Order and Chaos in some Deterministic Infinite Trigonometric Products

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Abstract

It is shown that the deterministic infinite trigonometric products

$$\prod_{n\in\mathbb{N}}\left[1-p+p\cos\left(n^{-s}t\right)\right]=:\mathrm{Cl}_{p;s}(t)$$

with parameters $p\in(0,1]$ & $s>\frac{1}{2}$, and variable $t\in\mathbb{R}$, are inverse Fourier transforms of the probability distributions for certain random series $\Omega_p^\zeta(s)$ taking values in the real ω line; i.e. the $\mathrm{Cl}_{p;s}(t)$ are characteristic functions of the $\Omega_p^\zeta(s)$. The special case $p=1=s$ yields the familiar random harmonic series, while in general $\Omega_p^\zeta(s)$ is a "random Riemann-ζ function," a notion which will be explained and illustrated — and connected to the Riemann hypothesis. It will be shown that $\Omega_p^\zeta(s)$ is a very regular random variable when $p\in(0,\frac{1}{2})$ & $s>\frac{1}{2}$, having an infinitely-often differentiable probability density function (PDF) on the ω line. More precisely, an elementary proof is given that when $p\in(0,\frac{1}{2})$ & $s>\frac{1}{2}$, then there exists $K_{p;s}>0$, and $\varepsilon_{p;s}(|t|)$ with $|\varepsilon_{p;s}(|t|)|\leq K_{p;s}|t|^{1/(s+1)}$, and $C_{p;s}:=-\frac{1}{s}\int_0^\infty\ln(1-p+p\cos\xi)\frac{1}{\xi^{1+1/s}}\mathrm{d}\xi$, so that

$$\forall\, t\in\mathbb{R}:\quad \mathrm{Cl}_{p;s}(t)=\exp\bigl(-C_{p;s}\,|t|^{1/s}+\varepsilon_{p;s}(|t|)\bigr);$$

the regularity of $\Omega_p^\zeta(s)$ follows. Incidentally, this theorem confirms a surmise by Benoit Cloitre, that $\ln\mathrm{Cl}_{1/3;2}(t)\sim -C\sqrt{t}\ (t\to\infty)$ for *some* $C>0$. Graphical evidence suggests that $\mathrm{Cl}_{1/3;2}(t)$ is a chaotic (empirically unpredictable) function of t. This is reflected in the rich structure of the pertinent PDF, the Fourier transform of $\mathrm{Cl}_{1/3;2}$, and illustrated by random sampling of the Riemann-ζ walks, whose branching rules allow the build-up of fractal-like structures.

1 Introduction and Summary

The Riemann hypothesis is perhaps the best-known open problem of mathematics. It hypothesizes that all non-real zeros of Riemann's zeta function $\zeta(s)$, $s \in \mathbb{C}$, lie on the straight line $\frac{1}{2} + i\mathbb{R}$, where $\zeta(s)$ is obtained from Euler's real (Dirichlet-)series

$$\zeta(s) = \sum_{n\in\mathbb{N}} \frac{1}{n^s}, \quad s > 1, \tag{1}$$

by analytic continuation to the complex plane; see [Edw74] for a good introduction. The importance of Riemann's hypothesis derives from the fact that its truth would confirm deep putative insights into the distribution of prime numbers amongst the natural numbers — a holy grail of number theory, and a feat which would have applications chiefly in encryption, i.e. security issues (to whom they may concern). It continues to fascinate the mathematical minds of professionals and amateurs alike.

The latter group of mathematicians includes Benoit Cloitre, who has been documenting his experimental approach to number theory in general, and to the Riemann hypothesis in particular, on his homepage [Clo16]. Some years ago he pondered ("for no particular reason")[1] the deterministic infinite trigonometric product

$$\prod_{n\in\mathbb{N}} \left[\frac{2}{3} + \frac{1}{3}\cos\left(\frac{t}{n^2}\right)\right] =: \mathrm{P}_{\mathrm{CL}}(t), \quad t \in \mathbb{R}, \tag{2}$$

which appears to be fluctuating chaotically about some monotone trend; see Fig. 1 and Fig. 2 below. Cloitre "guessed" that $\ln \mathrm{P}_{\mathrm{CL}}(t) \sim -C\sqrt{t}$ when $t \to \infty$ for *some* constant $C > 0$, which captures the trend asymptotically, and he asked us whether we can prove this. The proof requires only elementary undergraduate mathematics and will be given in section 5. But why does $\mathrm{P}_{\mathrm{CL}}(t)$ fluctuate apparently chaotically about its monotone trend? And what does this have to do with the Riemann hypothesis?

To answer these questions we note (see section 4) that any trigonometric product

$$\prod_{n\in\mathbb{N}} \left[1 - p + p\cos\left(\frac{t}{n^s}\right)\right] =: \mathrm{Cl}_{p;s}(t), \quad t \in \mathbb{R}, \quad p \in (0,1] \ \&\ s > \frac{1}{2}, \tag{3}$$

is *the characteristic function of a "random Riemann-*ζ *function"* $\Omega_p^\zeta(s)$, i.e. $\mathrm{Cl}_{p;s}(t) \equiv \mathrm{Exp}\big(\exp\big(it\Omega_p^\zeta(s)\big)\big) =: \Phi_{\Omega_p^\zeta(s)}(t)$, where "Exp" means *expected value.* Here,

$$\Omega_p^\zeta(s) := \sum_{n\in\mathbb{N}} R_p(n)\frac{1}{n^s}, \quad s > \frac{1}{2}, \quad p \in (0,1], \tag{4}$$

where $\{R_p(n) \in \{-1,0,1\}\}_{n\in\mathbb{N}}$ is a sequence of independent and identically distributed (i.i.d.) random coefficients, generated with the help of a pair of independent coins; see section 2. We draw heavily on the probabilistically themed publications

