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On a realization of $\{\beta\}$-expansion in QCD

S. V. Mikhailov[a]

[a] *Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Russia*

E-mail: mikhs@theor.jinr.ru

Abstract: We suggest a simple algebraic approach to fix the elements of the $\{\beta\}$-expansion for renormalization group invariant quantities, which uses additional degrees of freedom. The approach is discussed in detail for N^2LO calculations in QCD with the MSSM gluino – an additional degree of freedom. We derive the formulae of the $\{\beta\}$-expansion for the nonsinglet Adler D-function and Bjorken polarized sum rules in the actual N^3LO within this quantum field theory scheme with the MSSM gluino and the scheme with the second additional degree of freedom.

Keywords: Renormalization Group, QCD

Contents

1 Introduction

Here we consider the detailed structure of the QCD perturbation series for the renormalization group invariant (RGI) one-scale dependent quantities like the Bjorken polarized sum rules $S^{\rm Bjp}(Q^2, \mu^2)$,

$$S^{\rm Bjp}(Q^2) = \frac{g_A}{6}\left[C^{\rm Bjp}_{\rm NS}(Q^2/\mu^2, a_s(\mu^2)) + \left(\sum_i q_i\right) C^{\rm Bjp}_{\rm S}(Q^2/\mu^2, a_s(\mu^2))\right], \quad (1.1)$$

or the Adler function $D(Q^2, \mu^2)$,

$$D^{\rm EM}\left(\frac{Q^2}{\mu^2}, a_s(\mu^2)\right) = \left(\sum_i q_i^2\right) d_R D_{\rm NS}\left(\frac{Q^2}{\mu^2}, a_s(\mu^2)\right) + \left(\sum_i q_i\right)^2 d_R D_{\rm S}\left(\frac{Q^2}{\mu^2}, a_s(\mu^2)\right), (1.2)$$

where q_i is the electric charge of the quark, g_A – nucleon axial charge, d_R – the dimension of the quark color representation. Perturbative expression for the nonsinglet (NS) coefficient functions of both the quantities at the renormalization scale $\mu^2 = Q^2$ can be written down as

$$D_{\rm NS}(a_s(\mu^2)) = 1 + \sum_{n\geq 1} a_s^n(\mu^2)\ d_n,\ \ C^{\rm Bjp}_{\rm NS}(a_s(\mu^2)) = 1 + \sum_{l\geq 1} a_s^l(\mu^2)\ c_l; \quad (1.3)$$

they are calculable in the $\overline{\rm MS}$ -scheme, $a_s = \alpha_s/(4\pi)$. We use here only the NS parts of these quantities, $D = D_{\rm NS}$, $C^{\rm Bjp} = C^{\rm Bjp}_{\rm NS}$, omitting the corresponding notation further in the text. The perturbative coefficients d_n (c_n) in this case are the combinations of only the color coefficients. The $\{\beta\}$-expansion representation introduced in [1] prescribes to decompose d_n of RGI quantities in the following way:

$$d_1 = d_1[0]\,, \tag{1.4a}$$

$$d_2 = \beta_0\, d_2[1] + d_2[0]\,, \tag{1.4b}$$

$$d_3 = \beta_0^2\, d_3[2] + \beta_1\, d_3[0,1] + \beta_0\, d_3[1] + d_3[0]\,, \tag{1.4c}$$

$$d_4 = \beta_0^3\, d_4[3] + \beta_1\,\beta_0\, d_4[1,1] + \beta_2\, d_4[0,0,1] + \beta_0^2\, d_4[2] + \beta_1 d_4[0,1] + \beta_0\, d_4[1] + d_4[0]\,, \tag{1.4d}$$

$$\vdots$$

$$d_n = \beta_0^{n-1}\, d_n[n-1] + \cdots + d_n[0]\,, \tag{1.4e}$$

where β_i are the coefficients of the QCD β-function

$$\mu^2 \frac{d a_s(\mu^2)}{d\mu^2} = \beta(a_s) = -a_s^2(\mu^2) \sum_{i\geq 1} \beta_{i-1} a_s^{i-1}(\mu^2)\,, \tag{1.5}$$

and the explicit expressions for β_{0-3} are presented in Appendix A. The notation $i_0, i_1, \ldots$ of the arguments of $d_n[i_0, i_1, \ldots]$ denotes the powers of accompanying $\beta_0, \beta_1, \ldots$. The elements $d_n[\cdot]$ of the decomposition do not depend on the number of active quarks n_f at least up to the actual order $O(a_s^3)$ (n=3). The last important property of the $d_n[\cdot]$ will be discussed at the beginning of Sec.3 and in Sec.4. The decompositions in Eqs.(1.4) should contain the complete knowledge about strong charge renormalization by means of using there all of the possible β_i-terms [2, 3] (see for details the beginning of Sec.4). This kind of the detailed expansion is the essential part of the procedures for the optimization of perturbation series, see, *e.g.* , [3–5] and the references cited therein.

At NLO of QCD the decomposition in (1.4b) looks evident because the term proportional to $\frac{4}{3}T_{\rm R} n_f$ unambiguously marks the contribution of the term proportional to $\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_R n_f$ in d_2, see the discussion in Sec.3B in [3] and the result in Eq.(A.1b) in Appendix A. That decomposition was the starting point of the well-known BLM prescription [6] for the series optimization. How to fix the elements of the decomposition in higher orders? The consideration of color coefficients content of the d_n is not enough for this, so one need to find additional conditions. Let us consider the task to fix the decomposition elements in Eq.(1.4) algebraically, taking Eq.(1.4c) as an example. The "renormalon" term $d_3[2]$ (or $d_n[n-1]$ for any n) at the maximum power of β_0 can be identified by the maximum power of $\frac{4}{3}T_{\rm R} n_f$ (here it is proportional to $C_{\rm F}\ \left(\frac{4}{3}T_{\rm R} n_f\right)^2$), or even calculated independently, see [7]. The corresponding residual in the RHS of Eq.(1.4c) contains 5 Casimir coefficients $C_{\rm F}^3,\ C_{\rm F}^2 C_{\rm A},\ C_{\rm F} C_{\rm A}^2,\ C_{\rm F}^2 T_{\rm R} n_f,\ \ C_{\rm F} C_{\rm A} T_{\rm R} n_f$ that are distributed among three terms $d_3[\cdot]$ there. Finally we need to obtain these three unknown elements $d_3[0,1], d_3[1], d_3[0]$ in the RHS of Eq.(1.6)

