# A COMPUTER ALGEBRA PACKAGE FOR POLYNOMIAL SEQUENCE RECOGNITION 

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## Abstract

The software package developed in the thesis research implements functions for the intelligent guessing of polynomial sequence formulas based on user-defined expected sequence factors of the input coefficients. We present a specialized hybrid approach to finding exact representations for polynomial sequences that is motivated by the need for an automated procedures to discover the precise forms of these sums based on user guidance, or intuition, as to special sequence factors present in the formulas. In particular, the package combines the user input on the expected special sequence factors in the polynomial coefficient formulas with calls to the existing functions as subroutines that then process formulas for the remaining sequence terms already recognized by these packages.

The factorization-based approach to polynomial sequence recognition is unique to this package and allows the search functions to find expressions for polynomial sums involving Stirling numbers and other special triangular sequences that are not readily handled by other software packages. In contrast to many other sequence recognition and summation software, the package not provide an explicit proof, or certificate, for the correctness of these sequence formulas - only computationally guided educated guesses at a complete identity generating the sequence over all $n$. The thesis contains a number of concrete, working examples of the package that are intended to both demonstrate its usage and to document its current sequence recognition capabilities.

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## List of Symbols and Notation

## Notation and Conventions

$\mathbb{N} \quad$ The set of natural numbers, $\mathbb{N}=\{0,1,2,3,4, \ldots\}$.
$\mathbb{Z}^{+} \quad$ The set of positive integers, $\mathbb{Z}^{+}=\{1,2,3,4, \ldots\}$.
$\mathbb{Z}[x] \quad$ The ring of polynomials in $x$ with coefficients in the integers, $\mathbb{Z}$.
$\mathbb{Q}[x] \quad$ The ring of polynomials in $x$ with coefficients in the rational numbers, $\mathbb{Q}$.
$\mathbb{K}[x] \quad$ The ring of polynomials in $x$ with coefficients over the field $\mathbb{K}$.
$D_{z}^{(j)} \quad$ The derivative, or differential, operator with respect to $z$, i.e., where $D_{z}^{(j)}[F(z)] \equiv F^{(j)}(z)$ denotes the $j^{t h}$ derivative of $F(z)$, provided that the $j^{t h}$ derivative of the function exists.
$\binom{n}{k} \quad$ The binomial coefficients.
$\left[\begin{array}{l}n \\ k\end{array}\right] \quad$ The unsigned Stirling numbers of the first kind, also denoted by $(-1)^{n-k} s(n, k)$.
$\left\{\begin{array}{l}n \\ k\end{array}\right\} \quad$ The Stirling numbers of the second first kind, also denoted by $S(n, k)$.
$\left\langle\begin{array}{l}n \\ k\end{array}\right\rangle \quad$ The first-order Eulerian numbers.
$\left\langle\left\langle\begin{array}{l}n \\ k\end{array}\right\rangle\right\rangle$ The second-order Eulerian numbers.
$H_{n}^{(r)} \quad$ The $r$-order harmonic numbers, $H_{n}^{(r)}:=\sum_{k=1}^{n} k^{-r}$, where the first-order harmonic numbers are denoted in the shorthand notation, $H_{n} \equiv H_{n}^{(1)}$.
$B_{n} \quad$ The Bernoulli numbers.

## Chapter 1

## Introduction

### 1.1 Background and Motivation

The form of composite sequences involving the Stirling numbers of the first and second kinds are common in many applications. The Stirling number triangles arise naturally in formulas involving sums of factorial functions and in the symbolic, polynomial expansions of binomial coefficients and other factorial function variants. The Stirling and Eulerian number triangles also both frequently occur in applications involving finite sums and generating functions over non-negative powers of integers. These applications include finding closed-form expressions and formulas for generating functions over polynomial multiples of an arbitrary sequence.

### 1.1.1 Example I: Computing Derivatives of Stirling Number Generating Functions

If $p, k \in \mathbb{N}$, the following modified series for the generating functions for polynomial multiples of the unsigned Stirling numbers of the first kind, denoted by $\left[\begin{array}{l}n \\ k\end{array}\right]$, result in the expansions

$$
\begin{align*}
\sum_{n=0}^{\infty} n^{k} \cdot\left[\begin{array}{l}
n \\
p
\end{array}\right] \frac{z^{n}}{n!} & =\sum_{j=0}^{k}\left\{\begin{array}{l}
k \\
j
\end{array}\right\} z^{j} \cdot D_{z}^{(j)}\left[\frac{(-1)^{p}}{p!} \cdot \log (1-z)^{p}\right]  \tag{1.1}\\
\sum_{n=0}^{\infty} n^{k} \cdot\left[\begin{array}{l}
n+1 \\
p+1
\end{array}\right] \frac{z^{n}}{n!} & =\sum_{j=0}^{k}\left\{\begin{array}{l}
k \\
j
\end{array}\right\} z^{j} \cdot D_{z}^{(j)}\left[\frac{(-1)^{p}}{p!} \cdot \frac{\log (1-z)^{p}}{(1-z)}\right],
\end{align*}
$$

where the derivative operator, $D_{z}^{(j)}$, denotes the $j^{\text {th }}$ derivative with respect to $z$ of its input and the Stirling numbers of the second kind are denoted by $\left\{\begin{array}{l}n \\ k\end{array}\right\}$. Given enough familiarity with the Stirling numbers of the first kind and some trial and error, formulas for each of the $j^{t h}$ derivatives involved in the expansions of 1.1) are obtained by extrapolation from the first several cases of $j \in \mathbb{N}$ to obtain the finite sums

$$
\begin{align*}
& D_{z}^{(j)}\left[\frac{(-1)^{p}}{p!} \cdot \log (1-z)^{p}\right]=\sum_{i=0}^{j}\left[\begin{array}{c}
j \\
i
\end{array}\right] \frac{(-1)^{p-i}}{(p-i)!} \cdot \frac{\log (1-z)^{p-i}}{(1-z)^{j}}  \tag{1.2}\\
& D_{z}^{(j)}\left[\frac{(-1)^{p}}{p!} \cdot \frac{\log (1-z)^{p}}{(1-z)}\right]=\sum_{i=0}^{j}\left[\begin{array}{c}
j+1 \\
j+1-i
\end{array}\right] \frac{(-1)^{p+j-i}}{(1-z)^{j+1}} \cdot \frac{\log (1-z)^{p-j+i}}{(p-j+i)!}
\end{align*}
$$

where the formulas in 1.2 may be regarded as polynomials in the function, $\log (1-z)$. A proof of the correctness of these formulas is then later obtained formally by induction on $j$.

### 1.1.2 Example II: A More Challenging Application

A more challenging, and less straightforward, example arises in attempting to find an exact, closed-form representation for the expansions of the ordinary generating function for the Stirling number sequence
variant, $S_{k}^{(d)}(n)$, defined as in 1.3 with respect to each fixed $d, k \in \mathbb{Z}^{+}$.

$$
S_{k}^{(d)}(n):=\sum_{j=1}^{n}\binom{n}{j}\left[\begin{array}{l}
j+1  \tag{1.3}\\
k+1
\end{array}\right] \frac{(-1)^{j}}{j!\cdot(j+d)} \cdot \frac{(n+d)!}{n!}
$$

The first few examples of the ordinary generating function, $\widetilde{S}_{k}^{(d)}(z)$, over $d \geq 2$ for the sequence defined by (1.3) are provided for reference as follows:

$$
\begin{align*}
\widetilde{S}_{k}^{(2)}(z)= & -\frac{\log (1-z)^{k-1}}{(1-z)^{2}}\left[\frac{z}{(k-1)!}+\frac{(-1+z) \log (1-z)}{k!}\right]  \tag{1.4}\\
\widetilde{S}_{k}^{(3)}(z)= & \frac{\log (1-z)^{k-2}}{(1-z)^{3}}\left[\frac{z^{2}}{(k-2)!}+\frac{z(-4+3 z) \log (1-z)}{(k-1)!}+\frac{\left(2-4 z+2 z^{2}\right) \log (1-z)^{2}}{k!}\right] \\
\widetilde{S}_{k}^{(4)}(z)= & -\frac{\log (1-z)^{k-3}}{(1-z)^{4}}\left[\frac{z^{3}}{(k-3)!}+\frac{z^{2}(-9+6 z) \log (1-z)}{(k-2)!}+\frac{z\left(18-27 z+11 z^{2}\right) \log (1-z)^{2}}{(k-1)!}\right. \\
& \left.+\frac{\left(-6+18 z-18 z^{2}+6 z^{3}\right) \log (1-z)^{3}}{k!}\right] \\
\widetilde{S}_{k}^{(5)}(z)= & \frac{\log (1-z)^{k-4}}{(1-z)^{5}}\left[\frac{z^{4}}{(k-4)!}+\frac{z^{3}(-16+10 z) \log (1-z)}{(k-3)!}+\frac{z^{2}\left(72-96 z+35 z^{2}\right) \log (1-z)^{2}}{(k-2)!}\right. \\
& \left.+\frac{z\left(-96+216 z-176 z^{2}+50 z^{3}\right) \log (1-z)^{3}}{(k-1)!}+\frac{\left(24-96 z+144 z^{2}-96 z^{3}+24 z^{4}\right) \log (1-z)^{4}}{k!}\right]
\end{align*}
$$

Based observations of the first several cases of these generating functions in (1.4), we rewrite the expansions of these generating functions as the sum

$$
\begin{equation*}
\widetilde{S}_{k}^{(d)}(z):=\sum_{n=0}^{\infty} S_{k}^{(d)}(n) z^{n}=\left(\frac{(-1)^{d-1} \cdot \log (1-z)^{k+1-d}}{(1-z)^{d}}\right) \times \sum_{m=0}^{d-1} \frac{\log (1-z)^{m} \cdot z^{d-1-m}}{(k+1-d+m)!} \cdot g_{m}^{(d)}(z) \tag{1.5}
\end{equation*}
$$

It is clear from examining the sequence data in 1.4 that the formulas for the polynomials, $g_{m}^{(d)}(z)$, specified in 1.5 involve a sum over factors of the Stirling numbers of the first kind and the binomial coefficients. However, finding the precise sequence inputs in the formula for these polynomials with the correct corresponding multiplier terms in the sum is not immediately obvious from the first few example cases in 1.4. We then proceed forward seeking a formula for the polynomials, $g_{m}^{(d)}(z)$, in the general template form of

$$
\begin{equation*}
g_{m}^{(d)}(z)=\sum_{i} S_{1}(\cdot, \cdot) \cdot \operatorname{Binom}(\cdot, \cdot) \times \operatorname{RemSeq}_{1}(i) \cdot \operatorname{RemSeq}_{2}\left(m+m_{0}-i\right) \times z^{i} \tag{1.6}
\end{equation*}
$$

where the functions $S_{1}(\cdot, \cdot)$ and $\operatorname{Binom}(\cdot, \cdot)$ denote the Stirling numbers of the first kind and binomial coefficients, respectively, each over some unspecified index inputs to these sequence functions.