[1] Private communication by B.C. on 02.2016; we took the liberty to attach $_{\mathrm{CL}}$ at Cloitre's $\mathrm{P}(t)$.

by Kac [Kac59], Morrison [Mor95], and Schmuland [Sch03], in which *the random harmonic series* $\Omega^{\mathrm{harm}} \equiv \Omega_1^{\zeta}(1)$ is explored; in [Sch03] also the special case $\Omega_1^{\zeta}(2)$ is explored. We register that $p = 1$ and $s = 1$ in $\mathrm{Cl}_{p;s}(t)$ yields (cf. sect.5.2 in [Mor95])

$$\Phi_{\Omega^{\mathrm{harm}}}(t) = \prod_{n \in \mathbb{N}} \cos \frac{t}{n}, \tag{5}$$

while Cloitre's $\mathrm{P}_{\mathrm{CL}}(t)$ is the special case $p = \frac{1}{3}$ and $s = 2$ in $\mathrm{Cl}_{p;s}(t)$.

Both $\zeta(s)$ and $-\zeta(s)$ are possible outcomes for such random Riemann-ζ functions $\Omega_p^{\zeta}(s)$, namely the extreme cases in which each $R_p(n), n \in \mathbb{N}$, comes out 1, respectively -1. While this is trivial, we anticipate that also $1/\zeta(s)$ is a possible outcome for $\Omega_p^{\zeta}(s)$, which is nontrivial and going to be interesting!

After introducing the notion of *typicality* for the random walks associated to $\Omega_p^{\zeta}(s)$ we will ask how typical $\zeta(s)$ and $1/\zeta(s)$ are. It should come as no surprise that $\zeta(s)$ is an extremely atypical outcome of a random Riemann-ζ walk, and so is $-\zeta(s)$. However, for the particular value of $p = 6/\pi^2$, its reciprocal $1/\zeta(s)$ does exhibit several aspects of typicality. Intriguingly, as pointed out to us by Alex Kontorovich, *if* $1/\zeta(s)$ *also exhibits some particular aspect of typicality, then the Riemann hypothesis is true, and false if not*! This can be extracted from [Edw74], see section 3.

Which of the many aspects of typicality are exhibited by $1/\zeta(s)$ is an interesting open question which may go beyond settling the Riemann hypothesis. We will use a paradox to argue, though, that $1/\zeta(s)$ cannot possibly exhibit each and every aspect of typicality, i.e. $1/\zeta(s)$ cannot be a perfectly typical random Riemann-ζ walk.

Curiously, a well-known structure emerged unexpectedly during our inquiry into Cloitre's surmise that $\ln \mathrm{Cl}_{1/3;2}(t) \sim -C\sqrt{t}\ (t \to \infty)$ for some $C > 0$. Using elementary analysis we will prove in section 5 that if $p \in (0, \frac{1}{2})$ & $s > \frac{1}{2}$, then there exists $K_{p;s} > 0$, and $\varepsilon_{p;s}(|t|)$ with $|\varepsilon_{p;s}(|t|)| \leq K_{p;s}|t|^{1/(s+1)}$, such that

$$\forall\, t \in \mathbb{R}: \quad \mathrm{Cl}_{p;s}(t) = \exp\big(-C_{p;s}\, |t|^{1/s} + \varepsilon_{p;s}(|t|)\big) \tag{6}$$

with

$$C_{p;s} = -\frac{1}{s} \int_0^{\infty} \ln(1 - p + p\cos\xi) \frac{1}{\xi^{1+1/s}} \mathrm{d}\xi; \tag{7}$$

the integral will be evaluated in terms of a rapidly converging series expansion. This result not only vindicates Cloitre's surmise as a corollary (and more; see below), we note that the factor $\exp\big(-C_{p;s}\, |t|^{1/s}\big)$ at r.h.s.(6) in itself is a characteristic function — of so-called *stable laws*, first studied by Paul Lévy; see [PrRo69]. Stable laws exist for all $s \geq 1/2$, but here $s = 1/2$ is ruled out because $C_{p;1/2} = \infty$. Be that as it may, stable Lévy laws were discovered by answering a completely different question [PrRo69, GaFr03], and the probabilistic reason why they would feature in the analysis of our random Riemann-ζ functions is presently obscure. Some food for thought!

Lest the reader gets the wrong impression that random Riemann-ζ functions were small perturbations of Lévy random variables, we emphasize that they are not! Although the "error term" $\varepsilon_{p;s}(|t|)$ in (6) is small *relative to* $|t|^{1/s}$ when $|t|$ is large enough, $\exp(\varepsilon_{p;s}(|t|))$ is not approaching 1 and in fact responsible for relatively large chaotic fluctuations of $\mathrm{Cl}_{p;s}(t)$ about the Lévy trend; see Fig. 1 & Fig. 2.