$$\bar{d}_3(x) \equiv d_3(x) - \beta_0^2(x)\, d_3[2] = \beta_1(x)\, d_3[0,1] + \beta_0(x)\, d_3[1] + d_3[0]\,, \tag{1.6}$$

where we put variable $x = \frac{4}{3}T_R n_f$. Taking Eq.(1.6) at any three different values of x, $(x_1, x_2, x_3) = X$ and compiling the coupled system of linear equations we can obtain the unique solution of this system under the evident condition that the corresponding determinant Δ_3,

$$\Delta_3(X) = (\beta_0(x_2) - \beta_0(x_1))\,(\beta_1(x_1) - \beta_1(x_0)) - (\beta_0(x_1) - \beta_0(x_0))\,(\beta_1(x_2) - \beta_1(x_1))\,, \quad (1.7)$$

is *not zero*. The opposite condition $\Delta_3 = 0$ unambiguously means that the functions $\beta_0(x), \beta_1(x)$ are linear in x,

$$\frac{\beta_0(x_2) - \beta_0(x_1)}{\beta_0(x_1) - \beta_0(x_0)} = \frac{\beta_1(x_2) - \beta_1(x_1)}{\beta_1(x_1) - \beta_1(x_0)}\,,$$

this is just the case of QCD with only quark degrees of freedom (see the explicit expressions in Eq.(B.1)). Due to this reason one cannot untangle contributions from β_0 and β_1 in N^{2}LO without an additional constraint, see the discussion in [3]. In the case of an additional degree of freedom that contributes to both sides of Eq.(1.6), *i.e.* , to the coefficient d_3 and to β_0, β_1, one can obtain the unique solution. The goal of this note is to elaborate an algebraic formalism to obtain the decompositions in Eqs.(1.4) using additional degrees of freedom like $n_{\tilde{g}}$ – the number of MSSM gluino (we use $y = \dfrac{4}{3}\dfrac{C_\text{A}}{2}n_{\tilde{g}}$) and, may be, other fields that interact following the universal gauge principle and appear only in intrinsic loops. The net effect of this field will be parameterized by means of the parameter z. Further, we shall suggest that the coefficients of perturbation expansion, like d_n (c_n) in the LHS of (1.4), as well as the coefficients of the β-function in the RHS of (1.4) are calculated within the $\overline{\text{MS}}$ scheme and are known functions on the arguments x, y, z.

2 Algebraic approach for the $\{\beta\}$-expansion in N^{2}LO

Let us consider the Adler function $D(x, y)$ [9] as well as the β-coefficients $\beta_0(x, y), \beta_1(x, y)$ presented in Appendix B as functions on both the quark (x) and the MSSM gluinos (y) degrees of freedom (d.f.). In this notation $\beta_i(x, 0) = \beta_i(x)$, $d_n(x, 0) = d_n(x), \ldots$. The results for the decomposition presented below are valid also for the coefficient function $C^{\text{Bjp}}(x, y)$ up to the replacement of the notation and for *any RGI one-scale* quantities.

2.1 The formalism of decomposition for D-function

To simplify the system of equations (SE) based on Eq.(2.1) (the extended by the y d.f. Eq.(1.6)),

$$\bar{d}_3(x, y) \equiv d_3(x, y) - \beta_0^2(x, y)\, d_3[2] = \beta_1(x, y)\, d_3[0, 1] + \beta_0(x, y)\, d_3[1] + d_3[0]\,, \quad (2.1)$$

we take for the components of X: $x_0, x_1, (x_{01}, y_{01})$, the special values – the roots of equations

$$\beta_0(x_0) = 0,\ \beta_1(x_1) = 0,\ \{\beta_0(x_{01}, y_{01}) = 0,\ \beta_1(x_{01}, y_{01}) = 0\}. \quad (2.2)$$

For this X_3 the SE$_3$ looks like

$$\begin{cases} d_3(x_{01}, y_{01}) \;=\; d_3[0] \\ \quad d_3(x_0, 0) \;=\; \beta_1(x_0) d_3[0,1] + d_3[0] \\ \quad \bar{d}_3(x_1, 0) \;=\; \beta_0(x_1) d_3[1] + d_3[0]\,. \end{cases} \tag{2.3}$$

Now the value of the determinant $\Delta_3(X_3) = \beta_0(x_1)\beta_1(x_0)$ that follows from Eq.(1.7) or, can be obtained from the SE$_3$ (2.3) directly. The determinant $\Delta_3(X_3) \neq 0$, therefore the unique solution of the SE$_3$ exists, it is

$$d_3[0] \;=\; d_3(x_{01}, y_{01})\,, \tag{2.4a}$$

$$d_3[0,1] \;=\; \big(d_3(x_0) - d_3(x_{01}, y_{01})\big)/\beta_1(x_0)\,, \tag{2.4b}$$

$$d_3[1] \;=\; \big(\bar{d}_3(x_1) - d_3(x_{01}, y_{01})\big)/\beta_0(x_1)\,. \tag{2.4c}$$

These values were obtained first in [1] using another trick; here they are presented explicitly in Eq.(A.1) in Appendix A.

