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Patterns in Inversion Sequences II: Inversion Sequences Avoiding Triples of Relations

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Abstract

Inversion sequences of length n, $\mathbf{I}_n$, are integer sequences $(e_1, \ldots, e_n)$ with $0 \leq e_i < n$ for each i. The study of patterns in inversion sequences was initiated recently by Mansour-Shattuck and Corteel-Martinez-Savage-Weselcouch through a systematic study of inversion sequences avoiding words of length 3. We continue this investigation by generalizing the notion of a pattern to a fixed triple of binary relations (ρ_1, ρ_2, ρ_3) and consider the set $\mathbf{I}_n(\rho_1, \rho_2, \rho_3)$ consisting of those $e \in \mathbf{I}_n$ with no $i < j < k$ such that $e_i \rho_1 e_j$, $e_j \rho_2 e_k$, and $e_i \rho_3 e_k$. We show that "avoiding a triple of relations" can characterize inversion sequences with a variety of monotonicity or unimodality conditions, or with multiplicity constraints on the elements. We uncover several interesting enumeration results and relate pattern avoiding inversion sequences to familiar combinatorial families. We highlight open questions about the relationship between pattern avoiding inversion sequences and families such as plane permutations and Baxter permutations. For several combinatorial sequences, pattern avoiding inversion sequences provide a simpler interpretation than otherwise known.

1 Introduction

Pattern avoiding permutations have been studied extensively for their connections in computer science, biology, and other fields of mathematics. Within combinatorics they have proven their usefulness, providing an interpretation that relates a vast array of combinatorial structures. See the comprehensive survey of Kitaev [16].

The notion of pattern avoidance in inversion sequences was introduced in [13] and [21]. An *inversion sequence* is an integer sequence $(e_1, e_2, \ldots, e_n)$ satisfying $0 \leq e_i < i$ for all $i \in [n] = \{1, 2, \ldots, n\}$. There is a natural bijection $\Theta : \mathbf{S}_n \to \mathbf{I}_n$ from $\mathbf{S}_n$, the set of permutations of $[n]$, to $\mathbf{I}_n$, the set of inversion sequences of length n. Under this bijection, $e = \Theta(\pi)$ is obtained from a permutation $\pi = \pi_1 \pi_2 \ldots \pi_n \in \mathbf{S}_n$ by setting $e_i = |\{j \mid j < i \text{ and } \pi_j > \pi_i\}|$.

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The encoding of permutations as inversion sequences suggests that it could be illuminating to study patterns in inversion sequences in the same way that patterns have been studied in permutations. The paper [13] focused on the enumeration of inversion sequences that avoid words of length three and [21] treated permutations of length 3. For example, the inversion sequences $e \in \mathbf{I}_n$ that avoid the pattern 021 are those with no $i < j < k$ such that $e_i < e_j > e_k$ and $e_i < e_k$. We denote these by $\mathbf{I}_n(021)$. Similarly, $\mathbf{I}_n(010)$ is the set of $e \in \mathbf{I}_n$ with no $i < j < k$ such that $e_i < e_j > e_k$ and $e_i = e_k$. The results in [13, 21] related pattern avoidance in inversion sequences to a number of well-known combinatorlal sequences including the Fibonacci numbers, Bell numbers, large Schröder numbers, and Euler up/down numbers. They also gave rise to natural sequences that previously had not appeared in the On-Line Encyclopedia of Integer Sequences (OEIS) [15].

In this paper we consider a generalization of pattern avoidance to a fixed triple of binary relations (ρ_1, ρ_2, ρ_3) and study the set $\mathbf{I}_n(\rho_1, \rho_2, \rho_3)$ consisting of those $e \in \mathbf{I}_n$ with no $i < j < k$ such that $e_i \rho_1 e_j$, $e_j \rho_2 e_k$, and $e_i \rho_3 e_k$. For example, $\mathbf{I}_n(<, >, <) = \mathbf{I}_n(021)$ and $\mathbf{I}_n(<, >, =) = \mathbf{I}_n(010)$. Table 1 illustrates that "avoiding a triple of relations" can characterize inversion sequences with a variety of natural monotonicity or unimodality conditions, or with multiplicity constraints on the appearance of elements in the inversion sequence.

For this project, we considered all triples of relations in the set $\{<, >, \leq, \geq, =, \neq, -\}^3$. The relation "$-$" on a set S is all of $S \times S$; that is, x "$-$" y for all $x, y \in S$. There are 343 possible triples of relations (patterns). For each pattern, (ρ_1, ρ_2, ρ_3), we can consider the avoidance set, $\mathbf{I}_n(\rho_1, \rho_2, \rho_3)$, or the avoidance sequence $|\mathbf{I}_1(\rho_1, \rho_2, \rho_3)|, |\mathbf{I}_2(\rho_1, \rho_2, \rho_3)|, |\mathbf{I}_3(\rho_1, \rho_2, \rho_3)|, \ldots$. We say two patterns are *equivalent* if they give rise to the same avoidance sets and two patterns are *Wilf equivalent* if they yield the same avoidance sequence.

The 343 patterns partition into 98 equivalence classes of patterns. Additionally, we conjecture that there are 63 Wilf equivalence classes. In this paper, we enumerate a number of avoidance sets either directly or by relating them to familiar combinatorial structures. These relationships establish Wilf equivalence between a number of inequivalent patterns. However, even where enumeration is elusive, in many cases, Wilf equivalence can be proved via a bijection. This paper presents the results we have been able to prove, documents what has not yet been settled, and highlights the most intriguing open questions.

We uncovered several interesting enumeration results beyond those in [13, 21]. For example, the inversion sequences with no $i < j < k$ such that $\mathbf{e_i = e_j \leq e_k}$ are counted by the **Fibonacci numbers** (as are, e.g., **permutations avoiding the pair (321, 3412)**). Inversion sequences with no $i < j < k$ such that $\mathbf{e_i < e_j \leq e_k}$ are counted by **powers of two** (as are, e.g., **permutations avoiding (213, 312)**). Inversion sequences avoiding $(-, \neq, =)$ are counted by the **Bell numbers**; inversion sequences avoiding $(\geq, -, >)$ are counted by the **large Schröder numbers**; and inversion sequences avoiding $(-, \geq, <)$ are counted by the **Catalan numbers**. There are the same number of inversion sequences in $\mathbf{I}_n$ avoiding $(\neq, \neq, \neq)$ as there are **Grassmannian permutations** in $\mathbf{S}_n$. $\mathbf{I}_n(\neq, <, \leq)$ is counted by the number of **321-avoiding separable permutations** in $\mathbf{S}_n$. $\mathbf{I}_n(\neq, <, \neq)$ has the same number of elements as **the set of permutations in $\mathbf{S}_n$ avoiding both of the patterns 321 and 2143**.

In addition to results we could prove, several conjectures are suggested by our calculations, including the following. Inversion sequences avoiding $\mathbf{e_i > e_j \geq e_k}$ seem to have the same counting sequence as **permutations avoiding the barred pattern $\mathbf{21\bar{3}54}$**, also known as **plane permu-**

tations [6]. Inversion sequences avoiding $\mathbf{e_i} \geq \mathbf{e_j} \geq \mathbf{e_k}$ seem to have the same counting sequence as **set partitions avoiding enhanced 3-crossings**. The set $\mathbf{I}_n(\geq, \geq, >)$ appears to have the same counting sequence as the **Baxter permutations**.

Of the 63 conjectured Wilf equivalence classes, five classes are counted by sequences that are ultimately constant. In the remaining 58 classes, 30 have counting sequences that appear to match sequences in the OEIS and 28 do not.

In Sections 2.1 through 2.30 we present our results and conjectures for the 30 Wilf classes of pattern-avoiding inversion sequences that (appear to) match sequences in the OEIS. Table 2 gives an overview. Even for patterns with a "no" in this table, we are able to prove some Wilf equivalence results.

For the patterns whose counting sequence does not match a sequence in the OEIS we have some limited results on Wilf equivalence and counting. Table 3 gives an overview of the patterns in these 28 Wilf classes. Our results and conjectures for a few of these patterns are presented in Section 3.

In Tables 1, 2, and 3, each row represents an equivalence class of patterns whose identifier is given in the last column. A Wilf class of patterns is identified by the number a_7, the number of inversion sequences of length 7 avoiding a pattern in the class. Within a Wilf class, equivalence classes are labeled A,B,C, etc. So, for example, there are three equivalence classes of patterns counted by the Catalan numbers and these classes are labeled 429A, 429B, and 429C.

In the remainder of this section we give some definitions that will be needed throughout this paper, keeping the number of definitions to a minimum so that subsections can be somewhat self-contained. Finally, for completeness, we list the triples of relations whose avoidance sequences are ultimately constant.

Encodings of permutations

We compare Θ with a few other common encodings of permutations mentioned later in this paper: Lehmer codes, and *invcodes*, which are reverse Lehmer codes.

For a sequence $t = (t_1, \ldots, t_n)$, let $t^R = (t_n, t_{n-1}, \ldots, t_1)$, and for a set of sequences, T, let $T^R = \{t^R \mid t \in T\}$. For a permutation $\pi = (\pi_1, \ldots, \pi_n) \in \mathbf{S}_n$, let $\pi^C = (n+1-\pi_1, \ldots, n+1-\pi_n)$, and for a set of permutations P, let $P^C = \{\pi^C \mid \pi \in P\}$. We use the following encodings.

$\Theta : \mathbf{S}_n \to \mathbf{I}_n$

$$e = \Theta(\pi) \text{ iff } e_i = |\{j \mid j < i \text{ and } \pi_j > \pi_i\}|$$

$L : \mathbf{S}_n \to \mathbf{I}_n^R$

$$e = L(\pi) \text{ iff } e_i = |\{j \mid j > i \text{ and } \pi_j < \pi_i\}|$$

$invcode : \mathbf{S}_n \to \mathbf{I}_n$

$$e = invcode(\pi) \text{ iff } e^R = L(\pi).$$

Note that $invcode(\pi) = e$ if and only if $e = \Theta((\pi^C)^R)$. Additionally, notice that if $\Theta(\pi) = e$, then i is a *descent* of π, that is $\pi_i > \pi_{i+1}$, if and only if i is an *ascent* of e, that is $e_i < e_{i+1}$.

We will make use of another encoding, ϕ in Sections 2.24 and 2.29.

Operations on inversion sequences

For $e = (e_1, e_2, \ldots, e_n) \in \mathbf{I}_n$ and any integer t, define $\sigma_t(e) = (e'_1, e'_2, \ldots e'_n)$ where $e'_i = e_i$ if $e_i = 0$, and $e'_i = e_i + t$ otherwise. So σ_t adds t to the nonzero elements of a sequence (notice that t could be negative).

Concatenation is used to add an element to the beginning or end of an inversion sequence. For $e = (e_1, e_2, \ldots, e_n) \in \mathbf{I}_n$, $0 \cdot e = (0, e_1, e_2, \ldots, e_n) \in \mathbf{I}_{n+1}$ and, if $0 \le i \le n$, $e \cdot i = (e_1, e_2, \ldots, e_n, i) \in \mathbf{I}_{n+1}$. More generally, $x \cdot y$ denotes the concatenation of two sequences or two words x, y

Statistics on inversion sequences

In several cases, statistics on inversion sequences helped to prove or refine the results in Table 2 and make connections with statistics on other combinatorial families. These include the following, defined for an inversion sequence $e \in \mathbf{I}_n$:

$$
\begin{aligned}
\mathrm{asc}(e) &= |\{i \in [n-1] \mid e_i < e_{i+1}\}| \\
\mathrm{zeros}(e) &= |\{i \in [n] \mid e_i = 0\}| \\
\mathrm{dist}(e) &= |\{e_1, e_2, \ldots, e_n\}| \\
\mathrm{repeats}(e) &= |\{i \in [n-1] \mid e_i \in \{e_{i+1}, \ldots, e_n\}\}| = n - \mathrm{dist}(e) \\
\mathrm{maxim}(e) &= |\{i \in [n] \mid e_i = i-1\}| \\
\mathrm{maxx}(e) &= \max\{e_1, e_2, \ldots, e_n\} \\
\mathrm{last}(e) &= e_n.
\end{aligned}
$$

These statistics are, respectively, the number of ascents, the number of zeros, the number of distinct elements, the number of repeats, the number of maximal elements, the maximum element, and the last element of e.

Equivalence classes of patterns whose avoidance sequences are ultimately constant

These can be easily checked:

pattern	avoidance sequence
$(-,-,-)$	$1,2,0,0,0,0,\ldots$
$(\le,\le,-)$	$1,2,1,0,0,0,\ldots$
$(-,-,\ne)$	$1,2,2,1,1,1\ldots$
$(-,-,<)$	$1,2,2,2,2,2,\ldots$
$(-,-,<)$	$1,2,2,2,2,2,\ldots$
$(-,\ne,-)$	$1,2,2,2,2,2,\ldots$
$(-,\ne,\ne)$	$1,2,3,3,3,3,\ldots$

Inversion sequences e satisfying:	are those with no $i<j<k$ such that:	and are counted by:	a_7, equiv class	Section
Monotonicity constraints:				
$e_1=e_2=\ldots=e_{n-1}$	$e_i \neq e_j$	n	7,D	2.1
$\exists t$: $e_1=e_2=\ldots=e_t \leq e_{t+1}=e_{t+2}=\ldots=e_n$	$e_i<e_j \neq e_k$	$1+n(n-1)/2$	22,A	2.4
$\exists t$: $e_1=e_2=\ldots=e_t < e_{t+1}<e_{t+2}<\ldots<e_n$	$e_i<e_j \geq e_k$	2^{n-1}	64,C	2.6
$e_1 \leq e_2 \leq \ldots \leq e_{n-1}$	$e_i > e_j$	Binom$(2n-2,n-1)$	924	2.18
$e_1 \leq e_2 \leq \ldots \leq e_{n-1} \leq e_n$	$e_j > e_k$	Catalan number C_n	429,A	2.14
$e_1 \leq e_2 < e_3 < \ldots < e_n$	$e_j \geq e_k$	n	7,B	2.1
$e_1 < e_2 < \ldots < e_{n-1}$	$e_i = e_j$	n	7,C	2.1
$e_2 \geq e_3 \geq \ldots \geq e_n$	$e_j < e_k$	n	7,A	2.1
Unimodality constraints:				
$\exists t$: $e_1=e_2=\ldots=e_t \leq e_{t+1} \geq e_{t+2}=\ldots=e_n=0$	$e_i<e_j$ and $e_i<e_k$	$1+n(n-1)/2$	22,B	2.4
$\exists t$: $e_1=e_2=\ldots=e_t \leq e_{t+1} \geq e_{t+2} \geq \ldots \geq e_n$	$e_i \neq e_j < e_k$	Grassmannian perms	121,A	2.7
$\exists t$: $e_1=e_2=\ldots=e_t < e_{t+1} > e_{t+2} > \ldots > e_n$	$e_i \neq e_j \leq e_k$	$F_{n+2}-1$	33,A	2.5
$\exists t$: $e_1 \leq e_2 \leq \ldots \leq e_t > e_{t+1} \geq e_{t+2} \geq \ldots \geq e_n$	$e_i > e_j < e_k$	A033321	1265	2.20
$\exists t$: $e_1 \leq e_2 \leq \ldots \leq e_t > e_{t+1} > e_{t+2} > \ldots > e_n$	$e_i > e_j \leq e_k$	A071356	1064	2.19
$\exists t$: $e_1 \leq e_2 \leq \ldots \leq e_t \geq e_{t+1} = e_{t+2} = \ldots = e_n$	$e_i > e_j \neq e_k$	See Section 3	1079,A	3.1.3
$\exists t$: $e_1 < e_2 < \ldots < e_t \geq e_{t+1} = e_{t+2} = \ldots = e_n$	$e_i \geq e_j \neq e_k$	$1+n(n-1)/2$	22,C	2.4
$\exists t$: $e_1 < e_2 < \ldots < e_t \geq e_{t+1} \geq e_{t+2} \geq \ldots \geq e_n$	$e_i = e_j < e_k$	2^{n-1}	64,A	2.6
$\exists t$: $e_1 < e_2 < \ldots < e_t \geq e_{t+1} > e_{t+2} > \ldots > e_n$	$e_i = e_j \leq e_k$	F_{n+1}	21	2.3
$\exists (t \leq s)$: $e_1 < e_2 < \ldots < e_t = \ldots = e_s > \ldots > e_n$	$e_i \geq e_j \leq e_k$ and $e_i \neq e_k$	$F_{n+2}-1$	33B	2.5
Positive elements monotone:				
positive entries are strictly decreasing	$e_i < e_j \leq e_k$	2^{n-1}	64,B	2.6
positive entries are weakly decreasing	$e_i < e_j < e_k$	F_{2n-1}	233	2.12
positive entries are strictly increasing	$e_j \geq e_k$ and $e_i < e_k$	Catalan number C_n	429,B	2.14
positive entries are weakly increasing	$e_j > e_k$ and $e_i < e_k$	large Schröder number	1806,A	2.24
Multiplicity constraints:				
entries $e_2,\ldots,e_n$ are all distinct	$e_i \leq e_j = e_k$	2^{n-1}	64,D	2.6
$\lvert\{e_1,e_2,\ldots,e_n\}\rvert \leq 2$	$e_i \neq e_j \neq e_k$ and $e_i \neq e_k$	Grassmannian perms	121,B	2.7
positive entries are distinct	$e_i < e_j = e_k$	Bell numbers	877,A	2.17
no three entries equal	$e_i = e_j = e_k$	Euler up/down nos.	1385	2.22
only adjacent entries can be equal	$e_i \neq e_j \neq e_k$ and $e_i \neq e_k$	Bell numbers	877,C	2.17
$e_s = e_t \implies \lvert s-t\rvert \leq 1$	$e_i = e_k$	A229046	304	2.13

Table 1: Characterizations of inversion sequences avoiding triples of relations.