Fig. 1

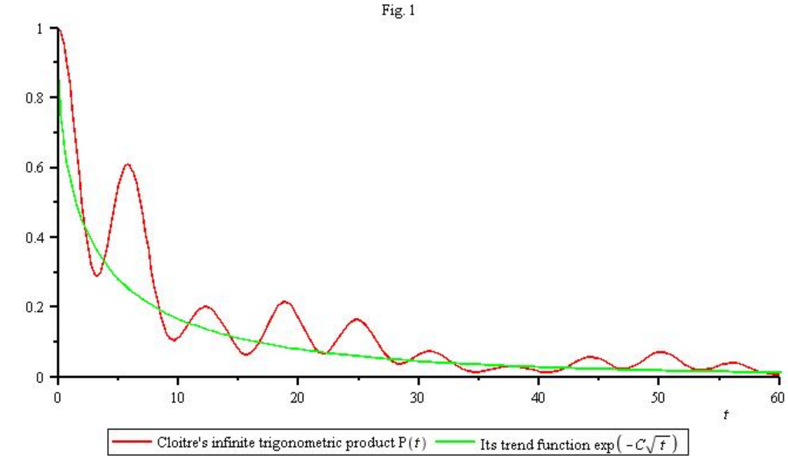
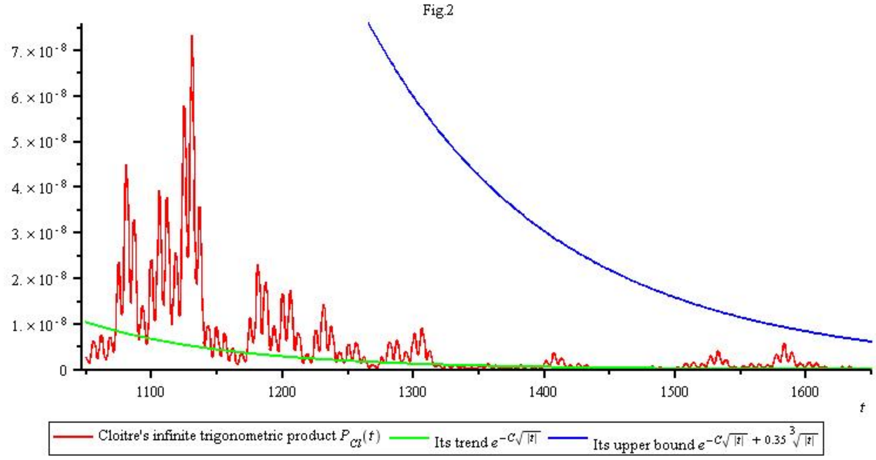


Fig.2



In section 6 we will see that the "empirically unpredictable" behavior of $\mathrm{Cl}_{1/3;2}(t)$ is reflected in a *fractal-like* structured Fourier transform $\varrho^{\zeta}_{1/3;s}(d\omega)$ of $\mathrm{Cl}_{1/3;2}(t)$ (section 4), the probability distribution of $\Omega^{\zeta}_{1/3}(s)$, also illustrated in section 2 by random sampling of the Riemann-ζ walks. We will show, though, that $\varrho^{\zeta}_{p;s}(d\omega)$ is not supported on a true fractal when $p \in (0, \frac{1}{2})$. Random variables supported on a fractal are discussed in [DFT94], [Mor95], and [PSS00]; see our Appendix on *power walks*.

The remainder of our paper supplies the details of our inquiry, and we conclude with a list of intriguing open questions.

2 Random Riemann-ζ functions

Since the random Riemann-ζ functions $\Omega_p^\zeta(s)$ considered in this article are of the type (4), with the random coefficients $R_p(n) \in \{-1, 0, 1\}$ generated by a two-coin tossing process, all that needs to be done to complete their formal definition is to explain this coin tossing process. In this vein, let's write $R_p(n) = \sigma(n)|R_p(n)|$, where $\sigma(n) \in \{-1, 1\}$ and $|R_p(n)| \in \{0, 1\}$. One now repeatedly tosses both, a generally loaded coin with $\text{Prob}(H) = p \in (0, 1]$ (where "H" means "head"), and a fair one, independently of each other and of all the previous tosses. The n-th toss of the generally loaded coin decides whether $|R_p(n)| = 0$ or $|R_p(n)| = 1$; let's stipulate that $|R_p(n)| = 1$ when H shows, which happens with probability p, and $|R_p(n)| = 0$ else. The concurrent and independent toss of the fair coin decides whether $\sigma(n) = +1$ or $\sigma(n) = -1$, either outcome being equally likely. Incidentally, we remark that the $R_{1/3}(n)$ can also be generated by rolling a fair die — if the n-th roll shows 1, then $R_{1/3}(n) = 1$, if it shows 6 then $R_{1/3}(n) = -1$, and $R_{1/3}(n) = 0$ otherwise (which is the case 2/3 of the time, in the long run). Also, it is clear that when $p = 1$ then the loaded coin is superfluous, i.e. $R_1(n) \in \{-1, 1\}$ is generated with a single, fair coin.

That's it. Now let us understand which type of objects we have defined.

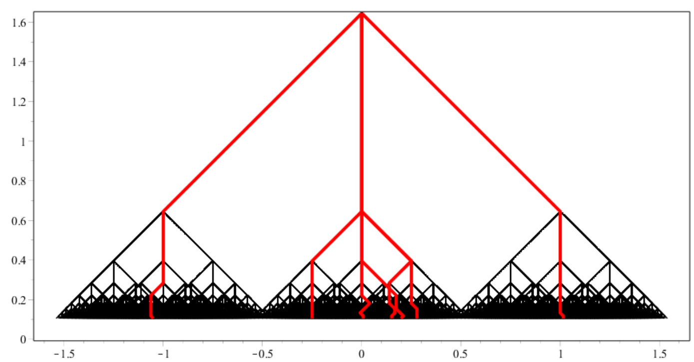
2.1 Random Riemann-ζ walks and their kin

Evaluating a random Riemann-ζ function $\Omega_p^\zeta(s)$ for given $p \in (0, 1]$ at any particular $s > \frac{1}{2}$ turns (4) into a numerical random series. Recalling that an infinite series is defined as the sequence of its partial sums, viz.

$$\Omega_p^\zeta(s) = \left\{ \sum_{n=1}^{N} R_p(n) \frac{1}{n^s} \right\}_{N \in \mathbb{N}}, \tag{8}$$

