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ON THE ARITHMETIC AND GEOMETRIC MEANS OF THE PRIME NUMBERS

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ABSTRACT. In this paper we establish explicit upper and lower bounds for the ratio of the arithmetic and geometric means of the prime numbers, which improve the current best estimates. Further, we prove several conjectures related to this ration stated by Hassani. In order to do this, we use explicit estimates for the prime counting function, Chebyshev's ϑ-function and the sum of the first n prime numbers.

1. Introduction

Let a_n be the arithmetic mean and g_n be the geometric mean of the first n positive integers, respectively. Stirling's approximation for $n!$ implies that

$$\lim_{n\to\infty} \frac{a_n}{g_n} = \frac{e}{2}.$$

In his paper [12], Hassani studied the arithmetic and geometric means of the prime numbers, i.e.

$$A_n = \frac{1}{n}\sum_{k\leq n} p_k, \quad G_n = (p_1 \cdot \ldots \cdot p_n)^{1/n}.$$

Here, as usual, p_k denotes the kth prime number. By setting

$$D(n) = \log p_n - \frac{\vartheta(p_n)}{n}, \quad R(n) = \frac{1}{n}\sum_{k\leq n} p_k - \frac{p_n}{2},$$

where Chebyshev's ϑ-function is defined by

$$\vartheta(x) = \sum_{p\leq x} \log p,$$

Hassani [12, p. 1595] derived the identity

$$\log \frac{A_n}{G_n} = D(n) + \log\left(1 + \frac{2R(n)}{p_n}\right) - \log 2 \tag{1.1}$$

for the ratio of A_n and G_n, which plays an important role in this paper. First, we establish asymptotic formulae for the quantities $D(n)$, G_n and A_n which help us to find the following asymptotic formula for the ratio of A_n and G_n. Here, let $r_t = (t-1)!(1-1/2^t)$ and the positive integers $k_1, \ldots, k_s$, where s is a positive integer, are defined by the recurrence formula

$$k_s + 1!k_{s-1} + 2!k_{s-2} + \ldots + (s-1)!k_1 = s \cdot s!.$$

Theorem 1.1 (See Theorem 2.8)**.** *For each positive integer m, we have*

$$\frac{A_n}{G_n} = e\left(\frac{1}{2} + \sum_{i=1}^{m} \frac{1}{\log^i p_n}\left(-r_{i+1} + r_i + \sum_{s=1}^{i-1} r_s k_{i-s}\right)\right) \cdot \exp\left(\sum_{j=1}^{m} \frac{k_j}{\log^j p_n}\right) + O\left(\frac{1}{\log^{m+1} p_n}\right).$$

One of Hassani's results [12, p. 1602] is that

$$\frac{A_n}{G_n} = \frac{e}{2} + O\left(\frac{1}{\log n}\right),$$

which implies that the ration of A_n and G_n also tends to $e/2$ for $n \to \infty$. Setting $m = 2$ in Theorem 1.1, we get the following more accurate asymptotic formula.

Corollary 1.2 (See Corollary 2.9)**.** *We have*

$$\frac{A_n}{G_n} = \frac{e}{2} + \frac{e}{4\log p_n} + \frac{e}{\log^2 p_n} + O\left(\frac{1}{\log^3 p_n}\right).$$

Using explicit estimates for the n-th prime number p_n and the prime counting function $\pi(x)$, which is defined for every $x \geq 0$ by

$$\pi(x) = \sum_{p \leq x} 1,$$

where p runs over primes not exceeding x, Hassani [12, Theorem 1.1] found that the ratio of A_n and G_n fulfills the inequalities

$$\frac{e}{2} - \frac{14.951}{\log n} < \frac{A_n}{G_n} < \frac{e}{2} + \frac{9.514}{\log n} \tag{1.2}$$

for every $n \geq 2$. The proof of the inequalities (1.2) consists of three steps. First, Hassani gave some explicit estimates for the quanities $D(n)$ and $\log(1 + 2R(n)/p_n)$ and then he used (1.1). We follow this method to improve the inequalities given in (1.2) by showing the following both results in the direction of Corollary 1.2.

Theorem 1.3 (See Corollary 7.2)**.** *For every $n \geq 47$, we have*

$$\frac{A_n}{G_n} > \frac{e}{2} + \frac{e}{4\log p_n} + \frac{2e}{5\log^2 p_n}.$$

Theorem 1.4 (See Theorem 7.4)**.** *For every positive integer n, we have*

$$\frac{A_n}{G_n} < \frac{e}{2} + \frac{e}{4\log p_n} + \frac{7e}{4\log^2 p_n}.$$

In particular, we prove several conjectures concerning $D(n)$, G_n and the ratio of A_n and G_n stated by Hassani [12] in 2013. For instance, we use Theorem 1.3 to show that the ratio of A_n and G_n is always greater than $e/2$.

2. Several asymptotic formulae

In this section, we give some asymptotic formulae for the quantities $D(n)$, G_n, A_n, the ratio of A_n and G_n and finally for $\log(1 + 2R(n)/p_n)$. Here, an asymptotic formula for the prime counting function plays an important role.

2.1. **Two asymptotic formulae for $D(n)$.** In order to find the first asymptotic formula for $D(n)$, we introduce the following definition.

Definition. Let m be a positive integer. The positive integers $k_1, \ldots, k_m$ are defined by the recurrence formula

$$k_m + 1!k_{m-1} + 2!k_{m-2} + \ldots + (m-1)!k_1 = m \cdot m!. \tag{2.1}$$

In particular, $k_1 = 1$, $k_2 = 3$, $k_3 = 13$ and $k_4 = 71$.

Then, we obtain the following result.

Proposition 2.1. *Let r be a non-negative integer. Then*

$$D(n) = 1 + \frac{k_1}{\log p_n} + \frac{k_2}{\log^2 p_n} + \ldots + \frac{k_r}{\log^r p_n} + O\left(\frac{1}{\log^{r+1} p_n}\right).$$

Proof. The proof of the required asymptotic formula for $D(n)$ consists of two steps. First, we find an asymptotic formula for $\log x$. Panaitopol [16] showed that

$$\pi(x) = \frac{x}{\log x - 1 - \frac{k_1}{\log x} - \frac{k_2}{\log^2 x} - \ldots - \frac{k_{r+1}}{\log^{r+1} x}} + O\left(\frac{x}{\log^{r+3} x}\right). \tag{2.2}$$

Hence,

$$\log x = \frac{x}{\pi(x)} + 1 + \frac{k_1}{\log x} + \frac{k_2}{\log^2 x} + \ldots + \frac{k_r}{\log^r x} + O\left(\frac{x}{\pi(x)\log^{r+2} x}\right). \tag{2.3}$$

Further, (2.2) implies that

$$\lim_{x \to \infty} \frac{\pi(x)}{x/\log x} = 1, \tag{2.4}$$

which is known as the Prime Number Theorem. So, we can simplify the error term in (2.3) and get

$$\log x = \frac{x}{\pi(x)} + 1 + \frac{k_1}{\log x} + \frac{k_2}{\log^2 x} + \ldots + \frac{k_r}{\log^r x} + O\left(\frac{1}{\log^{r+1} x}\right). \tag{2.5}$$

The next step is to find an asymptotic formula for Chebyshev's ϑ-function. A well-known result concering this function is that $\vartheta(x) = x + O\left(x \exp(-c \log^{1/10} x)\right)$, where c is an absolute positive constant (see, for example, Brüdern [7, p. 41]). Since $\exp(-c\log^{1/10} x) = O(1/\log^s x)$ for every positive integer s, we get

$$\vartheta(x) = x + O\left(\frac{x}{\log^{r+2} x}\right). \tag{2.6}$$

From (2.4) follows that

$$\lim_{n\to\infty} \frac{n}{p_n/\log p_n} = 1 \tag{2.7}$$

and combined with (2.6), we get

$$\frac{\vartheta(p_n)}{n} = \frac{p_n}{n} + O\left(\frac{1}{\log^{r+1} p_n}\right). \tag{2.8}$$

Together with (2.5) and the definition of $D(n)$ we conclude the proof. □

To prove a second asymptotic formula for the quantity $D(n)$, we first note two useful results of Cipolla [9] from 1902 concerning asymptotic formulae for the nth prime number p_n and $\log p_n$. Here, $lc(P)$ denotes the leading coefficient of a polynomial P.

Lemma 2.2 (Cipolla, [9])**.** *Let m be a positive integer. Then there exist uniquely determined polynomials $Q_1, \ldots, Q_m \in \mathbb{Z}[x]$ with $\deg(Q_k) = k$ and $lc(Q_k) = (k-1)!$, so that*

$$p_n = n\left(\log n + \log\log n - 1 + \sum_{k=1}^{m} \frac{(-1)^{k+1} Q_k(\log\log n)}{k! \log^k n}\right) + O\left(\frac{n(\log\log n)^{m+1}}{\log^{m+1} n}\right).$$

The polynomials Q_k can be computed explicitly. In particular, $Q_1(x) = x - 2$, $Q_2(x) = x^2 - 6x + 11$ and $Q_3(x) = 2x^3 - 21x^2 + 84x - 131$.