After a few hours of frustrating trial and error with Mathematica, we finally arrive at a formula for these polynomials in the form of

$$
g_{m}^{(d)}(z)=\sum_{i=0}^{m}\left[\begin{array}{c}
d-m+i  \tag{1.7}\\
d-m
\end{array}\right]\binom{d-1}{m-i}^{2}(-1)^{m-i}(m-i)!\cdot z^{i}
$$

The motivation for constructing the package routines in the thesis is to automate the eventual discovery of the formula in (1.7) based on user input as to the general template to the formula for these polynomials specified as in (1.6). The automated discovery of the first pair of less complicated formulas given in 1.2 is then also possible using the package.

### 1.2 High-Level Description of the Package

The Mathematica package GuessPolySequenceFormulas.m developed as a part of the thesis research implements software routines for the intelligent guessing of polynomial sequence formulas based on user input on the expected form of the sequence formulas. These functions for sequence recognition then rely on some degree of user intuition to correctly find closed-form formulas that represent the input polynomial sequence. The logic used to construct these routines is based on factorization data for the expected sequence factors of the input polynomial coefficient terms suggested by the user.

The template of the polynomial sequence formulas that the package aims to recognize satisfies an expansion of the general form outlined in where $j, j_{0} \in \mathbb{N}, r \in \mathbb{Z}^{+}$, and $x$ is some (formal) polynomial variable that may assume different forms in the sequences input to the package routines.

$$
\begin{equation*}
\operatorname{Poly}_{j}(x):=\sum_{i=0}^{j+j_{0}}\left(\prod_{i=1}^{r}\left\|\widetilde{u}_{i}(j)+u_{i} \cdot i\right\|_{i}(j)+\ell_{i} \cdot i \|_{i}\right) \times \operatorname{RS}_{1}(i) \operatorname{RS}_{2}\left(j+j_{0}-i\right) \cdot x^{i} . \tag{1.8}
\end{equation*}
$$

The product of (triangular) sequences in the first term of 1.8 correspond to the factors of the expected user-defined sequences in the polynomial coefficient terms where the functions $\widetilde{u}_{i}(j), \widetilde{\ell}_{i}(j)$ are prescribed functions of the sequence index $j$ and where the $u_{i}, \ell_{i} \in \mathbb{Z}$ are prescribed application-dependent multiples of the polynomial summation index $i$. The functions $\mathrm{RS}_{i}(\cdot)$ in the previous equation denote the coefficient remainder terms in these polynomial formulas, which should ideally correspond to comparatively simpler sequences that are already easily recognized by existing packages discussed in Section 1.4 of the thesis below. These existing packages may be called as subroutines to recognize the sequences corresponding to the remainder terms in the input polynomial coefficients after the forms of the sequence factors expected by the user are determined by the package routines.

### 1.3 Plan of Attack and Aims of the Thesis Research

A significant part of the work for the thesis is a "proof of concept" implementation of the logic to find polynomial sequence formulas of the form in 1.8 based on user input of the first several terms of the sequence. In this implementation, the focus of the package development is in constructing the logic to recognize the polynomial sequence formulas in the form of 1.8 . For example, in the absence of obvious, or known, algorithms for the factorization of an integer by an arbitrary sequence, the implementation of this part of the algorithm employed by the package is effectively treated as an oracle within the working source code. The construction of this type of integer factorization algorithm is motivated by the need for such algorithms in a more efficient implementation of this package. A more complete and detailed specification of these factorization routines is described in the future research topics outlined in Chapter 3

The plan is that later, once more of the machinery for generating the proposed polynomial sequence formulas is in place, optimizations to the code and the task of finding a more efficient implementation to generate the factorizations of a given integer over multiple sequence factors may be investigated further. Several examples of usage of the sequence recognition functions in the package, including figures of the Mathematica output, are given in Chapter 2 These examples provide both the working syntax of Mathematica programs that employ the package routines and serve to document the capabilities of the package current at the time of this thesis draft.

### 1.4 Software for Sequence Recognition

### 1.4.1 Software Packages for Sequence Recognition

There are a number of notable existing software packages and online resources geared towards guessing formulas for integer and semi-rational sequences based on the forms of the first few terms of a sequence. Notable and well-known examples include the gfun package for Maple [9], the Rate packages for Mathematico ${ }^{1}$ [5] Ap-

[^0]pendix A], the more recently updated Guess package ${ }^{2}$ for the FriCAS fork of Axiom which includes enhancements to the previous packages documented in [3], and a default, built-in function, FindSequenceFunction, in Mathematica. There are still other software packages designed to perform related operations aimed at recognizing auxiliary properties such as identifying recurrence relations and generating functions for sequences freely available onling ${ }^{3}$ The Online Encyclopedia of Integer Sequences, and its email-based SuperSeeker program, provide lookup access to a large database of integer sequences, including the integer-valued entries for the numerator and denominators of rational sequences such as the Bernoulli numbers. A more historical account of the development of software for sequence recognition is provided in [3, §2].

### 1.4.2 Polynomial and Summation Identities Involving the Stirling Numbers

Notice that in the absence of some underlying structure to a sequence (or satisfied by its generating function), guessing functions that attempt to find closed-form expressions for an arbitrary sequence by extrapolation from the input of its first few terms are inherently limited in obtaining a proof to verify the correctness of the formulas returned. The routines in many software packages and in the algorithms described in 7 are able to obtain computerized proofs or certificates for closed-form identities obtained from summations involving special functions. The correctness of formulas obtained by packages such as gfun follow if the generating function for a sequence is holonomic, or equivalently, if the sequence, say $S_{n}$, itself satisfies a $P$-recurrence of the form

$$
\begin{equation*}
\widehat{p}_{0}(n) \cdot S_{n}+\widehat{p}_{1}(n) \cdot S_{n+1}+\cdots+\widehat{p}_{r}(n) \cdot S_{n+r}=0 \tag{1.9}
\end{equation*}
$$

whenever $n \geq n_{0}$ for some fixed $n_{0}$, with $r \geq 1$, and where the coefficients, $\widehat{P}_{i}(n) \in \mathbb{C}[n]$, are polynomials for each $0 \leq i \leq r$ [1, §B.4].

As pointed out in [2] and in [4], unlike a number of other special sequences of interest in applications, the Stirling numbers are not holonomic, or do not satisfy a homogeneous recurrence relation of the form in 1.9, , so it is reasonable to expect that existing software to guess sequence formulas should be at least somewhat limited in recognizing the exact forms of summations involving factors these sequences. The Mathematica package Stirling.m by M. Kauers is still able to find recurrences satisfied by many polynomial-like sums involving the Stirling and Eulerian number triangles in cases of many known and new summation identities. However, the example cited in Kauers' article about the package shows a seemingly simple polynomial-like summation involving the Stirling numbers of the second kind for which a recurrence relation in the form of (1.9) fails to exist [4, See $\S 4$ ].

This behavior offers some explanation as to the deficiency of functions like FindSequenceFunction in recognizing formulas for sequences involving factors of the Stirling and Bernoulli numbers. We now restrict our attention to constructing software routines that recognize formulas for the class of polynomial sums of the form in 1.8 based on intelligent guesses as to the coefficient forms input by the user. In the context, the package is intended to quickly assist the user in the discovery of formulas that arise in practice, like those motivated by the examples from Section 1.1, which we then are able to prove correct later by separate methods.

### 1.4.3 Comparisons of the Packages to Existing Software Routines

The treatment of the user-defined expected sequence factors in finding formulas for input polynomial sequences is to consider these expected sequence terms as primitives in the matching formulas returned by the package. This treatment of the user-defined expected sequence factors as primitives in the search for matching formulas is analogous to the handling of the closed-form functions returned by FindSequenceFunction in Mathematica, such as for scalar or constant values, powers of (polynomials in) a variable $n$, factorial and gamma functions, or powers of a fixed constant, $c^{n}$.

For example, acceptable formulas returned by the package for the sequence of generating functions for polynomial powers of $n$ may correspond to either of the sums in the following equation involving the Stirling

[^1]numbers of the second kind, $\left\{\begin{array}{l}n \\ k\end{array}\right\}$, or the first-order Eulerian numbers, $\left\langle\begin{array}{l}n \\ m\end{array}\right\rangle$, when $p \in \mathbb{N}$ and $|z|<1$ [2], §7.4, §6.2]:

$$
\sum_{n=0}^{\infty} n^{p} z^{n}=\sum_{k=0}^{p}\left\{\begin{array}{l}
p \\
k
\end{array}\right\} \frac{k!\cdot z^{k}}{(1-z)^{k+1}}=\frac{1}{(1-z)^{p+1}} \sum_{i=0}^{p}\left\langle\begin{array}{c}
p \\
i
\end{array}\right\rangle z^{i+1}
$$

The forms of sequence factors of other standard sequences, including the Stirling numbers of the first and second kinds, common variants of the binomial coefficients, $\binom{n}{k}$ and $\binom{n+m}{m}$, the Eulerian number triangles, and other triangular sequences of interest in application-specific contexts are handled similarly as primitives in the desired formulas output by the package routines.

The factorization-based approach to determine factors of expected sequences by the user in this package differs from the methods employed to recognize sequence formulas by existing sequence recognition software. Since this method relies on user direction as to what terms the sequence formulas should contain, this approach is also useful in determining formulas involving factors of difficult sequence forms that are not easily recognized by existing software packages. The package then employs a hybrid of the complementary approaches noted in [3, §1] to the search for polynomial sequence formulas. Specifically, the package routines employ existing sequence recognition functions as a subroutine to process the reminder terms in the sequence after the expected special sequence factors are identified in the coefficients of the input polynomials.