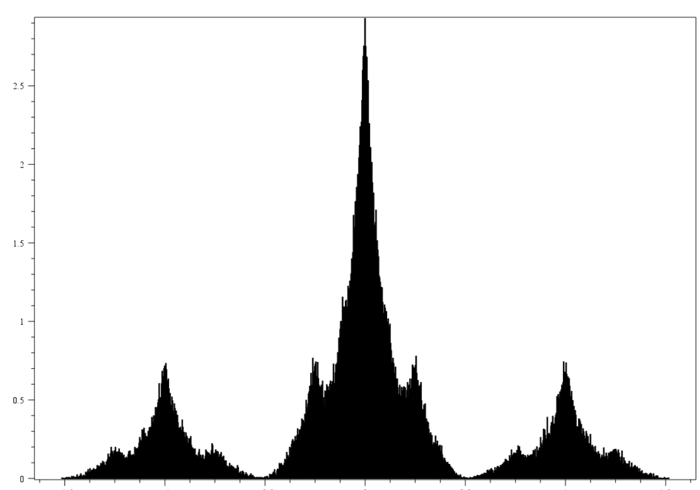
and interpreting $\sum_{n=1}^{N} R_p(n) \frac{1}{n^s}$ as the position of a walker after N random steps $R_p(n) \frac{1}{n^s}, n = 1, ..., N$, we can identify such an evaluation of $\Omega_p^\zeta(s)$ for given $p \in (0, 1]$ at a particular $s > \frac{1}{2}$ with a *random walk on the real ω line*. If the n-th toss of the pair of coins comes out on "move," the walker moves $1/n^s$ units in the direction determined by the fair coin; otherwise he stays put (note that such a "non-move" is called a "step," too). Starting at $\omega = 0$, he keeps carrying out these random steps ad infinitum. We call this a "random Riemann-ζ walk," and its outcome (whenever it converges) is a "random Riemann-ζ function" evaluated at s. Absolute convergence is guaranteed for each and every such walk when $s > 1$ (because the series (1) for $\zeta(s)$ converges absolutely for $s > 1$), and by a famous result of Rademacher conditional convergence holds with probability 1 when $s > \frac{1}{2}$, see [Kac59], [Mor95], and [Sch03]. Since the harmonic series diverges logarithmically, the outcome of the random walks with $\frac{1}{2} < s \leq 1$ is distributed over the whole real line; see [Sch03] for $s = 1$.

To have some illustrative examples, we first pick $s = 2$ and $p = \frac{1}{3}$. In Fig. 3 we display (in black) the fractal tree (cf. [Man77], chpt.16; note its self-similarity) of all possible walks for $s = 2$ when $p \in (0, 1)$, plotted top-down to resemble a Galton board figure. (The tree is truncated after 9 steps, for more steps would only produce a black band between the current cutoff and the finish line). Also shown (in red) is a computer-generated sample of 7 random Riemann-ζ walks with $p = \frac{1}{3}$ & $s = 2$.



Fractal tree & 7 walks)

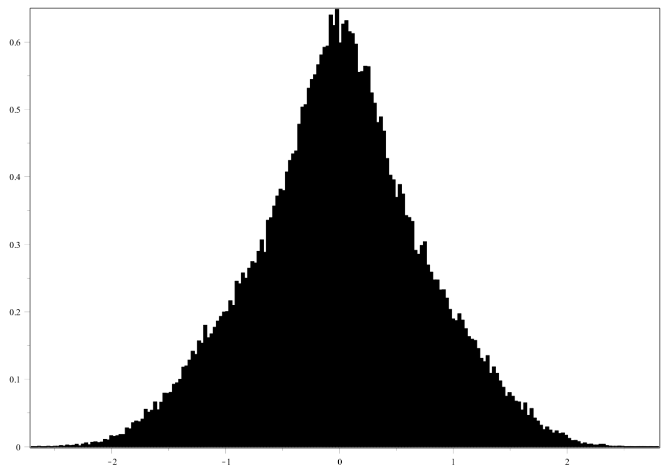
We also exhibit a histogram of the endpoints of 10^5 walks with 1000 steps (Fig. 4).



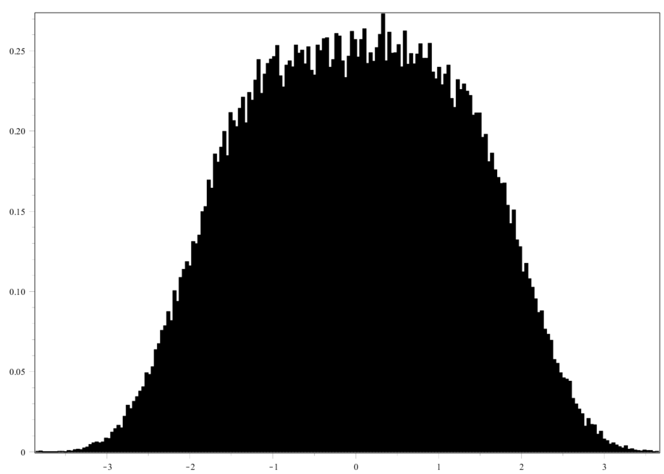
Histogram ($s = 2$, $p = \frac{1}{3}$)

We next pick $s = 1$ and two different choices of p, namely $p = \frac{1}{3}$ and $p = 1$. For $s = 1$ the random Riemann-ζ walks become so-called "random Harmonic Series," which have been studied by Kac [Kac59], Morrison [Mor95], and Schmuland [Sch03] in the special case that $p = 1$. When $p \neq 1$ these harmonic random walks are interesting variations on their theme. We refrain, though, from trying to display the infinitely long harmonic random walk tree, for it is difficult to illustrate it faithfully.

