## Closed-Form Expressions for the n-Queens Problem and Related Problems

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In this paper, we derive simple closed-form expressions for the n-queens problem and three related problems in terms of permanents of (0, 1) matrices. These formulas are the first of their kind. Moreover, they provide the first method for solving these problems with polynomial space that has a nontrivial time complexity bound. We then show how a closed form for the number of Latin squares of order n follows from our method. Finally, we prove lower bounds. In particular, we show that the permanent of Schur's complex-valued matrix is a lower bound for the toroidal semi-queens problem, or equivalently, the number of transversals in a cyclic Latin square.

#### 1 Introduction

The *n*-queens problem is to determine Q(n), the number of arrangements of *n* queens on an *n*-by-*n* chessboard such that no two queens attack. It is a generalization of the eight queens puzzle posed in 1848 by Max Bezzel, a German chess player. The *n*-queens problem has been widely studied since then, attracting the attention of Pólya and Lucas. It is now best known as a toy problem in algorithm design [1].

Despite this rich history, little is known of the general behavior of Q(n). Key results are that Q(n) > 1 for n > 3, and  $Q(n) > 4^{n/5}$  when gcd(n, 30) = 5. See [1] for a comprehensive survey. The only closed-form expression<sup>\*</sup> we are aware of was given in [2]. It is "very complicated" in the authors' own words, however.

The variants of the *n*-queens problem we consider are the *toroidal n*-queens problem T(n), the *semi-queens* problem S(n), and the *toroidal semi-queens* problem TS(n). As with Q(n), the general behavior of these functions is not well understood; asymptotic lower bounds are only known for TS(n) [3].

In this paper, we derive closed-form expressions for Q(n), T(n), S(n), and TS(n) in terms of permanents of (0, 1) matrices. The method we use is general and proceeds as follows. First, we come up with an *obstruction matrix* for a problem. Each entry in this matrix is a multilinear monomial. We then prove a formula for the sum of the coefficients of the terms containing some number of distinct variables in a polynomial. This is then used to obtain closed-form expressions for our problems. The expressions we obtain are very similar to those for the number of Latin squares of order n, such as those given in [9]. In fact, we show that one such formula is an immediate corollary of our method.

The permanent was previously considered by Rivin and Zabih to compute Q(n) and T(n) [7]. Similarly, in 1874 Gunther used the determinant to construct solutions to the *n*-queens problem for small values of n [1]. As far as we can tell however, no one has previously attempted to obtain closed-form expressions with this approach. The expressions we obtain in doing so can be evaluated in nontrivial time (i.e., better than the O(n!) brute-force approach) and with polynomial space. The only other algorithms for computing Q(n) and T(n) with nontrivial time complexity bounds were given in [8]; however, this approach requires exponential space. We are not aware of any previously known algorithms for computing S(n) and TS(n) with nontrivial complexity bounds.

Finally, we prove lower bounds for these problems in terms of determinants of (0,1) matrices. As a consequence, we show that the permanent of Schur's complex-valued matrix [4] provides a lower bound for the toroidal semi-queens problem.

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<sup>\*</sup>We would like to correct a misunderstanding in [1]. The authors state that there exists no closed-form expression for Q(n) because it was shown to be beyond the #P complexity class. However, the result referenced only shows that the *n*-queens problem is beyond #P because Q(n) can be more than polynomial in *n* [5]. A function can clearly be beyond #P for this reason and still have a closed-form expression; consider  $2^n$  for instance.

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# 2 **Preliminary Definitions**

The permanent of an *n*-by-*n* matrix  $\mathbf{A} = (a_{i,j})$  is given by

$$per(\mathbf{A}) = \sum_{\sigma \in S_n} \prod_{i=1}^n a_{i,\sigma(i)}$$

where  $S_n$  is the symmetric group on *n* elements. It is a well-known result in complexity theory that computing the permanent of a matrix is intractable, even when restricted to the set of (0, 1) matrices [11].

An obstruction matrix **A** is a square matrix whose entries are multilinear monomials. If **A** contains the variables  $x_1, x_2, \ldots, x_m$  and  $s = (s_i) \in \{0, 1\}^m$ , then **A**|s is the matrix obtained by substituting  $x_i = s_i$  for all *i*.

An *n*-by-*n* matrix  $\mathbf{M} = (m_{i,j})$  is diagonally constant if each northwest-southeast diagonal is constant; that is,  $m_{i,j} = m_{i+1,j+1}$ . A circulant matrix is a diagonally constant matrix with the property that each row is obtained by rotating the preceding row one position to the right, i.e.,  $m_{i,j} = m_{i+1,j+1 \mod n}$ .