Lemma 2.3 (Cipolla, [9])**.** *Let m be a positive integer. Then there exist uniquely determined polynomials $R_1, \ldots, R_m \in \mathbb{Z}[x]$ with $\deg(R_k) = k$ and $lc(R_k) = (k-1)!$, so that*

$$\log p_n = \log n + \log\log n + \sum_{k=1}^{m} \frac{(-1)^{k+1} R_k(\log\log n)}{k! \log^k n} + O\left(\frac{(\log\log n)^{m+1}}{\log^{m+1} n}\right).$$

The polynomials R_k can be computed explicitly. In particular, $R_1(x) = x - 1$, $R_2(x) = x^2 - 4x + 5$ and $R_3(x) = 2x^3 - 15x^2 + 42x - 47$.

Now, we give another asymptotic formula for the quantity $D(n)$.

Proposition 2.4. *Let r be a positive integer and let $T_k(x) = R_k(x) - Q_k(x)$ for $1 \leq k \leq r$. Then, $\deg(T_k) = k - 1$, $lc(T_k) = k!$ and*

$$D(n) = 1 + \sum_{k=1}^{r} \frac{(-1)^{k+1} T_k(\log\log n)}{k! \log^k n} + O\left(\frac{(\log\log n)^r}{\log^{r+1} n}\right).$$

In particular, $T_1(x) = 1$, $T_2(x) = 2x - 6$ and $T_3(x) = 6x^2 - 42x + 84$.

Proof. Let $1 \leq k \leq r$. Since $\deg(Q_k) = \deg(R_k) = k$ and $lc(Q_k) = lc(R_k) = (k-1)!$, we have $\deg(T_k) \leq k - 1$. Following Cipolla [9, p. 144], we write

$$Q_k(x) = (k-1)!x^k - a_{k,1}x^{k-1} + \sum_{j=2}^{k} (-1)^j a_{k,j} x^{k-j}$$

and

$$R_k(x) = (k-1)!x^k - b_{k,1}x^{k-1} + \sum_{j=2}^{k} (-1)^j b_{k,j} x^{k-j},$$

where $a_{i,j}, b_{i,j} \in Z$. Then

$$T_k(x) = -(b_{k,1} - a_{k.1})x^{k-1} + \sum_{j=2}^{k} (-1)^j (b_{k,j} - a_{k,j}) x^{k-j}.$$

By Cipolla [9, p. 150], we have $-(b_{k,1} - a_{k,1}) = k! \neq 0$. Hence, $\deg(T_k) = k-1$ and $lc(T_k) = k!$. By the definition of $D(n)$ and (2.8), we get

$$D(n) = \log p_n - \frac{p_n}{n} + O\left(\frac{1}{\log^{r+1} p_n}\right).$$

Let $m = r+1$. Then we substitute the asymptotic formulae given in Lemma 2.2 and Lemma 2.3 to obtain

$$D(n) = 1 + \sum_{k=1}^{r+1} \frac{(-1)^{k+1} T_k(\log\log n)}{k! \log^k n} + O\left(\frac{1}{\log^{r+1} p_n}\right).$$

Since $\deg(T_{r+1}) = r$ and $1/\log^{r+1} p_n = O(1/\log^{r+1} n)$, we conclude the proof. □

Remark. Proposition 2.4 implies that

$$D(n) = 1 + \frac{1}{\log n} - \frac{\log\log n - 3}{\log^2 n} + \frac{(\log\log n)^2 - 7\log\log n + 14}{\log^3 n} + O\left(\frac{(\log\log n)^3}{\log^4 n}\right), \tag{2.9}$$

which precises Hassani's [12] asymptotic formula for $D(n)$. He found that

$$D(n) = 1 + O\left(\frac{1}{\log n}\right).$$

2.2. **An asymptotic formula for G_n.** Next, we derive an asymptotic formula for G_n, the geometric mean of the prime numbers. Using the definition of G_n and $D(n)$, we obtain the identity

$$G_n = \frac{p_n}{e^{D(n)}}. \tag{2.10}$$

Proposition 2.1 implies that $\lim_{n\to\infty} D(n) = 1$. Hence,

$$G_n \sim \frac{p_n}{e} \qquad (n \to \infty), \tag{2.11}$$

which was conjectured by Vrba [17] in 2010 and was already proved by Sándor and Verroken [20, Theorem 2.1] in 2011. Using (2.10) and Proposition 2.1, we get the following refinement of (2.11). Here, the positive intergers $k_1, \ldots, k_r$ are defined by the recurrence formula (2.1).

Proposition 2.5. *Let r be a positive integer. Then,*

$$G_n = \frac{p_n}{\exp\left(1 + \frac{k_1}{\log p_n} + \frac{k_2}{\log^2 p_n} + \ldots + \frac{k_r}{\log^r p_n}\right)} + O\left(\frac{p_n}{\log^{r+1} p_n}\right). \tag{2.12}$$

Proof. We use (2.10), Proposition 2.1 and the fact that $\exp(c/x) = 1 + O(1/x)$ for every $c \in \mathbb{R}$ to get

$$G_n = \frac{p_n}{\exp\left(1 + \frac{k_1}{\log p_n} + \frac{k_2}{\log^2 p_n} + \ldots + \frac{k_r}{\log^r p_n}\right)} \cdot \left(1 + O\left(\frac{1}{\log^{r+1} p_n}\right)\right),$$

which completes the proof. □

Remark. The asymptotic formula (2.12) was independently found by Kourbatov [13, Remark (ii)] in 2016.

Remark. The asymptotic relation (2.7) and Proposition 2.5 imply that

$$G_n = \frac{p_n}{e} + O(n), \tag{2.13}$$

which was already obtained by Hassani [12, p. 1602] in 2013.

2.3. **Two asymptotic formulae for A_n.** We start with the following proposition concerning an asymptotic formula for A_n, the arithmetic mean of the prime numbers.

Proposition 2.6. *For each positive integer m, we have*

$$A_n = p_n - \sum_{k=1}^{m-1} (k-1)! \left(1 - \frac{1}{2^k}\right) \frac{p_n^2}{n \log^k p_n} + O\left(\frac{p_n^2}{n \log^m p_n}\right).$$

Proof. See [3, Theorem 2]. □

Another asymptotic formula for A_n is given as follows.

Proposition 2.7. *Let m be a positive integer. Then there exist unique monic polynomials $L_s \in \mathbb{Q}[x]$, where $1 \leq s \leq m$ and $\deg(L_s) = s$, such that*

$$A_n = \frac{n}{2}\left(\log n + \log\log n - \frac{3}{2} + \sum_{s=1}^{m} \frac{(-1)^{s+1} L_s(\log\log n)}{s \log^s n}\right) + O\left(\frac{n(\log\log n)^{m+1}}{\log^{m+1} n}\right).$$

In particular, $L_1(x) = x - 5/2$ and $L_2(x) = x^2 - 7x + 29/2$.

Proof. See [5, Theorem 1.1]. The polynomials L_s can be computed explicitly by Theorem 2.7 of [5]. □

2.4. **An asymptotic formula for the ratio of A_n and G_n.** Now, we use (2.10), Proposition 2.1 and Proposition 2.6 to prove our first main result Theorem 1.1 concerning an asymptotic formula for the ratio of A_n and G_n. Here we define r_i for every $1 \leq i \leq m+1$ by

$$r_i = (i-1)!\left(1 - \frac{1}{2^i}\right). \tag{2.14}$$

Theorem 2.8. *For each positive integer m, we have*

$$\frac{A_n}{G_n} = e\left(\frac{1}{2} + \sum_{i=1}^{m} \frac{1}{\log^i p_n}\left(-r_{i+1} + r_i + \sum_{s=1}^{i-1} r_s k_{i-s}\right)\right) \cdot \exp\left(\sum_{j=1}^{m} \frac{k_j}{\log^j p_n}\right) + O\left(\frac{1}{\log^{m+1} p_n}\right).$$

Proof. Form (2.10), Proposition 2.1 and Proposition 2.6 follow that

$$\frac{A_n}{G_n} = \left(1 - \sum_{i=1}^{m+1} \frac{r_i p_n}{n \log^i p_n} + O\left(\frac{p_n}{n \log^{m+2} p_n}\right)\right) \cdot \left(\exp\left(1 + \sum_{j=1}^{m} \frac{k_j}{\log^j p_n}\right) + O\left(\frac{1}{\log^{m+1} p_n}\right)\right).$$

Since $p_n \sim n \log p_n$ for $n \to \infty$, we get

$$\frac{A_n}{G_n} = e\left(1 - \sum_{i=1}^{m+1} \frac{r_i p_n}{n \log^i p_n} + O\left(\frac{1}{\log^{m+1} p_n}\right)\right) \cdot \exp\left(\sum_{j=1}^{m} \frac{k_j}{\log^j p_n}\right) + O\left(\frac{1}{\log^{m+1} p_n}\right). \tag{2.15}$$

