# Closed-Form Expressions for the n-Queens Problem and Related Problems 

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In this paper, we derive simple closed-form expressions for the $n$-queens problem and three related problems in terms of permanents of $(0,1)$ matrices. These formulas are the first of their kind. Moreover, they provide the first method for solving these problems with polynomial space that has a nontrivial time complexity bound. We then show how a closed form for the number of Latin squares of order $n$ follows from our method. Finally, we prove lower bounds. In particular, we show that the permanent of Schur's complex-valued matrix is a lower bound for the toroidal semi-queens problem, or equivalently, the number of transversals in a cyclic Latin square.

## 1 Introduction

The $n$-queens problem is to determine $Q(n)$, the number of arrangements of $n$ queens on an $n$-by- $n$ chessboard such that no two queens attack. It is a generalization of the eight queens puzzle posed in 1848 by Max Bezzel, a German chess player. The $n$-queens problem has been widely studied since then, attracting the attention of Pólya and Lucas. It is now best known as a toy problem in algorithm design [1].

Despite this rich history, little is known of the general behavior of $Q(n)$. Key results are that $Q(n)>1$ for $n>3$, and $Q(n)>4^{n / 5}$ when $\operatorname{gcd}(n, 30)=5$. See [1] for a comprehensive survey. The only closed-form expression* we are aware of was given in [2]. It is "very complicated" in the authors' own words, however.

The variants of the $n$-queens problem we consider are the toroidal $n$-queens problem $T(n)$, the semi-queens problem $S(n)$, and the toroidal semi-queens problem $T S(n)$. As with $Q(n)$, the general behavior of these functions is not well understood; asymptotic lower bounds are only known for $T S(n)$ [3].

In this paper, we derive closed-form expressions for $Q(n), T(n), S(n)$, and $T S(n)$ in terms of permanents of $(0,1)$ matrices. The method we use is general and proceeds as follows. First, we come up with an obstruction matrix for a problem. Each entry in this matrix is a multilinear monomial. We then prove a formula for the sum of the coefficients of the terms containing some number of distinct variables in a polynomial. This is then used to obtain closed-form expressions for our problems. The expressions we obtain are very similar to those for the number of Latin squares of order $n$, such as those given in [9]. In fact, we show that one such formula is an immediate corollary of our method.

The permanent was previously considered by Rivin and Zabih to compute $Q(n)$ and $T(n)$ [7]. Similarly, in 1874 Gunther used the determinant to construct solutions to the $n$-queens problem for small values of $n$ [1]. As far as we can tell however, no one has previously attempted to obtain closed-form expressions with this approach. The expressions we obtain in doing so can be evaluated in nontrivial time (i.e., better than the $O(n!$ ) brute-force approach) and with polynomial space. The only other algorithms for computing $Q(n)$ and $T(n)$ with nontrivial time complexity bounds were given in [8]; however, this approach requires exponential space. We are not aware of any previously known algorithms for computing $S(n)$ and $T S(n)$ with nontrivial complexity bounds.

Finally, we prove lower bounds for these problems in terms of determinants of $(0,1)$ matrices. As a consequence, we show that the permanent of Schur's complex-valued matrix [4] provides a lower bound for the toroidal semi-queens problem.

## 2 Preliminary Definitions

The permanent of an $n$-by- $n$ matrix $\mathbf{A}=\left(a_{i, j}\right)$ is given by

$$
\operatorname{per}(\mathbf{A})=\sum_{\sigma \in S_{n}} \prod_{i=1}^{n} a_{i, \sigma(i)}
$$

where $S_{n}$ is the symmetric group on $n$ elements. It is a well-known result in complexity theory that computing the permanent of a matrix is intractable, even when restricted to the set of $(0,1)$ matrices [11].

An obstruction matrix $\mathbf{A}$ is a square matrix whose entries are multilinear monomials. If $\mathbf{A}$ contains the variables $x_{1}, x_{2}, \ldots, x_{m}$ and $s=\left(s_{i}\right) \in\{0,1\}^{m}$, then $\mathbf{A} \mid s$ is the matrix obtained by substituting $x_{i}=s_{i}$ for all $i$.