Q(n) is the number of arrangements of *n* queens on an *n*-by-*n* chessboard such that no two attack; that is, lie on the same row, column, or diagonal [10, Sequence A000170].

S(n) is the number of arrangements of *n* nonattacking *semi-queens* on an *n*-by-*n* chessboard [10, Sequence A099152]. A semi-queen has the same moves as a queen except for the northeast-southwest diagonal moves. Note that  $S(n) \ge Q(n)$ .

T(n) is the number of arrangements of n nonattacking queens on a toroidal n-by-n chessboard [10, Sequence A051906]. The toroidal board is obtained by identifying the edges of the board as if it were a torus. As a result, the diagonals a queen can move along wrap around the board. Note that  $Q(n) \ge T(n)$ .

TS(n) is the number of arrangements of n nonattacking semi-queens on an n-by-n toroidal chessboard. TS(n) is also the number of transversals in a cyclic Latin square [10, Sequence A006717]. Note that  $S(n) \ge TS(n)$ .

## 3 Derivation of the Main Results

We begin by introducing the *n*-by-*n* obstruction matrices  $\mathbf{Q}_n$ ,  $\mathbf{T}_n$ ,  $\mathbf{S}_n$ , and  $\mathbf{Z}_n$ , which will be used to compute Q(n), T(n), S(n), and TS(n), respectively.

 $\mathbf{Q}_n$  contains the variables  $x_1, y_1, \ldots, x_{2n-1}, y_{2n-1}$ . The variable  $x_i$  corresponds to the *i*th northwestsoutheast diagonal (indexed from bottom left to top right), and  $y_i$  corresponds to the *i*th northeast-southwest diagonal (indexed from bottom right to top left). The (i, j)<sup>th</sup> entry of  $\mathbf{Q}_n$  is  $x_{n-i+j}y_{2n-i-j+1}$ .

 $\mathbf{T}_n$  contains the variables  $x_1, y_1, \ldots, x_{2n}, y_{2n}$ . The variable  $x_i$  corresponds to the *i*th northwest-southeast broken diagonal, and  $y_i$  corresponds to the *i*th northeast-southwest broken diagonal. The  $(i, j)^{\text{th}}$  entry of  $\mathbf{T}_n$  is  $x_{(n-i+j) \mod n} y_{(2n-i-j+1) \mod n}$ .

 $\mathbf{S}_n$  contains the variables  $x_1, x_2, \ldots, x_{2n-1}$ , and  $x_i$  corresponds to the *i*th northwest-southeast diagonal. The  $(i, j)^{\text{th}}$  entry of  $\mathbf{S}_n$  is  $x_{n-i+j}$ .

 $\mathbf{Z}_n$  contains the variables  $x_1, x_2, \ldots, x_n$ , and  $x_i$  corresponds to the *i*th northwest-southeast broken diagonal. The  $(i, j)^{\text{th}}$  entry of  $\mathbf{Z}_n$  is  $x_{(n-i+j) \mod n}$ .

**Example 3.1.** Obstruction matrices for Q(n), T(n), S(n), and TS(n).

$\mathbf{Q}_4 =$	$\begin{bmatrix} x_4y_7 & x_5y_6 \\ x_3y_6 & x_4y_5 \\ x_2y_5 & x_3y_4 \\ x_1y_4 & x_2y_3 \end{bmatrix}$	$egin{array}{cccccccccccccccccccccccccccccccccccc$	$egin{array}{c} x_7y_4 \ x_6y_3 \ x_5y_2 \ x_4y_1 \end{bmatrix}$	$\Gamma_4 = \begin{bmatrix} & & \\ & & \\ & & \\ & & \end{bmatrix}$	$egin{array}{cccc} x_4y_1 & x_1 \ x_3y_2 & x_4 \ x_2y_3 & x_5 \ x_1y_4 & x_5 \end{array}$	$1 \frac{y_2}{4 \frac{y_3}{3 \frac{y_4}{2 \frac{y_1}{2 \frac{y_1}{2$	$x_2y_3 \ x_1y_4 \ x_4y_1 \ x_3y_2$	$egin{array}{c} x_3y_4 \ x_2y_1 \ x_1y_2 \ x_4y_3 \end{bmatrix}$
$\mathbf{S}_4 =$	$\begin{bmatrix} x_4 & x_5 & x_6 \\ x_3 & x_4 & x_5 \\ x_2 & x_3 & x_4 \\ x_1 & x_2 & x_3 \end{bmatrix}$	$egin{array}{c} x_7 \ x_6 \ x_5 \ x_4 \end{bmatrix}$	2	$\mathbf{Z}_4 = \begin{bmatrix} \mathbf{z} \\ \mathbf{z} \end{bmatrix}$	$egin{array}{cccc} x_4 & x_1 \ x_3 & x_4 \ x_2 & x_3 \ x_1 & x_2 \end{array}$	$egin{array}{c} x_2 \ x_1 \ x_4 \ x_3 \end{array}$	$egin{array}{c} x_3 \ x_2 \ x_1 \ x_4 \end{bmatrix}$	