Yet the histograms of the endpoints of 10^5 harmonic walks with 10^3 steps when $p = \frac{1}{3}$ (Fig. 5) and $p = 1$ (Fig. 6) are easily generated.



Histogram ($s = 1$, $p = \frac{1}{3}$)



Histogram ($s = 1$, $p = 1$)

Our Fig. 6 resembles the smooth theoretical PDF of the endpoints of the harmonic walk with $p = 1$, displayed in Fig. 3 of [Mor95] and Fig. 1 of [Sch03], quite closely; cf. the histogram based on $5,000$ walks with 100 steps displayed in Fig. 4 of [Mor95]. When $p = 1$ one is always on the move, so the histogram is quite broad. Our Fig. 5 indicates that reducing p (in this case to $p = 1/3$) will lead to the build-up of a "central peak." The peak is even more pronounced in our Fig. 4 (where $p = 1/3$ and $s = 2$) which reveals a rich, conceivably self-similar structure with side peaks, and side peaks to the side peaks. Our Fig. 4 also makes one wonder whether the peaks, if not fractal, could indicate that the first or second derivative of a theoretical PDF might blow up. These questions will be investigated in section 5.

But first, after having introduced random Riemann-ζ functions, at this point it is appropriate to inject their truly intriguing relationship with the Riemann hypothesis.

3 Typicality and the Riemann Hypothesis

Loosely speaking, a *typical feature* of a random Riemann-ζ walk is a feature which ideally occurs "with probability 1" (strong typicality), or at least "in probability" (weak typicality); see below. A *(strongly or weakly) perfectly typical random Riemann-ζ walk* is an empirical outcome of a random Riemann-ζ function evaluated at s which *exhibits all (strong or weak) typical features.*

Since coin tosses are involved, for simplicity we look at the example of the set of all infinitely long sequences of fair coin tosses first.

3.1 Typicality for coin toss sequences

We identify the events H with 1 and T with 0, and introduce the Bernoulli random variable $B \in \{0, 1\}$, with $\text{Prob}(B = 1) = \frac{1}{2}$. Let B_n be an identical and independent copy of B. Then by the *strong law of large numbers* (see [PrRo69]) one has

$$\text{Prob}\left(\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} B_n = \frac{1}{2}\right) = 1 \tag{9}$$

whereas the *weak law of large numbers* (see [PrRo69]) says that for any $\epsilon > 0$,

$$\lim_{N\to\infty} \text{Prob}\left(\left|\frac{1}{N}\sum_{n=1}^{N} B_n - \frac{1}{2}\right| > \epsilon\right) = 0. \tag{10}$$

Let $b_n \in \{0, 1\}$ denote the outcome of the coin toss B_n. Then based on either the strong, or the weak law of large numbers we say that "$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} b_n = \frac{1}{2}$" is a *strongly, or weakly, typical feature* for such an empirical sequence of outcomes $\{b_n\}_{n\in\mathbb{N}}$. Of course, not every empirical sequence $\{b_n\}_{n\in\mathbb{N}}$ does exhibit this typical feature; take, for instance, $\{b_n\}_{n\in\mathbb{N}} = \{1, 1, 1, 1, ...\}$. We therefore say that $\{1, 1, 1, 1, ...\}$ is *an atypical empirical sequence* for the fair coin tossing process. More generally, *any* empirical sequence $\{b_n\}_{n\in\mathbb{N}}$ for which $\left|\frac{1}{N}\sum_{n=1}^{N} b_n - \frac{1}{2}\right| > \epsilon$ occurs infinitely often is said to be an *atypical empirical sequence* for this coin tossing process.

Next, consider the sequence $\{b_n\}_{n\in\mathbb{N}} = \{1, 0, 1, 0, 1, 0, ...\}$. Could this be a perfectly typical sequence? Clearly, $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} b_n = \frac{1}{2}$, but anyone who has ever flipped a coin a dozen times, again and again, knows that "typically" it doesn't happen to obtain six consecutive 1-0 pairs — here we borrow the common sense usage of "typicality;" indeed, on average the alternating pattern of six consecutive 1-0 pairs occurs less than once in 4,000 repetitions of a dozen coin tosses, and the likelihood of k 1-0 pairs decreases to zero with k increasing to infinity in a trial of length $2k$.

Yet, in an infinite sequence of coin tosses, with probability 1 the pattern of six consecutive 1-0 pairs occurs infinitely often; more generally, for *any* $k \in \mathbb{N}$, with probability 1 a pattern with k consecutive 1s, or a pattern with k consecutive 0s, as well as k consecutive 1-0 pairs, all occur infinitely often. Thus *recurrences of such k-patterns are strongly typical features* of this coin tossing process.

Let's look at one more strongly typical feature — a variation on this theme will turn out to be related to the Riemann hypothesis. Namely, since by either the weak or the strong law of large numbers we can informally say that when N is large enough then $\sum_{n=1}^{N} b_n \approx \frac{1}{2}N$, i.e. $\sum_{n=1}^{N}(2b_n - 1) \approx 0$ in a perfectly typical empirical sequence, we next ask for the typical size of the fluctuations about this theoretical mean, i.e. how large can they be, typically? *Khinchin's law of the iterated logarithm* states that for any $\epsilon > 0$, with probability 1 the event $|\sum_{n=1}^{N}(2B_n - 1)| > (1-\epsilon)\sqrt{2N \ln\ln N}$ will occur infinitely often, while the event $|\sum_{n=1}^{N}(2B_n - 1)| > (1+\epsilon)\sqrt{2N \ln\ln N}$ has probability 0 of occurring infinitely often in the sequence $\{B_n\}_{n\in\mathbb{N}}$. Thus,

$$\Big|\sum_{n=1}^{N}(2b_n - 1)\Big| > (1-\epsilon)\sqrt{2N \ln\ln N} \tag{11}$$

occurs for infinitely many N in a perfectly typical empirical sequence $\{b_n\}_{n\in\mathbb{N}}$, and

$$\Big|\sum_{n=1}^{N}(2b_n - 1)\Big| > (1+\epsilon)\sqrt{2N \ln\ln N} \tag{12}$$

will happen at most finitely many times.