## Chapter 2

## The Guess Polynomial Sequence Function Packages

### 2.1 Features of the Package

### 2.1.1 Overview

The GuessPolySequenceFormulas.m package is designed to recognize formulas for polynomial sequences in one variable based on input user observations on factors of the polynomial coefficients. The public function GuessPolynomialSequence provided by the package attempts to perform intelligent guessing of closed-form summation representations for a polynomial sequence of elements, $p_{j}(x) \in \mathbb{Z}[x]$, based on the user insights as to the coefficient factors in the end formula for the sequence and the first several polynomial terms passed as input to the function. Several particular concrete examples of uses of the package to obtain formulas and other identities involving the Stirling numbers and binomial coefficients are contained in the discussions of Section 2.2 and Section 2.3 of the thesis below.

### 2.1.2 Specification of the Package Routines and Polynomial Sequence Formulas

The primary package function GuessPolynomialSequence provided to the user is implemented in Mathematica code in such a way that it is able to handle multiple coefficient factors of sequences expected by the user. The focus of the examples provided as documentation for the package focus on the particular cases of "single-factor" and "double-factor" coefficient formulas for the input polynomials. In particular, the package search routines are of interest in obtaining sequence formulas corresponding to the following pair of summation formulas:

$$
\begin{align*}
& \operatorname{Poly}_{j}(x):=\sum_{i=0}^{j+j_{0}}\| \|_{\widetilde{u}_{1}(j)+u_{1} i}^{\widetilde{\ell}_{1}(j)+\ell_{1} i} \|_{1} \times \operatorname{RS}_{1}(i) \operatorname{RS}_{2}\left(j+j_{0}-i\right) \cdot x^{i}  \tag{2.1}\\
& \operatorname{Poly}_{j}(x):=\sum_{i=0}^{j+j_{0}}\left\|\begin{array}{l}
\widetilde{u}_{1}(j)+u_{1} i \\
\widetilde{\ell}_{1}(j)+\ell_{1} i
\end{array}\right\|_{1}\left\|\begin{array}{l}
\widetilde{u}_{2}(j)+u_{2} i \\
\widetilde{\ell}_{2}(j)+\ell_{2} i
\end{array}\right\|_{2} \times \operatorname{RS}_{1}(i) \operatorname{RS}_{2}\left(j+j_{0}-i\right) \cdot x^{i} . \tag{2.2}
\end{align*}
$$

The polynomials in 2.1 and 2.2 correspond to the single-factor and double-factor sequence formula templates, respectively.

In the previous equations, $j, j_{0} \in \mathbb{N}, u_{i}, \ell_{i} \in \mathbb{Z}$, the functions $\widetilde{u}_{i}(j)$ and $\widetilde{\ell}_{i}(j)$ denote some prescribed application-dependent functions of the sequence index, and the form of the remaining sequences in the polynomial coefficient formulas are denoted by the functions $\mathrm{RS}_{1}(\cdot)$ and $\mathrm{RS}_{2}(\cdot)$. The package formula search routines only currently handle linear functions of the summation index inputs. Also notice that it is assumed that at least one of the $R S_{i}(\cdot)$ sequence functions is identically one, and that a formula for the remaining function is either easily obtained by an existing sequence recognition routine such as Mathematica's FindSequenceFunction function, or may be later identified with a relevant entry in the Online Encyclopedia of Integer Sequences database [10].

### 2.1.3 Special Triangular Sequence Factors Supported by the Package

The built-in subpackage GuessSequenceData.m included with the current package source code provides an "out of the box" implementation of several triangular sequences of interest in my research and that are important in motivating the development of this package. In the current implementation of the package, these user-specified sequences identified in the package routines include factors of the (signed and unsigned) Stirling number triangles, variations of triangular sequences derived from the binomial coefficients, and the first and second-order Eulerian number triangles defined recursively as in [2, §6.1, 6.2] [6, c.f. §26.8, 26.14]. Each of these respective sequences correspond to special cases of the following triangular recurrence relation where $\alpha, \beta, \gamma, \alpha^{\prime}, \beta^{\prime}, \gamma^{\prime} \in \mathbb{Z}[2, \S 5, \S 6.1-6.2]$ :

$$
\left\|\begin{array}{l}
n \\
k
\end{array}\right\|=(\alpha n+\beta k+\gamma)\left\|\begin{array}{c}
n-1 \\
k
\end{array}\right\|+\left(\alpha^{\prime} n+\beta^{\prime} k+\gamma^{\prime}\right)\left\|\begin{array}{l}
n-1 \\
k-1
\end{array}\right\|+[n=k=0]_{\delta}
$$

Mathematica provides several standard, built-in functions for the (signed) Stirling numbers of the first and second kinds, and for the binomial coefficients. The related Mathematica package Stirling.m authored by Manuel Kauers ${ }^{1}$ further extends the default functions for the Stirling numbers and defines additional functions that implement the Eulerian number triangles of both orders 4].

### 2.1.4 Some Restrictions on the Form of the Input Polynomials

The package function GuessPolynomialSequence is designed to find formulas for polynomials, $p_{j}(x)$, whose coefficients are integer-valued. The guessing function is, however, able to find formulas for semi-rational polynomial sequences in $\mathbb{Q}[x]$ provided that the first several terms of the sequence input to the function GuessPolynomialSequence are normalized by a user guess function, $U_{\text {guess }}(j, i)$, as described in Section 2.2 .3 of the thesis below. The difficulties in handling formulas for polynomials with rational coefficients arise in determining strictly integer-valued factors of rational-valued coefficient forms. These implementation issues are outlined in Section 3.2.1. Several suggestions for transformations that pre-process polynomials with rational coefficients are also suggested in the section as features to be implemented in a future revision of the package.

### 2.2 Installation and Usage of the Package Routines

### 2.2.1 Installation

## Mathematica Package Installation

The package requires a working installation of Mathematica and a copy of the two source files GuessPolySequenceFormulas.m and GuessSequenceData.m provided on the SageMathCloud project page at the URL listed in the next section. To load the package under Linux, suppose that the package files are located in $\sim /$ guess-polys-pkg. The package is then loaded by running
<<"~/guess-polys-pkg/GuessPolySequenceFormulas.m"
A graphical summary of the short description and revision information for the package is printed when the package is successfully loaded from within a Mathematica notebook.

## Sage Package Installation

The Mathematica package routines accompanying the original Master's thesis manuscript from 2014 now have a counterpart in the open-source SageMath application. The Python source code to this updated software for the Sage environment is freely available for non-commercial usage online at https://github. com/maxieds/GuessPolynomialSequences. Provided that there is a correctly functioning version of Sage,

[^2]or a user account on the SageMathCloud servers, installation of the package is as simple as copying all of the Python, or *.py, files into the current working directory for Sage.

### 2.2.2 Typical Usage

The examples given in this section illustrate both the syntax and utility of the sequence recognition routines provided by the package functions. Notice that the formulas returned by the function are pure functions in Mathematica with three ordered parameters: 1) The polynomial sequence index; 2) An input variable that denotes the summation index of the formula; and 3) A parameter that specifies the polynomial variable. The graphical printing of the formula data provided in the figures given in this section is disabled by setting the runtime option PrintFormulas->False. The runtime option FSFFunction is also available to replace the default Mathematica function FindSequenceFunction by an alternate sequence handling function to process the formulas for the remaining sequences in the polynomial coefficient terms, as well as the formulas for the coefficient indices in the polynomial index $j$ and the upper index of summation in (2.1) and (2.2). The most common and useful of these option settings are documented in the examples below and in the sections of this chapter.

## Examples: Coefficient Factors Involving the Stirling Numbers of the First Kind

Consider the following pair of sums resulting from the expansions of the binomial coefficients as polynomials in $n$

$$
\begin{align*}
\binom{n}{k} & =\frac{n^{\underline{k}}}{k!}=\frac{1}{k!} \times n \cdot(n-1) \cdot(n-2) \cdots(n-k+1) \\
& =\frac{1}{k!} \times \sum_{i=0}^{k}\left[\begin{array}{l}
k \\
i
\end{array}\right](-1)^{k-i} n^{i}  \tag{2.3}\\
\binom{n+m}{m} & =\frac{n^{\overline{m+1}}}{n \cdot m!}=\frac{1}{m!} \times(n+1) \cdot(n+2) \cdots(n+m-1) \cdot(n+m) \\
& =\frac{1}{m!} \times \sum_{i=0}^{m}\left[\begin{array}{c}
m+1 \\
i+1
\end{array}\right] n^{i}, \tag{2.4}
\end{align*}
$$

where $n^{\underline{k}}$ denotes the falling factorial function and $n^{\bar{m}}$ is the rising factorial function in the respective expansions of the previous equations [2, §2.6; §5.1] [6, c.f. §26.1]. The sums for the binomial coefficient expansions involving the Stirling numbers in each of 2.3) and (2.4) are known closed-form identities for the rising and falling factorial functions, respectively, stated in [2, §6.1] [6] c.f. §26.8]. To see how the package can assist a user in rediscovering these identities, consider the respective Mathematica outputs given in Figure 2.1 and Figure 2.3 A related example involving the polynomial sequence in (2.3) is shown in Figure 2.11.

## Example: Multiple Polynomial Sequence Formulas Derived from Symmetric Sequences

The package sequence recognition routines are able to find formulas for polynomials involving the binomial coefficients, $\binom{n}{k}$, and the first-order Eulerian numbers, $\left\langle\begin{array}{l}n \\ m\end{array}\right\rangle$. These sequences both have symmetry in each row of the corresponding triangles that satisfy the following pair of reflection identities where $n, k, m \in \mathbb{N}$ [2. $\S 5, \S 6.2]:$

$$
\binom{n}{k}=\binom{n}{n-k} \quad \text { and } \quad\left\langle\begin{array}{c}
n \\
m
\end{array}\right\rangle=\left\langle\begin{array}{c}
n \\
n-1-m
\end{array}\right\rangle .
$$

The examples given in this section demonstrate the multiple formulas obtained by the package for polynomial sequences involving these triangles that result from the coefficient symmetry noted in the forms of the previous equation.

