ' avoids σ.

Thus by Theorem 6.1, there is some c such that there are at most c^{a+b-m} ways to choose the matching.

Summarizing this information in recursive form, we obtain that where $\pi = [\sigma]$,

$$S(n,m,\pi) \leq \sum_{a,b} S\left(\frac{n}{2},a,\pi\right) S\left(\frac{n}{2},b,\pi\right) \binom{a}{a+b-m}\binom{b}{a+b-m} c^{a+b-m}.$$

(In this recursion and until the end of the proof, when we sum over a and b, the bounds are $0 \leq a, b \leq \frac{n}{2}$.) To simplify the binomials, we use the following lemma.

Lemma 6.2. *For integers $w \geq x \geq 0$ and $y \geq z \geq 0$, $\binom{w}{x}\binom{y}{z} \leq \binom{w+y}{x+z}\binom{x+z}{x}$.*

Proof of Lemma. Cancelling out the factorials that can be cancelled and multiplying out, this is equivalent to

$$w!y!(w+y-x-z)! \leq (w-x)!(y-z)!(w+y)!,$$

which we can rearrange to

$$\binom{w+y-x-z}{w-x} \leq \binom{w+y}{w}.$$

This final statement holds because if we repeatedly recurse the term on the right side using the binomial identity $\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m}$, the term on the left side will be a summand. This proves the lemma. $\square$

Now, letting $w = a$, $y = b$, and $x = z = a+b-m$ in Lemma 6.2,

$$\binom{a}{a+b-m}\binom{b}{a+b-m} \leq \binom{a+b}{2(a+b-m)}\binom{2(a+b-m)}{a+b-m} \leq \binom{a+b}{2(a+b-m)} 4^{a+b-m},$$

where the last inequality simply comes from the inequality $\binom{n}{k} \leq 2^n$ for $0 \leq k \leq n$. Substituting this into our recursion,

$$S(n,m,\pi) \leq \sum_{a,b} S\left(\frac{n}{2},a,\pi\right) S\left(\frac{n}{2},b,\pi\right) \binom{a+b}{2(a+b-m)} (4c)^{a+b-m}.$$

Now, for any $n \geq k \geq 0$, $\binom{n}{2k} = \binom{n}{k}\binom{n-k}{k} \leq \binom{n}{k}^2$. For the binomial term in the recursion, we get $\binom{a+b}{2(a+b-m)} \leq \binom{a+b}{a+b-m}^2 = \binom{a+b}{m}^2$. Substituting this and multiplying through by $(4c)^m$, we obtain

$$(4c)^m S(n,m,\pi) \leq \sum_{a,b} (4c)^a S\left(\frac{n}{2},a,\pi\right) (4c)^b S\left(\frac{n}{2},b,\pi\right) \binom{a+b}{m}^2.$$

Let $f(n,m,\pi) = \sqrt{(4c)^m S(n,m,\pi)}$. We obtain that

$$f(n,m,\pi)^2 \leq \sum_{a,b} f\left(\frac{n}{2},a,\pi\right)^2 f\left(\frac{n}{2},b,\pi\right)^2 \binom{a+b}{m}^2 \leq \left(\sum_{a,b} f\left(\frac{n}{2},a,\pi\right) f\left(\frac{n}{2},b,\pi\right) \binom{a+b}{m}\right)^2,$$

simply using the inequality $(\sum x_i)^2 \geq \sum x_i^2$ for $x_i \geq 0$. Taking the square root of both sides,

$$f(n,m,\pi) \leq \sum_{a,b} f\left(\frac{n}{2},a,\pi\right) f\left(\frac{n}{2},b,\pi\right) \binom{a+b}{m}.$$

Thus if $g(n,m,\pi)$ is defined by $g(1,m,\pi) = f(1,m,\pi)$ and defined for n a power of 2 by the recursion

$$g(n,m,\pi) = \sum_{a,b} g\left(\frac{n}{2},a,\pi\right) g\left(\frac{n}{2},b,\pi\right) \binom{a+b}{m},$$

then $f(n,m,\pi) \leq g(n,m,\pi)$ for all n a power of 2.