2.2 How to obtain the $\{\beta\}$-expansion for C^{Bjp} from one for D

To relate the already known structure of d_3 (the solutions in (2.4)) to the corresponding $\{\beta\}$-expansion of c_3, we use the generalized Crewther relation (CR) [2, 3],

$$D(a_s) \cdot C^{\mathrm{Bjp}}(a_s) = 1\!\!1 + \beta(a_s)\, \cdot K(a_s)\,, \tag{2.5}$$

where $K(a_s) = \sum_{n=1} a_s^{n-1} K_n$ is a polynomial in a_s. In the case of the β-function having identically zero coefficients $\beta_i = 0$, the generalized CR (2.5) returns to its initial form [8] with only $1\!\!1$ in its RHS that expresses the unbroken conformal symmetry. The later condition relates the $d_n[0]$, $c_n[0]$ elements in every order (see definition (4.1) and Eq.(4.2) in [3])

$$c_n[0] \;=\; -d_n[0] - \sum_{l=1}^{n-1} d_l[0] c_{n-l}[0]\,. \tag{2.6}$$

The explicit solution of the relation (2.6) with respect to $c_k[0]$ is

$$c_k[0] = (-)^k \mathrm{det}[D_0^{(k)}] \equiv (-)^k \begin{vmatrix} d_1 & 1 & 0 & \dots & 0 \\ d_2 & d_1 & 1 & \dots & 0 \\ d_3 & d_2 & d_1 & \dots & 0 \\ \dots & & & d_1 & 1 \\ d_k & d_{k-1} & d_{k-2} & \dots & d_1 \end{vmatrix}\,, \tag{2.7}$$

here $D_0^{(k)}$ – matrix, which consists of $d_i \equiv d_i[0]$ elements. The knowledge of the element $c_3[0] = -d_3[0] + 2d_1 d_2[0] - (d_1)^3$, followed from (2.6), allows us to fix all the other elements of the expansion in this order, *i.e.* to disentangle the contributions from $c_3[1]\beta_0$ and $c_3[0,1]\beta_1$, which were discussed in detail in Sec.IV B in [3].

From another side one can use in the RHS of Eq.(2.5) the second term proportional to $\beta(a_s)$ that expresses the conformal symmetry-breaking. This leads to the series of relations [2, 3] for the elements of the different orders n at β_{n-1}, *e.g.* ,

$$d_2[1] + c_2[1] = d_3[0,1] + c_3[0,1] = \ldots = d_n[\underbrace{0,0,\ldots,1}_{n-1}] + c_n[\underbrace{0,0,\ldots,1}_{n-1}] = 3C_{\rm F}\left(\frac{7}{2} - 4\zeta_3\right). \quad (2.8)$$

In the third order this gives the equation $c_3[0,1] = d_2[1] + c_2[1] - d_3[0,1]$, which fixes $c_3[0,1]$ and also admits restoration of two other terms $c_3[1]$, $c_3[0]$.

Both of the ways provide the same results for the elements $c_3[0,1], c_3[1]$, $c_3[0]$ of the $\{\beta\}$-expansion [3]. As a byproduct of the procedure we predicted the contribution to $C^{\rm Bjp}$ from the new d.f., here the MSSM gluino, see Eq.(4.11) in [3], if the one is already known for D and vice versa.

3 The $\{\beta\}$-expansion for Bjorken polarized SR and D-function in N^{3}LO

In the 5 loop case, $d_4(x)$ was first obtained in [10] as the polynomial with numerical coefficients, then all the color coefficients in decomposition for $d_4(x)$ and $c_4(x)$ were presented in [11]. Following the $\{\beta\}$-expansion we propose for these coefficients the decomposition,

$$\begin{aligned}\bar{d}_4(x,y) &\equiv d_4(x,y) - \beta_0^3\, d_4[3] \\ &= \beta_1\,\beta_0\, d_4[1,1] + \beta_2\, d_4[0,0,1] + \beta_0^2\, d_4[2] + \beta_1 d_4[0,1] + \beta_0\, d_4[1] + d_4[0]\,. \end{aligned} \quad (3.1)$$

Here one has six unknown elements $d_4[1,1]$, $d_4[0,0,1]$, $d_4[2]$, $d_4[0,1]$, $d_4[1]$, $d_4[0]$, while the seventh element $d_4[3]$ can be directly identified. The ten Casimirs (here $T_f \equiv T_{\rm R} n_f$) $C_F^4\,, C_F^3 T_f\,, C_F^2 T_f^2\,, C_F T_f^3\,,\ C_F^3 C_A\,,\ C_F^2 T_f C_A, C_F T_f^2 C_A\,,\ C_F^2 C_A^2\,, C_F T_f C_A^2\,,\ C_F C_A^3$ are distributed among all of the $d_4[\cdot]$ elements, while the abelian elements of the box subgraphs with four gluon legs, related to color coefficients $n_f\, d_F^{abcd} d_F^{abcd}/d_R$ (quark box inside), $d_F^{abcd} d_A^{abcd}/d_R$ (gluon box inside), enter into $d_4[0]$. These terms do not contribute to the renormalization of the charge[1] a_s, see also the discussion of the subject in [12]. Although the corresponding 5-loop diagrams contain these one-loop boxes, further contraction of the subgraphs (see the discussion in [13]) do not contribute to β_0. Due to this reason $d_4[0]$ get (x,y)-dependence $d_4[0] \to d_4[0](x,y)$. We decompose it as $d_4[0](x,y) = \tilde{d}_4[0] + \delta d_4(x,y)$, where the (x,y)-dependent part $\delta d(x,y)$ is well recognized, while $\delta d_4(x,0)(\delta c_4(x,0))$ is already known from the result in [11] (see Eq.(A.7) in Appendix A). Therefore, the n_f $(n_{\tilde{g}})$-dependence becomes partly separated from the charge renormalization for the first time in N^{3}LO.

3.1 Decomposition with 2 degrees of freedom x, y

To obtain these $d_4[\cdot]$ elements, one can take six points X in the plane (x,y). Then one takes the set X as the arguments of $d_4(x,y)$ and $\beta_i(x,y)$ and compiles the system of linear equations (SE$_6$), based on Eq.(3.1), with respect to these six unknown elements of the β-expansion. Again, to simplify the calculation, we take for these six components of X_6:

[1] I thank A. Grozin for clarifying this subject

x_0, x_1, x_2, (x_{01}, y_{01}), (x_{02}, y_{02}), (x_{12}, y_{12}) the roots of the equations and the systems of equations

$$\beta_0(x_0, 0) = 0, \beta_1(x_1, 0) = 0, \beta_2(x_{2m(p)}, 0) = 0, \{\beta_0(x_{01}, y_{01}) = 0, \beta_1(x_{01}, y_{01}) = 0\}, \tag{3.2a}$$
$$\{\beta_0(x_{02}, y_{02}) = 0, \beta_2(x_{02}, y_{02}) = 0\}, \{\beta_1(x_{12}, y_{12}) = 0, \beta_2(x_{12}, y_{12}) = 0\}. \tag{3.2b}$$