The first example corresponds to an identity involving squares of the binomial coefficients and the derivative operator, $D^{(j)}[F(z)] \equiv F^{(j)}(z)$, of a function, $F(z)$, whose $j^{\text {th }}$ derivative with respect to $z$ exists for some $j \in \mathbb{N}$. In particular, suppose that the function $F(z)$ denotes the ordinary generating function of the

```
In[3]:= seq = Table [Expand [FunctionExpand [Binomial[n, k ] * Factorial [k]]], {k, 1, 8 }];
    GuessPolynomialSequence [seq, n ]
#== === === === Found Matching Formula #1 / 1: =============
Poly j (n) \mapsto}\mp@subsup{\sum}{i=0}{j-1}(-1\mp@subsup{)}{}{-i+j-1}\mp@subsup{n}{}{i+1}\mp@subsup{S}{1}{}(j,i+1
- Latex Formula Output: Null
- Remaining Sequence Data: {1, -1, 1, -1, 1, -1, 1, -1}
* User Function: U guess (j, i) = 1
- Formula Function: PolyFormula index =1 (j, i, n)
- Sequence Formula Diffs: {True, True, True, True, True, True, True, True} [\checkmark ]
```



Figure 2.1: Computing a Polynomial Formula for the Falling Factorial Function (Mathematica )

```
sage: ## Falling factorial polynomials }
sage: from GuessPolynomialSequenceFunction import * 
sage: n = var('n')
sage: poly_seq_func = lambda k: expand(simplify(binomial(n, k) 4
    * factorial(k)))
sage: pseq_data = map(poly_seq_func, range(1, 6)) 
sage: guess_polynomial_sequence(pseq_data, n, index_offset = 6
    1);
```

Figure 2.2: Computing a Polynomial Formula for the Falling Factorial Function (Sage)
sequence, $\left\langle f_{n}\right\rangle$, and the function has $j^{\text {th }}$ derivatives of orders $j \in[0, d] \subseteq \mathbb{N}$. Then for $d \in \mathbb{Z}^{+}$, the generating function for the modified sequence, $\left\langle\frac{(n+d)!}{n!} f_{n}\right\rangle$, satisfies the formula

$$
\begin{equation*}
\sum_{n=0}^{\infty} \frac{(n+d)!}{n!} f_{n} z^{n}=\sum_{n=0}^{\infty}(n+1) \cdots(n+d) \times f_{n} z^{n}=\sum_{i=0}^{d}\binom{d}{i}^{2}(d-i)!\times z^{i} D^{(i)}[F(z)] . \tag{2.5}
\end{equation*}
$$

Notice that a proof of the formula given in (2.5) follows easily by induction on $d \geq 1$. A user may obtain the first several values of this sequence empirically by evaluating Mathematica's GeneratingFunction for the modified sequence terms over the first few values of $d \geq 1$. Figure 2.5 shows a use of the package in guessing a formula for (2.5) where the polynomial variable ( $w^{i}$ in the figure listing) corresponds to the operator form of $z^{i} D^{(i)}$, and where the Pochhammer symbol, $(1)_{j-i} \equiv(j-i)!$.

The next example in this section corresponds to the sequence of ordinary generating functions for polynomial powers of $n, \sum_{n \geq 0} n^{m} z^{n}$ for $m \in \mathbb{N}$. These generating functions satisfy well-known polynomial identities involving the Stirling numbers of the second kind, $\left\{\begin{array}{l}n \\ k\end{array}\right\}$, and the first-order Eulerian numbers, $\left\langle\begin{array}{l}n \\ m\end{array}\right\rangle$, stated as follows [2, c.f. §7.4]:

$$
\sum_{n=0}^{\infty} n^{m} z^{n}=\sum_{k=0}^{m}\left\{\begin{array}{l}
m  \tag{2.6}\\
k
\end{array}\right\} \frac{k!\cdot z^{k}}{(1-z)^{k+1}}=\frac{1}{(1-z)^{m+1}} \sum_{i=0}^{m}\left\langle\begin{array}{c}
m \\
i
\end{array}\right\rangle z^{i+1} .
$$

```
\(\ln [13]:=\) seq \(=\) Table [Expand [FunctionExpand [Binomial [n + m, m]*Factorial [m]]], \{m, 0, 8\}];
    GuessPolynomialSequence [seq, \(n\), StartIndex \(\rightarrow 0\), PrintLaTeXFormulas \(\rightarrow\) False]
```



```
    Poly \(_{j}(n) \mapsto \sum_{i=0}^{j} n^{i} S_{1}(j+1, i+1)\)
    - Remaining Sequence Data: \(\{1,1,1,1,1,1,1,1,1\}\)
    - User Function: U guess (j, i) \(=1\)
    - Formula Function: PolyFormula \({ }_{\text {index }}=1(j, i, n)\)
    - Sequence Formula Diffs: \{True, True, True, True, True, True, True, True, True \} [ \(\checkmark\) ]
\(\operatorname{Out}[14]=\left\{\sum_{\# 2=0}^{\# 1} \operatorname{SeqFnS} 1[1+\# 1,1+\# 2] \# 3^{\# 2} \&\right\}\)
```

Figure 2.3: Computing a Polynomial Formula for the Rising Factorial Function (Mathematica )

```
sage: ## Rising factorial polynomials 
sage: from GuessPolynomialSequenceFunction import * 8
sage: n = var('n')
sage: poly_seq_func = lambda m: expand(simplify(binomial(n + m 10
    , m) * factorial(m)))
sage: pseq_data = map(poly_seq_func, range(1, 4)) 11
sage: guess_polynomial_sequence(pseq_data, n, index_offset = 12
    1);
```

Figure 2.4: Computing a Polynomial Formula for the Rising Factorial Function (Sage)

The second example cited in this section focuses on the second expansion in 2.6 given in terms of the Eulerian number triangle. Figure 2.7 shows the output of the package on the second polynomial sequence scaled by a multiple of $(1-z)^{m+1}$. As with the first example, the row-wise symmetry in the Eulerian number triangle results in the two separate formulas in the figure.

## Examples: Two-Factor Polynomial Sequence Formulas

The examples cited in this section correspond to the two-factor polynomial sequence formulas in the form of 2.2 . The first example given in Figure 2.9 shows the output of the package function
GuessPolynomialSequence for a formula involving the Stirling numbers of the first and second kinds. The second example given in Figure 2.10 shows the pair of formulas output for a sequence formula involving the Stirling numbers of the first kind and the binomial coefficients. The use of the runtime option IndexOffsetPairs in both of these examples is explained in more detail by Section 2.2.4.

### 2.2.3 User Guess Functions

The guessing routines implemented in the package rely on some intuition on the part of the user to determine a general template for the end formulas for an input polynomial sequence with coefficients over the integers. The user may specify an additional "user guess function" that is employed by the package to pre-process the coefficients of the polynomial sequence terms passed to the function GuessPolynomialSequence . This construction allows semi-rational, and even non-polynomial functions in the input variable to be processed
by the package functions.

## Example: A Second Formula for the Falling Factorial Function

A first example of the syntax for guessing the polynomial expansions of the binomial coefficient identity from (2.3) is provided in Figure 2.11. Notice that this example is similar to the first form of the sequence formula computed by the package in Figure 2.1, except that in this case the input sequence is not normalized by a factor of $k$ to make the polynomial coefficients strictly integer-valued. A similar computation is employed to discover an analogous sum for the non-normalized sequence formula corresponding to the rising factorial function from Figure 2.3 .

## Example: An Exponential Generating Function for the Binomial Coefficients

A sequence of exponential generating functions for the symmetric form of the binomial coefficients, $\binom{n+m}{m}$, taken over $m \in \mathbb{N}$ satisfies the formula given in the following equation [2, c.f. §7.2]:

$$
\begin{equation*}
\operatorname{EGF}_{z}\left(\frac{1}{(1-z)^{m+1}}\right) \equiv \sum_{n=0}^{\infty}\binom{n+m}{n} \frac{z^{n}}{n!} \equiv \sum_{s=0}^{m}\binom{m}{s} \frac{e^{z} \cdot z^{s}}{s!} \tag{2.7}
\end{equation*}
$$

A proof of this identity is given using Vandermonde's convolution identity for the binomial coefficients [2, $\S$ Table 174; $\S 5.2 ; c . f$. eq. (5.22)]. Figure 2.13 shows a use of the package to guess the formula in 2.7) by providing a user guess function that effectively removes the factor of $e^{z}$ in the expected formula, and that cancels out the coefficient factors of $1 / s$ ! to produce an input sequence with integer coefficients.

### 2.2.4 Troubleshooting Possible Issues

## Inputting an Insufficient Number of Sequence Elements

There are a couple of issues that can arise in running the package routines when too few values of the sequence are passed to the GuessPolynomialSequence function. The first of these is that FindSequenceFunction may require a lower bound on the number of sequence values necessary to compute formulas for the remaining sequence terms. This can occur, for example, when the remaining sequence is a polynomial in the summation index. Another quirk of Mathematica's built-in FindSequenceFunction is that it may return a sequence formula matching a recurrence relation that is actually accurate for the few sequence elements input to the function. An example of this behavior is illustrated by the output given in Figure 2.15. In most cases, the problem is resolved by simply passing more polynomials from the sequence, usually at least 6 , but possibly 8 or more elements from the sequence. The package is configured to warn users when less than 6 initial terms are input to the function with no matching formulas.

## Number of Rows for the Expected Triangular Sequence Factors

In some cases, the package functions may not be able to obtain a formula for an input sequence due to an insufficient setting for the number of rows to consider for the expected triangular sequence factors. The runtime option to change the number of rows used to detect the factors of the expected triangular sequence is TriangularSequenceNumRows (the current default setting is TriangularSequenceNumRows->12). Figure 2.16 provides an example involving the Stirling numbers of the second kind where the upper index of the sequence depends quadratically on the polynomial index $j$. In this example, the package routines are unable to obtain a formula when the runtime option is reset to TriangularSequenceNumRows->24, but correctly finds the sequence formula by setting the option to the higher value of TriangularSequenceNumRows->72. Notice that choosing a significantly higher default setting for this option may result in much slower running times, especially if the expected triangular sequence factors contain a large number of 1 -valued entries, for example, as in the Stirling numbers of the first kind, binomial coefficient, and first-order Eulerian number triangles.