Multiplying both sides by x^m and summing from $m = 0$ to n (and using the fact that $a, b \leq \frac{n}{2}$ so $a + b \leq n$), we obtain

$$\begin{aligned}
\sum_{m=0}^{n} g(n,m,\pi)x^m &= \sum_{a,b} g\left(\frac{n}{2},a,\pi\right) g\left(\frac{n}{2},b,\pi\right) \sum_{m=0}^{n} \binom{a+b}{m} x^m \\
&= \sum_{a,b} g\left(\frac{n}{2},a,\pi\right) g\left(\frac{n}{2},b,\pi\right) (x+1)^{a+b} \\
&= \left(\sum_{a=0}^{\frac{n}{2}} g\left(\frac{n}{2},a,\pi\right) (x+1)^a \right) \left(\sum_{b=0}^{\frac{n}{2}} g\left(\frac{n}{2},b,\pi\right) (x+1)^b \right).
\end{aligned}$$

Thus if $F_{n,\pi}(x) = \sum_{m=0}^{n} g(n,m,\pi)x^m$, then we have found the recursion

$$F_{n,\pi}(x) = \left(F_{\frac{n}{2},\pi}(x+1)\right)^2.$$

Writing $n = 2^k$ and applying the recursion k times, we obtain

$$F_{2^k,\pi}(x) = \left(F_{1,\pi}(x+k)\right)^{2^k}.$$

Looking back through the previous steps, we see that

$$\begin{aligned}
F_{1,\pi}(x) &= \sum_{m=0}^{1} g(1,m,\pi)x^m \\
&= \sum_{m=0}^{1} f(1,m,\pi)x^m \\
&= f(1,0,\pi) + f(1,1,\pi)x \\
&= \sqrt{S(1,0,\pi)} + \sqrt{4cS(1,1,\pi)}x \\
&= \sqrt{4c}x
\end{aligned}$$

(except in the trivial $\pi = \{\{1\}\}$ case). Therefore,

$$F_{2^k,\pi}(x) = (\sqrt{4c}(x+k))^{2^k}.$$

We now relate this back to our original problem; that of bounding $B_n(\pi)$. We have that for $n = 2^k$,

$$\begin{aligned}
B_n(\pi) &= \sum_{m=0}^{n} S(n,m,\pi) \\
&= \sum_{m=0}^{n} \frac{f(n,m,\pi)^2}{(4c)^m} \\
&\le \sum_{m=0}^{n} f(n,m,\pi)^2 \\
&\le \left(\sum_{m=0}^{n} f(n,m,\pi)\right)^2 \\
&\le \left(\sum_{m=0}^{n} g(n,m,\pi)\right)^2 \\
&= (F_{n,\pi}(1))^2 \\
&= (\sqrt{4c}(k+1))^{2^{k+1}} \\
&= (4c)^n(\lg n + 1)^{2n}.
\end{aligned}$$

Now consider the case where n is not necessarily a power of 2. Since $\pi = [\sigma]$ has no blocks of size 1, we can again add any number of blocks of size 1 to increase the length of a partition avoiding π, so $B_n(\pi)$ is at least weakly increasing. If n' is the smallest power of 2 greater than or equal to n, then $\lg n' < \lg n + 1$ and $n' < 2n$, so

$$B_n(\pi) \le B_{n'}(\pi) \le (4c)^{n'}(\lg n' + 1)^{2n'}(4c)^{2n}(\lg n + 2)^{4n}.$$

Letting $d(\pi) = 16c^2$, we have obtained the result

$$B_n(\pi) < d(\pi)^n(\lg n + 2)^{4n}.$$

Therefore,

$$\lim_{n\to\infty} \frac{\log B_n(\pi)}{n \log n} \le \lim_{n\to\infty} \frac{\log d(\pi) + 4\log(\lg n + 2)}{\log n} = 0,$$

and the proof is complete.

7 Implications of Conjecture 1 for Parallel Pattern Avoidance

This section will be dedicated to the asymptotics of parallel pattern avoidance. We make the following definition.