Inversion sequences with no $i<j<k$ such that:	appear to be counted by OEIS seq:	proven?	notes/OEIS description	a_7, equiv class	Section
$e_i \neq e_j$ and $e_i \neq e_k$	A004275	yes	$2(n-1)$ for $n>1$	12,A	2.2
$e_i \geq e_j$ and $e_i \neq e_k$	A004275	yes	$2(n-1)$ for $n>1$	12,B	2.2
$e_i = e_j \leq e_k$	A000045	yes	Fibonacci numbers, F_{n+1}	21	2.3
$e_i < e_j \neq e_k$	A000124	yes	Lazy caterer sequence	22,A	2.4
$e_i < e_j$ and $e_i < e_k$	A000124	yes	Lazy caterer sequence	22,B	2.4
$e_i \geq e_j \neq e_k$	A000124	yes	Lazy caterer sequence	22,C	2.4
$e_i \neq e_j \leq e_k$	A000071	yes	$F_{n+2}-1$	33,A	2.5
$e_i \geq e_j \leq e_k$ and $e_i \neq e_k$	A000071	yes	$F_{n+2}-1$	33,B	2.5
$e_i = e_j < e_k$	A000079	yes	$\mathbf{I}_n(001)$, 2^{n-1} (see [13])	64,A	2.6
$e_i < e_j \leq e_k$	A000079	yes	2^{n-1}	64,B	2.6
$e_i < e_j \geq e_k$	A000079	yes	2^{n-1}	64,C	2.6
$e_i \leq e_j = e_k$	A000079	yes	2^{n-1}	64,D	2.6
$e_i \neq e_j < e_k$	A000325	yes	Grassmannian permutations	121,A	2.7
$e_i \neq e_j \neq e_k$ and $e_i \neq e_k$	A000325	yes	Grassmannian permutations	121,B	2.7
$e_j \geq e_k$ and $e_i \neq e_k$	A000325	yes	Grassmannian permutations	121,C	2.7
$e_i \neq e_j < e_k$ and $e_i \leq e_k$	A034943	yes	321-avoiding separable perms	151	2.8
$e_i \neq e_j < e_k$ and $e_i \neq e_k$	A088921	yes	$\mathbf{S}_n(321, 2143)$	185	2.9
$e_i \geq e_k$	A049125	no	ordered trees, internal nodes adj. to ≤ 1 leaf	187	2.10
$e_i \leq e_j \geq e_k$ and $e_i \neq e_k$	A005183	yes	$\mathbf{S}_n(132, 4312)$, $n2^{n-1}+1$	193	2.11
$e_i < e_j < e_k$	A001519	yes	$\mathbf{I}_n(012)$, F_{2n-1} (see [13, 21])	233	2.12
$e_i = e_k$	A229046	no	recurrence $\rightarrow$ gf?	304	2.13
$e_j > e_k$	A000108	yes	Catalan numbers	429,A	2.14
$e_j \geq e_k$ and $e_i < e_k$	A000108	yes	Catalan numbers	429,B	2.14
$e_i \geq e_j$ and $e_i \geq e_k$	A000108	no	Catalan numbers	429,C	2.14
$e_i \neq e_j = e_k$	A047970	yes	$\mathbf{S}_n(\bar{3}\bar{1}542)$, nexus numbers	523	2.15
$e_j \leq e_k$ and $e_i \geq e_k$	A108307	no	set partitions avoiding enhanced 3-crossings	772,A	2.16
$e_i \geq e_j \geq e_k$	A108307	no	set partitions avoiding enhanced 3-crossings	772,B	2.16
$e_i < e_j = e_k$	A000110	yes	$\mathbf{I}_n(011)$ (see [13]), Bell numbers B_n	877,A	2.17
$e_i = e_j \geq e_k$	A000110	no	$\mathbf{I}_n(000, 110)$, B_n	877,B	2.17
$e_j \neq e_k$ and $e_i = e_k$	A000110	yes	$\mathbf{I}_n(010, 101)$, B_n	877,C	2.17
$e_i \geq e_j$ and $e_i = e_k$	A000110	no	$\mathbf{I}_n(000, 101)$, B_n	877,D	2.17
$e_i > e_j$	A000984	yes	central binomial coefficients	924	2.18
$e_i > e_j \leq e_k$	A071356	no	certain underdiagonal lattice paths	1064	2.19
$e_i > e_j < e_k$	A033321	yes	$\mathbf{S}_n(2143, 3142, 4132)$ (see [8])	1265	2.20
$e_i > e_j$ and $e_i \leq e_k$	A106228	no	$\mathbf{I}_N(101, 102)$, $\mathbf{S}_n(4123, 4132, 4213)$	1347	2.21
$e_i = e_j = e_k$	A000111	yes	$\mathbf{I}_n(000)$ (see [13]), Euler up/down numbers	1385	2.22
$e_i > e_j$ and $e_i < e_k$	A200753	yes	$\mathbf{I}_n(102)$, [21]	1694	2.23
$e_j > e_k$ and $e_i < e_k$	A006318	yes	$\mathbf{I}_n(021)$ [13, 21], large Schröder numbers R_{n-1}	1806,A	2.24
$e_i > e_j$ and $e_i \geq e_k$	A006318	yes	$\mathbf{I}_n(210, 201, 101, 100)$, R_{n-1}	1806,B	2.24
$e_i \geq e_j$ and $e_i > e_k$	A006318	yes	$\mathbf{I}_n(210, 201, 100, 110)$, R_{n-1}	1806,C	2.24
$e_i \geq e_j \neq e_k$ and $e_i \geq e_k$	A006318	yes	$\mathbf{I}_n(210, 201, 101, 110)$, R_{n-1}	1806,D	2.24
$e_i \geq e_j \geq e_k$ and $e_i > e_k$	A001181	no	Baxter permutations	2074	2.25
$e_i > e_j$ and $e_i > e_k$	A098746	no	$\mathbf{I}_n(210, 201, 100)$, $\mathbf{S}_n(4231, 42513)$	2549,A	2.26
$e_i > e_j \neq e_k$ and $e_i \geq e_k$	A098746	no	$\mathbf{I}_n(210, 201, 101)$, $\mathbf{S}_n(4231, 42513)$	2549,B	2.26
$e_i \geq e_j \neq e_k$ and $e_i > e_k$	A098746	no	$\mathbf{I}_n(210, 201, 110)$, $\mathbf{S}_n(4231, 42513)$	2549,C	2.26
$e_j < e_k$ and $e_i \geq e_k$	A117106	no	$\mathbf{I}_n(201, 101)$, $\mathbf{S}_n(21\bar{3}54)$	2958,A	2.27
$e_i > e_j \geq e_k$	A117106	no	$\mathbf{I}_n(210, 100)$, $\mathbf{S}_n(21\bar{3}54)$	2958,B	2.27
$e_i \geq e_j > e_k$	A117106	no	$\mathbf{I}_n(210, 110)$, $\mathbf{S}_n(21\bar{3}54)$	2958,C	2.27
$e_j \leq e_k$ and $e_i > e_k$	A117106	no	$\mathbf{I}_n(201, 100)$, $\mathbf{S}_n(21\bar{3}54)$	2958,D	2.27
$e_j < e_k$ and $e_i = e_k$	A113227	yes	$\mathbf{I}_n(101)$, $\mathbf{S}_n$(1-23-4), (see [13])	3207,A	2.28
$e_i = e_j > e_k$	A113227	yes	$\mathbf{I}_n(110)$, $\mathbf{S}_n$(1-23-4), (see [13])	3207,B	2.28
$e_i > e_j \neq e_k$ and $e_i > e_k$	A212198	yes	$\mathbf{I}_n(201, 210)$, $MMP(0,2,0,2)$-avoiding perms	3720	2.28

Table 2: Patterns whose avoidance sequences appear to match sequences in the OEIS. Those marked as "yes" are cited, if known, and otherwise are proven in this paper.

Inversion sequences with no $i < j < k$ such that:	comments	initial terms $a_1, \dots a_9$	a_7, equiv class
$e_j \geq e_k$ and $e_i \geq e_k$	$\mathbf{I}_n(000, 010, 011, 021)$	$1, 2, 4, 10, 26, 73, 214, 651, 2040$	214
$e_i \leq e_j$ and $e_i \geq e_k$	$\mathbf{I}_n(000, 010, 110, 120)$	$1, 2, 4, 10, 27, 79, 247, 816, 2822$	247
$e_j \geq e_k$ and $e_i = e_k$	$\mathbf{I}_n(000, 010)$	$1, 2, 4, 10, 29, 95, 345, 1376, 5966$	345
$e_j \neq e_k$ and $e_i \geq e_k$	Wilf eq. to 663B (Sec. 3.2)	$1, 2, 5, 15, 50, 178, 663, 2552, 10071$	663,A
$e_i \neq e_j$ and $e_i \geq e_k$	Wilf eq. to 663A (Sec. 3.2)	$1, 2, 5, 15, 50, 178, 663, 2552, 10071$	663,B
$e_i \neq e_j \neq e_k$ and $e_i \geq e_k$	$\mathbf{I}_n(010, 101, 120, 201)$	$1, 2, 5, 15, 51, 188, 733, 2979, 12495$	733
$e_j > e_k$ and $e_i \geq e_k$	Wilf eq. to 746B (Sec. 3.2	$1, 2, 5, 15, 51, 189, 746, 3091, 13311$	746,A
$e_i \neq e_j \geq e_k$ and $e_i \geq e_k$	Wilf eq. to 746A (Sec. 3.2	$1, 2, 5, 15, 51, 189, 746, 3091, 13311$	746,B
$e_i \leq e_j \neq e_k$ and $e_i \geq e_k$	$\mathbf{I}_n(010, 110, 120)$	$1, 2, 5, 15, 51, 190, 759, 3206, 14180$	759
$e_i \leq e_j > e_k$ and $e_i \neq e_k$	counted - See Section 3.1	1,2,6,20,68,233,805,2807,9879	805
$e_i \neq e_j > e_k$ and $e_i \geq e_k$	$\mathbf{I}_n(010, 120)$	$1, 2, 5, 15, 52, 200, 830, 3654, 16869$	830
$e_i < e_j$ and $e_i \geq e_k$	$\mathbf{I}_n(010, 120)$	$1, 2, 5, 15, 52, 201, 845, 3801, 18089$	845
$e_j > e_k$ and $e_i = e_k$	$\mathbf{I}_n(010)$ (A263779)	$1, 2, 5, 15, 53, 215, 979, 4922, 26992$	979
$e_i > e_j$ and $e_i \neq e_k$	counted - See Section 3.1	$1, 2, 6, 21, 76, 277, 1016, 3756, 13998$	1016
$e_i > e_j \neq e_k$	Wilf eq. to 1079B (Sec. 3.2)	$1, 2, 6, 21, 77, 287, 1079, 4082, 15522$	1079,A
$e_i < e_j > e_k$ and $e_i \neq e_k$	Wilf eq. to 1079A (Sec. 3.2)	$1, 2, 6, 21, 77, 287, 1079, 4082, 15522$	1079,B
$e_i > e_j \leq e_k$ and $e_i \neq e_k$	$\mathbf{I}_n(100, 102, 201)$	$1, 2, 6, 21, 78, 299, 1176, 4729, 19378$	1176
$e_i > e_j \neq e_k$ and $e_i \neq e_k$	$\mathbf{I}_n(210, 201, 102)$	$1, 2, 6, 22, 85, 328, 1253, 4754, 17994$	1253
$e_i \geq e_j = e_k$	$\mathbf{I}_n(000, 100)$	$1, 2, 5, 16, 60, 260, 1267, 6850, 40572$	1267
$e_i > e_k$	$\mathbf{I}_n(100, 110, 120, 210, 201)$	$1, 2, 6, 21, 81, 332, 1420, 6266, 28318$	1420
$e_i > e_j < e_k$ and $e_i \neq e_k$	$\mathbf{I}_n(102, 201)$	$1, 2, 6, 22, 87, 354, 1465, 6154, 26223$	1465
$e_j \geq e_k$ and $e_i > e_k$	$\mathbf{I}_n(100, 110, 210)$	$1, 2, 6, 21, 82, 343, 1509, 6893, 32419$	1509
$e_j \neq e_k$ and $e_i > e_k$	Wilf eq. to 1833B (Sec. 3.2)	$1, 2, 6, 22, 90, 396, 1833, 8801, 43441$	1833,A
$e_i \neq e_j$ and $e_i > e_k$	Wilf eq. to 1833A (Sec. 3.2)	$1, 2, 6, 22, 90, 396, 1833, 8801, 43441$	1833,B
$e_j > e_k$ and $e_i > e_k$	Wilf eq. to 1953B (Sec. 3.2)	$1, 2, 6, 22, 91, 409, 1953, 9763, 50583$	1953,A
$e_i \neq e_j \geq e_k$ and $e_i > e_k$	Wilf eq. to 1953A (Sec. 3.2)	$1, 2, 6, 22, 91, 409, 1953, 9763, 50583$	1953,B
$e_i \leq e_j > e_k$ and $e_i > e_k$	$\mathbf{I}_n(110, 120)$	$1, 2, 6, 22, 92, 423, 2091, 10950, 60120$	2091
$e_i > e_j \leq e_k$ and $e_i \geq e_k$	$\mathbf{I}_n(100, 101, 201)$	$1, 2, 6, 22, 92, 424, 2106, 11102, 61436$	2106
$e_i \neq e_j \neq e_k$ and $e_i > e_k$	$\mathbf{I}_n(120, 210, 201)$	$1, 2, 6, 23, 101, 484, 2468, 13166, 72630$	2468
$e_i \neq e_j > e_k$ and $e_i > e_k$	$\mathbf{I}_n(210, 120)$	$1, 2, 6, 23, 102, 499, 2625, 14601, 84847$	2625
$e_i < e_j$ and $e_i > e_k$	$\mathbf{I}_n(120)$ (A263778)	$1, 2, 6, 23, 103, 515, 2803, 16334, 100700$	2803
$e_i > e_j = e_k$	$\mathbf{I}_n(100)$ (A263780)	$1, 2, 6, 23, 106, 565, 3399, 22678, 165646$	3399
$e_j < e_k$ and $e_i > e_k$	$\mathbf{I}_n(201)$ (A263777)	$1, 2, 6, 24, 118, 674, 4306, 29990, 223668$	4306,A
$e_i > e_j > e_k$	$\mathbf{I}_n(210)$ Wilf eq to 4306A [13]	$1, 2, 6, 24, 118, 674, 4306, 29990, 223668$	4306,B

Table 3: The patterns whose avoidance sequences did not match sequences in the OEIS. (OEIS numbers in parentheses were newly assigned after [13] was posted to the arXiv.)

2 Patterns whose sequences appear in the OEIS

2.1 7(A,B,C,D): n

There are four equivalence classes of patterns whose avoidance sequences are counted by the positive integers. We characterize each, from which it is straightforward to prove that

$$|\mathbf{I}_n(-,<,-)| = |\mathbf{I}_n(-,\geq,-)| = |\mathbf{I}_n(=,-,-)| = |\mathbf{I}_n(\neq,-,-)| = n,$$

although these four patterns are not equivalent.

$\boxed{\textbf{7A}\text{: } e_j < e_k}$

$\mathbf{I}_n(-,<,-)$ is the set of $e \in \mathbf{I}_n$ satisfying $e_2 \geq e_3 \geq \ldots \geq e_n$.

$\boxed{\textbf{7B}\text{: } e_j \geq e_k}$

$\mathbf{I}_n(-,\geq,-)$ is the set of $e \in \mathbf{I}_n$ satisfying $e_1 \leq e_2 < e_3 < \ldots < e_n$.