An $n$-by- $n$ matrix $\mathbf{M}=\left(m_{i, j}\right)$ is diagonally constant if each northwest-southeast diagonal is constant; that is, $m_{i, j}=m_{i+1, j+1}$. A circulant matrix is a diagonally constant matrix with the property that each row is obtained by rotating the preceding row one position to the right, i.e., $m_{i, j}=m_{i+1, j+1} \bmod n$.
$Q(n)$ is the number of arrangements of $n$ queens on an $n$-by- $n$ chessboard such that no two attack; that is, lie on the same row, column, or diagonal [10, Sequence A000170].
$S(n)$ is the number of arrangements of $n$ nonattacking semi-queens on an $n$-by- $n$ chessboard [10, Sequence A099152]. A semi-queen has the same moves as a queen except for the northeast-southwest diagonal moves. Note that $S(n) \geq Q(n)$.
$T(n)$ is the number of arrangements of $n$ nonattacking queens on a toroidal $n$-by- $n$ chessboard [10, Sequence A051906]. The toroidal board is obtained by identifying the edges of the board as if it were a torus. As a result, the diagonals a queen can move along wrap around the board. Note that $Q(n) \geq T(n)$.
$T S(n)$ is the number of arrangements of $n$ nonattacking semi-queens on an $n$-by- $n$ toroidal chessboard. $T S(n)$ is also the number of transversals in a cyclic Latin square [10, Sequence A006717]. Note that $S(n) \geq$ $T S(n)$.

## 3 Derivation of the Main Results

We begin by introducing the $n$-by-n obstruction matrices $\mathbf{Q}_{n}, \mathbf{T}_{n}, \mathbf{S}_{n}$, and $\mathbf{Z}_{n}$, which will be used to compute $Q(n), T(n), S(n)$, and $T S(n)$, respectively.
$\mathbf{Q}_{n}$ contains the variables $x_{1}, y_{1}, \ldots, x_{2 n-1}, y_{2 n-1}$. The variable $x_{i}$ corresponds to the $i$ th northwestsoutheast diagonal (indexed from bottom left to top right), and $y_{i}$ corresponds to the $i$ th northeast-southwest diagonal (indexed from bottom right to top left). The $(i, j)^{\text {th }}$ entry of $\mathbf{Q}_{n}$ is $x_{n-i+j} y_{2 n-i-j+1}$.
$\mathbf{T}_{n}$ contains the variables $x_{1}, y_{1}, \ldots, x_{2 n}, y_{2 n}$. The variable $x_{i}$ corresponds to the $i$ th northwest-southeast broken diagonal, and $y_{i}$ corresponds to the $i$ th northeast-southwest broken diagonal. The $(i, j)^{\text {th }}$ entry of $\mathbf{T}_{n}$ is $x_{(n-i+j) \bmod n} y_{(2 n-i-j+1) \bmod n}$.
$\mathbf{S}_{n}$ contains the variables $x_{1}, x_{2}, \ldots, x_{2 n-1}$, and $x_{i}$ corresponds to the $i$ th northwest-southeast diagonal. The $(i, j)^{\text {th }}$ entry of $\mathbf{S}_{n}$ is $x_{n-i+j}$.
$\mathbf{Z}_{n}$ contains the variables $x_{1}, x_{2}, \ldots, x_{n}$, and $x_{i}$ corresponds to the $i$ th northwest-southeast broken diagonal. The $(i, j)^{\text {th }}$ entry of $\mathbf{Z}_{n}$ is $x_{(n-i+j)} \bmod n$.