## Handling Long Running Times with Multiple Sequence Factors

The package function GuessPolynomialSequence is able to return sequence formulas in the single-factor form given in 2.1 in a reasonable amount of running time. As suggested in the double-factor sequence examples of the form in 2.2 from Figure 2.9 and Figure 2.10, the runtime option Index0ffsetPairs is needed to speed-up the running time for the computations involved in these sequence cases. The IndexOffsetPairs option is defined as a list of lists of the form

$$
\begin{equation*}
\left\{\left\{u_{1}, \ell_{1}\right\},\left\{u_{2}, \ell_{2}\right\}, \ldots,\left\{u_{r}, \ell_{r}\right\}\right\} \tag{2.8}
\end{equation*}
$$

where $r \geq 1$ denotes the expected number of sequence factors involved in the search for the sequence formulas by GuessPolynomialSequence. In the examples cited in Figure 2.9, Figure 2.10, and in the template form of $(\sqrt{2.2})$, the value of $r$ corresponds to $r:=2$. For a fixed choice of $r \geq 1$, each element of the list defined by IndexOffsetPairs passed in the form of 2.8 corresponds to a search for a sequence formula of the form

$$
\operatorname{Poly}_{j}(x):=\sum_{i=0}^{j+j_{0}}\left(\prod_{i=1}^{r}\left\|\begin{array}{l}
\widetilde{u}_{i}(j)+u_{i} i \\
\tilde{\ell}_{i}(j)+\ell_{i} i
\end{array}\right\|_{i}\right) \times \operatorname{RS}_{1}(i) \operatorname{RS}_{2}\left(j+j_{0}-i\right) \cdot x^{i}
$$

Thus resetting the value of this option at runtime can speed-up the search for matching formulas in the cases of multiple expected sequence factors, especially compared to the number of index offset pairs resulting from the default enumeration of these pair values.

### 2.3 More Examples of Polynomial Sequence Types Recognized by the Packages

The examples cited in this section are intended to document further forms of the polynomial sequence types that the package is able to recognize. These examples include handling polynomial sequence formulas that depend on arithmetic progressions of indices, coefficients that contain symbolic data, and examples of sequence formulas obtained by the package routines when the expected sequence factors do not depend on the summation index, i.e., when the factors only depend on the polynomial sequence index.

### 2.3.1 Example: Arithmetic Progressions of Coefficient Indices

The package function GuessPolynomialSequence can be configured to search for sequence formulas involving arithmetic progressions of the summation index, $f(j)+a i$, for values besides $a:= \pm 1$ by resetting the runtime option IndexMultiples. The default setting of this option is IndexMultiples->\{0,1\}. Figure 2.17provides an example of recognizing sequence formulas involving squares of the binomial coefficients where the upper index of the triangle does not depend on the summation index (a setting of $a:=0$ ) and where the lower triangle index involves an arithmetic progression of the summation index with $a:= \pm 3$. Related sequence formulas are recognized by setting the runtime value of this option to a list of test values that is some subset of the natural numbers. Notice that if the list of values for the option IndexMultiples does not contain 0 , the package routines will not find formulas like those given in Figure 2.17 where the upper index of the expected triangle factors only depends on the polynomial sequence index ( $j$ in the figure examples).

### 2.3.2 Examples: Formulas Involving Symbolic Coefficient Data

The function GuessPolynomialSequence can be configured to search for formulas where the input coefficients of the polynomial sequence contain non-numeric factors of symbolic data through the runtime option AllowSymbolicData. Figure 2.18 and Figure 2.19 provide examples of sequence formulas involving nonnumeric, symbolic terms, named $a, b, c, q, r$, that are recognized by the package by passing AllowSymbolicData->True to the GuessPolynomialSequence function at runtime.

### 2.3.3 Examples: Recognition of Other Sequence Formulas with the Mathematica Package

Figure 2.20 and Figure 2.21 cite two additional examples of sequence formulas that the package is able to recognize when the triangular sequence factors expected by the user do not depend on the summation index, $i$, only the polynomial sequence index, $j$. In the first example given in Figure 2.20 the expected binomial coefficient factor corresponds to a polynomial in $j$. In the second example given in Figure 2.21, the expected Stirling number factor corresponds to an expansion in terms of $r$-order harmonic numbers, $H_{j+2}^{(r)}$, that is reported as the factor of the original Stirling number sequence.

### 2.3.4 Examples: Recogntion of Other Polynomial Sum Identities with the Sage Package Implementation

## Example: A Legendre Polynomial Identity Involving Squares of the Binomial Coefficients

A finite polynomial sum over the squared powers of the binomial coefficients is expressed through the Legendre polynomials, $P_{n}(x)$, and its ordinary generating function in two variables in the following forms [6, §18]:

$$
S_{n}(z):=\sum_{k=0}^{n}\binom{n}{k}^{2} z^{k}=(1-z)^{n} P_{n}\left(\frac{1+z}{1-z}\right)=\left[t^{n}\right]\left(\frac{1}{\sqrt{1-(1+z) t+(z-1)^{2} t^{2}}}\right)
$$

Alternately, we may obtain information about a closed-form sum for the Legendre polynomials over these polynomial inputs to the sequence through a known recurrence relation for the sums, $S_{n}(z)$, given by [2, p. 543]

$$
(n+1)(1-z)^{2} S_{n}(z)-(2 n+3)(z+1) S_{n+1}(z)+(n+2) S_{n+2}(z)=0 .
$$

Figure 2.22 provides a listing of Sage commands using the new package implementation in Python to obtain a sequence formula for the right-hand-side polynomial expansions.

## Example: An Exponential Generating Function for the Exponential Bell Polynomials

An exponential generating function for the Bell, or exponential, polynomials and the corresponding finite sum expansion over the Stirling numbers of the second kind is given in the next equation [8, §4.1.3].

$$
n!\cdot B_{n}(x)=\left[t^{n}\right] \exp \left(\left(e^{t}-1\right) x\right)=\sum_{k=0}^{n}\left\{\begin{array}{l}
n \\
k
\end{array}\right\} x^{k}
$$

Figure 2.23 provides a listing of the Sage commands needed to recognize the rightmost identity for this special sequence.

```
sumFn[d_, w_]:= Sum[(Binomial [d, i]^ 2) * Factorial [d - i] * Power [w, i], {i, 0, d}]
(****)
seq = Table [sumFn[d, w], {d, 1, 6}];
GuessPolynomialSequence [seq, w, SequenceFactors }->\mathrm{ {"Binom2 "}]
##========== Found Matching Formula #1 / 2: ============
Poly j (w) \mapsto [ < wi=0
- Latex Formula Output: Null
- Remaining Sequence Data: {1, 1, 2, 6, 24, 120, 720}
- User Function: U guess ( j, i) = 1
- Formula Function: PolyFormula index =1 (j, i, w)
- Sequence Formula Diffs: {True, True, True, True, True, True} [ }\sqrt{}{\prime}
```



```
Poly j (w) \mapsto [ \sum wi=0
- Latex Formula Output: Null
- Remaining Sequence Data: {1, 1, 2, 6, 24, 120, 720}
- User Function: U guess (j, i) = 1
- Formula Function: PolyFormula index =2(j, i, w)
- Sequence Formula Diffs: {True, True, True, True, True, True} [\checkmark ]
{ \sum1 Binomial [#1,#1-#2] 2 Pochhammer [1,#1-#2]#3#2 &,
\sum ## Binomial [#1, #2 ] 2 Pochhammer [1,#1-#2] #3 #2 &}
```

Figure 2.5: A Sum Involving Derivative Operators and Squares of the Binomial Coefficients (Mathematica )

```
sage: ## Binomial squared difference operator identity
sage: from GuessPolynomialSequenceFunction import *
sage: n, i, w = var('n}\mp@subsup{|}{\sqcup}{\prime}\mp@subsup{|}{\sqcup}{\prime}'
sage: poly_seq_func = lambda d: sum((binomial(d, i) ** 2) *
    factorial(d - i) * (n ** i), i, 0, d)
sage: pseq_data = map(poly_seq_func, range(1, 8)) 17
sage: guess_polynomial_sequence(pseq_data, n, seq_factors = [" 18 18
```

Figure 2.6: A Formula Involving Derivative Operators and Squares of the Binomial Coefficients (Sage)

```
sumFnNPowE1 [m_, z_]:=
    Expand [Simplify [Sum [Power [n, m] * Power [z, n], {n, 0, Infinity }] * Power [(1 - z), m + 1]]]
(****)
seq = Table [sumFnNPowE1 [m, w], {m, 1, 8}];
GuessPolynomialSequence [seq, w, SequenceFactors }->{"E1"}
Em========== Found Matching Formula #1 / 2: =============
Poly j (w) \mapsto 
```

- Latex Formula Output: Null
- Remaining Sequence Data: $\{1,1,1,1,1,1,1,1\}$
- User Function: $U_{\text {guess }}(j, i)=1$
- Formula Function: PolyFormula ${ }_{\text {index }}=1(j, i, w)$
- Sequence Formula Diffs: \{True, True, True, True, True, True, True, True \} [ $\checkmark$ ]

$\operatorname{Poly}_{j}(w) \mapsto \sum_{i=0}^{j-1} w^{i+1} E_{1}(j, i)$
- Latex Formula Output: Null
- Remaining Sequence Data: $\{1,1,1,1,1,1,1,1\}$
- User Function: $U_{\text {guess }}(j, i)=1$
- Formula Function: PolyFormula ${ }_{\text {index }}=2(j, i, w)$
- Sequence Formula Diffs: \{True, True, True, True, True, True, True, True \} [ $\checkmark$ ]
$\left\{\sum_{\# 2=0}^{-1+\# 1} \operatorname{SeqFnE1}[\# 1,-1+\# 1-\# 2] \# 3^{1+\# 2} \&, \sum_{\# 2=0}^{-1+\# 1} \operatorname{SeqFnE1}[\# 1, \# 2] \# 3^{1+\# 2} \&\right\}$

Figure 2.7: Ordinary Generating Functions of Polynomial Powers (Mathematica )

```
sage: ## First-order Eulerian numbers in the OGFs of the
    polylogarithm
sage: ## functions, Li_{-m}(z), for natural numbers m >= 1
sage: from GuessPolynomialSequenceFunction import *
sage: n, z = var(' }\mp@subsup{n}{\sqcup}{\prime
sage: poly_seq_func = lambda m: expand(factor(sum((n ** m) * (
    z ** n), n, 0, infinity)) * ((1 - z) ** (m + 1))).subs(z =
    n)
sage: pseq_data = map(poly_seq_func, range(1, 6))
sage: guess_polynomial_sequence(pseq_data, n, seq_factors = ["
    E1"], index_offset = 1);
```