We shall supply the solutions in Eqs.(3.2b) the subscripts: $x_{02m(p)}, y_{02m(p)}, x_{12m(p)}, y_{12m(p)}$, to separate the different roots $m(-), p(+)$ of quadratic equations for the cases where $\beta_2(x, y)$ are involved (see the expression in (B.1c)). The determinant $\Delta_6(X_6)$ of the corresponding SE_6 is

$$\Delta_6(X_6) = \beta_0(x_1)\beta_0(x_{2m})\beta_1(x_{2m})\mathbf{Re}\beta_0(x_{12m}, y_{12m})\left[\beta_0(x_1) - \mathbf{Re}\beta_0(x_{12m}, y_{12m})\right]\delta_6, \tag{3.3}$$
$$\delta_6 = \left[\beta_1(x_{02m}, y_{02m})\beta_2(x_0) - \beta_1(x_{02m}, y_{02m})\beta_2(x_{01}, y_{01}) + \beta_1(x_0)\beta_2(x_{01}, y_{01})\right]. \tag{3.4}$$

This value $\Delta_6(X_6) \neq 0$; therefore, the solution of this SE_6 exists and unique, and can be obtained like the solution of Eq.(2.4) for the N^2LO in Sec.2.1. Therefore, to derive the $\{\beta\}$-expansion for d_4, it is *enough to obtain one at an additional single* d.f. y, $d_4 \to d_4(x, y)$ together with the coefficients $\beta_{0,1,2}(x, y)$ (see Appendix B).

We present the solutions of SE_6 for a number of elements in the explicit form, taking the notation for the arguments (x_{ij}, y_{ij}) and the function Y_4 for shortness,

$$r_{ij} = (x_{ij}, y_{ij}), \quad Y_4(X_6) \equiv \bar{d}_4(X_6) - \delta d_4(X_6), \tag{3.5}$$

$$d_4[0,0,1] = \left[Y_4(r_{01})(\beta_1(x_0) - \beta_1(r_{02})) - Y_4(r_{02})\beta_1(x_0) + Y_4(x_0)\beta_1(r_{02})\right]/\delta_6, \tag{3.6a}$$
$$d_4[0,1] = \left[Y_4(r_{02})(\beta_2(x_0) - \beta_2(r_{01})) - Y_4(r_{01})\beta_2(x_0) + Y_4(x_0)\beta_2(r_{01})\right]/\delta_6, \tag{3.6b}$$
$$\tilde{d}_4[0] = \left[Y_4(r_{02})\beta_2(r_{01})\beta_1(x_0) + Y_4(r_{01})\beta_1(r_{02})\beta_2(x_0) - Y_4(x_0)\beta_2(r_{01})\beta_1(r_{02})\right]/\delta_6. \tag{3.6c}$$

Just these elements will be used for the relation with similar elements in C^{Bjp}.

Of course, one can take another set X'_6 and construct the corresponding SE'_6. The necessary condition is that its determinant $\Delta_6(X'_6) \neq 0$. In any case the solution for the elements $d_4[\cdot]$ should be the same. The usage of the roots in Eqs.(3.2) to construct X_6 leads to the simplification of the final SE.

3.2 Relations between the elements of D and C^{Bjp}

Suppose that the coefficient functions for the Adler D-function $d_4(x, y)$ and the Bjorken SR $c_4(x, y)$ are known. Then, based on two terms in the RHS of the Crewther relation, Eq.(2.5), one can obtain for the sum of these functions $d_4(x, y) + c_4(x, y)$ and their elements (see [2]) a series of the relations. In part, one can obtain from (2.6) the sum of "zero" elements,

$$d_4[0](x, y) + c_4[0](x, y) = \tilde{d}_4[0] + \tilde{c}_4[0] = 2d_1 d_3[0] - 3d_1^2 d_2[0] + d_2[0]^2 + d_1^4, \tag{3.7}$$

the term $\delta d_4(x, y)$ is cancel in the sum $d_4[0] + c_4[0]$. Generally speaking, for n-order case

$$c_n[0]+d_n[0]=-\sum_{l=1}^{n-1} d_{n-l}[0](-)^l\det[D_0^{(l)}]=(-)^n\begin{vmatrix} d_1 & 1 & 0 & \dots & 0\\ d_2 & d_1 & 1 & \dots & 0\\ d_3 & d_2 & d_1 & \dots & 0\\ \dots & & & d_1 & 1\\ 0 & d_{n-1} & d_{n-2} & \dots & d_1 \end{vmatrix}\,, \tag{3.8}$$

where $D_0^{(l)}$ is defined in (2.7). The LHS of Eq.(3.8) is of n-order, while its RHS dependents on the $d_l[0]$ elements of less orders $l \leqslant n-1$, therefore this equation can serve a good check for next order results. The others relations are:

$$(d_4+c_4)\,(x_{01},y_{01}) = \beta_2(x_{01},y_{01})\left(\underline{d_4[0,0,1]+c_4[0,0,1]}\right)+\underline{\underline{d_4[0]+c_4[0]}}= \tag{3.9a}$$

$$=\beta_2(x_{01},y_{01})\underline{3C_\mathrm{F}\left(\frac{7}{2}-4\zeta_3\right)}+\left(\underline{\underline{2d_1d_3[0]-3d_1^2d_2[0]+d_2[0]^2+d_1^4}}\right) \tag{3.9b}$$

$$=\beta_2(x_{01},y_{01})\underline{3C_\mathrm{F}\left(\frac{7}{2}-4\zeta_3\right)}+\underline{\underline{(3C_\mathrm{F})^2\left[\left(\frac{175}{6}-144\zeta_3\right)\mathrm{C}_\mathrm{A}^2+44\mathrm{C}_\mathrm{F}\mathrm{C}_\mathrm{A}-\frac{37}{4}\mathrm{C}_\mathrm{F}^2\right]}}$$