Example 3.1. Obstruction matrices for $Q(n), T(n), S(n)$, and $T S(n)$.

$$
\begin{array}{lll}
\mathbf{Q}_{4}=\left[\begin{array}{llll}
x_{4} y_{7} & x_{5} y_{6} & x_{6} y_{5} & x_{7} y_{4} \\
x_{3} y_{6} & x_{4} y_{5} & x_{5} y_{4} & x_{6} y_{3} \\
x_{2} y_{5} & x_{3} y_{4} & x_{4} y_{3} & x_{5} y_{2} \\
x_{1} y_{4} & x_{2} y_{3} & x_{3} y_{2} & x_{4} y_{1}
\end{array}\right] & \mathbf{T}_{4}=\left[\begin{array}{llll}
x_{4} & x_{5} & x_{6} & x_{7} \\
x_{4} y_{1} & x_{1} y_{2} & x_{2} y_{3} & x_{3} y_{4} \\
x_{3} y_{2} & x_{4} y_{3} & x_{1} y_{4} & x_{2} y_{1} \\
x_{2} y_{3} & x_{3} y_{4} & x_{4} y_{1} & x_{1} y_{2} \\
x_{1} y_{4} & x_{2} y_{1} & x_{3} y_{2} & x_{4} y_{3}
\end{array}\right] \\
\mathbf{S}_{4} & x_{5} & x_{6} \\
x_{2} & x_{3} & x_{4}
\end{array} x_{5}, ~ \mathbf{Z}_{4}
$$

Definition 3.2. Let $P$ be a polynomial, and let $k \in \mathbb{N}$. Then $g(P, k)$ is defined to be the sum of the coefficients of the terms in $P$ that are a product of exactly $k$ distinct variables.

Note that when $k=\operatorname{deg} P$, the terms whose coefficients are summed by $g(P, k)$ are multilinear. This leads to the following fact:

Lemma 3.3. $g\left(\operatorname{per}\left(\mathbf{Q}_{n}\right), 2 n\right)=Q(n), g\left(\operatorname{per}\left(\mathbf{T}_{n}\right), 2 n\right)=T(n), g\left(\operatorname{per}\left(\mathbf{S}_{n}\right), n\right)=S(n)$, and $g\left(\operatorname{per}\left(\mathbf{Z}_{n}\right), n\right)=T S(n)$.


Fig. 1. From top left to bottom right: The squares attacked by a queen, a semi-queen, a toroidal queen, and a toroidal semi-queen.

Proof. This follows immediately from the definition of the permanent and the structure of $\mathbf{Q}_{n}, \mathbf{T}_{n}, \mathbf{S}_{n}$, and $\mathbf{Z}_{n}$. Consider $\operatorname{per}\left(\mathbf{Q}_{n}\right)$ for instance. We can write this as a sum of $n!$ terms of degree $2 n$. Each term in this polynomial corresponds to a permutation matrix. If a term is square-free, then from the definition of $\mathbf{Q}_{n}$ no two elements in the corresponding permutation matrix lie along the same diagonal. Since a permutation matrix has no two nonzero entries on the same row or column, it follows that this permutation matrix corresponds to a solution for the $n$-queens problem.

Suppose that $P$ is a polynomial in $m$ variables. Let $S_{m, k}$ be the subset of $\{0,1\}^{m}$ that consists of the tuples containing $k$ ones; that is,

$$
S_{m, k}=\left\{\left(s_{1}, \ldots, s_{m}\right) \in\{0,1\}^{m}: \sum_{i=1}^{m} s_{i}=k\right\}
$$

Define

$$
f(P, k)=\sum_{\left(s_{1}, \ldots, s_{m}\right) \in S_{m, k}} P\left(s_{1}, \ldots, s_{m}\right) .
$$

The following fact is now used to derive an expression for $g$ in terms of $f$.
Fact 3.4. Let $m \geq k \geq l \geq 0$. Assume

$$
a(k)=\sum_{i=l}^{k} b(i)\binom{m-i}{k-i} .
$$

Then

$$
b(k)=\sum_{i=l}^{k} a(i)\binom{m-i}{k-i}(-1)^{k-i}
$$

Theorem 3.5. Let $P$ be a polynomial in $m$ variables, and let $1 \leq k \leq m$. Then

$$
\begin{equation*}
g(P, k)=\sum_{i=1}^{k}(-1)^{i+k} f(P, i)\binom{m-i}{k-i} \tag{1}
\end{equation*}
$$