Using (2.5) with $x = p_n$ and $r = m-1$, we get

$$\frac{p_n}{n} = \log p_n - 1 - \frac{k_1}{\log p_n} - \ldots - \frac{k_{m-1}}{\log^{m-1} p_n} + O\left(\frac{1}{\log^m p_n}\right).$$

Applying this asymptotic formula to (2.15), we get

$$\frac{A_n}{G_n} = e\left(1 - \sum_{i=1}^{m+1} \frac{r_i}{\log^i p_n}\left(\log p_n - 1 - \sum_{s=1}^{m-1} \frac{k_s}{\log^s p_n}\right) + O\left(\frac{1}{\log^{m+1} p_n}\right)\right) \cdot \exp\left(\sum_{j=1}^{m} \frac{k_j}{\log^j p_n}\right)$$
$$+ O\left(\frac{1}{\log^{m+1} p_n}\right).$$

Hence,

$$\frac{A_n}{G_n} = e\left(1 - \sum_{i=1}^{m+1} \frac{r_i}{\log^{i-1} p_n} + \sum_{i=1}^{m+1} \frac{r_i}{\log^i p_n} + \sum_{i=1}^{m+1} \frac{k_1 r_i}{\log^{i+1} p_n} + \ldots + \sum_{i=1}^{m+1} \frac{k_{m-1} r_i}{\log^{m-1+i} p_n} + O\left(\frac{1}{\log^{m+1} p_n}\right)\right)$$
$$\times \exp\left(\sum_{j=1}^{m} \frac{k_j}{\log^j p_n}\right) + O\left(\frac{1}{\log^{m+1} p_n}\right).$$

Separating the terms in the first brace, which are $O(1/\log^{m+1} p_n)$, we get

$$\frac{A_n}{G_n} = e\left(1 - \sum_{i=1}^{m+1} \frac{r_i}{\log^{i-1} p_n} + \sum_{i=1}^{m} \frac{r_i}{\log^i p_n} + \sum_{i=1}^{m-1} \frac{k_1 r_i}{\log^{i+1} p_n} + \ldots + \frac{k_{m-1} r_1}{\log^m p_n} + O\left(\frac{1}{\log^{m+1} p_n}\right)\right)$$
$$\times \exp\left(\sum_{j=1}^{m} \frac{k_j}{\log^j p_n}\right) + O\left(\frac{1}{\log^{m+1} p_n}\right).$$

An index offset in the first brace gives

$$\frac{A_n}{G_n} = e\left(1 - r_1 + \sum_{i=1}^{m} \frac{1}{\log^i p_n}\left(-r_{i+1} + r_i + \sum_{s=1}^{i-1} r_s k_{i-s}\right) + O\left(\frac{1}{\log^{m+1} p_n}\right)\right) \cdot \exp\left(\sum_{j=1}^{m} \frac{k_j}{\log^j p_n}\right)$$
$$+ O\left(\frac{1}{\log^{m+1} p_n}\right).$$

We conclude by using the facts that $r_1 = 1/2$ and that $\exp(c/x) = 1 + O(1/x)$ for every $c \in \mathbb{R}$. □

Setting $m = 2$ in Theorem 2.8, we get the following asymptotic formula for the ratio of A_n and G_n.

Corollary 2.9. *We have*

$$\frac{A_n}{G_n} = \frac{e}{2} + \frac{e}{4\log p_n} + \frac{e}{\log^2 p_n} + O\left(\frac{1}{\log^3 p_n}\right).$$

Proof. We set $m = 2$ in Theorem 2.8 to get

$$\frac{A_n}{G_n} = e\left(1 - r_1 + \frac{r_1 - r_2}{\log p_n} + \frac{k_1 r_1 + r_2 - r_3}{\log^2 p_n}\right) \cdot \exp\left(\sum_{j=1}^{2} \frac{k_j}{\log^j p_n}\right) + O\left(\frac{1}{\log^3 p_n}\right).$$

By (2.14), we have $r_1 = 1/2$, $r_2 = 3/4$ and $r_3 = 7/4$. Together with $k_1 = 1$, we get

$$\frac{A_n}{G_n} = e\left(\frac{1}{2} - \frac{1}{4\log p_n} - \frac{1}{2\log^2 p_n}\right) \cdot \exp\left(\sum_{j=1}^{2} \frac{k_j}{\log^j p_n}\right) + O\left(\frac{1}{\log^3 p_n}\right). \tag{2.16}$$

Since $\exp(1/x) = 1 + 1/x + 1/(2!x^2) + O(1/x^3)$, we obtain

$$\exp\left(\sum_{j=1}^{2} \frac{k_j}{\log^j p_n}\right) = 1 + \frac{1}{\log p_n} + \frac{7}{2\log^2 p_n} + O\left(\frac{1}{\log^3 p_n}\right),$$

since $k_1 = 1$ and $k_2 = 3$. Applying this to (2.16), we obtain that

$$\frac{A_n}{G_n} = e\left(\frac{1}{2} - \frac{1}{4\log p_n} - \frac{1}{2\log^2 p_n}\right) \cdot \left(1 + \frac{1}{\log p_n} + \frac{7}{2\log^2 p_n} + O\left(\frac{1}{\log^3 p_n}\right)\right) + O\left(\frac{1}{\log^3 p_n}\right),$$

which completes the proof. □

2.5. **An asymptotic formula for the quantity** $\log(1 + 2R(n)/p_n)$**.** Finally, in the next proposition we derive an asymptotic formula for $\log(1 + 2R(n)/p_n)$ for $n \to \infty$.

Proposition 2.10. *We have*

$$\log\left(1 + \frac{2R(n)}{p_n}\right) \sim -\frac{1}{2\log n} \qquad (n \to \infty). \tag{2.17}$$

Proof. We have $R(n) \sim -n/4$ by [5] and $p_n \sim n \log n$ for $n \to \infty$. Hence,

$$\log\left(1 + \frac{2R(n)}{p_n}\right) \sim \log\left(1 - \frac{1}{2\log n}\right).$$

Since $\log(1 + c/x) \sim c/x$ for $x \to \infty$ and every $c \in \mathbb{R}$, the proposition is proved. □

At the end of Section 6, we give a more accurate asymptotic formula for the quantity $\log(1+2R(n)/p_n)$.

3. New estimates for the quantity $D(n)$

After giving two asymptotic formulae for the quantity $D(n)$ in Subsection 2.1, we are interested in finding some explicit estimates for $D(n)$. We start with the following one.

Proposition 3.1. *For every $n \geq 126$, we have*

$$D(n) > 1 + \frac{1}{\log p_n} + \frac{2.3}{\log^2 p_n}. \tag{3.1}$$

Proof. The proof consists of two steps. First, we give a lower bound for $\log p_n$. In [4, Corollary 3.5] it is shown that the inequality

$$\pi(x) > \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{2.65}{\log^2 x}}$$

holds for every $x \geq 38168363 = p_{2328664}$. Setting $x = p_n$, we get that the inequality

$$\log p_n > \frac{p_n}{n} + 1 + \frac{1}{\log p_n} + \frac{2.65}{\log^2 p_n} \tag{3.2}$$

is fulfilled for every $n \geq 2328664$. Next, the present author [6, Proposition 2.4] found that $\vartheta(x) < x + 0.35x/\log^3 x$ for every $x > 1$. Together with the definition od $D(n)$ and (3.2), we get

$$D(n) > 1 + \frac{1}{\log p_n} + \frac{2.65}{\log^2 p_n} - \frac{0.35 p_n}{n \log^3 p_n} \tag{3.3}$$

for every $n \geq 2328664$. By Rosser and Schoenfeld [18, Corollary 1], we have $\pi(x) > x/\log x$ for every $x \geq 17$. Hence,

$$p_n \leq n \log p_n \tag{3.4}$$

for every $n \geq 7$. Applying this inequality to (3.3), we get that the inequality (3.1) holds for every $n \geq 2328664$. A computer check shows that inequality (3.1) also holds for every $126 \leq n \leq 2328663$. □

In the direction of the asymptotic formula given in Proposition 2.1, we derive the following lower bounds for $D(n)$, which improve the inequality (3.1) for all sufficiently large values of n.

Proposition 3.2. *For every positive integer n, we have*

$$D(n) > 1 + \frac{1}{\log p_n} + \frac{3}{\log^2 p_n} - \frac{2551}{\log^3 p_n} \tag{3.5}$$

and

$$D(n) > 1 + \frac{1}{\log p_n} + \frac{3}{\log^2 p_n} + \frac{13}{\log^3 p_n} - \frac{11013803}{\log^4 p_n}. \tag{3.6}$$

Proof. We start with the proof of (3.5). In [6, Proposition 2.5 and Theorem 1.2] it is shown that $|\vartheta(x) - x| < 1282x/\log^4 x$ for every $x \geq 2$ and that the inequality

$$\pi(x) > \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3}{\log^2 x} + \frac{1269}{\log^3 x}}$$

holds for every $x \geq 2$. Similar to the proof of Proposition 3.1, we get that the inequality (3.5) is fulfilled for every $n \geq 7$. A direct comuter check shows that the required inequality also holds for every $1 \leq n \leq 6$.