$\boxed{\textbf{7C}\text{: } e_i = e_j}$

$\mathbf{I}_n(=,-,-)$ is the set of $e \in \mathbf{I}_n$ satisfying $e_1 < e_2 < \ldots < e_{n-1}$.

$\boxed{\textbf{7D}\text{: } e_i \neq e_j}$

$\mathbf{I}_n(\neq,-,-)$ is the set of $e \in \mathbf{I}_n$ satisfying $e_1 = e_2 = \ldots = e_{n-1}$.

Note: Simion and Schmidt [24] showed that $|S_n(123, 132, 231)| = n$.

2.2 12(A,B): $2(n-1)$ for $n > 1$

We show that the following patterns are Wilf equivalent, but not equivalent.

$\boxed{\textbf{12A}\text{: } e_i \neq e_j \text{ and } e_i \neq e_k}$

$\boxed{\textbf{12B}\text{: } e_i \geq e_j \text{ and } e_i \neq e_k}$

Theorem 1. *$|\mathbf{I}_n(\neq,-,\neq)|$ and $|\mathbf{I}_n(\geq,-,\neq)|$ are both counted by 1 if $n = 1$ and by $2(n-1)$ for $n > 1$. However, $\mathbf{I}_n(\neq,-,\neq) \neq \mathbf{I}_n(\geq,-,\neq)$ for $n > 2$.*

Proof. For $n = 1$ this is clear. For $n > 1$ this follows by noting that any $e \in \mathbf{I}_n$ with no $i < j < k$ such that $e_i \neq e_j$ and $e_i \neq e_k$ can be $(0, 0, \ldots, 0)$ or can be of the form $(0, 0, \ldots, t, 0)$ or $(0, 0, \ldots, 0, s)$ where $t \in [n-2]$ and $s \in [n-1]$. On the other hand, any $e \in \mathbf{I}_n$ with no $i < j < k$ such that $e_i \geq e_j$ and $e_i \neq e_k$ must have the form $(0, 1, 2, \ldots, n-2, t)$ for $t = 0, \ldots, n-1$ or the form $(0, 1, 2, \ldots, t-1, t, t, \ldots, t)$ for $t = 0, \ldots, n-3$. $\square$

2.3 21: F_{n+1}

Let F_n be the n-th Fibonacci number, where $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The Fibonacci numbers count pattern-avoiding permutations such as $\mathbf{S}_n(123, 132, 213)$ [24].

$\boxed{\mathbf{21}\text{: } e_i = e_j \leq e_k}$

Observation 1. *The inversion sequences $e \in \mathbf{I}_n$ with no $i < j < k$ such that $e_i = e_j \leq e_k$ are those satisfying, for some $t \in [n]$,*

$$e_1 < e_2 < \ldots < e_t \geq e_{t+1} > \ldots > e_n. \tag{1}$$

Theorem 2. $|\mathbf{I}_n(=, \leq, -)| = F_{n+1}$

Proof. This is clear for $n = 1, 2$. For $n \geq 3$, any $e \in \mathbf{I}_n(=, \leq, -)$ must have the form $(0, e_1 + 1, \ldots, e_{n-1} + 1)$ for $(e_1, \ldots, e_{n-1}) \in \mathbf{I}_{n-1}(=, \leq, -)$ or $(0, e_1 + 1, \ldots, e_{n-2} + 1, 0)$ for $(e_1, \ldots, e_{n-2}) \in \mathbf{I}_{n-2}(=, \leq, -)$. Conversely, strings of either of these forms are in $\mathbf{I}_n(=, \leq, -)$. $\square$

Among the 343 patterns checked it can be shown that the six patterns whose avoidance sequence is counted by F_{n+1} are equivalent.

Observation 2. *All of the following patterns are equivalent to $(=, \leq, -)$: $(=, -, \leq)$, $(=, \leq, \leq)$, $(\geq, -, \leq)$, $(\geq, \leq, -)$, $(\geq, \leq, \geq)$,*

2.4 22(A,B,C): Lazy caterer sequence, $\binom{n}{2} + 1$

We show that there are three inequivalent patterns that are all counted by the Lazy caterer sequence, $\binom{n}{2} + 1$, which also counts $\mathbf{S}_n(132, 321)$ [24]

$\boxed{\mathbf{22A}\text{: } e_i < e_j \neq e_k}$

It is not hard to see that inversion sequences avoiding $e_i < e_j \neq e_k$ are characterized by the following.

Observation 3. *The inversion sequences $e \in \mathbf{I}_n$ with no $i < j < k$ such that $e_i < e_j \neq e_k$ are those satisfying, for some t where $1 \leq t \leq n$,*

$$0 = e_1 = e_2 = \ldots = e_{t-1} \leq e_t = e_{t+1} = \ldots = e_n. \tag{2}$$

That is, either $e = (0, 0, \ldots, 0)$ or, for some $t : 2 \leq t \leq n$ and $j : 1 \leq j \leq t - 1$, e consists of a string of $t - 1$ zeros followed by a string of $n - t + 1$ copies of j. This gives the following.

Theorem 3. $|\mathbf{I}_n(<, \neq, -)| = \binom{n}{2} + 1$.

The sequence whose nth entry is $\binom{n}{2} + 1$ is sequence A000124 in the OEIS, where it is called the *Lazy Caterer* sequence [15]. This is also the avoidance sequence for certain pairs of permutation patterns, as was shown in [24].

Theorem 4 (Simion-Schmidt [24]). $|\mathbf{S}_n(\alpha,\beta)| = \binom{n}{2} + 1$ *for any of the following pairs* (α,β) *of patterns:*

$$(132, 321),\ (123, 231),\ (123, 312),\ (213, 321).$$

We can relate these permutations to the inversion sequences in $\mathbf{I}_n(<,\neq,-)$. Recall the bijection $\Theta(\pi) : \mathbf{S}_n \to \mathbf{I}_n$ for $\pi = \pi_1 \dots \pi_n \in \mathbf{S}_n$ defined by $\Theta(\pi) = (e_1, e_2, \dots, e_n)$, where $e_i = |\{j \mid j < i \text{ and } e_j > e_i\}|$.

Theorem 5. $\mathbf{I}_n(<,\neq,-) = \Theta(\mathbf{S}_n(213, 321))$.

Proof. Note that $e \in \mathbf{I}_n$ satisfies (2) if and only if $\pi = \Theta^{-1}(e)$ satisfies

$$\pi_1 < \pi_2 < \dots < \pi_t > \pi_{t+1} < \pi_{t+2} < \dots < \pi_n,$$

where $\pi_t, \pi_{t+1}, \dots, \pi_n$ are consecutive integers. Such permutations are the ones that avoid both 213 and 321. □

The patterns $(<,-,<)$ and $(\geq,\neq,-)$ are Wilf-equivalent to the pattern $(<,\neq,-)$ on inversion sequences, although the three patterns are pairwise inequivalent. This is clear from the following characterizations.

22B: $e_i < e_j$ and $e_i < e_k$

Observation 4. *The inversion sequences* $e \in \mathbf{I}_n$ *with no* $i < j < k$ *such that* $e_i < e_j$ *and* $e_i < e_k$ *are those satisfying, for some* t *where* $1 \leq t \leq n$,

$$0 = e_1 = e_2 = \dots = e_{t-1} \leq e_t \geq e_{t+1} = \dots = e_n = 0. \tag{3}$$

22C: $e_i \geq e_j \neq e_k$

Observation 5. *The inversion sequences* $e \in \mathbf{I}_n$ *with no* $i < j < k$ *such that* $e_i \geq e_j \neq e_k$ *are those satisfying, for some* t *where* $1 \leq t \leq n$,

$$e_1 < e_2 < \dots < e_{t-1} \geq e_t = e_{t+1} = \dots = e_n. \tag{4}$$

2.5 33(A,B): $F_{n+2} - 1$

We show that **33A**: $(\neq,\leq,-)$ and **33B**: $(\geq,\leq,\neq)$ are inequivalent Wilf equivalent patterns whose avoidance sequences are counted by $F_{n+2} - 1$.

33A: $e_i \neq e_j \leq e_k$

Theorem 6. $|\mathbf{I}_n(\neq,\leq,-)| = F_{n+2} - 1$.

Proof. Observe that the inversion sequences $e \in \mathbf{I}_n$ with no $i < j < k$ such that $e_i \neq e_j \leq e_k$ are those satisfying, for some t where $1 \leq t \leq n+1$,

$$0 = e_1 = e_2 = \ldots = e_{t-1} < e_t > e_{t+1} > \ldots > e_n. \tag{5}$$

We can partition the inversion sequences in $\mathbf{I}_n(\neq, \leq, -)$ into three disjoint sets: $\{(0,0,\ldots,0)\}$, $A = \{e \in \mathbf{I}_n(\neq, \leq, -) \mid e_n \neq 0\}$, and $B = \{e \in \mathbf{I}_n(\neq, \leq, -) \mid e \neq 0, e_n = 0\}$. Any inversion sequence in A can be constructed by taking any $e' = (e_1, e_2, \ldots, e_{n-1}) \in \mathbf{I}_{n-1}(\neq, \leq, -)$ and letting e_t be the first nonzero entry (if there is no nonzero entry, set $e_t = e_n$). Then we can use the characterization 5 to verify that $(0, e_1, e_2, \ldots, e_{t-1}, e_t + 1, \ldots, e_n + 1)$ is an element of A.

Any element of B can be constructed by taking some $e'' = (e_1, e_2, \ldots, e_{n-2}) \in \mathbf{I}_{n-2}(\neq, \leq, -)$ and letting e_t be the first nonzero entry (again, if no such entry exists, set $e_t = e_n$). Then $(0, e_1, \ldots, e_{t-1}, e_t + 1, \ldots, e_{n-2} + 1, 0)$ is an element of B.

Setting $a_n = |\mathbf{I}_n(\neq, \leq, -)|$, this gives $a_n = a_{n-1} + a_{n-2} + 1$, with initial conditions $a_1 = 1$, $a_2 = 2$. So $a_n = F_{n+2} - 1$. □

33B: $e_i \geq e_j \leq e_k$ and $e_i \neq e_k$

Theorem 7. $|\mathbf{I}_n(\geq, \leq, \neq)| = F_{n+2} - 1$.

Proof. The inversion sequences $e \in \mathbf{I}_n$ with no $i < j < k$ such that $e_i \geq e_j \leq e_k$ and $e_i \neq e_k$ are those satisfying, for some t, s where $1 \leq t \leq s \leq n$,

$$e_1 < e_2 < \ldots = e_{t-1} = e_t = e_{t+1} = \ldots = e_s > e_{s+1} > \ldots > e_n. \tag{6}$$

The following is a bijection mapping $\mathbf{I}_n(\geq, \leq, \neq)$ to $\mathbf{I}_n(\neq, \leq, -)$. For $e \in \mathbf{I}_n(\geq, \leq, \neq)$, let s be the first index, if any, such that $e_s > e_{s+1}$. If there is such an s, set $e_i = 0$ for $i = 1, \ldots s-1$ to get an element of $\mathbf{I}_n(\neq, \leq, -)$. Otherwise $e = (0, 0, \ldots, 0)$, which maps to itself in $\mathbf{I}_n(\neq, \leq, -)$. □

2.6 64(A,B,C,D): 2^{n-1}

64A: $e_i = e_j < e_k$

In [13], $\mathbf{I}_n(=, <, -) = \mathbf{I}_n(001)$ was characterized as the set of $e \in \mathbf{I}_n$ satisfying, for some $t \in [n]$,

$$e_1 < e_2 < \ldots < e_t \geq e_{t+1} \geq e_{t+2} \geq \ldots \geq e_n.$$

It was shown that $|\mathbf{I}_n(001)| = 2^{n-1}$ by showing that the bijection $\Theta : \mathbf{S}_n \to \mathbf{I}_n$ restricts to a bijection $\mathbf{S}_n(132, 231) \to \mathbf{I}_n(001)$. Permutations avoiding both 132 and 231 were shown by Simion and Schmidt to be counted by 2^{n-1} in [24].

We show that three other patterns are Wilf equivalent, though inequivalent, to **64A**.

64B: $e_i < e_j \leq e_k$

Theorem 8. *The number of $e \in \mathbf{I}_n$ with no $i < j < k$ such that $e_i < e_j \le e_k$ is 2^{n-1}.*

Proof. First observe that the inversion sequences e with no $i < j < k$ such that $e_i < e_j \le e_k$ are those whose positive entries form a strictly decreasing sequence.

Let $B_n = \mathbf{I}_n(<, \le, -)$. Notice that $|B_1| = |\{(0)\}| = 1$; we will show that for $n > 1$, $|B_n| = 2|B_{n-1}|$. Recall that $\sigma_1(e)$ adds 1 to each positive element in e. Each $e \in B_{n-1}$ gives rise to two elements of B_n. The first is $0 \cdot \sigma_1(e)$, which contains no "1". The second, which does have a "1", is $0 \cdot e$ if e contains a "1" and $e \cdot 1$ otherwise. □

64C: $e_i < e_j \ge e_k$

Theorem 9. *The number of $e \in \mathbf{I}_n$ avoiding $e_i < e_j \ge e_k$ with $i < j < k$ is 2^{n-1}.*

Proof. The inversion sequences avoiding the pattern $e_i < e_j \ge e_k$ with $i < j < k$ are those $e \in \mathbf{I}_n$ satisfying, for some $t \in [n]$,

$$0 = e_1 = e_2 = \cdots = e_t < e_{t+1} < e_{t+2} < \cdots < e_n.$$

Map $e \in \mathbf{I}_n(<, \ge, -)$ to the set consisting of its nonzero values. Clearly this is a bijection from $\mathbf{I}_n(<, \ge, -)$ to $2^{[n-1]}$. □

In fact, we can show that $\mathbf{I}_n(<, \ge, -)$ is the image under Θ of $\mathbf{S}_n(213, 312)$.

Theorem 10. $\Theta(\mathbf{S}_n(213, 312)) = \mathbf{I}_n(<, \ge, -)$.

Proof. It is straightforward to prove that $\mathbf{S}_n(213, 312)$ consists of the unimodal permutations where

$$\pi_1 < \pi_2 < \cdots < \pi_t = n > \pi_{t+1} > \cdots > \pi_n.$$

The inversion sequences avoiding the pattern $e_i < e_j \ge e_k$ with $i < j < k$ are those $e \in \mathbf{I}_n$ satisfying, for some $t \in [n]$,

$$0 = e_1 = e_2 = \cdots = e_t < e_{t+1} < e_{t+2} < \cdots < e_n.$$

It immediately follows that $\Theta(\mathbf{S}_n(213, 312)) = \mathbf{I}_n(<, \ge, -)$. □

64D: $e_i \le e_j = e_k$

Theorem 11. *The number of $e \in \mathbf{I}_n$ avoiding $e_i \le e_j = e_k$ where $i < j < k$ is 2^{n-1}.*

Proof. The inversion sequences avoiding the pattern $e_i \le e_j = e_k$, where $i < j < k$, are those $e \in \mathbf{I}_n$ in which all of the entries $e_2, e_3, \ldots, e_n$ are distinct.

Let $D_n = \mathbf{I}_n(\le, =, -)$. Then $|D_1| = |\{(0)\}| = 1$. We show that for $n > 1$, $|D_n| = 2|D_{n-1}|$. Each $e \in D_{n-1}$ gives rise to two elements of D_n: the first is $e \cdot (n-1)$, and the second is $e \cdot d$, where d is the unique element in $\{0, 1, \ldots, n-2\} \setminus \{e_2, \ldots, e_{n-1}\}$. □

2.7 121(A,B,C): Grassmannian permutations, $2^n - n$

Permutations with at most one descent were called *Grassmannian* by Lascoux and Schützenberger in [19], who also characterized them in terms of their Lehmer codes. Grassmannian permutations of length n are counted by $2^n - n$ and relate to three equivalence classes of patterns for inversion sequences.

$\boxed{\textbf{121A}: e_i \neq e_j < e_k}$

Theorem 12. $|\mathbf{I}_n(\neq, <, -)| = 2^n - n$.

Proof. First observe that those $e \in \mathbf{I}_n$ with no $i < j < k$ such that $e_i \neq e_j < e_k$ are exactly those with at most one ascent.

Using the mapping $\Theta : \mathbf{S}_n \to \mathbf{I}_n$, recall that π has a descent in a position i if and only if $\Theta(\pi)$ has an ascent in position i. Thus Θ restricts to a bijection from Grassmannian permutations of $[n]$ to $\mathbf{I}_n(\neq, <, -)$. □

The inversion sequences in $\mathbf{I}_n(\neq, <, -)$ correspond to the Grassmannian Lehmer codes of [19] via the natural bijection (reversal) between inversion sequences and Lehmer codes.