Figure 2.8: Ordinary Generating Functions of Polynomial Powers (Sage)

```
In[72]:= seq = Table [Expand [Sum [Abs[StirlingS1[j + 3, j + 1 - i]] *
    StirlingS2[j + 3, i + 2] * Power [-5, i + 2] * Power [x, i], {i, 0, j}]], {j, 1, 8}];
    GuessPolynomialSequence [seq, x, StartIndex }->1,\mathrm{ SequenceFactors }->\mathrm{ {"S1 ", "S2 "},
    TriangularSequenceNumRows }->\mathrm{ 12, IndexOffsetPairs }->{{{0,-1},{0, 1}} }
```



```
Polyj (x) \mapsto 
    - Latex Formula Output: Null
    \ Remaining Sequence Data: {25, -125, 625, - 3125, 15 625, -78125, 390 625, -1 953125, 9765625}
    - User Function: U guess (j, i) = 1
    - Formula Function: PolyFormula index =1 (j, i, x)
    \ Sequence Formula Diffs: {True, True, <<4>>, True, True} [\checkmark ]
Out[73]={ {\sum1 #2=0
```

Figure 2.9: A Double-Factor Sequence Example Involving the Stirling Number Triangles

```
In[74]:= seq = Table [Expand [Sum[Abs[StirlingS1[j + 3, j + 1 - i]] *
    Binomial[j + 3, i + 2] * Power [-5, i + 2] * Power [x, i], {i, 0, j}]], {j, 1, 8}];
GuessPolynomialSequence [seq, x, StartIndex }->\mathrm{ 1, SequenceFactors }->{"S1", "Binom"}
    TriangularSequenceNumRows }->\mathrm{ 12,
    IndexOffsetPairs }->{{{0,-1},{0, 1}}, {{0,-1}, {0, - 1 }} }, DisplayVars -> {m, k}
Eニニ=ニニ =ニニ=ニ= Found Matching Formula #1 / 2: ====== ======
Polym}(x)\mapsto\mp@subsup{\sum}{m=0}{m}(-1\mp@subsup{)}{}{k}\mp@subsup{5}{}{k+2}\mp@subsup{x}{}{k}(\begin{array}{l}{m+3}\\{k+2}\end{array})\mp@subsup{S}{1}{}(m+3,-k+m+1
－Latex Formula Output：Null
－Remaining Sequence Data：\(\{25,-125,625,-3125,15625,-78125,390625,-1953125,9765625\}\)
－User Function：\(U_{\text {guess }}(m, k)=1\)
－Formula Function：PolyFormula index \(^{=}(\mathrm{m}, \mathrm{k}, \mathrm{x})\)
－Sequence Formula Diffs：\｛True，True，\(\ll 4 \gg\) ，True，True \} [ \(\checkmark\) ］
```


## 

```
Poly \(m(x) \mapsto \sum_{k=0}^{m}(-1)^{k} 5^{k+2} x^{k}\binom{m+3}{-k+m+1} S_{1}(m+3,-k+m+1)\)
－Latex Formula Output：Null
－Remaining Sequence Data：\(\{25,-125,625,-3125,15625,-78125,390625,-1953125,9765625\}\)
－User Function：\(U_{\text {guess }}(m, k)=1\)
－Formula Function：PolyFormula \({ }_{\text {index }}=2(\mathrm{~m}, \mathrm{k}, \mathrm{x})\)
－Sequence Formula Diffs：\｛True，True，\(\ll 4 \gg\) ，True，True \} [ \(\checkmark\) ］
Out［75］＝
\[
\begin{aligned}
& \left\{\sum_{\# 2=0}^{\# 1}(-1)^{\# 2} 5^{2+\# 2} \text { Binomial }[3+\# 1,2+\# 2] \text { SeqFnS1 }[3+\# 1,1+\# 1-\# 2] \# 3^{\# 2} \&,\right. \\
& \\
& \left.\sum_{\# 2=0}^{\# 1}(-1)^{\# 2} 5^{2+\# 2} \text { Binomial }[3+\# 1,1+\# 1-\# 2] \operatorname{SeqFnS1}[3+\# 1,1+\# 1-\# 2] \# 3^{\# 2} \&\right\}
\end{aligned}
\]
```

Figure 2．10：A Double－Factor Sequence Example Involving the Stirling Numbers of the First Kind and the Binomial Coefficients

```
ln[7]:= userGuessFn[k_, i_]:=(1/Factorial[k])
    seq = Table [Expand [FunctionExpand [Binomial [n, k]]], {k, 1, 8}];
    GuessPolynomialSequence [seq, n, UserGuessFunction -> userGuessFn]
    #===ユ==ユ==== Found Matching Formula #1 / 1: =============
    Poly j (n) \mapsto}\mp@subsup{\sum}{i=0}{j-1}\frac{(-1\mp@subsup{)}{}{-i+j-1}\mp@subsup{n}{}{i+1}\mp@subsup{S}{1}{}(j,i+1)}{j!
    - Latex Formula Output: Null
    - Remaining Sequence Data: {1, -1, 1, -1, 1, -1, 1, -1}
    - User Function: U guess (j, i)}=\frac{1}{j!
    - Formula Function: PolyFormula index =1 (j, i, n)
    - Sequence Formula Diffs: {True, True, True, True, True, True, True, True} [\checkmark]
Out[9]=
```



Figure 2.11: A Formula for the Falling Factorial Function by a User Guess Function (Mathematica )

```
sage: ## Another exponential falling factorial polynomial
    example
sage: ## with a user guess function
sage: from GuessPolynomialSequenceFunction import *
sage: n, z = var(' }\mp@subsup{\textrm{n}}{\sqcup}{
sage: poly_seq_func = lambda k: binomial(n, k)
sage: user_guess_func = lambda n, k: 1 / factorial(k)
sage: pseq_data = map(poly_seq_func, range(1, 6))
sage: guess_polynomial_sequence(pseq_data, n, user_guess_func }3
    = user_guess_func, index_offset = 1);
```

Figure 2.12: Computing a Formula for the Falling Factorial Function by a User Guess Function (Sage)

```
ln[33]:= userGuessFn[k_, s_] := (Exp [z]/Factorial [s])
    seq =
    Table [Simplify [Sum [Binomial [n + k, k] * Power [z, n]/Factorial[n], {n, 0, Infinity }]], {k, 0, 8}];
GuessPolynomialSequence [seq, z, StartIndex }->0\mathrm{ , SequenceFactors }->\mathrm{ {"Binom"},
    UserGuessFunction -> userGuessFn]
```



```
\(\operatorname{Poly}_{j}(z) \mapsto \sum_{i=0}^{j} \frac{e^{z} z^{i}\binom{j}{j-i}}{i!}\)
- Latex Formula Output: Null
- Remaining Sequence Data: \(\{1,1,1,1,1,1,1,1,1\}\)
- User Function: \(U_{\text {guess }}(j, i)=\frac{e^{z}}{i!}\)
- Formula Function: PolyFormula index \(^{=1}(j, i, z)\)
- Sequence Formula Diffs: \{True, True, \(\ll 5 \gg\), True, True \} [ \(\checkmark\) ]
```



```
Poly \(_{j}(z) \mapsto \sum_{i=0}^{j} \frac{e^{z} z^{i}\binom{j}{i}}{i!}\)
- Latex Formula Output: Null
- Remaining Sequence Data: \(\{1,1,1,1,1,1,1,1,1\}\)
- User Function: \(U_{\text {guess }}(j, i)=\frac{e^{z}}{i!}\)
- Formula Function: PolyFormula \({ }_{\text {index }}=2(j, i, z)\)
- Sequence Formula Diffs: \{True, True, \(\ll 5 \gg\), True, True \} [ \(\checkmark\) ]
Out[35] \(=\left\{\sum_{\# 2=0}^{\# 1} \frac{e^{z} \text { Binomial [\#1, \#1-\#2] \#3 } \# 2}{\# 2!} \&, \sum_{\# 2=0}^{\# 1} \frac{e^{z} \text { Binomial [\#1, \#2] \#3 } \# 2}{\# 2!} \&\right\}\)
```

Figure 2.13: An Exponential Generating Function for the Binomial Coefficients (Mathematica)

```
sage: ## Exponential generating functions for the symmetric--
    indexed
sage: ## binomial coefficients
sage: from GuessPolynomialSequenceFunction import *
sage: n, z = var(' }\mp@subsup{\textrm{n}}{\sqcup}{}\mp@subsup{\textrm{z}}{}{\prime}\mathrm{ ')
sage: poly_seq_func = lambda k: sum(binomial(n + k, k) * (z ** 38
    n) / factorial(n), n, 0, infinity) * exp(-z)
sage: user_guess_func = lambda n, k: 1 / factorial(k) 39
sage: pseq_data = map(poly_seq_func, range(1, 6))
sage: guess_polynomial_sequence(pseq_data, z, seq_factors = ["
    Binom2"], user_guess_func = user_guess_func, index_offset =
    1);
```

Figure 2.14: An Exponential Generating Function for the Binomial Coefficients (Sage)

```
ln[65]:= sumFnE1[j_, w_] := Sum [(SeqFnE1[j, i]) * Power[-1, j-i] * Power [w, i], {i, 0, j}]
(****)
seq = Table [sumFnE1[j, w], {j, 1, 6}];
GuessPolynomialSequence [seq, w, StartIndex }->\mathrm{ 1, SequenceFactors }->{"E1"}, LimitFormulaCount -> 2]
E== === === === Found Matching Formula #1 / 34: =============
Poly j(w) \mapsto}\mp@subsup{\sum}{i=0}{j-1}(-1\mp@subsup{)}{}{j-i}\mp@subsup{w}{}{i
E ( (DifferenceRoot [{\dot{y},\dot{~}
-i +j-1)
- Latex Formula Output: Null
- Remaining Sequence Data: \(\{-1,1,-1,1,-1,1\}\)
- User Function: \(U_{\text {guess }}(j, i)=1\)
- Formula Function: PolyFormula \({ }_{\text {index }}=1\) (j, i, w)
- Sequence Formula Diffs: \{True, True, \(\ll 2 \gg\), True, True \} [ \(\sqrt{ }\) ]
```