Next we give the proof of (3.6). In [6, Proposition 2.6 and Theorem 1.2], the present author found that the inequalities $|\vartheta(x) - x| < 5506937x/\log^5 x$ and

$$\pi(x) > \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3}{\log^2 x} - \frac{13}{\log^3 x} + \frac{5506866}{\log^4 x}}$$

hold for every $x \geq 2$. Similar to the proof of Proposition 3.1, we get that the inequality (3.6) holds for every $n \geq 7$. We use a computer to verify that the inequality (3.6) is valid for every $1 \leq n \leq 6$ as well. □

Since $k_1 = 1$ and $k_2 = 3$, Proposition 2.1 implies that there is a smallest positive integer N_0 so that

$$D(n) > 1 + \frac{1}{\log p_n} + \frac{3}{\log^2 p_n} \tag{3.7}$$

for every $n \geq N_0$. In the following corollary, we make a first progress in finding this N_0.

Corollary 3.3. *The inequality* (3.7) *holds at least for every* $264 \leq n \leq \pi(10^{19}) = 234057667276344607$ *and every* $n \geq \pi(e^{11013803/13}) + 1$.

Proof. The inequality (3.6) implies the validity of (3.7) for every $n \geq \pi(e^{11013803/13}) + 1$. So, it suffices to prove that the inequality (3.7) holds for every $264 \leq n \leq \pi(10^{19})$. By [6, Corollary 3.5], we have

$$\pi(x) > \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3}{\log^2 x}} \tag{3.8}$$

for every $p_{3863019} = 65405887 \leq x \leq 5.5 \cdot 10^{25}$ and every $x \geq e^{5506866/13}$. Büthe [8, Theorem 2] found that $\vartheta(x) < x$ for every $1 \leq x \leq 10^{19}$ and together with (3.8), we get, similar to the proof of Proposition 3.1, that the inequality (3.7) holds for every $3863019 \leq n \leq \pi(10^{19})$. Finally, we check the remaining cases with a computer. □

Based on Corollary 3.3 we state the following conjecture.

Conjecture 3.4. *The inequality* (3.7) *holds for every* $n \geq 264$.

Hassani [12, Proposition 1.6] showed that the inequality

$$D(n) > 1 - \frac{15}{5 \log n}$$

is valid for every $n \geq 2$. In view of (2.9), we improve this result as follows.

Proposition 3.5. *For every* $n \geq 275$, *we have*

$$D(n) > 1 + \frac{1}{\log n} - \frac{\log\log n - 2.14}{\log^2 n}. \tag{3.9}$$

In particular, for every $0 < \alpha < 1$ *there exists a positive integer* $n_0 = n_0(\alpha)$ *so that for every* $n \geq n_0$

$$D(n) > 1 + \frac{\alpha}{\log n}. \tag{3.10}$$

Proof. First, we consider the case $n > \pi(10^{19}) = 234057667276344607$. By [2, Korollar 2.7], we have

$$\frac{1}{\log p_m} \geq \frac{1}{\log m} - \frac{\log\log m}{\log^2 m} + \frac{(\log\log m)^2 - \log\log m + 1}{\log^2 m \log p_m} \tag{3.11}$$

for every $m \geq 71$, which implies that the weaker inequality

$$\frac{1}{\log p_m} \geq \frac{1}{\log m} - \frac{\log\log m}{\log^2 m} \tag{3.12}$$

also holds for every $m \geq 71$. After combining (3.11) and (3.12), we get

$$\frac{1}{\log p_m} \geq \frac{1}{\log m} - \frac{\log\log m}{\log^2 m} + \frac{(\log\log m)^2 - \log\log m + 1}{\log^2 m}\left(\frac{1}{\log m} - \frac{\log\log m}{\log^2 m}\right) \tag{3.13}$$

for every $m \geq 71$. Together with (3.1) and (3.12), we get

$$\begin{aligned} D(n) &> 1 + \frac{1}{\log n} - \frac{\log\log n}{\log^2 n} + \frac{(\log\log n)^2 - \log\log n + 1}{\log^3 n} - \frac{(\log\log n)^3 - (\log\log n)^2 + \log\log n}{\log^4 n} \\ &\quad + 2.3\left(\frac{1}{\log n} - \frac{\log\log n}{\log^2 n}\right)^2 \\ &= f(n) + \frac{0.16}{\log^2 n} + \frac{(\log\log n)^2 - 5.6\log\log n + 1}{\log^3 n} - \frac{(\log\log n)^3 - 3.3(\log\log n)^2 + \log\log n}{\log^4 n}, \end{aligned}$$

where $f(n)$ denotes the right-hand site of (3.9). Since

$$\frac{0.16}{\log^2 x} + \frac{(\log\log x)^2 - 5.6\log\log x + 1}{\log^3 x} - \frac{(\log\log x)^3 - 3.3(\log\log x)^2 + \log\log x}{\log^4 x} > 0$$

for every $x \geq 1.3 \cdot 10^{17}$, the required inequality holds for every $n > \pi(10^{19}) \geq 1.3 \cdot 10^{17}$.

Now, let $580752 \leq n \leq \pi(10^{19})$. Similarly to the case $n > \pi(10^{19})$, we combine (3.12), (3.13) and Corollary 3.3 to get

$$D(n) > f(n) + \frac{0.86}{\log^2 n} + \frac{(\log\log n)^2 - 7\log\log n + 1}{\log^3 n} - \frac{(\log\log n)^3 - 4(\log\log n)^2 + \log\log n}{\log^4 n}.$$

Since $0.86\log x + (\log\log x)^2 - 7\log\log x > 0$ for every $x \geq 580752$ and $\log x > (\log\log x)^3 - 4(\log\log x)^2 + \log\log x$ for every $x \geq 3$, we obtain that the inequality (3.9) also holds for every $580752 \leq n \leq \pi(10^{19})$. We verify the remaining cases with a computer. □

Remark. Hassani [12, Conjecture 1.7] conjectured that there exist a real number β with $0 < \beta < 5.25$ and a positive integer n_0, such that the inequality (3.10) is valid for every $n \geq n_0$. The second part of Proposition 3.5 proves this conjecture. The inequality (3.9) implies

$$D(n) > 1 \tag{3.14}$$

for every $n \geq 275$. A computer check shows that the last inequality also holds for every $10 \leq n \leq 274$. Therefore, the inequality (3.14) holds for every $n \geq 10$, which was also conjectured by Hassani [12, Conjecture 1.7].

Next, we establish some explicit upper bounds for $D(n)$. From Proposition 2.1 follows that for each $\varepsilon > 0$ there is a positive integer $N = N(\varepsilon)$, such that

$$D(n) < 1 + \frac{1}{\log p_n} + \frac{3+\varepsilon}{\log^2 p_n}$$

for every $n \geq N$. In this regard, we show the following

Proposition 3.6. *For every $n \geq 704569$, we have*

$$D(n) < 1 + \frac{1}{\log p_n} + \frac{4.18}{\log^2 p_n}, \tag{3.15}$$

and for every positive integer n, we have

$$D(n) < 1 + \frac{1}{\log p_n} + \frac{3}{\log^2 p_n} + \frac{2577}{\log^3 p_n}. \tag{3.16}$$

Proof. We start with the proof of (3.15). First, we consider the case $n \geq 66775686$. In [4, Corollary 3.4] it is shown that the inequality

$$\pi(x) < \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3.83}{\log^2 x}}$$

holds for every $x \geq 9.25$. It follows that

$$\log p_n < \frac{p_n}{n} + 1 + \frac{1}{\log p_n} + \frac{3.83}{\log^2 p_n}. \tag{3.17}$$

Further, the present author [6, Proposition 2.4] found that $\vartheta(x) > x - 0.35x/\log^3 x$ for every $x \geq 1332492593 = p_{66775686}$. Together with the definition of D_n and the inequality (3.17) we obtain that

$$D(n) < 1 + \frac{1}{\log p_n} + \frac{3.83}{\log^2 p_n} + \frac{0.35 p_n}{n \log^3 p_n}.$$

Now we use (3.4) to get that the inequality (3.15) holds for every $n \geq 66775686$. A computer check shows that the inequality (3.15) holds for every $704569 \leq n \leq 66775685$ as well.