$\boxed{\textbf{121B}: e_i \neq e_j \neq e_k \text{ and } e_i \neq e_k}$

Theorem 13. $|\mathbf{I}_n(\neq, \neq, \neq)| = 2^n - n$.

Proof. Note that inversion sequences with no $i < j < k$ such that $e_i \neq e_j \neq e_k$ and $e_i \neq e_k$ are those with at most 2 distinct entries; precisely, $|\{e_1, \ldots, e_n\}| \leq 2$.

The theorem is clear for $n = 1$. Now consider some $e \in \mathbf{I}_n(\neq, \neq, \neq)$ when $n > 1$. Note that $(e_1, \ldots, e_{n-1}) \in \mathbf{I}_{n-1}(\neq, \neq, \neq)$. It follows that either (1) $|\{e_1, \ldots, e_{n-1}\}| = 2$, and e_n is one of the two elements occurring in $(e_1, \ldots, e_{n-1})$; or (2) $|\{e_1, \ldots, e_{n-1}\}| = 1$, and $e_n \in \{0, 1, \ldots, n-1\}$. Furthermore, the only inversion sequence in $\mathbf{I}_{n-1}(\neq, \neq, \neq)$ where $|\{e_1, \ldots, e_{n-1}\}| = 1$ is the zero inversion sequence. This gives the recurrence $|\mathbf{I}_n(\neq, \neq, \neq)| = 2(|\mathbf{I}_{n-1}(\neq, \neq, \neq)| - 1) + n$ which has the claimed solution. □

$\boxed{\textbf{121C}: e_j \geq e_k \text{ and } e_i \neq e_k}$

Theorem 14. $|\mathbf{I}_n(-, \geq, \neq)| = 2^n - n$.

Proof. Inversion sequences with no $i < j < k$ such that $e_j \geq e_k$ and $e_i \neq e_k$ are those satisfying

$$e_1 = \ldots = e_{i-1} < e_i < \ldots < e_n$$

or with $e_{i+1} = 0$ and

$$e_1 = \ldots = e_{i-1} < e_i < e_{i+2} < \ldots < e_n,$$

for some i: $2 \leq i \leq n + 1$.

To count these, for each $t = 1, \ldots, n-1$, and for any t-element subset $x_1 < x_2 < \ldots < x_t$ of $[n-1]$, associate the length n inversion sequence $(0, 0, \ldots, 0, x_1, x_2, \ldots, x_t)$ and, unless $\{x_1, \ldots, x_t\} = \{n-t, n-t+1, \ldots n-1\}$, also associate the length n inversion sequence $(0, 0, \ldots, 0, x_1, 0, x_2, \ldots, x_t)$, giving $2^{n-1} + (2^{n-1} - n) = 2^n - n$. □

2.8 151: (321)-avoiding separable permutations

$\boxed{\textbf{151}\text{: } e_i \neq e_j < e_k \text{ and } e_i \leq e_k}$

We show that the avoidance sequence for this pattern satisfies the recurrence $a_n = 3a_{n-1} - 2a_{n-2} + a_{n-3}$ with initial conditions $a_1 = 1$, $a_2 = 2$, and $a_3 = 5$. This coincides with sequence A034943 in the OEIS, where, among other things, it is said to count 321-avoiding separable permutations (OEIS entry by Vince Vatter) [15]. A *separable* permutation is one that avoids 2413 and 3142. Moreover, we show that $(\neq, <, \leq)$-avoiding inversion sequences have a simple characterization.

Theorem 15. *Let $A_n = \mathbf{I}_n(\neq, <, \leq)$ and $a_n = |A_n|$. Then $a_n = 3a_{n-1} - 2a_{n-2} + a_{n-3}$ with initial conditions $a_1 = 1$, $a_2 = 2$, and $a_3 = 5$.*

Proof. First, it can be shown that the set of $e \in \mathbf{I}_n$ such that there is no $i < j < k$ for which $e_i \neq e_j < e_k$ and $e_i \leq e_k$ is the set of $e \in \mathbf{I}_n$ where the nonzero elements are weakly decreasing and equal nonzero elements are consecutive. That is, (1) if $e_i < e_j$, then $e_i = 0$ and (2) if $0 < e_i = e_j$ for some $i < j$, then $e_i = e_{i+1} = \ldots = e_j$.

Define X_n, Y_n, Z_n by

$$\begin{aligned} X_n &= \{e \in A_n \mid e_i \neq 1, \text{ for all } 1 \leq i \leq n\}, \\ Y_n &= \{e \in A_n \mid e_n = 1\}, \\ Z_n &= \{e \in A_n \mid e_n = 0 \text{ and } e_i = 1 \text{ for some } i < n\}. \end{aligned}$$

Then A_n is the disjoint union $A_n = X_n \cup Y_n \cup Z_n$. Recall that the operator σ_1 adds 1 to the positive elements of an inversion sequence. To get a recurrence, note that $|X_n| = |A_{n-1}| = a_{n-1}$ since $e \in A_{n-1}$ if and only if $0 \cdot \sigma_1(e) \in X_n$. Also, $(e_1, \ldots, e_{n-1}, 0) \in Z_n$ if and only if $(e_1, \ldots, e_{n-1}) \in Y_{n-1} \cup Z_{n-1} = A_{n-1} - X_{n-1}$; so $|Z_n| = |A_{n-1}| - |X_{n-1}| = a_{n-1} - a_{n-2}$. Finally, $(e_1, \ldots, e_{n-1}, 1) \in Y_n$ if and only if $(e_1, \ldots, e_{n-1}) \in A_{n-1} - Z_{n-1}$, so $|Y_n| = a_{n-1} - |Z_{n-1}| = a_{n-1} - (a_{n-2} - a_{n-3})$. Putting this together,

$$a_n = |A_n| = |X_n| + |Y_n| + |Z_n| = 3a_{n-1} - 2a_{n-1} + a_{n-3}$$

and the result follows by checking the initial conditions. □

2.9 185: 321-avoiding vexillary permutations, $2^{n+1} - \binom{n+1}{3} - 2n - 1$

Vexillary permutations, studied by Lascoux and Schützenberger in [19], are 2143-avoiding permutations. The 321-avoiding vexillary permutations arose in work of Billey, Jockush and Stanley [5] on the combinatorics of Schubert polynomials. It was shown that $|\mathbf{S}_n(321, 2143)| = 2^{n+1} - \binom{n+1}{3} - 2n - 1$ which is entry A088921 in the OEIS. In this entry, it is noted that the 321-avoiding vexillary permutations are exactly the Grassmannian permutations (see Section 2.7) and their inverses.

$\boxed{\textbf{185}\text{: } e_i \neq e_j < e_k \text{ and } e_i \neq e_k}$

We show that the $(\neq, <, \neq)$-avoiding inversion sequences are counted by the same function as the 321-avoiding vexillary permutations.

Lemma 1. $\mathbf{I}_n(\neq, <, \neq) = \mathbf{I}_n(\neq, \neq, \neq) \cup \mathbf{I}_n(\neq, <, -)$.

Proof. If $\mathbf{e} \in \mathbf{I}_n(\neq, <, \neq)$, then either $e \in \mathbf{I}_n(\neq, <, -)$ or for any $i < j < k$ such that $e_i \neq e_j < e_k$, $e_i = e_k$ and therefore $e \in \mathbf{I}_n(\neq, \neq, \neq)$.

Conversely, if, for some $i < j < k$, $e_i \neq e_j < e_k$ and $e_i \neq e_k$, then e contains both $(\neq, <, -)$ and $(\neq, \neq, \neq)$. □

Theorem 16. $|\mathbf{I}_n(\neq, <, \neq)| = 2^{n+1} - \binom{n+1}{3} - 2n - 1$.

Proof. By Theorem 13, $|\mathbf{I}_n(\neq, \neq, \neq)| = 2^n - n$ and by Theorem 12, $|\mathbf{I}_n(\neq, <, -) = 2^n - n$. From the characterizations of these sets in the proof of Theorems 12 and 13, $\mathbf{I}_n(\neq, \neq, \neq) \cap \mathbf{I}_n(\neq, <, -)$ is the set of inversion sequences with at most one ascent and at most two distinct elements, that is, the set of $e \in \mathbf{I}_n$ satisfying, for some $1 \leq t < a < b \leq n+1$:

$$0 = e_1 = \ldots = e_{a-1}; \quad t = e_a = \ldots = e_{b-1}; \quad 0 = e_b = \ldots = e_n,$$

which is counted by $\binom{n+1}{3}$, together with $(0, 0, \ldots, 0)$. Thus

$$|\mathbf{I}_n(\neq, \neq, \neq) \cap \mathbf{I}_n(\neq, <, -)| = \binom{n+1}{3} + 1$$

and the result follows. □

2.10 187: A049125?

$\boxed{\textbf{187}\text{: } e_i \geq e_k}$

It appears that the number of $e \in \mathbf{I}_n$ avoiding this pattern is given by A049125 in the OEIS, where it is described by David Callan to be the number of ordered trees with n edges in which every non-leaf non-root vertex has at most one leaf child. However, we have not yet proven it. We can prove a characterization of the avoidance set and, from that, derive a 4-parameter recurrence that allows us to check against A049125 for several terms.

Observation 6. *The sequences $e \in \mathbf{I}_n$ having no $i < j < k$ with $e_i \geq e_k$ are those for which $e_i > \max\{e_1, \ldots e_{i-2}\}$ for $i = 3, \ldots, n$. For $e \in \mathbf{I}_n$, this is equivalent to the conditions $e_3 > e_1$ and, for $4 \leq i \leq n$, $e_i > \max\{e_{i-2}, e_{i-3}\}$.*

2.11 193: $\mathbf{S}_n(132, 4312)$, $(n-1)2^{n-2}+1$

Sequence $(n-1)2^{n-2}+1$ appears as A005183 in the OEIS, where Pudwell indicates that it counts $\mathbf{S}_n(132, 4312)$ [15]. We show it also counts $\mathbf{I}_n(<, \geq, \neq)$.

$\boxed{\textbf{193}\text{: } e_i \leq e_j \geq e_k \text{ and } e_i \neq e_k}$

Theorem 17. $|\mathbf{I}_n(<, \geq, \neq)| = (n-1)2^{n-2}$

Proof. Observe that if $e \in \mathbf{I}_n$ has no $i < j < k$ such that $e_i \leq e_j \geq e_k$ and $e_i \neq e_k$ then e must have the form

$$e = (0, \ldots, 0, e_a, 0, \ldots, 0, e_{n-b+1}, e_{n-b+2}, \ldots, e_n\}$$

where $1 \leq a < n+1$ and $b < n-a+2$ and $1 \leq e_a < e_{n-b+1} < e_{n-b+2} < \ldots < e_n < n$.

If $e_a > 1$ then $e = 0 \cdot \sigma_1(e')$ for some $e' \in \mathbf{I}_{n-1}(<, \geq, \neq)$. Otherwise, $e_a = 1$ and e can be obtained by first choosing a b-element subset of $\{2, \ldots, n-1\}$ to place (sorted) in locations $n-b+1, \ldots, n$, and then choosing one of the locations $2, \ldots, n-b$ to be the location a such that $e_a = 1$. Thus the number of sequences containing a 1 is:

$$\sum_{b=0}^{n-2} \binom{n-2}{b}(n-1-b) = n2^{n-3}.$$

This gives the recurrence

$$|\mathbf{I}_n(<, \geq, \neq)| = |\mathbf{I}_{n-1}(<, \geq, \neq)| + n2^{n-3},$$

where $|\mathbf{I}_1(<, \geq, \neq)| = 1$, whose solution is as claimed in the theorem. $\square$

2.12 233: $\mathbf{I}_n(012)$, F_{2n-1}

$\boxed{\textbf{233}\text{: } e_i < e_j < e_k}$

It was shown in [13] that the inversion sequences $e \in \mathbf{I}_n(<, <, -) = \mathbf{I}_n(012)$ are those in which the positive elements of e are weakly decreasing. From that characterization, it was shown that

$$|\mathbf{I}_n(<, <, -)| = |\mathbf{I}_n(012)| = F_{2n-1}.$$

The sequence F_{2n-1} also counts the *Boolean permutations*, given by $\mathbf{S}_n(321, 3412)$ [27, 22].

2.13 304: A229046?

$\boxed{\textbf{304}\text{: } e_i = e_k}$

We derive a recurrence to count the $(-, -, =)$-avoiding inversion sequences. This sequence appears to be sequence A229046 in the OEIS. If true, this would give a combinatorial interpretation of A229046 which so far is defined only by a generating function and summation.

Note that $\mathbf{I}_n(-,-,=)$ is the set of $e \in \mathbf{I}_n$ with at most two copies of any entry and any equal entries must be adjacent.

Let $S_{n,k}$ be the set of $e \in \mathbf{I}_n(-,-,=)$ with k distinct elements; that is, $S_{n,k}$ consists of the inversion sequences $e = (e_1, e_2, \ldots, e_n) \in \mathbf{I}_n(-,-,=)$ such that $|\{e_1, \ldots, e_n\}| = k$. Let $s(n,k) = |S_{n,k}|$.

Theorem 18. *for* $1 \leq k \leq n$*,*

$$s(n,k) = (n-1+k)s(n-1,k-1) + (n-k)s(n-2,k-1),$$

with initial conditions $s(1,1) = s(2,1) = s(2,2) = 1$ *and otherwise* $s(n,k) = 0$ *for* $k = 1$ *or* $n \leq 2$*.*

Proof. Let $A_{n,k}$ be the subset of $S_{n,k}$ consisting of those e in which e_n is unrepeated. Let $B_{n,k} = S_{n,k} \setminus A_{n,k}$. We can extend some $e \in S_{n,k}$ to strings in $S_{n+1,k}$ and $S_{n+1,k+1}$ in the following ways.

If $e \in B_{n,k}$, then $e \cdot n \in A_{n+1,k+1}$. Additionally, if x is one of the $n-k$ values in $\{0, 1, \ldots, n-1\}$ not used in e, then $e \cdot x \in A_{n+1,k+1}$.

If $e \in A_{n,k}$, then $e \cdot n \in A_{n+1,k+1}$. Furthermore, if x is one of the $n-k$ values in $\{0, 1, \ldots, n-1\}$ not used in e, then $e \cdot x \in A_{n+1,k+1}$. Finally, if $e_n = y$, then $e \cdot y \in B_{n+1,k}$. Letting $a(n,k) = |A_{n,k}|$ and $b(n,k) = |B_{n,k}|$, we have

$$\begin{aligned} s(n,k) &= a(n,k) + b(n,k); \\ b(n+1,k) &= a(n,k); \\ a(n+1,k+1) &= (n-k+1)b(n,k) + (n-k+1)a(n,k) \\ &= (n-k+1)s(n,k). \end{aligned}$$

So,

$$\begin{aligned} s(n,k) &= a(n,k) + b(n,k) \\ &= a(n,k) + a(n-1,k) \\ &= (n-k+1)s(n-1,k-1) + (n-k)s(n-2,k-1). \end{aligned}$$

□

Then $|\mathbf{I}_n(-,-,=)| = s(n,1) + \ldots + s(n,n)$. Can we prove that this theorem gives (a refinement) of A229046? Is there a natural description of $\Theta^{-1}(\mathbf{I}_n(-,-,=))$?

2.14 429(A,B,C): Catalan numbers

It is known that for any $\pi \in \mathbf{S}_3$, $|\mathbf{S}_n(\pi)|$ is the Catalan number $C_n = \binom{2n}{n}/(n+1)$ [20, 24]. There are three inequivalent triples of relations $\rho = (\rho_1, \rho_2, \rho_3) \in \{\geq, \leq, <, >, =, \neq, -\}^3$ such that $|\mathbf{I}_n(\rho)| = C_n$. The first corresponds naturally under $\Theta : \mathbf{S}_n \to \mathbf{I}_n$ to a pattern $\pi \in \mathbf{S}_3$.

$\boxed{\textbf{429A}: e_j > e_k}$

Theorem 19. $\mathbf{I}_n(-,>,-) = \Theta(\mathbf{S}_n(213))$.

Proof. Observe that an $e \in \mathbf{I}_n$ has no $i < j < k$ with $e_j > e_k$ if and only if e is weakly increasing. Similarly, it can be checked that $\pi \in \mathbf{S}_n$ avoids 213 if and only if $\Theta(\pi)$ is weakly increasing. □

429B: $e_j \geq e_k$ and $e_i < e_k$

Theorem 20. $|\mathbf{I}_n(-,\geq,<)| = C_n$.

Proof. Observe that some $e \in \mathbf{I}_n$ has no $i < j < k$ with $e_i < e_k$ and $e_j \geq e_k$ if and only if the positive elements of e are strictly increasing.