Countlessly many more features occur with probability 1, many of them trivially (like $\mathrm{Prob}(\sum_{n=1}^{N} B_n < N+\epsilon) = 1$), but many others not, and some of them are deep. This makes it plain that it is impossible, or at least extremely unlikely, that anyone will ever give an *explicit* characterization of a *perfectly typical empirical sequence* of coin tosses. (It is even conceivable that no such sequence exists!) By contrast, once a particular feature has been proven to occur with probability 1 (the strong version), or in probability (the weak version), it is straightforward to ask whether a given empirical sequence exhibits this particular *aspect of typicality.*

We are now armed to address the connection of the Riemann hypothesis with the notion of typicality of random Riemann-ζ functions.

3.2 Typicality for random Riemann-ζ functions

We begin by listing a few typical features of random Riemann-ζ walks. Let $r_p(n) \in \{-1, 0, 1\}$ denote the outcome of the random variable $R_p(n)$, and for given $p \in (0, 1]$ and $s > \frac{1}{2}$ let $\omega_p^\zeta(s)$ denote the outcome for the random Riemann-ζ walk $\Omega_p^\zeta(s)$, i.e.

$$\omega_p^\zeta(s) = \left\{ \sum_{n=1}^{N} r_p(n) \frac{1}{n^s} \right\}_{N \in \mathbb{N}}. \tag{13}$$

Then the fair coin tossing process of the previous subsection now yields that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} r_p(n) = 0 \tag{14}$$

is a feature typically exhibited by an outcome $\omega_p^\zeta(s)$, independently of p and s. Next,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |r_p(n)| = p \tag{15}$$

is a p-dependent feature typically exhibited by an $\omega_p^\zeta(s)$, independently of s. Lastly, Rademacher's result mentioned above actually shows that typically

$$\lim_{N\to\infty} \sum_{n=1}^{N} r_p(n) \frac{1}{n^s} = \omega_p^\zeta(s) \tag{16}$$

exists on the real ω line whenever $s > \frac{1}{2}$. All these are strongly typical features.

We now inquire into the typicality of the following outcomes of random Riemann-ζ functions with $s > \frac{1}{2}$: Riemann's ζ-function (1) itself, viz. $\zeta(s) = \sum_{n \in \mathbb{N}} 1/n^s$ understood as a (not necessarily convergent) sequence of its partial sums; its reciprocal

$$\frac{1}{\zeta(s)} = \sum_{n \in \mathbb{N}} \mu(n) \frac{1}{n^s}, \tag{17}$$

where $\mu(n) \in \{-1, 0, 1\}$ is the Möbius function (see [Edw74]); and also

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n \in \mathbb{N}} \lambda(n) \frac{1}{n^s}, \tag{18}$$

where $\lambda(n) \in \{-1, 1\}$ is Liouville's λ-function (see [Slo64]). All are possible outcomes of a random Riemann-ζ walk with $s > \frac{1}{2}$, any[2] $p \in (0, 1)$. In terms of the outcomes $r_p(n)$ of the coin tossing process, Riemann's zeta function corresponds to $r_p(n) = 1$ for all $n \in \mathbb{N}$, its reciprocal to $r_p(n) = \mu(n)$, and the ratio $\zeta(2s)/\zeta(s)$ to $r_p(n) = \lambda(n)$. Can any of these $\omega_p^\zeta(s)$ be perfectly typical outcomes, at least for some p values?

As to $\zeta(s)$ itself, it is clear that it must be atypical, since $r_p(n) = 1$ for all $n \in \mathbb{N}$ manifestly violates the p- and s-independent typicality feature (14). Yet $\zeta(s)$ does not necessarily violate each and every aspect of typicality! For instance, if $p = 1$ then (15) holds for $\zeta(s)$ (though not for any other $p \in (0, 1)$). Moreover, while the sequence of its partial sums diverges to infinity when $s \in (\frac{1}{2}, 1]$ in violation of the typicality feature (16), this feature is verified by $\zeta(s)$ if $s > 1$. In any event, since $\zeta(s)$ is an *extreme outcome*, it is intuitively clear that it will violate most aspects of typicality — in this sense, we say that $\zeta(s)$ is *extremely atypical* for all $p \in (0, 1]$.

On to its reciprocal. It is known that the *Prime Number Theorem*[3] is equivalent to the actual frequencies of the values $\mu(n) = 1$ and $\mu(n) = -1$ being equal in the long run, so $1/\zeta(s)$ exhibits the typicality feature (14). It is also known that the actual frequency of values $\mu(n) \neq 0$ equals $1/\zeta(2)$ $(= 6/\pi^2)$ in the long run, so $1/\zeta(s)$ also exhibits the typicality feature (15) if $p = 1/\zeta(2)$ (though clearly not for any other p value). Furthermore, $1/\zeta(s)$ satisfies the typicality feature (16) for all $s > \frac{1}{2}$. Could $1/\zeta(s)$ perhaps be a perfectly typical random Riemann-ζ function for all $s > \frac{1}{2}$ when $p = 1/\zeta(2)$? Recall that this would mean that for each $s > \frac{1}{2}$ the pertinent actual walk is a *perfectly typical walk*, i.e. a walk which *exhibits all features* of the theoretical random-walk law *which occur with probability* 1 (or at least *in probability*).