Next, we establish the inequality (3.16). In the direction of (2.2), the present author [6, Theorem 1.1] found that

$$\pi(x) < \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3}{\log^2 x} - \frac{1295}{\log^3 x}}$$

for every $x \geq 563$. As already mentioned in the proof of Proposition 3.2, we have $|\vartheta(x) - x| < 1282x/\log^4 x$ for every $x \geq 2$. Now we argue as in the proof of Proposition 3.6. For the remaining cases, we use a computer. □

Using estimates for the n-th prime number and Chebyshev's ϑ-function, Hassani [12, Proposition 1.6] found that

$$D(n) < 1 + \frac{21}{4\log n}$$

for every $n \geq 2$. In view of (2.9), we give the following result, which leads to an improvement of the last inequality.

Proposition 3.7. *For every positive integer $n \geq 2$, we have*

$$D(n) < 1 + \frac{1}{\log n} - \frac{\log\log n - 4.56}{\log^2 n}.$$

In particular, for every $\beta \geq 1$ there exists an positive interger $n_1 = n_1(\beta)$ so that for every $n \geq n_1$

$$D(n) < 1 + \frac{\beta}{\log n}.$$

Proof. By [2, Korollar 2.21], we have

$$\frac{1}{\log p_n} \leq \frac{1}{\log n} - \frac{\log\log n}{\log^2 n} + \frac{(\log\log n)^2 - \log\log n + 1}{\log^2 n \log p_n} + \frac{P_8(\log\log n)}{2\log^3 n \log p_n} - \frac{P_9(\log\log n)}{2\log^4 n \log p_n} \tag{3.18}$$

for every $n \geq 2$, where $P_8(x) = 3x^2 - 6x + 5.2$ and $P_9(x) = x^3 - 6x^2 + 11.4x - 4.2$. Since $P_9(x) > 0$ for every $x \geq 0.5$, we get

$$\frac{1}{\log p_n} \leq \frac{1}{\log n} - \frac{\log\log n}{\log^2 n} + \frac{(\log\log n)^2 - \log\log n + 1}{\log^3 n} + \frac{P_8(\log\log n)}{2\log^4 n} \tag{3.19}$$

for every $n \geq 6$. Together with Proposition 3.6 and the inequality $4.18/\log^2 p_n \leq 4.18/\log^2 n$, we obtain that the inequality

$$D(n) < 1 + \frac{1}{\log n} - \frac{\log\log n - 4.18}{\log^2 n} + \frac{(\log\log n)^2 - \log\log n + 1}{\log^3 n} + \frac{P_8(\log\log n)}{2\log^4 n} \tag{3.20}$$

holds for every $n \geq 704569$. Notice that

$$\frac{(\log\log x)^2 - \log\log x + 1}{\log^3 x} + \frac{P_8(\log\log x)}{2\log^4 x} < \frac{0.38}{\log^2 x}$$

for every $x \geq 56615486$. Applying this inequality to (3.20), the claim follows for every $n \geq 56615486$. Finally, we use a computer to check that the required inequality also holds for every $2 \leq n \leq 56615485$. □

4. New estimates for the geometric mean of the prime numbers

In the following, we use the identity (2.10); i.e. $G_n = p_n/e^{D(n)}$, and the explicit estimates for $D(n)$ obtained in Section 3 to find new bounds for G_n, the geometric mean of the prime numbers in the direction of (2.11)–(2.13). First, we notice that (3.14) and (2.10) imply $G_n < p_n/e$ for every $n \geq 10$, which was already proved by Panaitopol [14] in 1999. In the direction of Proposition 2.5, Kourbatov [13, Theorem 2] used explicit estimates for the prime counting function $\pi(x)$ and Chebyshev's ϑ-function to show that the inequality

$$G_n < \frac{p_n}{\exp(1 + \frac{1}{\log p_n} + \frac{1.62}{\log^2 p_n})}$$

is fulfilled for every $p_n \geq 32059$; i.e. for every $n \geq 3439$. Actually, this inequality also holds for every $92 \leq n \leq 3438$ as well. In the next proposition, we give a sharper estimate for G_n.

Proposition 4.1. *If $n \geq 126$, then*

$$G_n < \frac{p_n}{\exp(1 + \frac{1}{\log p_n} + \frac{2.3}{\log^2 p_n})}.$$

Proof. The claim follows directly from (2.10) and Proposition 3.1. □

Proposition 3.2 implies the following upper bounds for G_n, which improve the inequality obtained in Proposition 4.1 for all sufficiently large values of n.

Proposition 4.2. *For every positive integer n, we have*

$$G_n < \frac{p_n}{\exp(1 + \frac{1}{\log p_n} + \frac{3}{\log^2 p_n} - \frac{2551}{\log^3 p_n})}$$

and

$$G_n < \frac{p_n}{\exp(1 + \frac{1}{\log p_n} + \frac{3}{\log^2 p_n} + \frac{13}{\log^3 p_n} - \frac{11013803}{\log^4 p_n})}.$$

Proof. We apply the inequalities obtained in Proposition 3.2 to the identity (2.10). □

Next, we use Corollary 3.3 to get the following upper bound.

Proposition 4.3. *For every $264 \leq n \leq \pi(10^{19}) = 234057667276344607$ and every $n \geq \pi(e^{11013803/13})+1$, we have*

$$G_n < \frac{p_n}{\exp(1 + \frac{1}{\log p_n} + \frac{3}{\log^2 p_n})}. \tag{4.1}$$

Proof. We combine (2.10) with Corollary 3.3. □

Remark. Under the assumption that Conjecture 3.4 is true, we get that the inequality (4.1) holds for every $n \geq 264$.

After finding some upper bounds for G_n in the direction of (2.12), we establish now upper bounds for G_n in view of the asymptotic formula (2.13). In order to do this, we first notice the following result.

Proposition 4.4. *If $n \geq 47$, then*

$$G_n < \frac{p_n}{e}\left(1 - \frac{1}{\log p_n}\right).$$

Proof. Using Proposition 4.1 and the inequality $e^x \geq 1 + x$, which holds for every $x \in \mathbb{R}$, we get

$$G_n < \frac{p_n}{e}\left(1 - \frac{\log p_n + 2.3}{\log^2 p_n + \log p_n + 2.3}\right)$$

for every $n \geq 126$. Since $1.3 \log p_n > 2.3$ for every $n \geq 4$, we obtain the required inequality for every $n \geq 126$. For the remaining cases of n we use a computer. □

Using (3.17) and Proposition 4.4, we find the following upper bound for G_n in the direction of (2.13).

Corollary 4.5. *If $n \geq 31$, then*

$$G_n < \frac{p_n}{e} - \frac{n}{e}\left(1 - \frac{1}{\log p_n} - \frac{1}{\log^2 p_n} - \frac{3.83}{\log^3 p_n}\right).$$

In particular, for every $0 < \gamma < 1/e$ there is a positive integer $n_2 = n_2(\gamma)$ so that for every $n \geq n_2$

$$G_n < \frac{p_n}{e} - \gamma n.$$

Proof. We use (3.17) and Proposition 4.4 to obtain that

$$G_n < \frac{p_n}{e} - \frac{p_n}{e \log p_n} < \frac{p_n}{e} - \frac{n}{e}\left(1 - \frac{1}{\log p_n} - \frac{1}{\log^2 p_n} - \frac{3.83}{\log^3 p_n}\right)$$

for every $n \geq 47$. For every $31 \leq n \leq 46$ we check the required inequality with a computer. □

Remark. The second part of Corollary 4.5 proves a conjecture stated by Hassani [12, Conjecture 4.3].

Next, we find new lower bounds for G_n. In the direction of Proposition 2.5, Kourbatov [13, Theorem 2] found that the inequality

$$G_n > \frac{p_n}{\exp(1 + \frac{1}{\log p_n} + \frac{4.83}{\log^2 p_n})}$$

holds for every $n \geq 3439$ by using explicit estimates for the prime counting function $\pi(x)$ and Chebyshev's ϑ-function. In the next proposition, we give two sharper lower bounds for G_n.

Proposition 4.6. *For every $n \geq 704569$, we have*

$$G_n > \frac{p_n}{\exp(1 + \frac{1}{\log p_n} + \frac{4.18}{\log^2 p_n})}, \tag{4.2}$$

and for every positive integer n, we have

$$G_n > \frac{p_n}{\exp(1 + \frac{1}{\log p_n} + \frac{3}{\log^2 p_n} + \frac{2577}{\log^3 p_n})}.$$

Proof. We use (2.10) and Proposition 3.6 to obtain the required inequalities. □

In order to derive a lower bound for G_n in the direction of (2.13), we first establish the following result.