Let $I(x) = \sum_{n\geq 0} \mathbf{I}_n(-,\geq,<)x^n$. We will show that

$$I(x) = 1 + xI^2(x), \tag{7}$$

which has the solution $\frac{1-\sqrt{1-4x}}{2x}$; recall that this is the generating function for C_n.

Given any $e \in \mathbf{I}_n(-,\geq,<)$, consider the last maximal entry e_t; this is the largest t such that $e_t = t-1$. The string $(e_1, e_2, \ldots, e_{t-1})$ is an element of $\mathbf{I}_{t-1}(-,\geq,<)$. Additionally, it is straightforward to show that the string $\sigma_{1-t}(e_{t+1}, e_{t+2}, \ldots, e_n)$ (where $t-1$ is subtracted from each positive value) is an element of $\mathbf{I}_{n-t}(-,\geq,<)$. Conversely, any element of $\mathbf{I}_n(-,\geq,<)$ with last maximal entry in position t is of the form $e' \cdot (t-1) \cdot \sigma_{1-t}(e'')$ where $e' \in \mathbf{I}_{t-1}(-,\geq,<)$ and $e'' \in \mathbf{I}_{n-t}(-,\geq,<)$. This accounts for the "$xI^2(x)''$ term of equation 7. Since this construction doesn't account for the length 0 inversion sequence, we must also add a "1." □

Alternatively, it can be checked that the following map from $\mathbf{I}_n(-,>,-)$ to $\mathbf{I}_n(-,\geq,<)$ is a bijection. Send $e \in \mathbf{I}_n(-,>,-)$ to e', defined by $e'_i = 0$ if $e_i \in \{e_1, \ldots, e_{i-1}\}$ and otherwise $e'_i = e_i$.

429C: $e_i \geq e_j$ and $e_i \geq e_k$

Although we have not proven it, it appears from our computations that $|\mathbf{I}_n(\geq,-,\geq)| = C_n$. In fact computations suggest, but do not prove, all of the following:

- The number of $e \in \mathbf{I}_n(\geq,-,\geq)$ with $\text{last}(e) = k$ is equal to the number of standard tableaux of shape $(n-1, k)$ (ballot numbers A009766 in OEIS [15]).
- The number of $e \in \mathbf{I}_n(\geq,-,\geq)$ with $\text{asc}(e) = n-1-k$ is equal to the number of ordered trees with n edges and with k interior vertices (non-leaf, non-root) adjacent to a leaf (A108759).
- The number of $e \in \mathbf{I}_n(\geq,-,\geq)$ with $\text{repeats}(e) = k$ is equal to the number of ordered trees with n edges such that exactly k nodes have at least two children. (A091156 in the OEIS [15]). (A *repeat* in e is an i such that $e_i \in \{e_1, \ldots, e_{i-1}\}$.)
- The number of $e \in \mathbf{I}_n(\geq,-,\geq)$ with $\text{dist}(e) = k$ is equal to the number of $\pi \in \mathbf{S}_n(123)$ with $k-1$ descents (A166073 in the OEIS [15]). (The number of *distinct elements* is $\text{dist}(e) = |\{e_1, \ldots, e_n\}|$.)

2.15 523: $\mathrm{S}_n(\bar{3}\bar{1}542)$ and the nexus numbers

In this section we show that inversion sequences avoiding the pattern $(\neq, =, -)$ are equinumerous with permutations avoiding $\bar{3}\bar{1}542$. (A permutation π avoids the pattern $\bar{3}\bar{1}542$ if any occurrence of 542 in π is contained in an occurrence of 31542.) We do this by proving that the $(\neq, =, -)$-avoiding inversion sequences with k distinct entries are counted by the nexus numbers, $(n+1-k)^k - (n-k)^k$.

523: $e_i \neq e_j = e_k$

Observe that the sequences $e \in \mathbf{I}_n$ with no $i < j < k$ satisfying $e_i \neq e_j = e_k$ are those in which the nonzero elements are distinct and once a nonzero element has occurred, at most one more 0 can appear in e. We use this characterization to show that $\mathbf{I}_n(\neq, =, -)$ is counted by the sequence A047970 which counts diagonal sums of nexus numbers [15]. This sequence also counts permutations in $\mathbf{S}_n$ avoiding the barred pattern $\bar{3}\bar{1}542$ (conjectured by Pudwell, and proved by Callan in [11]).

Let $T_{n,k}$ be the set of $e \in \mathbf{I}_n(\neq, =, -)$ with k distinct elements. We prove the following refinement, which gives a new combinatorial interpretation of the *nexus numbers*, $(n+1-k)^k - (n-k)^k$ (see A047969 in the OEIS [15]).

Theorem 21. *For $1 \leq k \leq n$, $|T_{n,k}| = (n+1-k)^k - (n-k)^k$.*

Proof. We count $T_{n,k}$ directly. When $k = 1$, $|T_{n,k}| = 1$ and the result follows. When $k \geq 2$, any $e \in T_{n,k}$ will contain some e_t such that $0 = e_1 = e_2 = \ldots = e_{t-1} < e_t$ and there are no repeated values among $e_t, e_{t+1}, \ldots, e_n$. Therefore, if e has k distinct values, there are two cases: (1) e begins with $n-k+1$ zeros and contains no other zeros; or (2) e begins with $n-k$ zeros and contains one further zero after e_{n-k+1} (which is the first nonzero entry).

For Case (1), the values $e_{n-k+2}, e_{n-k+3}, \ldots, e_n$ must all be distinct and nonzero. So, there are $n-k+1$ possibilities for each, giving $(n-k+1)^{k-1}$ inversion sequences.

For Case (2), e_{n-k+1} must be nonzero, so there are $n-k$ choices for this entry. Additionally, each of $e_{n-k+1}, e_{n-k+2}, e_{n-k+3}, \ldots, e_n$ must be distinct, though zero could appear after e_{n-k+1}. In total, this gives $(n-k)(n-k+1)^{k-1}$ possible inversion sequences. Finally, we must remove any inversion sequence that does not include a zero among $e_{n-k+2}, e_{n-k+3}, \ldots, e_n$; there are $(n-k)^k$ such sequences. As a result, there are $(n-k)(n-k+1)^{k-1} - (n-k)^k$ inversion sequences that are part of Case (2).

Adding Cases (1) and (2), we have $|T_{n,k}| = (n-k+1)^{k-1} + (n-k)(n-k+1)^{k-1} - (n-k)^k = (n-k+1)^k - (n-k)^k$, as desired. $\square$

2.16 772(A,B): set partitions avoiding enhanced 3-crossings?

The avoidance sets for the patterns $(-, \leq, \geq)$ and $(\geq, \geq, -)$ appear to be counted by A108307 but we have not proved this. If true, this gives new simple combinatorial interpretations of A108307 in the OEIS [15]. It was shown by Bousquet-Mélou and Xin that A108307 gives the number of set partitions of $[n]$ avoiding enhanced 3-crossings [7].

We *can* show that there is a bijection that not only proves Wilf equivalence of the patterns 772A and 772B below, but also preserves a number of statistics.

$\boxed{\textbf{772A}\text{: } e_j \leq e_k \text{ and } e_i \geq e_k}$

$\boxed{\textbf{772B}\text{: } e_i \geq e_j \geq e_k}$

Observation 7. *The inversion sequences with no $i < j < k$ such that $e_i \geq e_j \geq e_k$ are precisely those that can be partitioned into two increasing subsequences.*

Proof. Suppose e has such a partition $e_{a_1} < e_{a_2} < \cdots < e_{a_t}$ and $e_{b_1} < e_{b_2} < \cdots < e_{b_{n-t}}$. If there exists $i < j < k$ such that $e_i \geq e_j \geq e_k$, then no two of i, j, k can both be in $\{a_1, \ldots, a_t\}$ or both be in $\{b_1, \ldots, b_{n-t}\}$, so e avoids $(\geq, \geq, -)$. Conversely, if e avoids $(\geq, \geq, -)$, let $a = (a_1, \ldots, a_t)$ be the sequence of left-to-right maxima of e. Then $e_{a_1} < e_{a_2} < \cdots < e_{a_t}$. Consider $i, j \notin \{a_1, \ldots, a_t\}$ where $i < j$. The fact that e_i is not a left-to-right maxima implies there exists some e_s such that $s < i$ and $e_s \geq e_i$. Thus to avoid $(\geq, \geq, -)$, we must have $e_i < e_j$. □

Observation 8. *Let $(e_1, e_2, \ldots, e_n) \in \mathbf{I}_n$. Additionally, for any $i \in [n]$, let $M_i = \max(e_1, e_2, \ldots, e_{i-1})$. Then $e \in \mathbf{I}_n(-, \leq, \geq)$ if and only if for every $i \in [n]$, the entry e_i is a left-to-right maximum, or for every j where $i < j$, $j > i$, we have $e_i > e_j$ or $M_i < e_j$.*

Proof. Let $e \in \mathbf{I}_n$ satisfy the conditions of Observation 8 and, to obtain a contradiction, assume there exist $i < j < k$ such that $e_j \leq e_k$ and $e_i \geq e_k$ (that is $e_j \leq e_k \leq e_i$). Notice that $M_j = \max\{e_1, e_2, \ldots, e_{j-1}\} \geq e_i$. It follows that $M_j \geq e_k \geq e_j$, which contradictions our assumption.

Conversely, if $(e_1, e_2, \ldots, e_n) \in \mathbf{I}_n(-, \leq, \geq)$, consider any e_i. If e_i is not a left-to-right maximum, then there exists some maximum value $M_i = e_s$ such that $s < i$ and $e_s \geq e_i$. Therefore, in order to avoid a 201 pattern, any e_j where $j > i$ must have $e_i > e_j$ or $e_j > M_i = e_s$. □

Theorem 22. *For $n \geq 1$, $|\mathbf{I}_n(\geq, \geq, -)| = |\mathbf{I}_n(-, \leq, \geq)|$.*

Proof. We exhibit a bijection based on the characterizations in Observations 7 and 8.

Given $e \in \mathbf{I}_n(\geq, \geq, -)$, define $f \in \mathbf{I}_n(-, \leq, \geq)$ as follows. Let $e_{a_1} < e_{a_2} < \cdots < e_{a_t}$ be the sequence of left-to-right maxima of e and let $e_{b_1} < e_{b_2} < \cdots < e_{b_{n-t}}$ be the subsequence of remaining elements of e.

For $i = 1, \ldots, t$, set $f_{a_i} = e_{a_i}$. For each $j = 1, 2, \ldots, n-t$, we extract an element of the multiset $B = \{e_{b_1}, e_{b_2}, \ldots, e_{b_{n-t}}\}$ and assign it to $f_{b_1}, f_{b_2}, \ldots, f_{b_{n-t}}$ as follows:

$$f_{b_j} = \max\{k \mid k \in B - \{f_{b_1}, f_{b_2}, \ldots, f_{b_{j-1}}\} \text{ and } k < \max(e_1, \ldots, e_{b_j - 1})\}.$$

By definition, f will satisfy the characterization property in Observation 8 of $\mathbf{I}_n(-, \leq, \geq)$. □

Corollary 1. *All of the following statistics have the same distribution over $\mathbf{I}_n(-, \leq, \geq, -)$ and $\mathbf{I}_n(\geq, \geq, -)$.*

- *the number of locations i such that $e_i = i - 1$;*

- *the largest entry of* e*;*
- *the number of zeros of* e *(there can be at most two in either class);*
- *the number of distinct elements of* e *(and therefore the number of repeats in* e*);*
- *the number of left-to-right maxima of* e*.*

It would also be interesting to relate these statistics on the classes 772A and 772B of inversion sequences to corresponding statistics on set partitions avoiding enhanced 3-crossings.

2.17 877(A,B,C,D): Bell numbers and Stirling numbers

The Bell number B_n is the number of partitions of the set $[n]$ into nonempty blocks. The Stirling number of the second kind, $S_{n,k}$ in the number of partitions of $[n]$ into k blocks.

Among the triples of relations under consideration in this paper, four equivalence classes of patterns have avoidance sets that appear to be counted by the Bell numbers. We have shown this to be true for the classes 877A and 877C, whose inversion sequences have a similar character. We have not confirmed this for the classes 877B and 877D, nor have we confirmed that 877B and 877D are Wilf equivalent, even though our experiments show that there is likely a bijection that preserves several statistics.

Note: $B(n)$ counts permutations avoiding $4\bar{1}32$ and several other barred patterns of length 4, as shown by Callan [9]. Can any of these be related to one of the four patterns 877(A,B,C,D)?

$\boxed{\textbf{877A}\text{: } e_i < e_j = e_k}$

These are the 011-avoiding sequences. In [13] it was observed that these are the $e \in \mathbf{I}_n$ in which the positive elements of e are distinct. It was shown that 011-avoiding sequences in $\mathbf{I}_n$ with k zeros are counted by the Stirling number of the second kind, $S_{n,k}$. (This also appears in [25].) Thus $\mathbf{I}_n(011)$ is counted by the Bell numbers.

$\boxed{\textbf{877B}\text{: } e_i = e_j \geq e_k}$

$\mathbf{I}_n(=, \geq, -)$ is the set of $e \in \mathbf{I}_n$ such that no element appears more than twice and if an element x is repeated, all elements following the second occurrence of x must be larger than x.

From our calculations, it appears that $\mathbf{I}_n(=, \geq, -)$ is counted by the Bell numbers and, in fact, that the number of $e \in \mathbf{I}_n(=, \geq, -)$ with k repeats is given by A124323, the number of set partitions of $[n]$ with k blocks of size larger than 1, but we have not proven this.

$\boxed{\textbf{877C}\text{: } e_i \neq e_j \neq e_k \text{ and } e_i = e_k}$

Observe that these are the $e \in \mathbf{I}_n$ in which only adjacent elements of e can be equal.

Theorem 23. *The number of* $e \in \mathbf{I}_n$ *in which only adjacent elements of* e *can be equal is* B_n*, the* n*th Bell number.*

Proof. It can be checked that the following map from $\mathbf{I}_n(\neq,\neq,=)$ to $\mathbf{I}_n(011)$ is a bijection. Send $e \in \mathbf{I}_n(\neq,\neq,=)$ to e', defined by $e'_i = 0$ if $e_i \in \{e_1,\ldots,e_{i-1}\}$ and otherwise $e'_i = e_i$. □

877D: $e_i \geq e_j$ and $e_i = e_k$

$\mathbf{I}_n(\geq,-,=)$ is the set of $e \in \mathbf{I}_n$ such that no element appears more than twice and if an element x is repeated, all elements between the two occurrences of x must be larger than x (Note the similarity to 877B).

From our calculations, it appears that $\mathbf{I}_n(\geq,-,=)$ is also counted by the Bell numbers. Moreover, it appears that all of the following statistics are equally distributed over the classes 877B and 877D:

- the number of locations i such that $e_i = i-1$;
- the largest entry of e;
- the number of zeros of e (there can be at most two in either class);
- the number of distinct elements of e (and therefore the number of repeats in e, which appears to be A124323).

2.18 924: Central binomial coefficients

924: $e_i > e_j$

Theorem 24. $|\mathbf{I}_n(>,-,-)| = \binom{2n-2}{n-1}$.

Proof. $\mathbf{I}_n(>,-,-)$ is the set of $e \in \mathbf{I}_n$ with $e_1 \leq \ldots \leq e_{n-1}$ (counted by the Catalan number C_{n-1} as shown in Section 2.14) and with e_n chosen arbitrarily from $\{0,\ldots,n-1\}$. Thus

$$|\mathbf{I}_n(>,-,-)| = nC_{n-1} = n\left(\frac{1}{n}\binom{2n-2}{n-1}\right) = \binom{2n-2}{n-1}.$$

□

2.19 1064: A071356?

1064: $e_i > e_j \leq e_k$

These are the inversion sequences $e \in \mathbf{I}_n$ satisfying, for some t such that $1 < t \leq n$,

$$e_1 \leq \ldots \leq e_t > e_{t+1} > \ldots > e_n.$$

Our experiments suggest that these are counted by A071356 in the OEIS, which Emeric Deutsch notes counts the number of underdiagonal lattice paths from $(0,0)$ to the line $x = n$ using only steps $R = (1,0)$, $V = (0,1)$, and $D = (1,2)$ [15].

It also appears from our experiments that the distribution of the number of distinct elements of e is symmetric and unimodal on $\mathbf{I}_n(<,\leq,-)$. The number of $e \in \mathbf{I}_n(<,\leq,-)$ with $\mathrm{dist}(e) = k$ is given in the table below for $n = 1, \ldots 7$:

1					
1	1				
1	4	1			
1	9	9	1		
1	16	38	16	1	
1	25	110	110	25	1
1	36	480	255	36	1

If these observations are true in general, this provides a new simple combinatorial interpretation for A071356 with a natural refinement via a symmetric statistic.