Similarly, the Prime Number Theorem is equivalent to the actual frequencies of the values $\lambda(n) = 1$ and $\lambda(n) = -1$ being equal in the long run, so also the ratio $\zeta(2s)/\zeta(s)$ exhibits the typicality feature (14). Furthermore, if (and only if) $p = 1$ then $\zeta(2s)/\zeta(s)$ exhibits the typicality feature (15). Lastly, $\zeta(2s)/\zeta(s)$ also exhibits the typicality feature (16) for all $s > \frac{1}{2}$. Could also $\zeta(2s)/\zeta(s)$ perhaps be a perfectly typical random Riemann-ζ function for all $s > \frac{1}{2}$ when $p = 1$?

A moment of reflection reveals that this would be *truly paradoxical*: if $1/\zeta(s)$ and / or $\zeta(2s)/\zeta(s)$ are perfectly typical random Riemann-ζ functions for the mentioned p-values, then one can learn a lot about them by studying what is typical for random walks with those p-values, without ever looking at $1/\zeta(s)$ or $\zeta(2s)/\zeta(s)$. Of course, if one learns something about $1/\zeta(s)$ and / or $\zeta(2s)/\zeta(s)$, then one also learns something about $\zeta(s)$ — but how can one learn something about an extremely atypical

[2] $\zeta(s)$ and $\zeta(2s)/\zeta(s)$ are possible outcomes also when $p = 1$, while $1/\zeta(s)$ is not.

[3] This is the statement that the number of primes less than x is asymptotically given by $\int_2^x \frac{d\xi}{\ln \xi}$, with relative error going to zero as $x \to \infty$; see [Edw74].

random Riemann-ζ function by studying what is typical for such random walks? The obvious way out of this dilemma is to conclude:

Neither $1/\zeta(s)$ *nor* $\zeta(2s)/\zeta(s)$ *can be perfectly typical random Riemann-*ζ *functions!*

The upshot is that both $1/\zeta(s)$ and $\zeta(2s)/\zeta(s)$ *must feature* some *atypical empirical statistics*, encoded in the sequences $\{\mu(n)\}_{n\in\mathbb{N}}$ and $\{\lambda(n)\}_{n\in\mathbb{N}}$. Obviously these atypical features must be inherited from the correlations in the distribution of prime numbers; recall that the coin tossing process, by contrast, is correlation-free. Since the Riemann hypothesis about the location of the non-real zeros of $\zeta(s)$ is equivalent to some detailed knowledge about the distribution of and correlations amongst prime numbers, it may well be that some particular atypical empirical feature of $1/\zeta(s)$ and $\zeta(2s)/\zeta(s)$ will be equivalent to the Riemann hypothesis. Which kind of feature, if any, remains anybody's best guess — to the best of our knowledge.

Surprisingly, and indeed intriguingly, it is known though that *a certain typical feature*, if indeed exhibited by the $1/\zeta(s)$ walk for $p = 1/\zeta(2)$, beyond the agreement of empirical and theoretical frequencies, *is equivalent to the Riemann hypothesis*! We are grateful to Alex Kontorovich for having pointed this out to us.

Namely, let us extend the definition of the random Riemann-ζ walk $1/\zeta(s)$ to $s = 0$, *not* by analytic extension, but in terms of the sequence of its partial sums:

$$\frac{1}{\zeta(0)} := \left\{ \sum_{n=1}^{N} \mu(n) \right\}_{N\in\mathbb{N}}. \tag{19}$$

Note that for $s \le \frac{1}{2}$ the $1/\zeta(s)$ random walk may well wander off to infinity, but the rate at which this happens is crucial (recall Khinchin's law of the iterated logarithm which we mentioned in subsection 3.1). As explained in [Edw74], chpt.12.1, Littlewood proved the equivalence:

$$\forall\epsilon > 0: \lim_{N\to\infty} N^{-\frac{1}{2}-\epsilon}\Big|\sum_{n=1}^{N} \mu(n)\Big| = 0 \quad \leftrightarrow \quad \text{The Riemann hypothesis is true.} \tag{20}$$

And as explained in [Edw74], chpt.12.3, Denjoy noted that if one assumes that the ± 1 values of $\mu(n)$ are distributed as if they were generated by fair and independent coin flips, then the central limit theorem implies that $\lim_{N\to\infty} N^{-\frac{1}{2}-\epsilon}|\sum_{n=1}^{N}\mu(n)| = 0$ holds with probability 1. Of course, $\mu(n) = 0$ is still determined by its formula, but the empirical frequency of $\mu(n) = 0$ occurrences is $1 - 6/\pi^2$ in the long run, and by adopting Denjoy's reasoning one can show that for $p = 6/\pi^2$ one has that

$$\forall\epsilon > 0: \text{ Prob}\left(\lim_{N\to\infty} N^{-\frac{1}{2}-\epsilon}\Big|\sum_{n=1}^{N} R_{6/\pi^2}(n)\Big| = 0 \right) = 1. \tag{21}$$

Thus l.h.s.(20) would be a typical feature exhibited by the $1/\zeta(0)$ walk at $p = 6/\pi^2$. Paraphrasing Morrison: The Riemann Hypothesis is endlessly fascinating. Isn't it?