$$(d_4+c_4)\,(x_{02},y_{02}) = \beta_1(x_{02},y_{02})\left(\underline{d_4[0,1]+c_4[0,1]}\right)+d_4[0]+c_4[0]= \tag{3.9c}$$

$$=\beta_1(x_{02},y_{02})\underline{3C_\mathrm{F}}\Big[\underline{C_\mathrm{A}\left(\frac{47}{9}-\frac{16}{3}\zeta_3\right)+C_\mathrm{F}\left(-\frac{397}{18}-\frac{136}{3}\zeta_3+80\zeta_5\right)+}$$
$$\underline{3C_\mathrm{F}\left(\frac{40}{3}-12\zeta_3\right)}\Big]+(3C_\mathrm{F})^2\left[\left(\frac{175}{6}-144\zeta_3\right)\mathrm{C}_\mathrm{A}^2+44\mathrm{C}_\mathrm{F}\mathrm{C}_\mathrm{A}-\right.$$
$$\left.\frac{37}{4}\mathrm{C}_\mathrm{F}^2\right]\,, \tag{3.9d}$$

$$(d_4+c_4)\,(x_0) = \beta_2(x_0)\,(d_4[0,0,1]+c_4[0,0,1])+\beta_1(x_0)\,(d_4[0,1]+c_4[0,1])+$$
$$d_4[0]+c_4[0]= \tag{3.9e}$$
$$=3\mathrm{C}_\mathrm{F}\left[-\frac{111}{4}\mathrm{C}_\mathrm{F}^3+\mathrm{C}_\mathrm{A}\mathrm{C}_\mathrm{F}^2\left(-\frac{1661}{36}+\frac{2618}{3}\zeta_3-880\zeta_5\right)+\right.$$
$$\left.\mathrm{C}_\mathrm{A}^2\mathrm{C}_\mathrm{F}\left(-\frac{3337}{18}+\frac{896}{3}\zeta_3-3516\zeta_5\right)+\mathrm{C}_\mathrm{A}^3\left(-\frac{28931}{144}+\frac{1351}{6}\zeta_3\right)\right]. \tag{3.9f}$$

The RHSs of (3.9a, 3.9c, 3.9e) are presented by mean of the already known results for d_3 and c_3 – (3.9b, 3.9d), see [3] and Appendix A here. The last Eq.(3.9e), suggested in [2], has already been verified and is put here for illustration and comparison with the two previous equations. Let us conclude,

(i) For the values of d_4+c_4 on (x_{01},y_{01}) and (x_{02},y_{02}) Eqs.(3.9) provide the simple check – the RHS of (3.9b) and (3.9d), respectively.

(ii)The equalities of the underlined terms in Eqs.(3.9) are realized independently following Eqs.(2.8) and (2.6). Therefore, the second equalities in (3.9a) and (3.9c) allow one to get $d_4[0,0,1], d_4[0,1], d_4[0]$ through their partners $c[\cdot]$ and vice versa using the solutions in Eq.(3.6), see Eqs.(30), (31)[2] in [2]. But, one cannot restore all the elements of C in N^{3}LO based only on CR and the known D opposite to the case of N^{2}LO, see Sec.2.2.

Let us mention thereupon an alternative approach to fix $d_4[\cdot]$, $c_4[\cdot]$ without additional d.f., which was suggested in [13] and was inspired by the structure of the RHS of CR (2.5). The idea is based on the specific proposition that the perturbation series for D and $C^{\rm Bjp}$ can be expanded in powers $(\beta(a_s)/a_s)^n$ similar to that had been proposed for the "conformal symmetry braking term" $\beta(a_s)K(a_s)$ in the RHS of CR, see the presentation in Eq.(6) in [2]. The results for the elements obtained within this approach differ from ours.

3.3 What can we get at 3 degrees of freedom x, y, z

Let us imagine that we have an additional third "intrinsic" d.f. that manifests itself as the parameter z. In this case $d_n = d_n(x,y,z)$, $\beta_i = \beta_i(x,y,z)$; therefore, one can use the points in (x,y,z) space to construct the set X:

$$\{\beta_0(x_{012},y_{012},z_{012}) = 0, \beta_1(x_{012},y_{012},z_{012}) = 0, \beta_2(x_{012},y_{012},z_{012}) = 0\} \tag{3.10}$$

instead of the later constraint in Eq.(3.2b) (if the solution of SE (3.10) exists). Let us call this solution $r_{012} = (x_{012},y_{012},z_{012})$ and $\beta_i(x) = \beta_i(x,0,0)$, $\beta_i(x,y) = \beta_i(x,y,0)$, $d_n(x) = d_n(x,0,0)$, $d_n(x,y) = d_n(x,y,0), \ldots$ for shortness. From Eq.(3.1) and Eq.(3.10) it immediately follows that

$$d_4(r_{012}) = \tilde{d}_4[0] + \delta d(r_{012}), \tag{3.11}$$

$$c_4(r_{012}) + d_4(r_{012}) = \tilde{d}_4[0] + \tilde{c}_4[0] = 2d_1d_3[0] - 3d_1^2d_2[0] + d_2[0]^2 + d_1^4. \tag{3.12}$$

Equation (3.12) provides an independent test for the c_4, d_4 results. In the case of constraint (3.10), the procedure for obtaining the $\{\beta\}$-expansion is simplified significantly. Let us show the list of the evident solutions of SE$_6$ taking into account the definition of Y_4 in (3.5)

$$d_4[0,0,1] = \left(Y_4(x_{01},y_{01}) - Y_4(r_{012})\right)/\beta_2(x_{01},y_{01}), \tag{3.13a}$$

$$d_4[0,1] = \left(Y_4(x_{02},y_{02}) - Y_4(r_{012})\right)/\beta_1(x_{02},y_{02}), \tag{3.13b}$$

$$d_4[0,1] = \left(Y_4(x_0) - Y_4(r_{012}) - \beta_2(x_0)d_4[0,0,1]\right)/\beta_1(x_0). \tag{3.13c}$$

Here Eq.(3.13c) together with Eq.(3.13a) admit checking of Eq.(3.13b). The solutions in (3.11,3.13) for $d_4[0], d_4[0,1], d_4[0,0,1]$ look evidently easier than ones in (3.6) obtained with only single additional d.f. The solutions for the elements $d_4[1]$, $d_4[2]$ can also be easily obtained, but they look rather cumbersome and we do not show them. The solutions presented here exhaust the problem of fixing the elements of the $\{\beta\}$-expansion in N^{3}LO.