Proof. Consider a term in $P$ that is a product of $i$ distinct variables where $i \leq k$. It follows from the definition of $f$ that the coefficient of this term is counted by $f(P, k)$ a total of $\binom{m-i}{k-i}$ times. Therefore

$$
f(P, k)=\sum_{i=1}^{k} g(P, i)\binom{m-i}{k-i}
$$

Then by applying Fact 3.4 with $a(k)=f(P, k), b(k)=g(P, k)$, and $l=1$, equation (1) follows.
The following expressions follow directly from Lemma 3.3 and Theorem 3.5, and the fact that $\operatorname{per}\left(\mathbf{Q}_{n}\right)$, $\operatorname{per}\left(\mathbf{T}_{n}\right), \operatorname{per}\left(\mathbf{S}_{n}\right)$, and $\operatorname{per}\left(\mathbf{Z}_{n}\right)$ are polynomials in $4 n-2,2 n, 2 n-1$, and $n$ variables, respectively.

Theorem 3.6. Let $S_{m, k}$ be the subset of $\{0,1\}^{m}$ that consists of the tuples containing $k$ ones, $U_{n}$ the set of all $n$-by- $n(0,1)$ diagonally constant matrices, and $V_{n}$ the set of all $n$-by- $n(0,1)$ circulant matrices. Then the following identities hold:

$$
\begin{aligned}
Q(n) & =\sum_{i=1}^{2 n}(-1)^{i}\binom{4 n-i-2}{2 n-i} \sum_{s \in S_{4 n-2, i}} \operatorname{per}\left(\mathbf{Q}_{n} \mid s\right), \\
T(n) & =\sum_{i=1}^{2 n}(-1)^{i+n} \sum_{s \in S_{2 n, i}} \operatorname{per}\left(\mathbf{T}_{n} \mid s\right), \\
S(n) & =\sum_{\mathbf{M} \in U_{n}}(-1)^{\gamma(\mathbf{M})+n} \operatorname{per}(\mathbf{M})\binom{2 n-\gamma(\mathbf{M})-1}{n-\gamma(\mathbf{M})}, \\
T S(n) & =\sum_{\mathbf{M} \in V_{n}}(-1)^{\sigma(\mathbf{M})+n} \operatorname{per}(\mathbf{M}),
\end{aligned}
$$

where $\gamma(\mathbf{M})$ is the number of nonzero diagonals in $\mathbf{M}$, and $\sigma(\mathbf{M})$ is the number of ones in the first row of $\mathbf{M}$.
Note that multiple $(0,1)$ variable assignments to $\mathbf{Q}_{n}$ and $\mathbf{T}_{n}$ can correspond to the same $(0,1)$ matrix. As a result, one can think of the formulas for $Q(n)$ and $T(n)$ as summing over multisets of $(0,1)$ matrices. In the cases of $\mathbf{S}_{n}$ and $\mathbf{Z}_{n}$, there is a one-to-one relationship between $(0,1)$ variable assignments and $(0,1)$ matrices, so we can write $S(n)$ and $T S(n)$ as sums over sets of $(0,1)$ matrices.

### 3.1 Complexity Analysis

The above expressions are impractical to evaluate even for small values of $n$; however, they do provide nontrivial time complexity bounds.

Corollary 3.7. $Q(n), T(n), S(n)$, and $T S(n)$ can be computed in quadratic space and in time $O\left(n 32^{n}\right), O\left(n 8^{n}\right)$, $O\left(n 8^{n}\right)$, and $O\left(n 4^{n}\right)$, respectively.