**Definition 3.2.** Let *P* be a polynomial, and let  $k \in \mathbb{N}$ . Then g(P, k) is defined to be the sum of the coefficients of the terms in *P* that are a product of exactly *k* distinct variables.

Note that when  $k = \deg P$ , the terms whose coefficients are summed by g(P, k) are multilinear. This leads to the following fact:

**Lemma 3.3.** 
$$g(per(\mathbf{Q}_n), 2n) = Q(n), g(per(\mathbf{T}_n), 2n) = T(n), g(per(\mathbf{S}_n), n) = S(n), \text{ and } g(per(\mathbf{Z}_n), n) = TS(n).$$



Fig. 1. From top left to bottom right: The squares attacked by a queen, a semi-queen, a toroidal queen, and a toroidal semi-queen.

**Proof.** This follows immediately from the definition of the permanent and the structure of  $\mathbf{Q}_n, \mathbf{T}_n, \mathbf{S}_n$ , and  $\mathbf{Z}_n$ . Consider  $\operatorname{per}(\mathbf{Q}_n)$  for instance. We can write this as a sum of n! terms of degree 2n. Each term in this polynomial corresponds to a permutation matrix. If a term is square-free, then from the definition of  $\mathbf{Q}_n$  no two elements in the corresponding permutation matrix lie along the same diagonal. Since a permutation matrix has no two nonzero entries on the same row or column, it follows that this permutation matrix corresponds to a solution for the *n*-queens problem.

Suppose that P is a polynomial in m variables. Let  $S_{m,k}$  be the subset of  $\{0,1\}^m$  that consists of the tuples containing k ones; that is,

$$S_{m,k} = \{(s_1, \dots, s_m) \in \{0, 1\}^m : \sum_{i=1}^m s_i = k\}.$$

Define

$$f(P,k) = \sum_{(s_1,...,s_m) \in S_{m,k}} P(s_1,...,s_m).$$

The following fact is now used to derive an expression for g in terms of f.

Fact 3.4. Let  $m \ge k \ge l \ge 0$ . Assume

$$a(k) = \sum_{i=l}^{k} b(i) \binom{m-i}{k-i}.$$

Then

$$b(k) = \sum_{i=l}^{k} a(i) \binom{m-i}{k-i} (-1)^{k-i}.$$

**Theorem 3.5.** Let P be a polynomial in m variables, and let  $1 \le k \le m$ . Then

$$g(P,k) = \sum_{i=1}^{k} (-1)^{i+k} f(P,i) \binom{m-i}{k-i}.$$
(1)

**Proof.** Consider a term in P that is a product of i distinct variables where  $i \leq k$ . It follows from the definition of f that the coefficient of this term is counted by f(P,k) a total of  $\binom{m-i}{k-i}$  times. Therefore

$$f(P,k) = \sum_{i=1}^{k} g(P,i) \binom{m-i}{k-i}.$$

Then by applying Fact 3.4 with a(k) = f(P, k), b(k) = g(P, k), and l = 1, equation (1) follows.

The following expressions follow directly from Lemma 3.3 and Theorem 3.5, and the fact that  $per(\mathbf{Q}_n)$ ,  $per(\mathbf{T}_n)$ ,  $per(\mathbf{S}_n)$ , and  $per(\mathbf{Z}_n)$  are polynomials in 4n - 2, 2n, 2n - 1, and n variables, respectively.