Definition. If $\sigma_1, \ldots, \sigma_k$ are permutations of some $[m]$, we say that $S_n^k(\sigma_1, \ldots, \sigma_k)$ is the number of k-tuples of permutations $(\sigma'_1, \ldots, \sigma'_k)$ with $\sigma'_i \in S_n$ such that $(\sigma'_1, \ldots, \sigma'_k)$ avoids $(\sigma_1, \ldots, \sigma_k)$ in parallel.

Let $\sigma_1, \ldots, \sigma_k$ be permutations, say in S_m. Then for every $(\sigma'_1, \ldots, \sigma'_k)$ that avoids $(\sigma_1, \ldots, \sigma_k)$, we have a corresponding set partition $[\sigma'_1, \ldots, \sigma'_k]$ avoiding $[\sigma_1, \ldots, \sigma_k]$ by Proposition 2.1. Thus Conjecture 1 should imply a corresponding bound on parallel permutation pattern avoidance. This is summarized in the following conjecture.

Conjecture 2. *Let $\sigma_1, \ldots, \sigma_k \in S_m$ be permutations. Then the following hold.*

- *(Weak Form)* $\lim_{n\to\infty} \dfrac{\log S_n^k(\sigma_1, \ldots, \sigma_k)}{n \log n} \le \dfrac{k^2-1}{k}$.
- *(Strong Form) There exists a constant c_2 (depending on the σ_i) such that $S_n^k(\sigma_1, \ldots, \sigma_k) \le c_2^n n^{n\frac{k^2-1}{k}}$ for all n.*

If these bounds hold, they are tight for all $(\sigma_1, \ldots, \sigma_k)$ (as long as they are not of length 1), as we will see later in this section. Again, it is obvious that the strong form implies the weak form. The strong form for $k = 1$ is simply Theorem 6.1.

The following proposition formalizes the discussion in the paragraph preceding the conjecture. As such, the proof is quite simple.

Proposition 7.1. *The forms of Conjecture 1 imply the corresponding forms of Conjecture 2. In particular, the forms of Conjecture 1 on $[\sigma_1, \ldots, \sigma_k]$ imply the corresponding forms of Conjecture 2 for $(\sigma_1, \ldots, \sigma_k)$.*

Proof. Each element of $S_n^k(\sigma_1, \ldots, \sigma_k)$ gives an element of $B_{(k+1)n}([\sigma_1, \ldots, \sigma_k])$ by Proposition 2.1, and these elements are clearly different. Thus $S_n^k(\sigma_1, \ldots, \sigma_k) \le B_{(k+1)n}([\sigma_1, \ldots, \sigma_k])$. Since $[\sigma_1, \ldots, \sigma_k]$ has permutability k, substituting into Conjecture 1 yields exactly 2. $\square$

The main theorem of this section is the following.

Theorem 7.2. *Let $m > 1$ be an integer, and consider the trivial permutation $12\cdots m \in S_m$. There exists (m-dependent) constants $c_2 > c_1 > 0$ such that for $n \in \mathbb{N}$,*

$$c_1^n n^{\frac{k^2-1}{k}n} \le S_n^k(12\cdots m, \ldots, 12\cdots m) \le c_2^n n^{\frac{k^2-1}{k}n}.$$

In particular, Conjecture 2 holds for many copies of the trivial permutation.

The proof of Theorem 7.2 follows relatively simply from previous results and ideas.

Proof of Theorem 7.2. We first translate to the language of probabilities. Let $q_{m,k}(n)$ be the probability that randomly chosen $\sigma_1, \ldots, \sigma_k \in S_n$ will have $(\sigma_1, \ldots, \sigma_k)$ avoiding $(12\cdots m, \ldots, 12\cdots m)$. Note that since there are $n!^k$ ways to choose k permutations in S_n, $q_{m,k}(n) = \frac{S_n^k(12\cdots m, \ldots, 12\cdots m)}{n!^k}$.

We know that $n!$ is within an exponential factor of n^n by Stirling approximation, so if we divide the desired statement by $n!^k$, we obtain that we want to show

$$c_1^n n^{-\frac{n}{k}} \leq q_{m,k}(n) \leq c_2^n n^{-\frac{n}{k}}$$

for all $n \in \mathbb{N}$ for some constants $c_2 > c_1 > 0$.