Proof. We can compute $Q(n)$ as follows. There are $O\left(2^{4 n}\right)(0,1)$-tuples to enumerate in the summation. For each such tuple $s$, we compute $\mathbf{Q}_{n} \mid s$ in $O\left(n^{2}\right)$ time and space, and compute the permanent of this matrix in $O\left(n 2^{n}\right)$ time and with $O\left(n^{2}\right)$ space using Ryser's formula [6], which states that

$$
\operatorname{per}(\mathbf{A})=\sum_{S \subseteq\{1, \ldots, n\}}(-1)^{|S|+n} \prod_{i=1}^{n} \sum_{j \in S} a_{i j}
$$

Thus $Q(n)$ can be computed in $O\left(n 32^{n}\right)$ time using $O\left(n^{2}\right)$ space. The other bounds are obtained similarly.
The only other algorithms we know of for $Q(n)$ and $T(n)$ with nontrivial complexity bounds run in time $O\left(f(n) 8^{n}\right)$ where $f(n)$ is a low-order polynomial [8]. However, these algorithms require $O\left(n^{2} 8^{n}\right)$ space, whereas we only require $O\left(n^{2}\right)$ space. We do not know of any algorithms with nontrivial complexity bounds for the other two problems.

### 3.2 Extension: Latin Squares

A Latin square of order $n$ is an arrangement of $n$ copies of the integers $1,2, \ldots, n$ in an $n$-by- $n$ grid such that every integer appears exactly once in each row and column. We now show how an expression for $L_{n}$, the number of Latin squares of order $n$, follows naturally from the method used above.

Lemma 3.8. Let $\mathbf{B}_{n}$ be the $n$-by- $n$ obstruction matrix containing the variables $\left(x_{1}, x_{2}, \ldots, x_{n^{2}}\right)$ defined by $\left(\mathbf{B}_{n}\right)_{i, j}=x_{i+n(j-1)}$. Let $\mathbf{A}_{n}$ be the $n^{2}$-by- $n^{2}$ block diagonal matrix

$$
\mathbf{A}_{\mathbf{n}}=\left[\begin{array}{cccc}
\mathbf{B}_{n} & 0 & \cdots & 0 \\
0 & \mathbf{B}_{n} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \mathbf{B}_{n}
\end{array}\right]
$$

Then $L_{n}=g\left(\operatorname{per}\left(\mathbf{A}_{n}\right), n^{2}\right)$.

Proof. A Latin square of order $n$ can be thought of as an ordered set of $n$ disjoint permutation matrices of order $n$. On the other hand, a term in $\operatorname{per}\left(\mathbf{A}_{n}\right)$ can be thought of as an ordered set of $n$ permutation matrices of order $n$, one along each copy of $\mathbf{B}_{n}$. If this term contains $n^{2}$ distinct variables, these permutation matrices must be disjoint. Therefore the sum of the coefficients of the terms in $\operatorname{per}\left(\mathbf{A}_{n}\right)$ containing $n^{2}$ distinct variables is exactly $L_{n}$.

Theorem 3.9. Let $L_{n}$ be the number of Latin squares of order $n$. Then

$$
L_{n}=\sum_{\mathbf{M} \in M_{n}}(-1)^{\sigma(\mathbf{M})+n} \operatorname{per}(\mathbf{M})^{n}
$$

where $M_{n}$ is the set of all $(0,1) n$-by- $n$ matrices, and $\sigma(\mathbf{M})$ is the number of nonzero entries in $\mathbf{M}$.

Proof. From Lemma 3.8 and Theorem 3.5, it follows that

$$
\begin{aligned}
L_{n} & =\sum_{i=1}^{n^{2}}(-1)^{i+n^{2}} f\left(\operatorname{per}\left(\mathbf{A}_{n}\right), i\right) \\
& =\sum_{i=1}^{n^{2}}(-1)^{i+n} \sum_{s \in S_{n^{2}, i}} \operatorname{per}\left(\mathbf{A}_{n} \mid s\right) \\
& =\sum_{i=1}^{n^{2}}(-1)^{i+n} \sum_{s \in S_{n^{2}, i}} \operatorname{per}\left(\mathbf{B}_{n} \mid s\right)^{n}
\end{aligned}
$$

where the last step follows from the fact that $\operatorname{per}\left(\mathbf{A}_{n}\right)=\operatorname{per}\left(\mathbf{B}_{n}\right)^{n}$. Because $\mathbf{B}_{n}\left|u \neq \mathbf{B}_{n}\right| v$ if $u \neq v$, we can rewrite this as

$$
L_{n}=\sum_{\mathbf{M} \in M_{n}}(-1)^{\sigma(\mathbf{M})+n} \operatorname{per}(\mathbf{M})^{n}
$$

This formula was first given in [9].