**Theorem 3.6.** Let  $S_{m,k}$  be the subset of  $\{0,1\}^m$  that consists of the tuples containing k ones,  $U_n$  the set of all n-by-n (0,1) diagonally constant matrices, and  $V_n$  the set of all n-by-n (0,1) circulant matrices. Then the following identities hold:

$$Q(n) = \sum_{i=1}^{2n} (-1)^i \binom{4n-i-2}{2n-i} \sum_{s \in S_{4n-2,i}} \operatorname{per}(\mathbf{Q}_n | s),$$
  

$$T(n) = \sum_{i=1}^{2n} (-1)^{i+n} \sum_{s \in S_{2n,i}} \operatorname{per}(\mathbf{T}_n | s),$$
  

$$S(n) = \sum_{\mathbf{M} \in U_n} (-1)^{\gamma(\mathbf{M})+n} \operatorname{per}(\mathbf{M}) \binom{2n-\gamma(\mathbf{M})-1}{n-\gamma(\mathbf{M})},$$
  

$$TS(n) = \sum_{\mathbf{M} \in V_n} (-1)^{\sigma(\mathbf{M})+n} \operatorname{per}(\mathbf{M}),$$

where  $\gamma(\mathbf{M})$  is the number of nonzero diagonals in  $\mathbf{M}$ , and  $\sigma(\mathbf{M})$  is the number of ones in the first row of  $\mathbf{M}$ .

Note that multiple (0, 1) variable assignments to  $\mathbf{Q}_n$  and  $\mathbf{T}_n$  can correspond to the same (0, 1) matrix. As a result, one can think of the formulas for Q(n) and T(n) as summing over multisets of (0, 1) matrices. In the cases of  $\mathbf{S}_n$  and  $\mathbf{Z}_n$ , there is a one-to-one relationship between (0, 1) variable assignments and (0, 1) matrices, so we can write S(n) and TS(n) as sums over sets of (0, 1) matrices.

## 3.1 Complexity Analysis

The above expressions are impractical to evaluate even for small values of n; however, they do provide nontrivial time complexity bounds.

**Corollary 3.7.** Q(n), T(n), S(n), and TS(n) can be computed in quadratic space and in time  $O(n32^n), O(n8^n)$ ,  $O(n8^n)$ , and  $O(n4^n)$ , respectively.

**Proof.** We can compute Q(n) as follows. There are  $O(2^{4n})$  (0, 1)-tuples to enumerate in the summation. For each such tuple s, we compute  $\mathbf{Q}_n|s$  in  $O(n^2)$  time and space, and compute the permanent of this matrix in  $O(n2^n)$  time and with  $O(n^2)$  space using Ryser's formula [6], which states that

$$per(\mathbf{A}) = \sum_{S \subseteq \{1, \dots, n\}} (-1)^{|S|+n} \prod_{i=1}^{n} \sum_{j \in S} a_{ij}.$$

Thus Q(n) can be computed in  $O(n32^n)$  time using  $O(n^2)$  space. The other bounds are obtained similarly.

The only other algorithms we know of for Q(n) and T(n) with nontrivial complexity bounds run in time  $O(f(n)8^n)$  where f(n) is a low-order polynomial [8]. However, these algorithms require  $O(n^28^n)$  space, whereas we only require  $O(n^2)$  space. We do not know of any algorithms with nontrivial complexity bounds for the other two problems.

#### 3.2 Extension: Latin Squares

A Latin square of order n is an arrangement of n copies of the integers 1, 2, ..., n in an n-by-n grid such that every integer appears exactly once in each row and column. We now show how an expression for  $L_n$ , the number of Latin squares of order n, follows naturally from the method used above.

**Lemma 3.8.** Let  $\mathbf{B}_n$  be the *n*-by-*n* obstruction matrix containing the variables  $(x_1, x_2, \ldots, x_{n^2})$  defined by  $(\mathbf{B}_n)_{i,j} = x_{i+n(j-1)}$ . Let  $\mathbf{A}_n$  be the  $n^2$ -by- $n^2$  block diagonal matrix

$$\mathbf{A_n} = \begin{bmatrix} \mathbf{B}_n & 0 & \cdots & 0 \\ 0 & \mathbf{B}_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{B}_n \end{bmatrix}.$$

Then  $L_n = g(\operatorname{per}(\mathbf{A}_n), n^2).$ 

**Proof.** A Latin square of order n can be thought of as an ordered set of n disjoint permutation matrices of order n. On the other hand, a term in per( $\mathbf{A}_n$ ) can be thought of as an ordered set of n permutation matrices of order n, one along each copy of  $\mathbf{B}_n$ . If this term contains  $n^2$  distinct variables, these permutation matrices must be disjoint. Therefore the sum of the coefficients of the terms in per( $\mathbf{A}_n$ ) containing  $n^2$  distinct variables is exactly  $L_n$ .

**Theorem 3.9.** Let  $L_n$  be the number of Latin squares of order n. Then

$$L_n = \sum_{\mathbf{M} \in M_n} (-1)^{\sigma(\mathbf{M}) + n} \operatorname{per}(\mathbf{M})^n$$

where  $M_n$  is the set of all (0,1) *n*-by-*n* matrices, and  $\sigma(\mathbf{M})$  is the number of nonzero entries in  $\mathbf{M}$ .