We now translate the problem into the language of random $k+1$-dimensional orderings as follows. Let $p_1, \ldots, p_n$ be random points (in the usual sense) in $[0,1]^{k+1}$. We can sort them by their first coordinates. Once this is done, looking at the ordering of the i^{th} coordinates of all n points for some fixed $2 \leq i \leq k+1$ will generate a permutation, so we get k permutations $\sigma_1, \ldots, \sigma_k$ given by these orderings. It is easy to see that these permutations are independently and uniformly randomly chosen.

Now, we consider the (random) poset, also known as the random $k+1$-dimensional order $P_{k+1}(n)$, on these points as follows. We say that $p_i < p_j$ if and only if all coordinates of p_i are less than those of p_j. Suppose p_i has the a_i-th smallest first coordinate, and similarly p_j has the a_j-th smallest. Then the condition that $p_i < p_j$ corresponds to (looking at the first coordinate) the condition that $a_i < a_j$, and (looking at the other k coordinates) the condition that $\sigma_\ell(a_i) < \sigma_\ell(a_j)$ for all $\ell \in [k]$. This idea of relating sets of k permutations to random $k+1$-dimensional orderings seems to go back to Winkler. [9]

By definition, $(\sigma_1, \ldots, \sigma_k)$ avoids $(12\cdots m, \ldots, 12\cdots m)$ if and only if there is no $b_1 < \cdots < b_m$ with $\sigma_i(b_1) < \cdots < \sigma_i(b_m)$ for all i, and we can see by the previous paragraph that this is in turn equivalent to there being no length-m chain in $P_{k+1}(n)$. To summarize, we have shown that $q_{m,k}(n)$ is the probability that $P_{k+1}(n)$ contains no length-m chain.

By Mirsky's Theorem, this is the same as the probability that $P_{k+1}(n)$ can be partitioned into $m-1$ (possibly empty) disjoint antichains. We casework on the partition of the elements into antichains. Since there are $m-1$ choices for each element, there are at most $(m-1)^n$ ways that we can choose the sets for the antichains (it is not equality as some of these maybe the same upon for example permuting the sets). If these sets have $a_1, \ldots, a_{m-1}$ elements each, the probability that the i^{th} set is an antichain is the probability that it has no chains of size at least 2; that is, it is $q_{2,k}(a_i)$. Summing and bounding the sum by the number of terms times the maximal element, we see that

$$q_{m,k}(n) \leq (m-1)^n \max_{a_1+\cdots+a_{m-1}=n} \prod_{i=1}^{m-1} q_{2,k}(a_i).$$

Suppose we have shown the result for $m=2$. Since $q_{m,k}(n) \geq q_{2,k}(n)$, we have shown the lower bound for all m. Now, since $n!$ is within an exponential factor of n^n asymptotically, we have that there exists c such that $q_{2,k}(n) \leq c^n n!^{-\frac{1}{k}}$ for all n. Then substituting into the equation above shows

$$\begin{aligned} q_{m,k}(n) &\leq (m-1)^n \max_{a_1+\cdots+a_{m-1}=n} \prod_{i=1}^{m-1} q_{2,k}(a_i) \\ &\leq (m-1)^n \max_{a_1+\cdots+a_{m-1}=n} \prod_{i=1}^{m-1} c^{a_i} a_i!^{-\frac{1}{k}} \end{aligned}$$

$$
\begin{aligned}
&= (m-1)^n c^n \max_{a_1+\cdots+a_{m-1}=n} \frac{1}{\left(\prod_{i=1}^{m-1} a_i!\right)^{\frac{1}{k}}} \\
&= (m-1)^n c^n n!^{-\frac{1}{k}} \max_{a_1+\cdots+a_{m-1}=n} \binom{n}{a_1,\ldots,a_{m-1}}^{\frac{1}{k}} \\
&\leq (m-1)^n c^n n!^{-\frac{1}{k}} (m-1)^{\frac{n}{k}} \\
&= \left(c(m-1)^{1+\frac{1}{k}}\right)^n n!^{-\frac{1}{k}},
\end{aligned}
$$

which is within an exponential factor of $\left(c(m-1)^{1+\frac{1}{k}}\right)^n n^{-\frac{n}{k}}$ (again by Stirling approximation), so the upper bound also follows from all m. (The second to last step in the previous chain of inequalities holds as the sum of all such multinomial coefficients is $(m-1)^n$.)