2.20 1265: $\mathbf{S}_n(2143, 3142, 4132)$

$\boxed{\textbf{1265}\text{: } e_i > e_j < e_k}$

Observe that $\mathbf{I}_n(>,<,-)$ is the set of $e \in \mathbf{I}_n$ satisfying, for some t with $1 < t \leq n$,

$$e_1 \leq e_2 \leq \ldots \leq e_t > e_{t+1} \geq \ldots \geq e_n.$$

Our experiments suggested that $\mathbf{I}_n(>,<,-)$ is counted by A0333321, which counts $\mathbf{S}_n(2143, 3142, 4132)$, as well as permutations avoiding several other triples of 4-permutations. Burstein and Stromquist confirmed this by recognizing a natural bijection between $\mathbf{S}_n(2143, 3142, 4132)$ and $\mathbf{I}_n(>,<,-)$ in [8]. Their theorem is as follows. Recall from Section 1 that *invcode* : $\mathbf{S}_n \to \mathbf{I}_n$ is the reverse of the Lehmer code.

Theorem 25 (Burstein, Stromquist [8])**.** *For $n \geq 1$, invcode$(\mathbf{S}_n(2143, 3142, 4132)) = \mathbf{I}_n(>,<,-)$.*

From Section 1, *invcode*$(\pi) = e$ if and only if $e = \Theta((\pi^C)^R)$, giving the following.

Corollary 2. $\Theta(\mathbf{S}_n(2143, 3142, 3241)) = \mathbf{I}_n(>,<,-)$.

2.21 1347: $\mathbf{S}_n(4123, 4132, 4213)$?

$\boxed{\textbf{1347}\text{: } e_i > e_j \text{ and } e_i \leq e_k}$

Our calculations suggest that $\mathbf{I}_n(>,-,\leq)$ is counted by A106228 in the OEIS, which was recently shown to count $\mathbf{S}_n(4123, 4132, 4213)$ by Albert, Homberger, Pantone, Shar and Vatter [3]. We have not been able to confirm that our avoidance sequence is A106228.

2.22 1385: $\mathbf{I}_n(000)$ Euler up/down numbers

$\boxed{\textbf{1385}\text{: } e_i = e_j = e_k}$

$\mathbf{I}_n(=,=,-)$ is the set of inversion sequences avoiding the pattern '000'. It was shown in [13] that $|\mathbf{I}_n(000)| = E_{n+1}$, where E_n is the Euler up/down number which counts the number of $\pi \in \mathbf{S}_n$ such that $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \cdots$. The proof was via a bijection with n-vertex 0-1-2 increasing trees, which are also counted by E_{n+1}.

Another family of permutations counted by E_{n+1} is the number of simsun permutations of $[n]$, introduced by Simion and Sundaram [26]. A simsun permutation is one with no double descents, even after the removal of the elements $\{n, n-1, \ldots, k\}$ for any k. It was shown in [13] that the number of $e \in \mathbf{I}_n(000)$ with $n-k$ distinct elements is the number of simsun permutations of n with k descents. The method of proof was to show that they satisfy the same recurrence.

It would nice to have a natural bijection between $\mathbf{I}_n(000)$ and up-down (or down-up) permutations of $[n+1]$. For example, our calculations suggest that the number of $e \in \mathbf{I}_n(000)$ with $e_n = k-1$ is the number of down-up permutations π of $[n+1]$ with $\pi_1 = k+1$.

2.23 1694: $\mathbf{I}_n(102)$

$\boxed{\textbf{1694}\text{: } e_i > e_j \text{ and } e_i < e_k}$

As mentioned in [13], our calculations suggested that $\mathbf{I}_n(>,-,<)$ is counted by A200753 in the OEIS [15], a sequence defined by the generating function

$$A(x) \quad = \quad 1 + (x - x^2)(A(x))^3. \tag{8}$$

This was confirmed by Mansour and Shattuck in [21] where they derive an explicit formula for $|\mathbf{I}_n(102)|$.

Theorem 26 (Mansour-Shattuck [21])**.** *The generating function* $\sum_{n\geq 0} |\mathbf{I}_n(102)| x^n$ *satisfies* (8).

It would be interesting to find a direct combinatorial argument.

2.24 1806(A,B,C,D): large Schröder numbers

The *large Schröder number* R_n is the number of Schröder n-paths; that is, the number of paths in the plane from $(0,0)$ to $(2n,0)$ never going below the x-axis, and using only the steps $(1,1)$ (up), $(1,-1)$ (down) and $(2,0)$ (flat).

In the area of pattern avoiding permutations, R_{n-1} counts the separable permutations $\mathbf{S}_n(2413, 3142)$, as well as $\mathbf{S}_n(\alpha, \beta)$ for many other pairs (α, β) of patterns of length 4 [18]. We have four inequivalent triples of relations whose avoidance sets are counted by the large Schröder numbers, two of which (1806B and 1806D) correspond in natural ways to a pair of patterns of length 4.

$\boxed{\textbf{1806A}\text{: } e_j > e_k \text{ and } e_i < e_k}$

These are the sequences avoiding 021. It was shown in [13] that $e \in \mathbf{I}_n$ avoids 021 if and only if its positive entries are weakly increasing. It was also proven in [13] that $|\mathbf{I}_n(021)| = R_{n-1}$.

The following refinements were shown:

- The number of 021-avoiding inversion sequences e in $\mathbf{I}_n$ with k positions i such that $e_i = i-1$ is equal to the number of Schröder $(n-1)$-paths with $k-1$ initial up steps.
- The number of 021-avoiding inversion sequences e in $\mathbf{I}_n$ with k zeros is equal to the number of Schröder $(n-1)$-paths with $k-1$ peaks (or $k-1$ flat steps).

It was also shown in [13] that the ascent polynomial for $\mathbf{I}_n(021)$ is palindromic and corresponds to sequence A175124 in the OEIS.

Is there a nice characterization of $\Theta^{-1}(\mathbf{I}_n(021))$?

$\boxed{\textbf{1806B}\text{: } e_i > e_j \text{ and } e_i \geq e_k}$

It is known that $\mathbf{S}_n(2134, 2143)$ is counted by R_{n-1} [18]. This is a member of "Class VI" in Kitaev's book [16]; we use this fact to count $\mathbf{I}_n(>, -, \geq)$.

Theorem 27. $|\mathbf{I}_n(>, -, \geq)| = R_{n-1}$.

Proof. We show that $\Theta(\mathbf{S}_n(2134, 2143)) = \mathbf{I}_n(>, -, \geq)$.

Let $e \in \mathbf{I}_n$ satisfy $e_i > e_j$ and $e_i \geq e_k$ for some $i < j < k$. Let $\pi = \Theta^{-1}(e)$. Then $\min\{\pi_j, \pi_k\} > \pi_i$ and, since $e_i > e_j$, there must exist $a < i$ such that both $\pi_a > \pi_i$ and $\min\{\pi_j, \pi_k\} > \pi_a$. Thus $\pi_a\pi_i\pi_j\pi_k$ forms either a 2134 or a 2143.

Conversely, suppose, for some $\pi \in \mathbf{S}_n$, that $\pi_a\pi_i\pi_j\pi_k$ is one of the patterns 2134 or 2143 and let $e = \Theta(\pi)$. Let j' be the smallest index larger than i for which $\pi_{j'} > \pi_a$. Then $\pi_{i+1}, \ldots, \pi_{j'-1}$ are all smaller than $\pi_a < \pi_{j'}$ and so $e_i > e_{j'}$. Let k' be the smallest index larger than j' such that $\pi_{k'} > \pi_a$. Then, with the possible exception of $\pi_{j'}$, all of $\pi_{i+1}, \ldots, \pi_{k'-1}$ are smaller than $\pi_{k'}$ (since these entries are necessarily smaller than π_a). In addition, since $\pi_i < \pi_a < \pi_{k'}$, we have $e_i \geq e_{k'}$. Thus e has the pattern $(>, -, \geq)$. $\square$

Note. From our calculations, it appears that the ascent polynomial for $\mathbf{I}_n(>, -, \geq)$ is the same as that for 1806A, which was palindromic. Since Θ sends descents to ascents, if true, this would imply that the descent polynomial for $\mathbf{S}_n(2134, 2143)$ is palindromic. This is not true in general for permutations avoiding pairs of patterns of length 4, even those counted by the large Schröder numbers. For example, it is not true of $\mathbf{S}_n(1234, 2134)$ or $\mathbf{S}_n(1324, 2314)$. However Fu, Lin, and Zeng have recently shown that the descent polynomial for the separable permutations $\mathbf{S}_n(2413, 3142)$ is γ-positive and therefore palindromic [14].

$\boxed{\textbf{1806C}\text{: } e_i \geq e_j \text{ and } e_i > e_k}$

Theorem 28. $|\mathbf{I}_n(\geq, -, >)| = R_{n-1}$.

Proof. We will construct a generating function for $\mathbf{I}_n(\geq,-,>)$.

Let $E(x)=\sum_{i=1}^{\infty}|\mathbf{I}_n(\geq,-,>)|x^n$. We will show that $E(x)$ satisfies

$$E(x)=x+xE(x)+E^2(x),$$

whose solution is

$$E(x)=\frac{1-x-\sqrt{x^2-6x+1}}{2}.$$

This implies that $|\mathbf{I}_n(\geq,-,>)|$ is the $(n-1)$th large Schröder number.

Let $e=e_1e_2\dots e_n\in\mathbf{I}_n(\geq,-,>)$. Let e_t be the latest maximal entry of e; that is, $\max\{i\mid e_i=i-1\}$. If $t=1$, then either $e=(0)$ or $e=(0,f_1,f_2,\dots,f_{n-1})$ for some $(f_1,f_2,\dots f_{n-1})\in\mathbf{I}_{n-1}(\geq,-,>)$.

Now consider the case where $t>1$. Notice that $e_{t+1}\leq t-1$ since e_{t+1} cannot be maximal. This implies that $(e_1,e_2,\dots,e_{t-1},e_{t+1})$ is an inversion sequence of length t. Furthermore, it is straightforward to show that $(e_1,e_2,\dots,e_{t-1},e_{t+1})\in\mathbf{I}_n(\geq,-,>)$.

Additionally, for all e_j where $j>t+1$, we must have $e_j\geq t-1$; if this is not the case, and there exists some j where $e_j<t-1$, then we have $t<t+1<j$ where $e_t\geq e_{t+1}$ and $e_t>e_j$. Therefore $(e_t-t+1,e_{t+2}-t+1,\dots,e_n-t+1)\in\mathbf{I}_{n-t}(\geq,-,>)$.

Conversely, for any sequences $(e_1,e_2,\dots,e_t)\in\mathbf{I}_t(\geq,-,>)$ and $(f_1,f_2,\dots,f_{n-t})\in\mathbf{I}_{n-t}(\geq,-,>)$, we can construct the inversion sequence

$$(e_1,e_2,\dots,e_{t-1},f_1+t-1,e_t,f_2+t-1,f_3+t-1,\dots,f_{n-t}+t-1).$$

It is straightforward to show that this inversion sequence is in $\mathbf{I}_n(\geq,-,>)$ and the last maximal entry is in the t-th position. ∎

1806D: $e_i\geq e_j\neq e_k$ and $e_i\geq e_k$

In [18], it was shown that $|\mathbf{S}_n(4321,4312)|=R_{n-1}$. We can prove that $|\mathbf{I}_n(\geq,\neq,\geq)|$ is also enumerated by R_{n-1} by constructing a bijection between $\mathbf{S}_n$ and $\mathbf{I}_n$ that restricts to a bijection between $\mathbf{S}_n(4321,4312)$ and $\mathbf{I}_n(\geq,\neq,\geq)$. This bijection is useful for another class of inversion sequences: it will be used later to prove results related to $\mathbf{I}_n(>,\neq,>)$ (classified as 3720).

Definition 1. *Let $\pi\in S_n$ and define $\phi(\pi)=e_1e_2\dots e_n\in\mathbf{I}_n$ as follows, starting with e_n and defining entries in reverse order.*

1. $e_n=\pi_n-1$
2. *For* $1\leq i<n$,
   (a) *If $\pi_i\leq i$, then $e_i=\pi_i-1$.*
   (b) *Otherwise, if π_i is the k-th largest element of $\{\pi_1,\dots,\pi_i\}$ then e_i is the k-th smallest element of the set $\{e_{i+1},...,e_n\}$..*

Lemma 2. *For $\pi\in\mathbf{S}_n$, $\phi(\pi)\in\mathbf{I}_n$.*

Proof. To show that $\phi(\pi) \in \mathbf{I}_n$, we need to prove that $0 \leq e_i \leq i-1$ for every $i \in [n]$. We will use an inductive argument, starting with e_n, to show this. We defined $e_n = \pi_n - 1$; since $1 \leq \pi_n \leq n$, it follows that $0 \leq e_n \leq n-1$, as desired. Now consider e_i and assume that for all e_j among $e_{i+1}e_{i+2}\dots e_n$, $0 \leq e_j \leq j-1$. If $\pi_i \leq i$, then $e_i = \pi_i - 1$ and it immediately follows that $0 \leq e_i \leq i-1$.

If instead $\pi_i > i$, assume that π_i is in the k-th largest element of $\{\pi_1, \dots, \pi_i\}$. Notice that each value of $\{e_{i+1}, ..., e_n\}$ corresponds to an entry π_j where $i < j$ and $\pi_j \leq j$ (any entry π_j where $\pi_j > j$ will repeat a value). So, there are $n - \pi_i - k + 1$ entries π_j such that $i < j$ and $\pi_j > \pi_i$; in turn, this implies that there are $(n-i) - (n - \pi_i - k + 1) = \pi_i - (i+1) + k$ entries π_j such that $i < j$ and $\pi_j < \pi_i$. At a maximum, $\pi_i - (i+1)$ of these entries are greater than i; this leaves k entries occurring after π_i that are less than or equal to i. Each of these entries corresponds to a value in $\{e_{i+1}, ..., e_n\}$ that is less than or equal to $i-1$. It follows, that $e_i \leq i-1$, as desired. □

Lemma 3. *$\phi : \mathbf{S}_n \to \mathbf{I}_n$ is a bijection.*

Proof. Let $e = e_1 e_2 \dots e_n \in \mathbf{I}_n$. We can define the inverse image of e, $\phi^{-1}(e) = \pi_1 \pi_2 \dots \pi_n$ in reverse order, starting with π_n so that $\pi_n = e_n + 1$. For $1 \leq i < n$, if $e_i \neq e_j$ for all j where $i < j \leq n$, then $\pi_i = e_i + 1$; otherwise, if e_i is the k-th smallest value of $\{e_{i+1}, \dots, e_n\}$, π_i is the k-th largest value of $[n]$ that does not appear among $\pi_{i+1}, \dots, \pi_n$. □

It is interesting to note that for any $\pi \in \mathbf{S}_n$, $\mathrm{exc}(\pi) = \mathrm{repeats}(\phi(\pi))$, where $\mathrm{exc}(\pi)$ is the number of positions i such that $\pi_i > i$.

Now, we show that ϕ restricts to a bijection between $\mathbf{S}_n(4321, 4312)$ and $\mathbf{I}_n(\geq, \neq, \geq)$ by proving the following:

Theorem 29. $\phi(\mathbf{S}_n(4321, 4312)) = \mathbf{I}_n(\geq, \neq, \geq)$

Proof. Consider some $\pi \in \mathbf{S}_n$ that contains an occurrence of 4321 or 4312. So, there exists some $a < i < j < k$ such that $\pi_a \pi_i \pi_j \pi_k$ form a 4321 of 4312 pattern. We will show that there exists an occurrence of the pattern $(\geq, \neq, \geq)$ in $\phi(\pi) = e$.

We must consider two cases. If $\pi_i \leq i$, then $j > \pi_i > \pi_j$ and $k > \pi_i > \pi_k$. Therefore, $e_i = \pi_i - 1$, $e_j = \pi_j - 1$ and $e_k = \pi_k - 1$. So, since $\pi_i > \max\{\pi_j, \pi_k\}$ and $\pi_j \neq \pi_k$, e_i, e_j, e_k forms an occurrence of $(\geq, \neq, \geq)$.