Proposition 4.7. *For every positive integer n, we have*

$$G_n > \frac{p_n}{e}\left(1 - \frac{1}{\log p_n} - \frac{5.14}{\log^2 p_n}\right).$$

Proof. First, we consider the case $n \geq 64881104 = \pi(e^{20.98}) + 1$. It is easy to see that

$$e^t < 1 + t + \frac{2t^2}{3} \tag{4.3}$$

for every $t < \log(4/3)$. Hence, we obtain

$$\exp\left(\frac{1}{x} + \frac{4.18}{x^2}\right) < 1 + \frac{1}{x} + \frac{14.54}{3x^2} + \frac{16.72}{3x^3} + \frac{34.9448}{3x^4}$$

for every $x \geq 6$. Now, if $x \geq 20.98$, then $16.72/(3x) + 34.9448/(3x^2) < 0.293$ and we get

$$\exp\left(\frac{1}{x} + \frac{4.18}{x^2}\right) < 1 + \frac{1}{x} + \frac{5.14}{x^2} \tag{4.4}$$

for every $x \geq 20.98$. Since $\log p_n \geq 20.98$, it follows from (4.2) and the inequality (4.4) that

$$G_n > \frac{p_n}{e}\left(1 - \frac{\log p_n + 5.14}{\log^2 p_n + \log p_n + 5.14}\right).$$

Since the right-hand side of the last inequality is greater then the right-hand side of the required inequality, the corollary is proved for every $n \geq 64881104$. A computer check shows that the asserted inequality holds for every $1 \leq n \leq 64881103$ as well. □

In view of (2.13), Hassani [12, Corollary 4.2] found that

$$G_n > \frac{p_n}{e} - 2.37n$$

for every positive integer n. The following corollary improves this inequality.

Corollary 4.8. *If $n \geq 3$, then*

$$G_n > \frac{p_n}{e} - \frac{n}{e}\left(1 + \frac{4.14}{\log p_n} - \frac{6.14}{\log^2 p_n} - \frac{7.79}{\log^3 p_n}\right).$$

In particular, for every $\delta > 1/e$ there is a positive integer $n_3 = n_3(\delta)$ so that for every $n \geq n_3$,

$$G_n > \frac{p_n}{e} - \delta n.$$

Proof. First, we consider the case $n \geq 2328664$. We use (3.2) and the inequality obtained in Proposition 4.7 to get that

$$G_n > \frac{p_n}{e} - \frac{n}{e}\left(1 - \frac{1}{\log p_n} - \frac{1}{\log^2 p_n} - \frac{2.65}{\log^3 p_n}\right) - \frac{5.14p_n}{e\log^2 p_n}. \tag{5.3}$$

The inequality (3.2) implies that

$$-\frac{5.14p_n}{e\log^2 p_n} > -\frac{5.14n}{e\log p_n}\left(1 - \frac{1}{\log p_n} - \frac{1}{\log^2 p_n}\right).$$

We apply this inequality to (5.3) and obtain the required inequality. For every $3 \leq n \leq 2328663$ we check the required inequality with a computer. □

Remark. Corollary 4.5 and Corollary 4.8 yield a more accurate asymptotic formula for G_n than in (2.13), namely that

$$G_n = \frac{p_n}{e} - \frac{n}{e} + O\left(\frac{n}{\log p_n}\right).$$

Remark. Panaitopol [14, Theorem, p. 34] and Sándor [19, Theorem 2.1] showed another kind of inequality for G_n, namely that for every $n \geq 2$,

$$G_n \geq p_{n+1}^{1-\pi(n)/n}.$$

5. Estimates for the arithmetic mean of the prime numbers

Although we will not use them below, we note in this section, for the sake of completeness, the best known estimates concerning the arithmetic mean of the prime numbers A_n. In view of Proposition 2.6, it is shown in [3, Theorem 3 and Theorem 4] that the inequality

$$A_n \leq p_n - \frac{p_n^2}{2n\log p_n} - \frac{3p_n^2}{4n\log^2 p_n} - \frac{7p_n^2}{4n\log^3 p_n} - \frac{\Theta(n)}{n}$$

holds for every $n \geq 52703656$, where

$$\Theta(n) = \frac{43.6p_n^2}{8\log^4 p_n} + \frac{90.9p_n^2}{4\log^5 p_n} + \frac{927.5p_n^2}{8\log^6 p_n} + \frac{5620.5p_n^2}{8\log^7 p_n} + \frac{39537.75p_n^2}{8\log^8 p_n},$$

and that

$$A_n \geq p_n - \frac{p_n^2}{2n\log p_n} - \frac{3p_n^2}{4n\log^2 p_n} - \frac{7p_n^2}{4n\log^3 p_n} - \frac{\Omega(n)}{n}$$

for every positive integer n, where

$$\Omega(n) = \frac{46.4p_n^2}{8\log^4 p_n} + \frac{95.1p_n^2}{4\log^5 p_n} + \frac{962.5p_n^2}{8\log^6 p_n} + \frac{5809.5p_n^2}{8\log^7 p_n} + \frac{59424p_n^2}{8\log^8 p_n}.$$

In the direction of Proposition 2.7, the present author [5, Theorem 1.5 and Theorem 1.6] found that

$$A_n < \frac{n}{2}\left(\log n + \log\log n - \frac{3}{2} + \frac{\log\log n - 5/2}{\log n} - \frac{(\log\log n)^2 - 7\log\log n + 12.373}{2\log^2 n}\right)$$

for every $n \geq 355147$, and that the inequality

$$A_n > \frac{n}{2}\left(\log n + \log\log n - \frac{3}{2} + \frac{\log\log n - 5/2}{\log n} - \frac{(\log\log n)^2 - 7\log\log n + 17.067}{2\log^2 n}\right).$$

holds for every $n \geq 2$.

6. New estimates for the quantity $\log(1 + 2R(n)/p_n)$

As already mentioned in the introduction, the quantity $R(n)$ is defined by

$$R(n) = \frac{1}{n}\sum_{k \leq n} p_k - \frac{p_n}{2}.$$

Following Hassani's proof of (1.2), we next establish new estimates for the quantity $\log(1 + 2R(n)/p_n)$. Hassani [12, Corollary 1.5] proved that

$$-\frac{15}{2\log n} < \log\left(1 + \frac{2R(n)}{p_n}\right) < -\frac{5}{36\log n}, \tag{6.1}$$

where the left-hand side inequality holds for every $n \geq 2$, and the right-hand side inequality holds for every $n \geq 10$. In Proposition 2.10, we gave a more suitable approximation for the quantity $\log(1 + 2R(n)/p_n)$ for $n \to \infty$. In the direction of this approximation, we improve now the inequalities (6.1). The first proposition is about a lower bound for $\log(1 + 2R(n)/p_n)$.

Proposition 6.1. *For every positive integer $n \geq 2$, we have*

$$\log\left(1 + \frac{2R(n)}{p_n}\right) > -\frac{1}{2\log n} + \frac{\log\log n - 2.25}{2\log^2 n} - \frac{(\log\log n)^2 - 4.5\log\log n + 24.91/3}{2\log^3 n}. \tag{6.2}$$

Proof. First, we note a result proved by Dusart [10] concerning a lower bound for p_n, namely that

$$p_n \geq r_1(n) \tag{6.3}$$

for every $n \geq 2$, where $r_1(x) = x(\log x + \log\log x - 1)$. We set

$$s_1(x) = -\frac{x}{4} - \frac{x}{4\log x} + \frac{x(\log\log x - 5.22)}{4\log^2 x}$$

and by [5, Corollary 4.10] and Hassani [12, Corollary 1.5], we obtain that

$$s_1(n) < R(n) < 0 \tag{6.4}$$

for every $n \geq 26220$. Now, we define

$$h_1(x) = \log\left(1 + \frac{2s_1(x)}{r_1(x)}\right)$$

and we show that $h(x)$ is greater than the right-hand side of (6.2). For this, we set

$$\begin{aligned} f(y) = {} & (2\log^3 y - 13\log^2 y + 34.69\log y - 31.9575)y^3 \\ & + (1.5\log^4 y - 11.5\log^3 y + 35.83\log^2 y - 43.9575\log y + 10.2075)y^2 \\ & + (-0.5\log^3 y + 1.32\log^2 y + 5.19\log y - 23.85005)y \\ & + 0.75\log^4 y - 8.54\log^3 y + 35.37\log^2 y - 65.96005\log y + 38.38005 \end{aligned}$$

and

$$g_1(y) = y^3 + y^2\log y - 1.5y^2 - 0.5y + 0.5\log y - 2.61.$$

Since $2\log^3 y - 13\log^2 y + 34.69\log y - 31.9575 \geq 4$ for every $y \geq e^{2.4}$ and $1.5\log^4 y - 11.5\log^3 y + 35.83\log^2 y - 43.9575\log y + 10.2075 \geq 0$ for every $y \geq 10$, we get

$$\begin{aligned} f(y) \geq {} & 4y^3 + (-0.5\log^3 y + 1.32\log^2 y + 5.19\log y - 23.85005)y \\ & + 0.75\log^4 y - 8.54\log^3 y + 35.37\log^2 y - 65.96005\log y + 38.38005 \end{aligned}$$

for every $y \geq e^3$. Notice that $y^2 \geq 2\log^3 y$ for every $y \geq 1$. Hence,

$$\begin{aligned} f(y) \geq {} & (15.5\log^3 y + 1.32\log^2 y + 5.19\log y - 23.85005)y \\ & + 0.75\log^4 y - 8.54\log^3 y + 35.37\log^2 y - 65.96005\log y + 38.38005 \end{aligned}$$

for every $y \geq e^{2.4}$. Now we apply the inequality $y \geq \log y$, which holds for every $y > 0$, to get

$$f(y) \geq 16.25\log^4 y - 7.22\log^3 y + 40.56\log^2 y - 89.8101\log y + 38.38005$$

for every $y \geq e^{2.4}$. Now, it is clear that the right-hand side of the last inequality is positive for every $y \geq e^2$. Hence $f_1(y) > 0$ for every $y \geq e^{2.4}$. Similarly, we get that $g_1(y) > 0$ for every $y \geq e^{2.4}$. Therefore,

$$\left(h_1(x) + \frac{1}{2\log x} - \frac{\log\log x - 5/2}{2\log^2 x} + \frac{(\log\log x)^2 - 4.5\log\log x + 24.91/3}{2\log^3 x}\right)' = -\frac{f_1(\log x)}{g_1(\log x) r_1(x)\log^4 x} < 0$$

for every $x \geq \exp(\exp(2.4))$. In addition, we have

$$\lim_{x\to\infty}\left(h_1(x) + \frac{1}{2\log x} - \frac{\log\log x - 5/2}{2\log^2 x} + \frac{(\log\log x)^2 - 4.5\log\log x + 24.91/3}{2\log^3 x}\right) = 0.$$