## 

```
Poly \({ }_{j}(w) \mapsto \sum_{i=0}^{j-1}(-1)^{j-i} w^{i} E_{1}(j,-i+j-1)\)
- Latex Formula Output: Null
- Remaining Sequence Data: \(\{-1,1,-1,1,-1,1\}\)
- User Function: \(U_{\text {guess }}(j, i)=1\)
- Formula Function: PolyFormula \({ }_{\text {index }}=2(j, i, w)\)
- Sequence Formula Diffs: \{True, True, <<2>>, True, True \} [ \(\checkmark\) ]
Out[67] \(=\left\{\sum_{\# 2=0}^{-1+\# 1}(-1)^{\# 1-\# 2}\right.\)
\[
\text { SeqFnE1 }[\text { DifferenceRoot }[\text { Function }[\{\dot{y}, \dot{n}\},\{-1-\dot{y}[\dot{n}]+\dot{y}[1+\dot{n}]=0, \dot{y}[1]=0, \dot{y}[2]=2\}]][\# 1] \text {, }
\]
\[
\left.-1+\# 1-\# 2] \# 3^{\# 2} \&, \sum_{\# 2=0}^{-1+\# 1}(-1)^{\# 1-\# 2} \operatorname{SeqFnE1}[\# 1,-1+\# 1-\# 2] \# 3^{\# 2} \&\right\}
\]
```

Figure 2.15: Troubleshooting an Insufficient Number of Sequence Elements

```
ln[76]:= seq = Table [Expand [
            Sum[StirlingS2[j^2 + 5, 3 * j + 4] * Power [-5, i + 4] * Power [x, i], {i, 0, j + 1}]], {j, 3, 8}];
        Short [seq]
        GuessPolynomialSequence [seq, x, StartIndex }->\mathrm{ 3,
            SequenceFactors }->\mathrm{ {"S2"}, TriangularSequenceNumRows }->\mathrm{ 24]
    GuessPolynomialSequence [seq, x, StartIndex }->\mathrm{ 3,
    SequenceFactors }->\mathrm{ {"S2 "}, TriangularSequenceNumRows }->\mathrm{ 72]
Out[77]//Short= {56875-284375x+1421875 x m}-7109375\mp@subsup{x}{}{3}+35546875\mp@subsup{x}{}{4},<<1>>,<<1>>,<<1>>,<<1>>,<<1>>
    Out[78]= {}
#== ========== Found Natching Formula #1 / 1: === ===========
Poly j (x) \mapsto [ < (-1) i
- Latex Formula Output: Null
- Remaining Sequence Data:
\(\{625,-3125,15625,-78125,390625,-1953125,9765625,-48828125,244140625,-1220703125\}\)
- User Function: U guess \((j, i)=1\)
- Formula Function: PolyFormula index \(=1\) (j, i, x)
- Sequence Formula Diffs: \{True, True, <<2>>, True, True\} [ \(\checkmark\) ]
Out[79] \(=\left\{\sum_{\# 2=0}^{1+\# 1}(-1)^{\# 2} 5^{4+\# 2} \# 3^{\# 2}\right.\) StirlingS2 \(\left.\left[5+\# 1^{2}, 4+3 \# 1\right] \&\right\}\)
```

Figure 2.16: Troubleshooting Runtime Settings of the TriangularSequenceNumRows Option

```
ln[195]:= seq = Table [
            Expand [Sum[(Binomial[3 j, 3 i + 2]^ 2) * Power [8, i] * Power[t, i], {i, 0, j + 1}]], {j, 1, 8}];
        GuessPolynomialSequence [seq, t, StartIndex }->1\mathrm{ , SequenceFactors }->\mathrm{ {"Binom2 "},
    IndexMultiples }->{0,3},\mathrm{ TriangularSequenceNumRows }->\mathrm{ 36]
    #===== === === Found Matching Formula #1 / 2: === ==========
```



```
    - Latex Formula Output: Null
    - Remaining Sequence Data: {1, 8, 64, 512, 4096, 32 768, 262144, 2097152}
    - User Function: U guess (j, i) = 1
    - Formula Function: PolyFormula index =1 (j, i, t)
    - Sequence Formula Diffs: {True, True, <<4>>, True, True} [\checkmark ]
E== === === === Found Matching Formula #2 / 2: === ==========
```



```
- Latex Formula Output: Null
- Remaining Sequence Data: {1, 8, 64, 512, 4096, 32 768, 262144, 2097152}
- User Function: U guess (j, i) = 1
- Formula Function: PolyFormula index =2(j, i, t)
- Sequence Formula Diffs: {True, True, <<4>>, True, True} [\checkmark ]
```



Figure 2.17: Handling Arithmetic Progressions of Indices

```
\(\ln [102]:=\) seq \(=\) Table [Expand [Sum [Binomial [ \(j+3, j+1-i] *\) Power [ -5 *a, \(i+2]\) *
            \(\operatorname{Power}[6 * b, i+1] * \operatorname{Power}[c, i+3] * \operatorname{Power}[z, i],\{i, 0, j\}]],\{j, 3,8\}]\);
        GuessPolynomialSequence [seq, z, StartIndex \(\rightarrow 3\), SequenceFactors \(\rightarrow\) \{"Binom"\},
    DisplayVars \(\rightarrow\{\mathrm{n}, \mathrm{k}\}\), AllowSymbolicData \(\rightarrow\) True, LimitFormulaCount \(\rightarrow 1]\)
    E== ==ニ === === Found Matching Formula \#1 / 2: === =========
    \(\operatorname{Poly} y_{n}(z) \mapsto \sum_{k=0}^{n} a^{2} b c^{3} 5^{k+2} 6^{k+1} z^{k}\binom{n+3}{-k+n+1}\left(\begin{array}{lll}-a b c\end{array}\right)^{k}\)
    - Latex Formula Output: Null
    - Remaining Sequence Data: \(\left\{150 a^{2} b^{3},-4500 a^{3} b^{2} c^{4}\right.\),
    \(135000 \mathrm{a}^{4} \mathrm{~b}^{3} \mathrm{c}^{5},-4050000 \mathrm{a}^{5} \mathrm{~b}^{4} \mathrm{c}^{6}, 121500000 \mathrm{a}^{6} \mathrm{~b}^{5} \mathrm{c}^{7},-3645000000 \mathrm{a}^{7} \mathrm{~b}^{6} \mathrm{c}^{8}\),
    \(\left.109350000000 \mathrm{a}^{8} \mathrm{~b}^{7} \mathrm{c}^{9},-3280500000000 \mathrm{a}^{9} \mathrm{~b}^{8} \mathrm{c}^{10}, 98415000000000 \mathrm{a}^{10} \mathrm{~b}^{9} \mathrm{c}^{11}\right\}\)
    - User Function: \(U_{\text {guess }}(\mathrm{n}, \mathrm{k})=1\)
    - Formula Function: PolyFormula index \(=1(\mathrm{n}, \mathrm{k}, \mathrm{z})\)
    - Sequence Formula Diffs: \{True, True, True, True, True, True \} [ \(\sqrt{ }\) ]
Out[103]= \(\left\{\sum_{\# 2=0}^{\# 1} 5^{2+\# 2} 6^{1+\# 2} \mathrm{a}^{2} \mathrm{~b} \mathrm{c}^{3}(-\mathrm{a} \quad \mathrm{b} \quad \mathrm{c})^{\# 2}\right.\) Binomial \(\left.[3+\# 1,1+\# 1-\# 2] \# 3^{\# 2} \&\right\}\)
```

Figure 2.18: Recognizing Sequence Formulas Involving Symbolic Coefficients

```
In[104]:= seq = Table [Expand [Sum [StirlingS2[j + 3, j + 1 - i] *
            Power[-5 * r, i + 2] * Power [q, i^ 2] * Power[y, i], {i, 0, j}]], {j, 1, 8}];
        GuessPolynomialSequence [seq, y, SequenceFactors -> {"S2"}, FullSimplifyFormulas -> True,
        DisplayVars }->\mathrm{ { n, k}, AllowSymbolicData }->\mathrm{ True ]
    E== ====== === Found Matching Formula #1 / 1: =============
    Polyn}(y)\mapsto\mp@subsup{\sum}{k=0}{n}(-1\mp@subsup{)}{}{k}\mp@subsup{5}{}{k+2}\mp@subsup{q}{}{k}\mp@subsup{|}{}{2}\mp@subsup{r}{}{k+2}\mp@subsup{y}{}{k}\mp@subsup{S}{n+3}{(-k+n+1)
    - Latex Formula Output: Null
```




```
    - User Function: U guess (n, k) = 1
    - Formula Function: PolyFormula index =1 (n, k, y)
    \ Sequence Formula Diffs: {True, True, True, True, True, True, True, True } [` ]
```



```
    #2=0
```

Figure 2.19: A Second Formula Involving Square Index Powers of Symbolic Coefficients

```
|n[193]:= seq =
            Table [Expand [Sum [Binomial [j + 3, 3] * Power [-4, i + 4] * Power [x, i], {i, 0, j + 1}]], {j, 4, 8}];
    GuessPolynomialSequence [seq, x, StartIndex }->4\mathrm{ , SequenceFactors }->\mathrm{ {"Binom"}, LimitFormulaCount }->\mathrm{ 1]
E===== === === Found Matching Formula #1 / 2: =============
```



```
- Latex Formula Output: Null
- Remaining Sequence Data
{256, -1024, 4096, -16384, 65 536, - 262 144, 1048 576, -4 194 304, 16777 216, -67108864}
- User Function: U guess (j, i) = 1
- Formula Function: PolyFormula index =1 (j, i, x)
- Sequence Formula Diffs: {True, True, True, True, True} [`]
Out[194]={ { \sum
```

Figure 2.20: Expected Sequence Factors Independent of the Sum Index

```
In[191]:= seq = Table [Expand [Sum [StirlingS1[j + 3, 3] * Power[2, i] * Power [x, i], {i, 0, j + 1}]], {j, 3, 8}];
GuessPolynomialSequence [seq, x, StartIndex }->\mathrm{ 3, SequenceFactors }->{"s1"}
=== === === === Found Matching Formula #1 / 1: === === === ===
```



```
- Latex Formula Output: Null
- Remaining Sequence Data: {1, 2, 4, 8, 16, 32, 64, 128, 256, 512}
- User Function: U guess (j, i) = 1
- Formula Function: PolyFormula index =1 (j, i, x)
- Sequence Formula Diffs: {True, True, <<2>>, True, True} [\checkmark ]
```