Thus it suffices to show the result when $m = 2$. We know that $q_{2,k}(n)$ is the probability that $P_{k+1}(n)$ is itself an antichain. However, this follows directly from prior results–Brightwell showed in Theorem 1 of [2] that the probability that $P_{k+1}(n)$ is an antichain is at most $\left(2^{\frac{1}{k+1}}\left(k+1\right)^{\frac{k+2}{k}}\right)^n n^{-\frac{n}{k}}$ (shifting k by one from the exact result stated there), and Crane and Georgiou proved in Section 1.5 of [3] that this probability is also at least $\left((\frac{1}{e}+o(1)\right)^n n^{-\frac{n}{k}}$. These results together prove the desired bound for $m = 2$, and thus finish the proof. $\square$

A nice corollary is that, as promised, it follows from Theorem 7.2 that the bounds in Conjecture 2 are sharp.

Corollary 7.3. *Let $m > 1$ be an integer and $\sigma_1, \ldots, \sigma_k \in S_m$ be permutations. Then there exists $c_1 > 0$ such that $S_n^k(\sigma_1, \ldots, \sigma_k) \geq c_1^n n^{\frac{k^2-1}{k}n}$.*

Proof. Let $\tau_1, \ldots, \tau_k \in S_2$ be such that τ_i has the same relative ordering as the first two terms of σ_i. Then $(\sigma_1, \ldots, \sigma_k)$ contains $(\tau_1, \ldots, \tau_k)$, so $S_n^k(\sigma_1, \ldots, \sigma_k) \geq S_n^k(\tau_1, \ldots, \tau_k)$, as any k-tuple avoiding the latter must avoid the former. Thus it suffices to show the problem for $\tau_1, \ldots, \tau_k \in S_2$. But some permutation $\pi \in S_n$ contains the pattern 21 at exactly the pairs of indices where the complement of π (the permutation obtained replacing each i in by $n+1-i$, so for example 1243 becomes 4312) contains 12. So replacing all permutations in the places where $\tau_k = 21$ by their complement gives a bijection showing $S_n^k(\tau_1, \ldots, \tau_k) = S_n^k(12, \ldots, 12)$. Therefore, it suffices to show that $S_n^k(12, \ldots, 12) \geq c_1^n n^{\frac{k^2-1}{k}n}$ for some $c_1 > 0$. But this is simply the lower bound from the $m = 2$ case of Theorem 7.2, finishing the proof of the corollary. $\square$

8 Further Directions

There are several possible directions to attempt to extend these results. The most obvious of these comes in the form of Conjectures 1 and 2. To the author's knowledge, little is known in this area except for the results described in this paper. For example, the simplest case where the permutations are nontrivial is $S_n^2(123, 132)$, and even this does not appear to have been studied before. Even proving Conjecture 2 in this case does not seem obvious.

One obstacle to progress on Conjecture 1 is that the recursions become more complicated as $k \geq 2$. The corresponding way to proceed would be to let n be a power of $k+1$ and partition $[n]$ into $\{1, \ldots, \frac{n}{k+1}\}, \ldots, \{\frac{kn}{k+1} + 1, \ldots, n\}$, and then to look at the restrictions of a permutation to each of these sets and ways of combining them. However, the ways of combining them are more complicated, as one block of the partition can strech over anywhere from 1 to $k+1$ of these partitions. Although the blocks that stretch over $k+1$ partitions can possibly be dealt with using Conjecture 2 (in an attempt to reduce Conjecture 1 to Conjecture 2), the blocks that stretch over k partitions have no obvious restrictions (for example, any permutation consisting of blocks of size k or less avoids $[\sigma_1, \ldots, \sigma_k]$). Simply allowing all blocks of size k does not appear to yield a strong enough recursion. However, it is possible that the recursion may generalize in some other way, perhaps involving casework on some other statistic besides the number of blocks.

Another natural notion for further study is that of the permutability statistic and its distribution. To the author's knowledge, this statistic has not explicitly appeared before in the literature, and given its seemingly strong connection to asymptotics, it may be worthwhile to study.

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