[2]There in the RHS of Eq.(31) was missed the term $+(-47/48+\zeta_3)C_{\rm F}C_{\rm A}$

4 What can we expect for $\{\beta\}$-expansion in N^{4}LO

Let us consider the structure of a 6 loop result in order a_s^5, *i.e.* , at $n=5$,

$$\begin{aligned} d_5(x,\dots) = {} & \beta_0^4\, d_5[4] + \beta_2\beta_0\, d_1[1,0,1] + \beta_1^2\, d_5[0,2] + \beta_1\beta_0^2\, d_5[2,1] + \beta_3\, d_5[0,0,0,1] + \\ & \beta_0^3\, d_5[3] + \beta_1\,\beta_0 d_5[1,1] + \beta_2\, d_5[0,0,1] + \\ & \beta_0^2\, d_5[2] + \beta_1\, d_5[0,1] + \beta_0\, d_5[1] + d_5[0]. \end{aligned} \tag{4.1}$$

The number of new elements in this order, counting the elements starting with β_0^4 up to β_3 in the first line of Eq.(4.1), coincides with the number of partitions $p(5-1)=5$. The other terms in (4.1) repeat the structure of the result in previous order at $n=4$. In general, for the term of the order n, d_n, one should count new terms from $\beta_0^{(n-1)}$ up to $\beta_{(n-1)-1}$ that gives their number $p(n-1)$, while the complete number $N(n)$ of all the terms is the sum $N(n)=\sum_{l=0}^{(n-1)} p(l)$ that leads to series $\{1,2,4,7,\mathbf{12}\ (\text{here}),19,30,45,\dots\}$ sequentially in each order, see, *e.g.* , [15][3], and the terms in Eqs.(1.4), for an example.

The coefficient $d_5(x,y)$ is formed by the variety of 6-loop diagrams that get contributions from the intrinsic box- and pentagon-subgraphs with gluon legs that introduce into d_5 a specific $(n_f, n_{\tilde{g}})$-dependence that does not relate to the charge renormalization. Indeed, the new color coefficients $\dfrac{n_{\tilde{g}}}{2}\, d_F^{abcde} d_A^{abcde}/d_R$ (gluino pentagon inside), $n_f\, d_F^{abcde} d_F^{abcde}/d_R$ (quark pentagon inside) enter into $d_5[0](x,y)$ together with the contributions from the box-graphs, which was already mentioned in Sec.3. The contributions from the latter box-graphs, $n_{\tilde{g}} d_F^{abcd} d_A^{abcd}/d_R$, $n_f d_F^{abcd} d_F^{abcd}/d_R$ enter[4] now into the element $d_5[1] \to d_5[1](x,y)$. All these contributions, proportional to n_f, $n_{\tilde{g}}$, are well recognized and can be accumulated in the specific term $\delta d_5(x,y)$, like it was done for $\delta d_4(x,y)$ in Sec.3. The element $\beta_0^4\, d_5[4]$ should also be well recognized; therefore, one has $11 = \mathbf{12}-1$ unknown elements $d_5[\cdot]$. By analogy with the previous lower orders procedure one can compile SE$_{11}$ based on Eq.(4.1) with the rearranged LHS

$$Y_5(X) = \bar{d}_5(X) - \delta d_5(X) = d_5(X) - \beta_0^4(X)\, d_5[4] - \delta d_5(X)\,,$$

take the equation with the arguments at 11 points (X_{11}) on the plane (x,y). The SE$_{11}$ constructed in this way has unique solution with respect to the $d_5[\cdot]$ elements under the condition the corresponding determinant of the system $\Delta_{11}(X_{11}) \neq 0$.

Let us take for these 11 components of X_{11} the roots of the equations

$$\beta_0(x_0)=0,\ \beta_1(x_1)=0,\ \beta_2(x_{2m})=0,\ \beta_2(x_{2p})=0, \tag{4.2a}$$

$$\{\beta_0(x_{01},y_{01})=0, \beta_1(x_{01},y_{01})=0\}, \{\beta_0(x_{02m},y_{02m})=0, \beta_2(x_{02m},y_{02m})=0\}, \tag{4.2b}$$

$$\{\beta_0(x_{02p},y_{02p})=0, \beta_2(x_{02p},y_{02p})=0\}, \tag{4.2c}$$

$$\{\beta_1(x_{12},y_{12})=0, \beta_2(x_{12},y_{12})=0\}, \{\beta_0(x_{03},y_{03})=0, \beta_3(x_{03},y_{03})=0\}, \tag{4.2d}$$

$$\{\beta_1(x_{13},y_{13})=0, \beta_3(x_{13},y_{13})=0\}, \{\beta_2(x_{23},y_{23})=0, \beta_3(x_{23},y_{23})=0\}. \tag{4.2e}$$

[3] I thank N. Volchanskiy who paid my attention to this ref.

[4] the definition of the color elements $d_R^{a_1 a_2 \dots a_n}$ is presented, *e.g.* , in [16]

This choice of X_{11} simplifies the set of equations, as we made sure in the previous cases of the constructing sets $X_{3,6}$ in Eqs(2.2, 3.2), respectively. The determinant corresponding to SE_{11} $\Delta_{11}(X_{11}) \neq 0$ but looks too cumbersome to demonstrate it here.