Figure 2.21: Another Example of Expected Sequence Factors Independent of the Sum Index

```
sage: ## A Legendre polynomial identity involving squared
sage: ## powers of the binomial coefficients
sage: from GuessPolynomialSequenceFunction import *
sage: n, z = var(' }\mp@subsup{\textrm{n}}{\sqcup}{}\mp@subsup{\textrm{z}}{}{\prime}
sage: poly_seq_func = lambda m: expand( factor( legendre_P(m, }4
    (1+z)/(1-z))*((1-z)**m) ) )
sage: pseq_data = map(poly_seq_func, range(1, 6))
sage: guess_polynomial_sequence(pseq_data, z, seq_factors = [" 48
    Binom2"]);
```

Figure 2.22: An Exponential Generating Function for the Binomial Coefficients (Sage)

```
sage: ## Finite summation formula for the non-exponential
sage: ## Bell polynomials
sage: from GuessPolynomialSequenceFunction import * }
sage: def series_coefficient_zpow(f, fvar, ncoeff): return f. }
    taylor(fvar, 0, ncoeff) - f.taylor(fvar, 0, ncoeff - 1)
sage: def series_coefficient(f, fvar, ncoeff): return 53
    series_coefficient_zpow(f, fvar, ncoeff).subs_expr(fvar ==
    1)
```



```
sage: n, x, t= var('n}\mp@subsup{\textrm{n}}{\sqcup\sqcup\textrm{x}')}{\prime
sage: spoly_ogf = exp( (exp(t) - 1) * x)
sage: poly_seq_func = lambda n: series_coefficient(spoly_ogf, }5
    t, n) * factorial(n)
sage: pseq_data = map(poly_seq_func, range(1, 6)) 
sage: guess_polynomial_sequence(pseq_data, x, seq_factors = [" 58
    S2"]);
```

Figure 2.23: An Exponential Generating Function for the Binomial Coefficients (Sage)

## Chapter 3

## Conclusions

### 3.1 Concluding Remarks

The package source code portion of the thesis provides a successful "proof of concept" implementation of the logic employed by the approach to the package to recognize polynomial sequence formula types of the noted forms in 2.1 and 2.2. The primary deficiency of the package implementation is current as of this writing is the long running time of the package function GuessPolynomialSequence when processing double-factor and multiple-factor sequence formulas of the form outlined in 1.8 . Single-factor polynomial sequence formulas in the form of (2.1) like those cited in (1.1) of the introduction are already somewhat easy, though not trivial, to guess by the user. For the package to be really useful in practice, the sequence recognition routines provided through the wrapper function GuessPolynomialSequence should be able to guess double-factor formulas of the form in (2.2) fairly quickly and efficiently out-of-the-box.

The examples given in Chapter 2 provide several non-trivial uses of the package for recognizing singlefactor polynomial formulas of the first sequence form in 2.1. These and related applications corresponding to polynomials that satisfy a single-factor formula of this variety are easily and fairly quickly recognized by the package given an accurate user-defined setting of the SequenceFactors runtime option to GuessPolynomialSequence.

For polynomial sequences that satisfy a double-factor formula of the second form in 2.2 , and more generally a multiple-factor formula in the form stated in 1.8 where $r \geq 3$, the current package implementation is unable to quickly search for matching formulas without a somewhat manual limited setting of the IndexOffsetPairs option provided at runtime. The sample output for the examples given in Figure 2.9 and Figure 2.10 show the usage of the package for handling double-factor sequence formulas with an appropriate setting of this option. In future revisions of the package, it should ideally be possible for the package to quickly obtain formulas for these sequence cases without the user manually resetting the default search options used with the GuessPolynomialSequence function provided by the package.

### 3.2 Future Features in the Package

### 3.2.1 Processing Polynomial Sequences with Rational Coefficients

One approach to extending the package functionality to recognize formulas for polynomial sequences in $\mathbb{Q}[x]$ is to pre-process the rational-valued coefficients to transform the sequence into the polynomials over the integers already handled by the package routines. Variations of these pre-processing transformations include normalizing the polynomials, its coefficients, or both by exponential factors to clear the denominators of the rational-valued input sequence. For example, let the polynomial $p_{j}(x):=\sum_{i} c_{i} x^{i}$. Then these transformations are formulated as obtaining the modified polynomials, $\widetilde{p}_{j}(x)$, as $\widetilde{p}_{j}(x):=j!\cdot p_{j}(x)$, as $\tilde{p}_{j}(x):=\sum_{i} i!\cdot c_{i} x^{i}$, or in the combined form of $\tilde{p}_{j}(x):=\sum_{i} j!\cdot i!\cdot c_{i} x^{i}$, whenever the resulting modified polynomial sequences are in $\mathbb{Z}[x]$.

Another transformation option is applied to rationalize the polynomial sequences, $S_{m}(n)$, in $n$ defined
through the following sums where $B_{n}$ denotes the (rational) sequence of Bernoulli numbers [2, §6.5]:

$$
S_{m}(n):=\sum_{k=0}^{n-1} k^{m}=\sum_{k=0}^{m}\binom{m+1}{m-k} \frac{B_{m-k}}{(m+1)} \cdot n^{k+1}
$$

The sequences in the previous equation are normalized by multiplying each polynomial, $S_{m}(n)$, by the least common multiple of the denominators of each coefficient of $n^{k+1}$ in the formula. Then assuming access to the lookup capabilities of the Online Encyclopedia of Integer Sequences database, which contains sequence entries for both integer sequences of the numerators and denominators of the Bernoulli numbers, obvious factors, of say 691 , are recognized to process the full formulas for the sequences of $S_{m}(n)$ over $\mathbb{Q}[n]$.

### 3.2.2 Polynomial Expansions With Respect to a Suitable Basis

The discussion given in [5, Appendix A] related to the implementation of the Rate package for Mathematica states a useful observation that may be adapted to the polynomial formula searches local to this package. Specifically, expressing input polynomial sequences with respect to a "suitable" basis, like shifted factorial functions or polynomial terms expressed by binomial coefficients, allows for recognition of sequence formulas that are not apparent in the default expansions of the polynomial sequence variable. Several examples relevant to adapting this idea in the context of the factorization-based approach in this package include the following polynomial sequence expansions [2, Ex. 6.78; §6.2; Ex. 6.68]:

$$
\begin{aligned}
\binom{2 n}{n} \frac{B_{n}}{(n+1)} & =\sum_{k=0}^{n}\left\{\begin{array}{c}
n+k \\
k
\end{array}\right\}\binom{2 n}{n+k} \frac{(-1)^{k}}{(k+1)} \\
x^{n} & =\sum_{k=0}^{n}\left\langle\begin{array}{c}
n \\
k
\end{array}\right\rangle\binom{ x+k}{n} \\
{\left[\begin{array}{c}
x \\
x-n
\end{array}\right] } & =\sum_{k \geq 0}\left\{\begin{array}{c}
n \\
k
\end{array}\right\}\binom{x+k}{2 n}, n \geq 0 \\
\left\langle\left\langle\begin{array}{c}
n \\
m
\end{array}\right\rangle\right\rangle & =\sum_{k=0}^{m}\binom{2 n+1}{k}\left\{\begin{array}{c}
n+m+1-k \\
m+1-k
\end{array}\right\}(-1)^{k}, n>m \geq 0
\end{aligned}
$$

These sequences provide applications related to the polynomial expansions of the Catalan numbers (in $n$ ), the Stirling convolution polynomials, $\sigma_{n}(x)$, and the second-order Eulerian numbers, $\left\langle\left\langle\begin{array}{l}n \\ m\end{array}\right\rangle\right\rangle$, respectively.

### 3.3 Future Research Topics

The next sections discuss several topics for future research suggested by the implementation of the software package for the thesis. These future research topics include a new variation of integer factorization algorithms motivated by the factorization-based approach to handling the user-defined expected sequence factors in the package routines, as well as additional topics for future exploration to extend the current capabilities of the univariate polynomial sequence recognition in the package. The extension of the current package functionality to recognizing polynomials in a single variable with rational-valued coefficients is already considered in Section 3.2 of the thesis above.

### 3.3.1 Sequence-Based Integer Factorization Algorithms

The treatment of the user-defined expected sequence factors as "primitives" in the formulas returned by the package functions motivates the construction of a class of integer factorization algorithms formulated briefly in the discussion below. Much like computing the prime factorization of an arbitrary integer, this class of algorithms should compute the decomposition of an integer into a product of elements over some specified
set of integer sequences where the elements of these sequences are treated as "atoms" in the factorization returned by the procedure.

Stated more precisely: given an integer $i$ (or some set of integer-valued polynomial coefficients) and a list of $k$ integer sequences, $\left\{S_{1}, S_{2}, \ldots, S_{k}\right\}$, we seek the most efficient way to decompose the integer into all possible products of integer factors of the form

$$
\begin{equation*}
i:=f_{1} \cdot f_{2} \cdots f_{k} \times r \tag{3.1}
\end{equation*}
$$

where the factor $f_{i}$ belongs to the sequence $S_{i}$ (for each $1 \leq i \leq k$ ), and where the remaining factor term, $r$, is reserved for later processing. The computation of the list of all factors of the form in (3.1) can be computed over some specified number of elements of each sequence, or a fixed number of rows for the case of a triangular sequence, $S_{i}$. It seems reasonable to expect that such an algorithms must employ the prime factorizations of the individual factor sequences, $F_{i}$. We also seek a solution in the general case, though of course it may be possible to derive sequence-specific procedures, say, to recognize factors of the Stirling number or binomial coefficient triangles.

The need for this type of factorization is apparently new, as searches for such subroutines to employ within the package returned no useful known results, though it is possible that there are existing prime factorization algorithms that may be especially well-suited, or adapted, to this purpose. This required factorization procedure is handled as an inefficient implementation of an oracle of sorts within the current implementation of the Mathematica package.

## References

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[^0]:    ${ }^{1}$ See http://www.mat.univie.ac.at/~kratt/rate/rate.html

[^1]:    ${ }^{2}$ See http://axiom-wiki.newsynthesis.org/GuessingFormulasForSequences
    ${ }^{3}$ See the complete list of Algorithmic Combinatorics Software on the RISC website at http://www.risc.jku.at/research/ combinat/software/

[^2]:    ${ }^{1}$ See also the Mathematica package documentation at http://www.risc.jku.at/research/combinat/software/ergosum/ RISC/Stirling.html