## 4 Lower Bounds

In the last section, we showed that sums of coefficients in the permanents of the obstruction matrices $\mathbf{Q}_{n}, \mathbf{T}_{n}, \mathbf{S}_{n}$, and $\mathbf{Z}_{n}$ correspond to the values of $Q(n), T(n), S(n)$, and $T S(n)$, respectively. We then gave a closed-form expression for the function $g$ that computes these sums. More precisely, $g(P, k)$ was the sum of the coefficients of the terms in the polynomial $P$ containing $k$ distinct variables.

Now since each entry in $\mathbf{Q}_{n}$ is a monomial with coefficient 1, the coefficient of a term in $\operatorname{det}\left(\mathbf{Q}_{n}\right)$ is at most the coefficient of the corresponding term in $\operatorname{per}\left(\mathbf{Q}_{n}\right)$. Therefore $\left|g\left(\operatorname{det}\left(\mathbf{Q}_{n}\right), 2 n\right)\right| \leq g\left(\operatorname{per}\left(\mathbf{Q}_{n}\right), 2 n\right)=Q(n)$. The same argument applies to the other problems. As a result we have the following corollary:

Corollary 4.1. Let $S_{m, k}$ be the subset of $\{0,1\}^{m}$ that consists of the tuples containing $k$ ones, $U_{n}$ the set of all $n$-by- $n(0,1)$ diagonally constant matrices, and $V_{n}$ the set of all $n$-by- $n(0,1)$ circulant matrices. Then the following inequalities hold:

$$
\begin{aligned}
Q_{\operatorname{det}}(n) & :=\left|\sum_{i=1}^{2 n}(-1)^{i}\binom{4 n-i-2}{2 n-i} \sum_{s \in S_{4 n-2, i}} \operatorname{det}\left(\mathbf{Q}_{n} \mid s\right)\right| \leq Q(n), \\
T_{\operatorname{det}}(n) & :=\left|\sum_{i=1}^{2 n}(-1)^{i+n} \sum_{s \in S_{2 n, i}} \operatorname{det}\left(\mathbf{T}_{n} \mid s\right)\right| \leq T(n), \\
S_{\operatorname{det}}(n) & :=\left|\sum_{\mathbf{M} \in U_{n}}(-1)^{\gamma(\mathbf{M})} \operatorname{det}(\mathbf{M})\binom{2 n-\gamma(\mathbf{M})-1}{n-\gamma(\mathbf{M})}\right| \leq S(n), \\
T S_{\operatorname{det}}(n) & :=\left|\sum_{\mathbf{M} \in V_{n}}(-1)^{\sigma(\mathbf{M})} \operatorname{det}(\mathbf{M})\right| \leq T S(n),
\end{aligned}
$$

where $\gamma(\mathbf{M})$ is the number of nonzero diagonals in $\mathbf{M}$, and $\sigma(\mathbf{M})$ is the number of ones in the first row of $\mathbf{M}$.
We now show that $T S_{\text {det }}(n)$ is the permanent of Schur's matrix of order $n$; see [10, Sequence A003112].
Let $\mathbf{M}_{n}=\left(\epsilon^{j k}\right)$ be an $n$-by- $n$ matrix where $\epsilon$ is an $n$th root of unity, and let $P_{n}=\operatorname{per}\left(\mathbf{M}_{n}\right)$. The matrix $\mathbf{M}_{n}$ is known as Schur's matrix of order $n$. It has been of interest in number theory, statistics, and coding theory. Its permanent is the topic of [4].

Theorem 4.2. For all $n,\left|P_{n}\right| \leq T S(n) \leq S(n)$.
Proof. From Corollary 4.1, it suffices to show that $T S_{\text {det }}(n)=\left|P_{n}\right|$. This follows immediately from the fact that $P_{n}=g\left(\operatorname{det}\left(\mathbf{Z}_{n}\right), n\right)$ [4].

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