5 Conclusion

We suggest an algebraic approach to fix the elements of the $\{\beta\}$-expansion for renormalization group invariant quantities using additional degrees of freedom. This approach is discussed in detail for $\mathrm{N}^2\mathrm{LO}$ calculations of the nonsinglet Adler D-function and for Bjorken polarized sum rules C^{Bjp} within QCD with the MSSM gluino – an additional degree of freedom. We derive the explicit formulae for the elements of the $\{\beta\}$-expansion for these quantities, $d_n[\cdot]$ and $c_n[\cdot]$ respectively, in the actual $\mathrm{N}^3\mathrm{LO}$ within the aforementioned quantum field theory scheme with the MSSM gluino. This fixed structure together with the explicit elements $d_4[\cdot](c_4[\cdot])$ can be considered as a prediction for additional degrees of freedom to be included into consideration. Really, these degrees of freedom enter into either the well-known β_i, the coefficients of the β-function, or the well-recognized terms of the structure. Another kind of predictions is provided by the relation between the elements $d_n[\cdot]$ and $c_n[\cdot]$ in virtue of the Crewther relation. We constructed also the fixation procedure for the case of two additional degrees of freedom. Finally, we discussed the structure and properties of $\{\beta\}$-expansion for higher orders considering the $\mathrm{N}^4\mathrm{LO}$ with the MSSM gluino degree of freedom as an example, which also admits the fixation of the expansion elements.

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A Explicit formulas for the elements of D and C

For the Adler function $D^{\rm NS}$ the corresponding elements read [2, 3][5]

$$
\begin{aligned}
d_1 &= 3\mathrm{C_F}; && \text{(A.1a)}\\
d_2[1] &= d_1\left(\frac{11}{2}-4\zeta_3\right); \qquad d_2[0] = d_1\left(\frac{\mathrm{C_A}}{3}-\frac{\mathrm{C_F}}{2}\right); && \text{(A.1b)}\\
d_3[2] &= d_1\left(\frac{302}{9}-\frac{76}{3}\zeta_3\right);\ d_3[0,1] = d_1\left(\frac{101}{12}-8\zeta_3\right); && \text{(A.1c)}\\
d_3[1] &= d_1\left(\mathrm{C_A}\left(-\frac{3}{4}+\frac{80}{3}\zeta_3-\frac{40}{3}\zeta_5\right)-\mathrm{C_F}\left(18+52\zeta_3-80\zeta_5\right)\right); && \text{(A.1d)}\\
d_3[0] &= d_1\left(\left(\frac{523}{36}-72\zeta_3\right)\mathrm{C_A^2}+\frac{71}{3}\mathrm{C_A C_F}-\frac{23}{2}\mathrm{C_F^2}\right). && \text{(A.1e)}
\end{aligned}
$$

$$
\begin{aligned}
c_1 &= -3\ \mathrm{C_F}; && \text{(A.2a)}\\
c_2[1] &= 2c_1; c_2[0] = c_1\left(\frac{1}{3}\mathrm{C_A}-\frac{7}{2}\mathrm{C_F}\right); && \text{(A.2b)}
\end{aligned}
$$

$$
\begin{aligned}
c_3[2] &= \frac{115}{18}c_1;\ c_3[0,1] = c_1\left(\frac{59}{12}-4\zeta_3\right); && \text{(A.2c)}\\
c_3[1] &= -c_1\left(\left(\frac{166}{9}-\frac{16}{3}\zeta_3\right)\mathrm{C_F}+\left(\frac{215}{36}-32\zeta_3+\frac{40}{3}\zeta_5\right)\mathrm{C_A}\right); && \text{(A.2d)}\\
c_3[0] &= c_1\left(\left(\frac{523}{36}-72\zeta_3\right)\mathrm{C_A^2}+\frac{65}{3}\mathrm{C_F C_A}+\frac{\mathrm{C_F^2}}{2}\right) && \text{(A.2e)}
\end{aligned}
$$

$$
d_3[0,1]-c_3[0,1] = d_1\left(\frac{40}{3}-12\zeta_3\right) \qquad \text{(A.3)}
$$

$$
\begin{aligned}
(d_4[0]+c_4[0])(x,y) &= \tilde{d}_4[0]+\tilde{c}_4[0] = 2d_1d_3[0]-3d_1^2d_2[0]+d_2[0]^2+d_1^4 =\\
&= d_1^2\left[\left(\frac{175}{6}-144\zeta_3\right)\mathrm{C_A^2}+44\mathrm{C_F C_A}-\frac{37}{4}\mathrm{C_F^2}\right] && \text{(A.4)}
\end{aligned}
$$

From the results in [11] it follows that

$$
\begin{aligned}
d_4[3] &= d_1\left(\frac{6131}{27}-\frac{406}{3}\zeta_3-60\zeta_5\right),\ c_4[3] = c_1\left(\frac{2}{27}\right), && \text{(A.5)}\\
\delta d_4(x,y) &= \left[\frac{y}{2C_A}\frac{d_A^{abcd}d_F^{abcd}}{d_R}+x\frac{d_F^{abcd}d_F^{abcd}}{d_R}\right]3\cdot\left(-104-128\zeta_3+320\zeta_5\right), && \text{(A.6)}\\
\delta c_4(x,y) &= -\delta d_4(x,y). && \text{(A.7)}
\end{aligned}
$$

[5] we had a missprint in the expression for $d_3[1]$ in articles [2, 3]: in the first parenthesis at C_A should be $-\frac{3}{4}$, see Eq.(A.1d), instead of $+\frac{3}{4}$ there.

B The β-function coefficients

The required β-function coefficients with the Minimal Supersymmetric Model (MSSM) light gluinos $n_{\tilde{g}}$ [14], and the number n_f of quark flavors, calculated in the $\overline{\mathrm{MS}}$ scheme are

$$\beta_0\,(x,y) = \frac{11}{3}C_\mathrm{A} - x - y = \frac{11}{3}C_\mathrm{A} - \frac{4}{3}\left(T_R n_f + \frac{n_{\tilde{g}}C_\mathrm{A}}{2}\right)\,; \tag{B.1a}$$

$$\begin{aligned}\beta_1\,(x,y) &= \frac{34}{3}C_\mathrm{A}^2 - (5C_\mathrm{A}+3C_\mathrm{F})\,x - 8C_\mathrm{A}y \\ &= \frac{34}{3}C_\mathrm{A}^2 - \frac{20}{3}C_\mathrm{A}\left(T_R n_f + \frac{n_{\tilde{g}}C_\mathrm{A}}{2}\right) - 4\left(T_R n_f C_\mathrm{F} + \frac{n_{\tilde{g}}C_\mathrm{A}}{2}C_\mathrm{A}\right);\end{aligned} \tag{B.1b}$$