**Proof**. From Lemma 3.8 and Theorem 3.5, it follows that

$$L_n = \sum_{i=1}^{n^2} (-1)^{i+n^2} f(\operatorname{per}(\mathbf{A}_n), i)$$
  
=  $\sum_{i=1}^{n^2} (-1)^{i+n} \sum_{s \in S_{n^2, i}} \operatorname{per}(\mathbf{A}_n | s)$   
=  $\sum_{i=1}^{n^2} (-1)^{i+n} \sum_{s \in S_{n^2, i}} \operatorname{per}(\mathbf{B}_n | s)^n$ 

where the last step follows from the fact that  $per(\mathbf{A}_n) = per(\mathbf{B}_n)^n$ . Because  $\mathbf{B}_n | u \neq \mathbf{B}_n | v$  if  $u \neq v$ , we can rewrite this as

$$L_n = \sum_{\mathbf{M} \in M_n} (-1)^{\sigma(\mathbf{M}) + n} \operatorname{per}(\mathbf{M})^n.$$

This formula was first given in [9].

#### 4 Lower Bounds

In the last section, we showed that sums of coefficients in the permanents of the obstruction matrices  $\mathbf{Q}_n, \mathbf{T}_n, \mathbf{S}_n$ , and  $\mathbf{Z}_n$  correspond to the values of Q(n), T(n), S(n), and TS(n), respectively. We then gave a closed-form expression for the function g that computes these sums. More precisely, g(P, k) was the sum of the coefficients of the terms in the polynomial P containing k distinct variables.

Now since each entry in  $\mathbf{Q}_n$  is a monomial with coefficient 1, the coefficient of a term in det $(\mathbf{Q}_n)$  is at most the coefficient of the corresponding term in per $(\mathbf{Q}_n)$ . Therefore  $|g(\det(\mathbf{Q}_n), 2n)| \leq g(\operatorname{per}(\mathbf{Q}_n), 2n) = Q(n)$ . The same argument applies to the other problems. As a result we have the following corollary:

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**Corollary 4.1.** Let  $S_{m,k}$  be the subset of  $\{0,1\}^m$  that consists of the tuples containing k ones,  $U_n$  the set of all n-by-n (0,1) diagonally constant matrices, and  $V_n$  the set of all n-by-n (0,1) circulant matrices. Then the following inequalities hold:

$$Q_{\det}(n) := \left| \sum_{i=1}^{2n} (-1)^i \binom{4n-i-2}{2n-i} \sum_{s \in S_{4n-2,i}} \det(\mathbf{Q}_n|s) \right| \le Q(n),$$
  

$$T_{\det}(n) := \left| \sum_{i=1}^{2n} (-1)^{i+n} \sum_{s \in S_{2n,i}} \det(\mathbf{T}_n|s) \right| \le T(n),$$
  

$$S_{\det}(n) := \left| \sum_{\mathbf{M} \in U_n} (-1)^{\gamma(\mathbf{M})} \det(\mathbf{M}) \binom{2n-\gamma(\mathbf{M})-1}{n-\gamma(\mathbf{M})} \right| \le S(n),$$
  

$$TS_{\det}(n) := \left| \sum_{\mathbf{M} \in V_n} (-1)^{\sigma(\mathbf{M})} \det(\mathbf{M}) \right| \le TS(n),$$

where  $\gamma(\mathbf{M})$  is the number of nonzero diagonals in  $\mathbf{M}$ , and  $\sigma(\mathbf{M})$  is the number of ones in the first row of  $\mathbf{M}$ .

We now show that  $TS_{det}(n)$  is the permanent of Schur's matrix of order n; see [10, Sequence A003112].

Let  $\mathbf{M}_n = (\epsilon^{jk})$  be an *n*-by-*n* matrix where  $\epsilon$  is an *n*th root of unity, and let  $P_n = \text{per}(\mathbf{M}_n)$ . The matrix  $\mathbf{M}_n$  is known as Schur's matrix of order *n*. It has been of interest in number theory, statistics, and coding theory. Its permanent is the topic of [4].

**Theorem 4.2.** For all  $n, |P_n| \leq TS(n) \leq S(n)$ .

**Proof.** From Corollary 4.1, it suffices to show that  $TS_{det}(n) = |P_n|$ . This follows immediately from the fact that  $P_n = g(det(\mathbf{Z}_n), n)$  [4].

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