Now assume $\pi_i > i$. Recall that e_i is the t-th smallest element of $\{e_{i+1}, \dots, e_n\}$ if π_i is the t-th largest element of $\{\pi_1, \pi_2, \dots, \pi_i\}$. Since π_a is larger than and occurs before π_i, we know that t is at least 2. So, if $e_{j'}$ and $e_{k'}$ are the two smallest distinct values in the set $\{e_{i+1}, \dots, e_n\}$, we are guaranteed that $e_i \geq e_{j'}$ and $e_i \geq e_{k'}$; so, $e_i, e_{j'}, e_{k'}$ form the pattern $(\geq, \neq, \geq)$. □

Our calculations suggest that the ascent polynomial for the inversion sequences in $\mathbf{I}_n(\geq, \neq, \geq)$ is the same as the (symmetric) ascent polynomial for 1806A. It also appears that the number of these inversion sequences with k "repeats" is counted by A090981, the number of Schröder paths with k ascents.

2.25 2074: Baxter permutations?

2074: $e_i \geq e_j \geq e_k$ and $e_i > e_k$

From our calculations, it appears that the avoidance sequence for the pattern $(\geq, \geq, >)$ is A001181 in the OEIS, which counts the Baxter permutations, a result of Chung et al in [12]. A *Baxter permutation* π is one that avoids the vincular patterns 3-14-2 and 2-41-3, that is, there is no $i < j < k$ such that $\pi_j < \pi_k < \pi_i < \pi_{j+1}$ or $\pi_j > \pi_k > \pi_i > \pi_{j+1}$. We have not yet proven this.

Note that the Baxter permutations contain the separable permutations $\mathbf{S}_n(3142, 2413)$ which are counted by the large Schröder numbers. Similarly, $\mathbf{I}_n(\geq, \geq, >)$ contains the inversion sequences $\mathbf{I}_n(\geq, -, >)$ (which define class 1806C) which are also counted by the large Schröder numbers. It would be nice to find a bijection between $(\geq, \geq, >)$-avoiding inversion sequences and Baxter permutations that restricts to a bijection between $(\geq, -, >)$-avoiding inversion sequences and separable permutations.

2.26 2549(A,B,C): A098746, $\mathbf{S}_n(4231, 42513)$?

In [1], Albert et al showed that

$$|\mathbf{S}_n(4231, 42513)| = \sum_{i=0}^{n} \frac{n-i}{2i+n} \binom{2i+n}{i},$$

which is sequence A098746 in the OEIS [15].

It appears from our calculations that the three inequivalent patterns below have the same avoidance sequence A098746. We will show at least that the classes 2549A and 2549C are Wilf equivalent.

2549A: $e_i > e_j$ and $e_i > e_k$

2549B: $e_i > e_j \neq e_k$ and $e_i \geq e_k$

2549C: $e_i \geq e_j \neq e_k$ and $e_i > e_k$

Theorem 30. *The patterns* $(>, -, >)$ *and* $(\geq, \neq, >)$*, defining classes* 2549A *and* 2549C *respectively, are Wilf equivalent.*

The proof relies on Theorem 33 in the next section and will be given there.

2.27 2958(A,B,C,D): Plane permutations $\mathbf{S}_n(21\bar{3}54)$?

Although it has not yet been proven, it appears from our experiments that four different equivalence classes of patterns have avoidance sets equinumerous with the *plane permutations*, $\mathbf{S}_n(21\bar{3}54)$. These are permutations in which every occurrence of the pattern 2154 is contained in an occurrence of 21354.

In response to a challenge of Bousquet-Mélou and Butler in [6] to find a formula for the number of plane permutations, David Bevan, in his thesis [4], derived a functional equation. Although he could not solve it, he used it to extract coefficients and compute the number of plane permutations of n through $n = 37$. This sequence appears as A117106 in the OEIS [15].

In this section we show that the classes 2958(B,C,D) are Wilf equivalent. We give a characterization of $\mathbf{I}_n(100, 210)$, the avoidance set for 2958B, and use it to derive a recurrence to count $\mathbf{I}_n(100, 210)$. The recurrence allows us to compute up to $n = 200$ in a few minutes. Our results agree with Bevan's calculation of the number of plane permutations up to $n = 37$ and agree up through $n = 200$ with the conjectured recurrence of van Hoeij (see A117106 in the OEIS [15]) and Bevan's related Conjecture 13.3 [4].

$\boxed{\textbf{2958A}\text{: } e_j < e_k \text{ and } e_i \geq e_k}$

David Bevan has verified that inversion sequences avoiding the pattern $(-, <, \geq)$ (class 2958A) are equinumerous with plane partitions through $n = 36$ [4].

$\boxed{\textbf{2958B}\text{: } e_i > e_j \geq e_k}$

Note that $\mathbf{I}_n(>, \geq, -) = \mathbf{I}_n(210, 100)$. It was shown in [13] that the inversion sequences avoiding 210 (class 4306A) have a useful characterization.

Define a *weak left-to-right maximum* in an inversion sequence e to be a position j such that $e_i \leq e_j$ for all $i \in [j-1]$.

For $e \in \mathbf{I}_n$, let $a_1 < a_2 < \cdots < a_t$ be the sequence of weak left-to-right maxima of e. Let $b_1 < b_2 < \cdots < b_{n-t}$ be the sequence of remaining indices in $[n]$. Let $e^{top} = (e_{a_1}, e_{a_2}, \ldots, e_{a_t})$ and $\mathrm{top}(e) = e_{a_t}$. Let $e^{bottom} = (e_{b_1}, e_{b_2}, \ldots, e_{b_{n-t}})$ and $\mathrm{bottom}(e) = e_{b_{n-t}}$. If every entry of e is a weak left-to-right maxima, then e^{bottom} is empty and we set $\mathrm{bottom}(e) = -1$.

Observation 9 ([13])**.** *The inversion sequence e avoids 210 if and only if e^{top} and e^{bottom} are weakly increasing sequences.*

This characterization was used to derive a recurrence.

Theorem 31 ([13])**.** *Let $T_{n,a,b}$ be the number of $e \in \mathbf{I}_n(201)$ with $\mathrm{top}(e) = a$ and $\mathrm{bottom}(e) = b$. Then*

$$T_{n,a,b} = \sum_{i=-1}^{b} T_{n-1,a,i} + \sum_{j=b+1}^{a} T_{n-1,j,b}, \tag{9}$$

with initial conditions $T_{n,a,b} = 0$ if $a \geq n$ and $T_{n,a,-1} = \frac{n-a}{n}\binom{n-1+a}{a}$.

We use the same approach to enumerate $\mathbf{I}_n(210, 100)$.

Observation 10. *The inversion sequence e avoids both 210 and 100 if and only if e^{top} is weakly increasing and and e^{bottom} is strictly increasing.*

Theorem 32. *Let $S_{n,a,b}$ be the number of $e \in \mathbf{I}_n(201, 100)$ with $\mathrm{top}(e) = a$ and $\mathrm{bottom}(e) = b$. Then*

$$S_{n,a,b} = \sum_{i=-1}^{b-1} S_{n-1,a,i} + \sum_{j=b+1}^{a} S_{n-1,j,b},$$

with initial conditions $S_{n,a,b} = 0$ *if* $a \geq n$ *and* $S_{n,a,-1} = \frac{n-a}{n}\binom{n-1+a}{a}$.

From Theorem 32 we get:

$$|\mathbf{I}_n(210, 100)| \;=\; \sum_{a=0}^{n-1}\sum_{b=-1}^{a-1} S_{n,a,b} \;=\; \frac{1}{n+1}\binom{2n}{n} + \sum_{a=0}^{n-1}\sum_{b=0}^{a-1} S_{n,a,b}. \tag{10}$$

We conjecture that this is the number of plane permutations in $\mathbf{S}_n$. The first 50 values are:

```
1, 2, 6, 23, 104, 530, 2958, 17734, 112657, 750726, 5207910, 37387881, 276467208, 2097763554,
16282567502, 128951419810, 1039752642231, 8520041699078, 70840843420234, 596860116487097,
5089815866230374, 43886435477701502, 382269003235832006, 3361054683237796748, 29808870409714471629,
266506375018970260798, 2400594944788679086246, 21775140746921451807813, 198809340676892441106504,
1826282268703405468306242, 16872997989167207310526350, 156733628752383523517966154,
1463331781095592078766081067, 13728102975517134576035012166, 129375450056444890453148475138,
1224510110939244929853284519565, 11637123841882863436079893510908,
111022911072651911246209688239494, 1063116093852524285500741644638322,
10215821273522500820260330677099486, 98496126104298745718970566070156495,
952690698366216517516116694690619898, 9242930766428561890110747307163780874,
89936218036703072446114434758174384847, 877551360693740954054799745948902842226,
8585641010767276382316313171527991801182, 84214629478944993778155385601015021260758,
828079895548521670881282794955097090105284, 8161776754206403112036885185391701912831107,
80627888029113149463127387185375117917090526.
```

We will next show that the class 2958B is Wilf equivalent to 2958C.

$\boxed{\textbf{2958C}\text{: } e_i \geq e_j > e_k}$

Theorem 33. *The patterns* $(-, <, \geq)$ *and* $(\geq, >, -)$*, defining classes 2958B and 2958C respectively, are Wilf equivalent.*

Proof. The avoidance sets for classes 2958B and 2958C are $\mathbf{I}_n(100, 210)$ and $\mathbf{I}_n(110, 210)$, repspectively. We describe a bijection

$$\alpha : \mathbf{I}_n(110, 210) \to \mathbf{I}_n(100, 210).$$

For $e \in \mathbf{I}_n(110, 210)$, let $\alpha(e) = e' = (e'_1, \ldots, e'_n)$ where for $1 \leq j \leq n$

$$e'_j = \begin{cases} \max\{e_1, \ldots, e_j\} & \text{if } \ e_j = e_k \text{ for some } k > j \\ e_j & \text{otherwise} \end{cases} \tag{11}$$

Note that for $1 \leq j \leq n$,

$$e'_j \;\leq\; \max\{e_1, \ldots, e_j\} \;=\; \max\{e'_1, \ldots, e'_j\} \tag{12}$$

and if $e'_j \neq e_j$ then

$$e'_j \;=\; \max\{e_1, \ldots, e_j\} \;\geq e'_i \text{ for } 1 \leq i < j. \tag{13}$$

To see that e' avoids 100, suppose $e'_i > e'_j = e'_k$ for some $i < j < k$. Then by (13), we must have $e'_j = e_j$ and $e'_k = e_k$. But from the definition of α (11), if $e_j = e_k$ where $j < k$ then $e'_j = \max\{e_1, \ldots, e_j\} \geq e'_i$, a contradiction.

To see that e' avoids 210, suppose that $e'_i > e'_j > e'_k$ for some $i < j < k$. Again by (13), $e'_j = e_j$ and $e'_k = e_k$. Since e avoids 210, $e'_i \neq e_i$ so $e'_i = \max\{e_1, \ldots, e_i\} = e_s$ for some $s \in [i]$. But then $e_s > e_j > e_k$ is a 210 in e.

Thus $e' = \alpha(e) \in \mathbf{I}_n(100, 210)$. To show that α is a bijection, we define its inverse β.

First we make an observation. Consider some $e \in \mathbf{I}_n(110, 210)$ and an entry e_j such that $e'_j \neq e_j$ in $\alpha(e) = e'$. This implies that there exists some index k with $j < k$ and $e_j = e_k = m$ for some value m. Additionally, it must be the case that $m < M = \max\{e_1, \ldots, e_{j-1}\} = e_i$ where $i \in [j-1]$ (else, $e'_j = m$). Then, since e avoids 210, we must have $m = \min\{e_j, \ldots, e_n\}$. Thus, for $e' = \alpha(e)$ we have

$$e'_i = e_i = M = e'_j > e_j = m = \min\{e_j, \ldots, e_n\} = \min\{e'_j, \ldots, e'_n\}.$$

Thus, we can reconstruct e from e' by defining $\beta : \mathbf{I}_n(100, 210) \to \mathbf{I}_n(110, 210)$ as follows.

For $e \in \mathbf{I}_n(100, 210)$, let $\beta(e) = e' = (e'_1, \ldots, e'_n)$ where for $1 \leq j \leq n$

$$e'_j = \begin{cases} \min\{e_j, \ldots, e_n\} & \text{if } \ e_i = e_j \text{ for some } i < j \\ e_j & \text{otherwise} \end{cases}$$

Then $\beta(\alpha(e)) = e$. We can check similarly that $\beta(e) \in \mathbf{I}_n(110, 210)$ and $\alpha(\beta(e)) = e$. □

We now return to the proof of the Wilf equivalence of the classes 2549A and 2549C from the previous section.

Proof of Theorem 30

Observe that 2549C $\preceq$ 2958C and 2549A $\preceq$ 2958B in the following sense:

$$\text{2549C} : \mathbf{I}_\text{n}(\geq, \neq, >) = \mathbf{I}_\text{n}(110, 210, 201) \subseteq \mathbf{I}_\text{n}(110, 210) \subseteq \mathbf{I}_\text{n}(\geq, >, -) : \text{2958C}$$

$$\text{2549A} : \mathbf{I}_\text{n}(>, -, >) = \mathbf{I}_\text{n}(100, 210, 201) \subseteq \mathbf{I}_\text{n}(100, 210) \subseteq \mathbf{I}_\text{n}(>, \geq, -) : \text{2958B}$$

We check that the mapping α (11) restricts to a bijection between inversion sequences in class 2549C and in class 2549A.

Let $e \in \mathbf{I}_n(110, 210)$ and let $e' = \alpha(e)$. Suppose e avoids the pattern 201, but that for some $i < j < k$, $e'_i > e'_j < e'_k$ and $e'_i > e'_k$. Then by (13), $e'_j = e_j$ and $e'_k = e_k$. But, since e avoids 201, $e'_i \neq e_i$. Then, by definition of α, there is an $s \in [i-1]$ such that $e'_i = e_s$. But then $e_s e_j e_k$ forms a 201 pattern in e, a contradiction.

2958D: $e_j \leq e_k$ and $e_i > e_k$

Theorem 34. *The patterns* $(>, \geq, -)$ *and* $(-, \leq, >)$*, defining classes* 2958B *and* 2958D *respectively, are Wilf equivalent.*

Proof. The inversion sequences in classes 2958B and 2958D are defined by $\mathbf{I}_n(210, 100)$ and $\mathbf{I}_n(201, 100)$, respectively. It was shown in [13], Theorem 5, that the following gives a bijection from $\mathbf{I}_n(210)$ to $\mathbf{I}_n(201)$. It can be checked that this mapping preserves 100-avoidance and therefore restricts to a bijection from $\mathbf{I}_n(210, 100)$ to $\mathbf{I}_n(201, 100)$.

Given $e \in \mathbf{I}_n(210)$, define $f \in \mathbf{I}_n(201)$ as follows. Let $e_{a_1} \leq e_{a_2} \leq \cdots \leq e_{a_t}$ be the sequence of weak left-to-right maxima of e and let $e_{b_1}, e_{b_2}, \cdots, e_{b_{n-t}}$ be the subsequence of remaining elements of e. Since e avoids both 210 and100, $e_{b_1} < e_{b_2} < \cdots < e_{b_{n-t}}$.

For $i = 1, \ldots, t$, set $f_{a_i} = e_{a_i}$. For each $j = 1, 2, \ldots, n-t$, we extract an element of the multiset $B = \{e_{b_1}, e_{b_2}, \ldots, e_{b_{n-t}}\}$ and assign it to $f_{b_1}, f_{b_2}, \ldots, f_{b_{n-t}}$ as follows:

$$f_{b_j} = \max\{k \mid k \in B - \{f_{b_1}, f_{b_2}, \ldots, f_{b_{j-1}}\} \text{ and } k < \max(e_1, \ldots, e_{b_j - 1})\}.$$

□

This is the same mapping that was used in Section 2.16 to show that the patterns 772A and 772B are Wilf equivalent. Note that

$$772\text{B} : \mathbf{I}_n(210, 110, 100, 000) \subseteq \mathbf{I}_n(210, 100) : 2958\text{B};$$

$$772\text{A} : \mathbf{I}_n(201, 101, 100, 000) \subseteq \mathbf{I}_n(201, 100) : 2958\text{D}.$$

Note: Lara Pudwell in her thesis [23] found some other Wilf equivalent barred permutation patterns of length 5 that may be helpful to settle the open questions in this subsection.

2.28 3207: $\mathbf{I}_n(101)$, $\mathbf{I}_n(110)$

3207A: $e_j < e_k$ and $e_i = e_k$

3207B: $e_i = e_j > e_k$

It was shown in [13] that both $\mathbf{I}_n(<, -, =) = \mathbf{I}_n(101)$ and $\mathbf{I}_n(=, >, -) = \mathbf{I}_n(110)$ are counted by the sequence A113227 in the OEIS [15], where it is said to count $\mathbf{S}_n$(1-23-4). $\mathbf{S}_n$(1-23-4) is the set of permutations with no $i < j < k < \ell$ such that $\pi_i < \pi_j < \pi_k < \pi_\ell$ and $k = j + 1$.