$$\begin{aligned}\beta_2\,(x,y) &= \frac{2857}{54}C_\mathrm{A}^3 - x\left(\frac{1415}{36}C_\mathrm{A}^2 + \frac{205}{12}C_\mathrm{A}C_\mathrm{F} - \frac{3}{2}C_\mathrm{F}^2\right) + x^2\left(\frac{11}{4}\mathrm{C_F} + \frac{79}{24}C_\mathrm{A}\right) - \\ &\quad \frac{494}{9}C_\mathrm{A}^2 + 2xy\left(\frac{11}{8}C_\mathrm{F} + \frac{14}{3}C_\mathrm{A}\right)C_\mathrm{A} + y^2\frac{145}{24}C_\mathrm{A}; \\ &= \frac{2857}{54}C_\mathrm{A}^3 - n_f T_R\left(\frac{1415}{27}C_\mathrm{A}^2 + \frac{205}{9}C_\mathrm{A}C_\mathrm{F} - 2C_\mathrm{F}^2\right) + (n_f T_R)^2\left(\frac{44}{9}\mathrm{C_F} + \frac{158}{27}C_\mathrm{A}\right) - \\ &\quad \frac{988}{27}n_{\tilde{g}}C_\mathrm{A}(C_\mathrm{A}^2) + n_{\tilde{g}}C_\mathrm{A}n_f T_R\left(\frac{22}{9}C_\mathrm{A}C_\mathrm{F} + \frac{224}{27}C_\mathrm{A}^2\right) + (n_{\tilde{g}}C_\mathrm{A})^2\frac{145}{54}C_\mathrm{A}\,,\end{aligned} \tag{B.1c}$$

where we have introduced appropriate rescaled variables $x = \dfrac{4}{3}T_R n_f$ and $y = \dfrac{4}{3}\dfrac{C_\mathrm{A}}{2}n_{\tilde{g}}$ after the first equality to simplify the expressions. The N^{3}LO coefficient $\beta_3\,(n_f, n_{\tilde{g}})$ has been obtained recently in [16, 17],

$$\begin{aligned}\beta_3\,(n_f,n_{\tilde{g}}) &= -\left(\frac{150653}{486} - \frac{44}{9}\zeta_3\right)C_\mathrm{A}^4 + \left(\frac{80}{9} - \frac{704}{3}\zeta_3\right)\frac{d_A^{abcd}d_A^{abcd}}{N_A} \\ &\quad + n_{\tilde{g}}\left[\left(\frac{68507}{243} - \frac{52}{9}\zeta_3\right)C_\mathrm{A}^4 - \left(\frac{256}{9} - \frac{832}{3}\zeta_3\right)\frac{d_A^{abcd}d_A^{abcd}}{N_A}\right] \\ &\quad - n_{\tilde{g}}^2\left[\left(\frac{26555}{486} - \frac{8}{9}\zeta_3\right)C_\mathrm{A}^4 + \left(\frac{176}{9} - \frac{128}{3}\zeta_3\right)\frac{d_A^{abcd}d_A^{abcd}}{N_A}\right] - \frac{23}{27}C_\mathrm{A}(n_{\tilde{g}}C_\mathrm{A})^3 \\ &\quad + n_f T_\mathrm{R}\left[-46C_\mathrm{F}^3 + \left(\frac{4204}{27} - \frac{352}{9}\zeta_3\right)C_\mathrm{A}C_\mathrm{F}^2 - \left(\frac{7073}{243} - \frac{656}{9}\zeta_3\right)C_\mathrm{A}^2C_\mathrm{F}\right. \\ &\quad \left. + \left(\frac{39143}{81} - \frac{136}{3}\zeta_3\right)C_\mathrm{A}^3\right] - n_f\left(\frac{512}{9} - \frac{1664}{3}\zeta_3\right)\frac{d_F^{abcd}d_A^{abcd}}{N_A} \\ &\quad + (n_f T_\mathrm{R})^2\left[-\left(\frac{1352}{27} - \frac{704}{9}\zeta_3\right)C_\mathrm{F}^2 - \left(\frac{17152}{243} + \frac{448}{9}\zeta_3\right)C_\mathrm{A}C_\mathrm{F}\right. \\ &\quad \left. - \left(\frac{7930}{81} + \frac{224}{9}\zeta_3\right)C_\mathrm{A}^2\right] + n_f^2\left(\frac{704}{9} - \frac{512}{3}\zeta_3\right)\frac{d_F^{abcd}d_F^{abcd}}{N_A} \\ &\quad - (n_f T_\mathrm{R})^3\left[\frac{1232}{243}C_\mathrm{F} + \frac{424}{243}C_\mathrm{A}\right]\end{aligned}$$

$$
\begin{aligned}
&+n_{\tilde{g}}C_{\rm A}(n_f T_{\rm R})\left[\left(\frac{152}{27}+\frac{64}{9}\zeta_3\right)C_{\rm F}^2-\left(\frac{23480}{243}-\frac{352}{9}\zeta_3\right)C_{\rm A}C_{\rm F}\right.\\
&\left.-\left(\frac{30998}{243}+\frac{128}{3}\zeta_3\right)C_{\rm A}^2\right]+n_f n_{\tilde{g}}\left(\frac{704}{9}-\frac{512}{3}\zeta_3\right)\frac{d_F^{abcd}d_A^{abcd}}{N_A}\\
&-(n_{\tilde{g}}C_{\rm A})^2 n_f T_{\rm R}\left[\frac{308}{243}C_{\rm F}+\frac{934}{243}C_{\rm A}\right]-n_{\tilde{g}}C_{\rm A}(n_f T_{\rm R})^2\left[\frac{1232}{243}C_{\rm F}+\frac{1252}{243}C_{\rm A}\right]\,, \qquad \text{(B.2)}
\end{aligned}
$$

where $T_{\rm R}=\dfrac{1}{2}$, $C_{\rm F}=\dfrac{N_c^2-1}{2N_c}$, $C_A=N_c$, $N_{\rm A}=2C_{\rm F}C_{\rm A}=N_c^2-1$.

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