So, we get

$$h_1(x) > -\frac{1}{2\log x} + \frac{\log\log x - 5/2}{2\log^2 x} - \frac{(\log\log x)^2 - 4.5\log\log x + 24.91/3}{2\log^3 x}$$

for every $x \geq \exp(\exp(2.4))$. Together with (6.3) and (6.4), it follows that

$$\log\left(1 + \frac{2R(n)}{p_n}\right) > -\frac{1}{2\log n} + \frac{\log\log n - 2.25}{2\log^2 n} - \frac{(\log\log n)^2 - 4.5\log\log n + 24.91/3}{2\log^3 n}$$

for every $n \geq \exp(\exp(2.4))$. The remaining cases are checked with a computer. □

Corollary 6.2. *For every $n \geq 2194$, we have*

$$\log\left(1 + \frac{2R(n)}{p_n}\right) > -\frac{1}{2\log p_n} - \frac{2.25}{2\log^2 n}.$$

Proof. First, we consider the case $n \geq 9423108$. Using (3.11) and Proposition 6.1, we get

$$\log\left(1 + \frac{2R(n)}{p_n}\right) > -\frac{1}{2\log p_n} - \frac{2.25}{2\log^2 n} + \frac{(\log\log n)^2 - \log\log n + 1}{2\log^2 n \log p_n} - \frac{(\log\log n)^2 - 4.5\log\log n + 24.91/3}{2\log^3 n}. \tag{6.5}$$

Since $(\log\log n)^2 - \log\log n + 1 \geq 0$ for every $n \geq 2$, we apply (3.12) to (6.5) and get

$$\begin{aligned}\log\left(1 + \frac{2R(n)}{p_n}\right) &> -\frac{1}{2\log p_n} - \frac{2.25}{2\log^2 n} + \frac{(\log\log n)^2 - \log\log n + 1}{2\log^2 n}\left(\frac{1}{\log n} - \frac{\log\log n}{\log^2 n}\right) \\ &\quad - \frac{(\log\log n)^2 - 4.5\log\log n + 24.91/3}{2\log^3 n} \\ &= -\frac{1}{2\log p_n} - \frac{2.25}{2\log^2 n} + \frac{3.5\log\log n - 21.91/3}{2\log^3 n} - \frac{(\log\log n)^3 - \log\log n + \log\log n}{2\log^4 n}.\end{aligned}$$

Notice that $(3.5\log\log x - 21.91/3)\log x - ((\log\log x)^3 - \log\log x + \log\log x) \geq 0$ for every $x \geq 9423108$. So, we get that the required inequality holds for every $n \geq 9423108$. We check the remaining cases with a computer. □

Before we derive an upper bound for $\log(1 + 2R(n)/p_n)$, we prove the following lemma.

Lemma 6.3. *Let $P_8(x) = 3x^2 - 6x + 5.2$. Then, for every $n \geq 3$, we have*

$$\frac{1}{2\log n} + \frac{3.2}{2\log^2 n} \geq \frac{(\log\log n)^2 + \log\log n}{2\log^2 n} + \frac{P_8(\log\log n)}{4\log^3 n}.$$

Proof. First, we notice that $(\log\log n)^2 + \log\log n \leq \log n$ for every $n \geq 2$. Further, we have $P_8(\log\log n) \leq 3(\log\log n)^2 + 5.2 \leq 6.4\log n$ for every $n \geq 3$, which completes the proof. □

Now, we give an upper bound for $\log(1 + 2R(n)/p_n)$.

Proposition 6.4. *For every $n \geq 2701$, we have*

$$\log\left(1 + \frac{2R(n)}{p_n}\right) < -\frac{1}{2\log p_n} - \frac{1}{\log^2 p_n} - \frac{1}{\log^3 p_n}.$$

Proof. First, we consider the case $n \geq 348247$. We define

$$s_2(x) = -\frac{x}{4} - \frac{x}{4\log x} + \frac{x(\log\log x - 2.1)}{4\log^2 x}.$$

By [5, Corollary 4.10] and the definition of $R(n)$, we obtain $R(n) < s_2(n) < 0$. Hence,

$$\log\left(1 + \frac{2R(n)}{p_n}\right) < \log\left(1 + \frac{2s_2(n)}{p_n}\right).$$

Since $2s_2(n)/p_n > -1$, we apply the inequality $\log(1+x) \leq x$, which holds for every $x > -1$, to get that

$$\log\left(1 + \frac{2R(n)}{p_n}\right) < -\frac{n}{2p_n} - \frac{n}{2p_n \log n} + \frac{n(\log\log n - 2.1)}{2p_n \log^2 n}.$$

Now, we use a lower bound for the prime counting function given by Dusart [11, Theorem 6.9], namely that $\pi(x) \geq x/\log x + x/\log^2 x$ for every $x \geq 599$, with $x = p_n$ to obtain that

$$\log\left(1 + \frac{2R(n)}{p_n}\right) < -\frac{1}{2\log p_n} - \frac{1}{2\log^2 p_n} - \frac{1}{2\log n \log p_n} - \frac{1}{2\log n \log^2 p_n} + \frac{n(\log\log n - 2.1)}{2p_n \log^2 n}.$$

Further, from Dusart [11, Theorem 6.9] follows that $\pi(x) \leq x/\log x + 2x/\log^2 x$ for every $x > 1$. Hence,

$$\log\left(1 + \frac{2R(n)}{p_n}\right) < -\frac{1}{2\log p_n} - \frac{1}{2\log^2 p_n} - \frac{1}{2\log n \log p_n} - \frac{1}{2\log n \log^2 p_n} + \frac{\log\log n - 2.1}{2\log^2 n \log p_n} + \frac{\log\log n - 2.1}{\log^2 n \log^2 p_n}. \tag{6.6}$$

By (3.18), we have

$$-\frac{1}{\log n} \leq -\frac{1}{\log p_n} - \frac{\log\log n}{\log^2 n} + \frac{(\log\log n)^2 - \log\log n + 1}{\log^2 n \log p_n} + \frac{P_8(\log\log n)}{2\log^3 n \log p_n} - \frac{P_9(\log\log n)}{2\log^4 n \log p_n}$$

for every $n \geq 2$, where $P_8(x) = 3x^2 - 6x + 5.2$ and $P_9(x) = x^3 - 6x^2 + 11.4x - 4.2$. Since $P_9(x) > 0$ for every $x \geq 0.5$, we get

$$-\frac{1}{\log n} \leq -\frac{1}{\log p_n} - \frac{\log\log n}{\log^2 n} + \frac{(\log\log n)^2 - \log\log n + 1}{\log^2 n \log p_n} + \frac{P_8(\log\log n)}{2\log^3 n \log p_n}.$$

Applying this inequality to (6.6), we get

$$\log\left(1 + \frac{2R(n)}{p_n}\right) < -\frac{1}{2\log p_n} - \frac{1}{\log^2 p_n} + \frac{(\log\log n)^2 + \log\log n - 3.2}{2\log^2 n \log^2 p_n} + \frac{P_8(\log\log n)}{4\log^3 n \log^2 p_n} - \frac{1}{2\log n \log^2 p_n} - \frac{2.1}{2\log^2 n \log p}.$$

Finally, we use Lemma 6.3 to obtain that the required inequality holds for every $n \geq 348247$. For every $2701 \leq n \leq 348247$, we check the the asserted inequality with a computer. □

In the direction of (2.17), we find the following upper bound for $\log(1 + 2R(n)/p_n)$, which leads to an improvement of the right-hand side of (6.1).