It was proven by David Callan in [10], that $\mathbf{S}_n$(1-23-4) is in bijection with increasing ordered trees with $n+1$ vertices whose leaves, taken in preorder, are also increasing. He showed that if $u_{n,k}$ is the number of such trees with $n+1$ vertices in which the root has k children then

$$u_{n,k} = u_{n-1,k-1} + k \sum_{j=k}^{n-1} u_{n-1,j} \tag{14}$$

with initial conditions $u_{0,0} = 1$ and $u_{n,k} = 0$ if $k > n$, or $n > 0$ and $k = 0$.

It was shown in [13] that the number of $e \in \mathbf{I}_n(101)$ with exactly k zeros is $u_{n,k}$, as is the number of $e \in \mathbf{I}_n(110)$ with exactly k zeros. As a consequence, 101 and 110 are Wilf equivalent and both avoidance sets are counted by A113227.

2.29 3720: Quadrant Marked Mesh Patterns

In this section we prove that $\mathbf{I}_n(>, \neq, >)$ is counted by the sequence A212198 in the OEIS [15] where it is said to count permutations avoiding a particular marked mesh pattern.

3720: $e_i > e_j \neq e_k$ and $e_i > e_k$

In [17], Kitaev and Remmel introduced the idea of quadrant marked mesh patterns, a definition of which is given below.

Definition 2. *Let $\pi = \pi_1\pi_2 \dots \pi_n \in S_n$. Consider the graph of π, $G(\pi)$, consisting of the points (i, π_i) for all $i \in [n]$. The entry π_i is said to* match *the quadrant marked mesh pattern $MMP(a, b, c, d)$, where a, b, c, d are nonnegative integers, if in $G(\pi)$ there are at least a points to the northeast of (i, π_i), at least b points to the northwest of (i, π_i), at least c points to the southwest of (i, π_i), and at least d points to the southeast of (i, π_i).*

Let $S_n(MMP(a, b, c, d))$ denote the set of permutations of length n where no π_i matches $MMP(a, b, c, d)$. We will prove that $|\mathbf{S}_n(MMP(0, 2, 0, 2))| = |\mathbf{I}_n(>, \neq, >)|$ for all n. By symmetry established in [17], this implies that $|\mathbf{I}_n(>, \neq, >)| = |\mathbf{S}_n(MMP(2, 0, 2, 0))|$. In our proof, we make use of the bijection ϕ from Section 2.24, whose definition was given in Definition 1. Specifically, we can prove the following:

Theorem 35. *For all n, $\phi(\mathbf{S}_n(MMP(0, 2, 0, 2))) = \mathbf{I}_n(>, \neq, >)$.*

Proof. This proof is very similar to the proof of Theorem 29. Consider some $\pi \in \mathbf{S}_n$ such that there exists some π_i that matches $MMP(0, 2, 0, 2)$. This implies that there exist indices $a < b < i < j < k$ such that $\min\{\pi_a, \pi_b\} > \pi_i > \max\{\pi_j, \pi_k\}$. We will show that there exists an occurrence of the pattern $(>, \neq, >)$ in $\phi(\pi) = e$.

We must consider two cases. If $\pi_i \leq i$, then $j > \pi_i > \pi_j$ and $k > \pi_i > \pi_k$. Therefore, $e_i = \pi_i - 1$, $e_j = \pi_j - 1$ and $e_k = \pi_k - 1$. So, since $\pi_i > \max\{\pi_j, \pi_k\}$ and $\pi_j \neq \pi_k$, e_i, e_j, e_k forms an occurrence of $(>, \neq, >)$.

Now assume $\pi_i > i$. Recall that e_i is the t-th smallest element of $\{e_{i+1}, \dots, e_n\}$ if π_i is the t-th largest element of $\{\pi_1, \pi_2, \dots, \pi_i\}$. Since π_a, π_b are larger than and occur before π_i, we know that t is at least 3. If $e_{j'}$ and $e_{k'}$ are the two smallest distinct values in the set $\{e_{i+1}, \dots, e_n\}$, we are guaranteed that $e_i > e_{j'}$ and $e_i > e_{k'}$; so, $e_i, e_{j'}, e_{k'}$ form the pattern $(>, \neq, >)$. □

The bijection ϕ turns out to be a versatile tool, giving interesting results when restricted to $\mathbf{S}_n(MMP(k, 0, k, 0))$ for any positive integer k. In this case, $\phi(\mathbf{S}_n(MMP(k, 0, k, 0)))$ maps to inversion sequences that avoid a particular set of length $k + 1$ patterns.

2.30 5040: $n!$

The last equivalence class of patterns in this section is the set of those avoided by all inversion sequences. There are 41 such patterns among our 343, including the representative below.

5040: $e_i = e_j = e_k$ and $e_i \neq e_k$

3 Results about patterns whose sequences don't appear in the OEIS [15]

Table 3 lists all equivalence classes of the patterns $\rho \in \{<,>,\leq,\geq,=,\neq,-\}^3$ whose avoidance sequences do not appear in the OEIS. We were able to derive the avoidance sequences for a few of these patterns and prove Wilf equivalence of some others. In this section we describe our limited results and leave identification of the avoidance sequences of the remaining patterns in Table 3 as questions for future study.

3.1 Counting results

3.1.1 805: sum of Catalan numbers

805: $e_i \leq e_j > e_k$ and $e_i \neq e_k$

Conjecture 1. $|\mathbf{I}_n(\leq,>,\neq)| = C_{n+1} - 1 - \sum_{i=0}^{n} C_i$, *where* $C_n = \frac{1}{n+1}\binom{2n}{n}$.

(Differences of successive terms appeared in the OEIS as A002057: 1, 4, 14, 48, 165, 572, 2002, 7072. Second differences appeared as A026016: 3, 10, 34, 117, 407, 1430, 5070.)

3.1.2 1016

1016: $e_i > e_j$ and $e_i \neq e_k$

We can show that the counting sequence is as follows. We omit the details since we hope to find a simpler formula and nicer explanation.

$$|\mathbf{I}_n(>,-,\neq)| = \binom{2(n-1)}{n-1} + \sum_{k=2}^{n-2}\sum_{i=1}^{k-1}\sum_{u=1}^{i}\sum_{d=0}^{u-1} \frac{i-d+1}{i+1}\binom{i+d}{d}.$$

3.1.3 1079(A,B): sum of binomial coefficients

1079A: $e_i > e_j \neq e_k$

These sequences have a nice unimodality characterization. They are the inversion sequences $e \in \mathbf{I}_n$ satisfying for some t:

$$e_1 \leq e_2 \leq \ldots \leq e_t \geq e_{t+1} = e_{t+2} = \ldots = e_n.$$

From this characterization, we can show the following.

Theorem 36. $|\mathbf{I}_n(>, \neq, -)| = 1 + \sum_{i=1}^{n-1} \binom{2i}{i-1}$.

$\boxed{\textbf{1079B}\text{: } e_i < e_j > e_k \text{ and } e_i \neq e_k}$

These are the inversion sequences whose positive elements are weakly increasing (like 1806A) but where $e_i = 0$ implies $\mathrm{dist}(e_1, \ldots, e_{i-1}) \leq 2$. We have not yet shown pattern 1079B to be Wilf equivalent to 1079A.

3.1.4 4306A,B: $\mathbf{I}_n(210)$, $\mathbf{I}_n(201)$

$\boxed{\textbf{4306A}\text{: } e_j < e_k \text{ and } e_i > e_k\text{: } 201}$

$\boxed{\textbf{4306B}\text{: } e_1 > e_j > e_k\text{: } 210}$

These two patterns were shown to be Wilf equivalent in [13] via a bijection, which we made use of in Section 2.27. The recurrence (31) was also derived. See [21] for an alternate approach.

3.2 Wilf equivalence results

In the remainder of the section we show that the bijection α described in (11) of Section 2.27 proves Wilf equivalence of all of the following pairs of patterns: 663A,B; 746A,B; 1833A,B; and 1953A,B.

$\boxed{\textbf{1953A}\text{: } e_j > e_k \text{ and } e_i > e_k}$

$\boxed{\textbf{1953B}\text{: } e_i \neq e_j \geq e_k \text{ and } e_i > e_k}$

Theorem 37. *The patterns $(-, >, >)$ and $(\neq, \geq, >)$, defining classes* 1953A *and* 1953B *respectively, are Wilf equivalent.*

Proof. The inversion sequences in classes 1953A and 1953B are those in $\mathbf{I}_n(110, 210, 120)$ and $\mathbf{I}_n(100, 210, 120)$, respectively. Notice that these are the inversion sequences in $\mathbf{I}_n(110, 210)$ (class 2958C) and $\mathbf{I}_n(100, 210)$ (class 2958A), respectively, that avoid 120. So it suffices to show that both $\alpha : \mathbf{I}_n(110, 210) \to \mathbf{I}_n(100, 210)$ and $\beta = \alpha^{-1}$ preserves 120-avoidance.

Suppose $e \in \mathbf{I}_n(110, 210)$ avoids 120, but for $e' = \alpha(e)$ there exist $i < j < k$ such that $e'_i < e'_j > e'_k$ and $e'_i > e'_k$. By (13), $e'_k = e_k$. Notice that we cannot have both $e'_i = e_i$ and $e'_j = e_j$, since this would create a 120 in e.

Suppose first that $e'_j = e_j$. Since e avoids 120, $e'_i \neq e_i$ so, by definition of α, there is an $s \in [i-1]$ such that $e_s = e'_i$. But then $e_s e_j e_k$ forms a 120 in e.

So, assume that $e'_i = e_i$. Then, since e avoids 120, $e'_j \neq e_j$. So, there must be a $t \in [j-1]$ such that $e_t = e'_j$. If $i < t < k$ then $e_i e_t e_k$ is a 120 in e. Otherwise, $t < i < k$ and $e_t e_i e_k$ is a 210 in e, which is impossible.

Finally, if both $e'_i \neq e_i$ and $e'_j \neq e_j$, then let s and t be as above. If $s < t$ then $e_s e_t e_k$ forms a 102 in e. Otherwise, $e_t e_s e_k$ forms a 210 in e. Both cases lead to a contradiction.

If both $e'_i \neq e_i$ and $e'_j \neq e_j$, then let $s \in [i-1]$ and $j \in [j-1]$ be indices such that $e'_i = e_s$ and $e'_j = e_t$. Since $e'_i = \max\{e_1, e_2, \ldots, e_i\}$ and $e'_j = \max\{e_1, e_2, \ldots, e_j\}$, and we have $e'_i < e'_j$, it must be the case the $e_s < e_t$ and $s < t$. Therefore $e_s e_t e_k$ is an occurrence of 120 in e, giving a contradiction.

It follows that α preserves 210-avoidance. Showing that β preserves 210-avoidance is similar. $\square$

1833A: $e_j \neq e_k$ and $e_i > e_k$

1833B: $e_i \neq e_j$ and $e_i > e_k$

Theorem 38. *The patterns* $(-, \neq, >)$ *and* $(\neq, -, >)$*, defining classes* 1833A *and* 1833B *respectively, are Wilf equivalent.*

Proof. The inversion sequences in classes 1833A and 1833B are those in $\mathbf{I}_n(110, 210, 120, 201)$ and $\mathbf{I}_n(100, 210, 120, 201)$, respectively. It was shown in Section 2.27 that the bijection $\alpha : \mathbf{I}_n(110, 210) \to \mathbf{I}_n(100, 210)$ preserves 201-avoidance, as does its inverse β. It follows from Theorem 37 that α and β preserves 120-avoidance as well. So $\alpha(\mathbf{I}_n(-, \neq, >)) = \mathbf{I}_n(\neq, -, >)$. $\square$

746A: $e_j > e_k$ and $e_i \geq e_k$

746B: $e_i \neq e_j \geq e_k$ and $e_i \geq e_k$

Theorem 39. *The patterns* $(-, >, \geq)$ *and* $(\neq, \geq, \geq)$*, defining classes* 746A *and* 746B *respectively, are Wilf equivalent.*

Proof. The inversion sequences in classes 746A and 746B are those in $\mathbf{I}_n(110, 210, 120, 010)$ and $\mathbf{I}_n(100, 210, 120, 010)$, respectively. These are, in turn, the inversion sequences in $\mathbf{I}_n(110, 210, 120)$ (class 1953A) and $\mathbf{I}_n(100, 210, 120)$ (class 1953B), respectively, that avoid 010. By Theorem 37, $\alpha : \mathbf{I}_n(110, 210) \to \mathbf{I}_n(100, 210)$ and $\alpha^{-1} = \beta$ preserve 120-avoidance. So it suffices to show that both $\alpha : \mathbf{I}_n(110, 210) \to \mathbf{I}_n(100, 210)$ and $\alpha^{-1} = \beta$ preserves 010-avoidance.

Suppose $e \in \mathbf{I}_n(110, 210)$ avoids 010, but for $e' = \alpha(e)$ there exist $i < j < k$ such that $e'_i < e'_j > e'_k$ and $e'_i = e'_k$. By (13), $e'_k = e_k$. Since e avoids 010, it follows that we cannot have both $e'_i = e_i$ and $e'_j = e_j$.

Suppose first that $e'_j = e_j$. Since e avoids 010, $e'_i \neq e_i$ so, by definition of α there is an $s \in [i-1]$ such that $e_s = e'_i$. But then $e_s e_j e_k$ forms a 010 in e.

So, assume that $e'_i = e_i$. Then, since e avoids 010, $e'_j \neq e_j$. So, there must be a $t \in [j-1]$ such that $e_t = e'_j$. If $i < t < k$ then $e_i e_t e_k$ is a 010 in e. Otherwise, $t < i < k$ and $e_t e_i e_k$ is a 100 in e, which is impossible.

If both $e'_i \neq e_i$ and $e'_j \neq e_j$, then let $s \in [i-1]$ and $t \in [j-1]$ be indices such that $e'_i = e_s$ and $e'_j = e_t$. Since $e'_i = \max\{e_1, e_2, \ldots, e_i\}$ and $e'_j = \max\{e_1, e_2, \ldots, e_j\}$, and we have $e'_i < e'_j$, it must be the case the $e_s < e_t$ and $s < t$. Additionally, $e_s = e'_i = e'_k = e_k$. Therefore $e_s e_t e_k$ is an occurrence of 010 in e, giving a contradiction.

It follows that α preserves 010-avoidance. Showing that β preserves 010-avoidance is similar. $\square$

663A: $e_j \neq e_k$ and $e_i \geq e_k$

663B: $e_i \neq e_j$ and $e_i \geq e_k$

Theorem 40. *The patterns* $(-, \neq, \geq)$ *and* $(\neq, -, \geq)$*, defining classes* 663A *and* 6633B *respectively, are Wilf equivalent.*

Proof. The inversion sequences in classes 663A and 663B are those in $\mathbf{I}_n(110, 210, 120, 201, 010)$ and $\mathbf{I}_n(100, 210, 120, 201, 010)$, respectively. It was shown in Section 2.27 that the bijection α : $\mathbf{I}_n(110, 210) \to \mathbf{I}_n(100, 210)$ preserves 201-avoidance, as does its inverse. It follows from Theorem 37 that α, β preserve 120-avoidance and from Theorem 39 that they preserve 010-avoidance as well. So $\alpha(\mathbf{I}_n(-, \neq, \geq)) = \mathbf{I}_n(\neq, -, \geq)$. $\square$

4 Concluding remarks

The results in this work demonstrate that inversion sequences avoiding a triple of relations provide a rich and unifying paradigm for modeling combinatorial sequences.

Several interesting questions remain for future work, such those highlighted in Table 2 with a "no" in column 3. This includes, for example, showing that the number of inversion sequences $e \in \mathbf{I}_n$ with no $i < j < k$ such that $e_i > e_j \geq e_k$ is the same as the number of plane permutations in $\mathbf{S}_n$ (see Section 2.27). Another fascinating open question is whether the number of Baxter permutations in $\mathbf{S}_n$ is the same as the number of $e \in \mathbf{I}_n$ with no $i < j < k$ such that both $e_i \geq e_j \geq e_k$ and $e_i > e_k$ (Section 2.25). There are also a number of lingering open enumeration problems: can enumeration formulas be found for some of the avoidance sets in Table 3, such as $\mathbf{I}_n(010)$, $\mathbf{I}_n(100)$, $\mathbf{I}_n(120)$, or $\mathbf{I}_n(201) = \mathbf{I}_n(210)$?

In ongoing work, we construct and examine the partially ordered set defined on the base set of patterns $\{<, >, \leq, \geq, =, \neq, -\}^3$, where for $\rho, \rho' \in \{<, >, \leq, \geq, =, \neq, -\}^3$, $\rho \preceq \rho'$ if and only if for all $n \geq 1$, $\mathbf{I}_n(\rho) \subseteq \mathbf{I}_n(\rho')$.

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