Corollary 6.5. *For every $n \geq 259$, we have*

$$\log\left(1 + \frac{2R(n)}{p_n}\right) < -\frac{1}{2\log n} + \frac{\log\log n - 2}{2\log^2 n} + \frac{2\log\log n}{\log^3 n}.$$

Proof. First we consider the case $n \geq 2701$. Proposition 6.4 implies that

$$\log\left(1 + \frac{2R(n)}{p_n}\right) < -\frac{1}{2\log p_n} - \frac{1}{\log^2 p_n}. \tag{6.7}$$

From (3.13) follows that

$$-\frac{1}{\log p_n} \leq -\frac{1}{\log n} + \frac{\log\log n}{\log^2 n}. \tag{6.8}$$

Applying this to (6.7), we get that the inequality

$$\log\left(1 + \frac{2R(n)}{p_n}\right) < -\frac{1}{2\log n} + \frac{\log\log n}{2\log^2 n} - \frac{1}{\log n \log p_n} + \frac{\log\log n}{\log^2 n \log p_n}.$$

Again we use (6.8) to obtain that

$$\log\left(1+\frac{2R(n)}{p_n}\right) < -\frac{1}{2\log n}+\frac{\log\log n}{2\log^2 n}-\frac{1}{\log^2 n}+\frac{\log\log n}{\log^3 n}+\frac{\log\log n}{\log^2 n\log p_n},$$

which concludes the proof for every $n \geq 2701$. For every $259 \leq n \leq 2700$, we check the the required inequality with a computer. ☐

Compared with Proposition 2.10 we establish the following more precise result.

Corollary 6.6. *We have*

$$\log\left(1+\frac{2R(n)}{p_n}\right) = -\frac{1}{2\log n}+\frac{\log\log n}{2\log^2 n}+O\left(\frac{1}{\log^2 n}\right).$$

Proof. The asymptotic formula follows directly from Proposition 6.1 and Corollary 6.5. ☐

7. New bounds for the ratio of A_n and G_n

First, we recall the asymptotic formula for the ratio of A_n and G_n given in Corollary 2.9, namely

$$\frac{A_n}{G_n} = \frac{e}{2}+\frac{e}{4\log p_n}+\frac{e}{\log^2 p_n}+O\left(\frac{1}{\log^3 p_n}\right) \tag{7.1}$$

and the identity (1.1); i.e.

$$\log\frac{A_n}{G_n} = D(n)+\log\left(1+\frac{2R(n)}{p_n}\right)-\log 2.$$

Now we use this identity together with the explicit estimates for the quantities $D(n)$ and $\log(1+2R(n)/p_n)$ obtained in Section 3 and Section 6 to derive upper and lower bounds for the ratio of A_n and G_n in the direction of (7.1), which improve the inequalities given in (1.2). We start with the following result.

Theorem 7.1. *For every $n \geq 76$, we have*

$$\frac{A_n}{G_n} > \frac{e}{2}+\frac{e}{4\log n}-\frac{e(\log\log n-2)}{4\log^2 n}. \tag{7.2}$$

Proof. By (1.1), Proposition 3.5 and Proposition 6.1, we get that the inequality

$$\frac{A_n}{G_n} > \frac{e}{2}\cdot\exp\left(\frac{1}{2\log n}-\frac{\log\log n-2.03}{2\log^2 n}-\frac{(\log\log n)^2-4.5\log\log n+24.91/3}{2\log^3 n}\right)$$

holds for every $n \geq 275$. Since $0.23\log x > (\log\log x)^2-4.5\log\log x+24.91/3$ for every $x \geq 3281492$, we get that the inequality

$$\frac{A_n}{G_n} > \frac{e}{2}\cdot\exp\left(\frac{1}{2\log n}-\frac{\log\log n-1.8}{2\log^2 n}\right)$$

holds for every $n \geq 3281492$. Now we use the inequality $e^x \geq 1+x+x^2/2$, which holds for every real x, to get

$$\frac{A_n}{G_n} > \frac{e}{2}\cdot\left(1+\frac{1}{2\log n}-\frac{\log\log n-2}{2\log^2 n}+\frac{0.025}{\log^2 n}-\frac{\log\log n}{4\log^3 n}+\frac{0.45}{\log^3 n}\right)$$

for every $n \geq 3281492$. Notice that the function $t \mapsto 4\cdot 0.025-(\log\log t-4\cdot 0.45)/(4\log t)$ is positive for every $t \geq 2$. Hence, we obtain that the required lower bound for the ratio of A_n and G_n holds for every $n \geq 3281492$. For every $76 \leq n \leq 3281491$ we check the inequality (7.2) with a computer. ☐

Remark. Theorem 7.1 proves a conjecture stated by Hassani [12] in 2013, namely that there exists a real number α with $0 < \alpha < 9.514$ and a positive integer n_0 such that

$$\frac{A_n}{G_n} > \frac{e}{2}+\frac{\alpha}{\log n}$$

for every $n \geq n_0$.

Next, we derive the inequality stated in Theorem 1.3. Here, we use (3.19) and Theorem 7.1.

Corollary 7.2. *For every $n \geq 47$, we have*

$$\frac{A_n}{G_n} > \frac{e}{2}+\frac{e}{4\log p_n}+\frac{2e}{5\log^2 p_n}.$$

Proof. First, we consider the case $n \geq 2992205$. Using (7.2) and (3.19), we get

$$\frac{A_n}{G_n} > \frac{e}{2} + \frac{e}{4\log p_n} + \frac{e}{4}\left(\frac{2}{\log^2 n} - \frac{(\log\log n)^2 - \log\log n + 1}{\log^3 n} - \frac{P_8(\log\log n)}{2\log^4 n}\right),$$

where $P_8(x) = 3x^2 - 6x + 5.2$. Notice that the inequality

$$\frac{(\log\log x)^2 - \log\log x + 1}{\log^3 x} + \frac{P_8(\log\log x)}{2\log^4 x} < \frac{0.4}{\log^2 x} \tag{7.3}$$

holds for every $x \geq 2992205$. Hence, we get that the inequality

$$\frac{A_n}{G_n} > \frac{e}{2} + \frac{e}{4\log p_n} + \frac{2e}{5\log^2 n}$$

holds, which implies that required inequality holds for every $n \geq 2992205$. We verify the remaining cases with a computer. □

The following corollary confirms that the ratio of the arithmetic and geometric means of the prime numbers is always greater than $e/2$, as conjectured by Hassani [12].

Corollary 7.3. *For every positive integer n, we have*

$$\frac{A_n}{G_n} > \frac{e}{2}.$$

Proof. Corollary 7.2 implies the validity of the required inequality for every $n \geq 47$. We verify the remaining cases with a computer. □

Next, we use Proposition 3.6 and Proposition 6.4 to find the following upper bound for the ratio of A_n and G_n, stated in Theorem 1.4.

Theorem 7.4. *For every positive integer n, we have*

$$\frac{A_n}{G_n} < \frac{e}{2} + \frac{e}{4\log p_n} + \frac{7e}{4\log^2 p_n}.$$

Proof. First, let $n \geq 949552$. By (1.1), Proposition 6.4 and the inequality (3.15), we obtain that

$$\frac{A_n}{G_n} < \frac{e}{2} \cdot \exp\left(\frac{1}{2\log p_n} + \frac{3.18}{\log^2 p_n}\right).$$

Using (4.3), we get

$$\frac{A_n}{G_n} < \frac{e}{2} \cdot \left(1 + \frac{1}{2\log p_n} + \frac{10.04}{3\log^2 p_n} + \frac{2.12}{\log^3 p_n} + \frac{6.7416}{\log^4 p_n}\right).$$

Since $\log p_n \geq 16.5$, we have $10.04/3 + 2.12/\log p_n + 6.7416/\log^2 p_n < 3.5$, which concludes the proof for every $n \geq 949552$. We check the remaining cases with a computer. □

Now we use Theorem 7.4 and the inequality (3.19) to prove the following result.

Corollary 7.5. *For every $n \geq 2$, we have*

$$\frac{A_n}{G_n} < \frac{e}{2} + \frac{e}{4\log n} - \frac{e(\log\log n - 7.4)}{4\log^2 n}.$$

Proof. By Theorem 7.4, the inequality (3.19) and the inequality $7e/(4\log^2 p_n) \leq 7e/(4\log^2 n)$, we obtain

$$\frac{A_n}{G_n} < \frac{e}{2} + \frac{e}{4\log n} - \frac{e(\log\log n - 7)}{4\log^2 n} + \frac{e((\log\log n)^2 - \log\log n + 1)}{4\log^3 n} + \frac{eP_8(\log\log n)}{8\log^4 n} \tag{7.4}$$

for every $n \geq 6$. Applying (7.3) to (7.4), the claim follows for every $n \geq 2992205$. A computer check shows the validity of the required inequality for every $2 \leq n \leq 2992204$. □

Remark. One of the conjectures stated by Hassani [12] is still open, namely that the inequality

$$\frac{A_{n+1}}{G_{n+1}} < \frac{A_n}{G_n}$$

holds for every $n \geq 226$.

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