4 The Characteristic Function of $\Omega_p^\zeta(s)$

We now show that the infinite trigonometric products $\mathrm{Cl}_{p;s}(t)$ given in (3) are characteristic functions of the $\Omega_p^\zeta(s)$, i.e. $\mathrm{Cl}_{p;s}(t) = \mathrm{Exp}\big(\exp\big(it\Omega_p^\zeta(s)\big)\big)$, where "Exp" is *expected value* (not to be confused with the exponential function exp). Since $\Omega_p^\zeta(s)$ is an infinite sum of independent random variables $R_p(n)/n^s$ (see (4)), $\exp\big(it\Omega_p^\zeta(s)\big)$ is an infinite product of independent random variables $\exp\big(itR_p(n)/n^s\big)$, and by a well-known theorem in probability theory, expected values of products of independent random variables are products of the their individual expected values. And so we have (temporarily ignoring issues of convergence)

$$\begin{aligned}\mathrm{Exp}\Big(\exp\big(it\Omega_p^\zeta(s)\big)\Big) &= \mathrm{Exp}\Big(\prod_{n\in\mathbb{N}}\exp\big(itR_p(n)\tfrac{1}{n^s}\big)\Big) = \prod_{n\in\mathbb{N}}\mathrm{Exp}\Big(\exp\big(itR_p(n)\tfrac{1}{n^s}\big)\Big)\\ &= \prod_{n\in\mathbb{N}}\Big(\tfrac12 pe^{-it/n^s} + (1-p) + \tfrac12 pe^{it/n^s}\Big)\\ &= \prod_{n\in\mathbb{N}}\Big(1-p+p\cos\big(\tfrac{t}{n^s}\big)\Big) \equiv \mathrm{Cl}_{p;s}(t),\end{aligned} \tag{22}$$

where we have used Euler's formula to rewrite $\frac12\big(e^{it/n^s} + e^{-it/n^s}\big) = \cos\big(t/n^s\big)$.

That was straightforward. Next we explain the relationship between the characteristic functions $\mathrm{Cl}_{p;s}(t)$ of $\Omega_p^\zeta(s)$ and the probability distribution $\varrho_{p;s}^\zeta(d\omega)$ of the endpoints of these random Riemann-ζ walks on the ω line. Formally this is accomplished by realizing that $\mathrm{Cl}_{p;s}(t)$ is the inverse Fourier transform of $\varrho_{p;s}^\zeta(d\omega)$, viz.

$$\mathrm{Exp}\Big(\exp\big(it\Omega_p^\zeta(s)\big)\Big) = \int_{\mathbb{R}} e^{it\omega}\varrho_{p;s}^\zeta(d\omega). \tag{23}$$

Therefore we obtain $\varrho_{p;s}^\zeta(d\omega)$ by taking the Fourier transform of $\mathrm{Cl}_{p;s}(t)$. As recalled in [Mor95], the Fourier transform of a product equals the *convolution product* ("$*$", see below) of the Fourier transforms of its factors, and so we find

$$\begin{aligned}\varrho_{p;s}^\zeta(d\omega) &= \Big(*\prod_{n\in\mathbb{N}}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega t}\Big[\tfrac12 pe^{-it/n^s} + (1-p) + \tfrac12 pe^{it/n^s}\Big]dt\Big)(d\omega)\\ &= \Big(*\prod_{n\in\mathbb{N}}\Big[\tfrac12 p\delta_{-\frac{1}{n^s}} + (1-p)\delta_0 + \tfrac12 p\delta_{\frac{1}{n^s}}\Big]\Big)(d\omega);\end{aligned} \tag{24}$$

here, "$*\prod$" denotes repeated convolution (cf. [Mor95]), and δ_{ω_k} is a Dirac measure.[4]

This distribution looks intimidating, but it only conveys what we know already! Namely, formally (24) is the limit $N \to \infty$ of the N-fold partial convolution products[5]

$$\varrho_{p;s}^{(N)}(d\omega) := \Big(* \prod_{n=1}^{N} \Big[\tfrac{1}{2} p \delta_{-\frac{1}{n^s}} + (1-p)\delta_0 + \tfrac{1}{2} p \delta_{\frac{1}{n^s}} \Big] \Big)(d\omega). \tag{25}$$

Now recall that the convolution product, which for two integrable functions f and g is defined by $(f*g)(\omega) = \int f(\omega')g(\omega-\omega')d\omega'$, extends to delta measures where it acts as follows: $\delta_a * \delta_b = \delta_{a+b}$ (see [Mor95]). Therefore, by multiplying out the convolution product at r.h.s.(25), using the distributivity of "$*$" one finds that $\varrho_{p;s}^{(N)}(d\omega)$ is a weighted sum of point measures at the possible outcomes

$$\omega_p^{(N)}(s) := \sum_{n=1}^{N} r_p(n) \frac{1}{n^s} \in \mathcal{L}_p^{(N)}(s), \quad r_p(n) \in \{-1, 0, 1\}, \tag{26}$$

of the random walk truncated after N steps,

$$\Omega_p^{(N)}(s) := \sum_{n=1}^{N} R_p(n) \frac{1}{n^s}. \tag{27}$$

The set of locations $\mathcal{L}_p^{(N)}(s) \subset \mathbb{R}$ is finite, and generically[6] consists of 3^N distinct real points if $p \in (0,1)$, and of 2^N distinct real points if $p = 1$. Thus, $\varrho_{p;s}^{(N)}(d\omega)$ becomes

$$\varrho_{p;s}^{(N)}(d\omega) = \sum_{\mathcal{L}_p^{(N)}(s)} \mathrm{P}\big(\omega_p^{(N)}(s)\big) \delta_{\omega_p^{(N)}(s)}(d\omega); \tag{28}$$

the sum runs over all $\omega_p^{(N)}(s) \in \mathcal{L}_p^{(N)}(s)$, and $\mathrm{P}\big(\omega_p^{(N)}(s)\big) := \mathrm{Prob}\big(\Omega_p^{(N)}(s) = \omega_p^{(N)}(s)\big)$. These probabilities $\mathrm{P}\big(\omega_p^{(N)}(s)\big)$ are readily computed from the tree diagram in Fig. 3, or by inspecting (26): if in order to reach $\omega_p^{(N)}(s)$ you need to move $m \leq N$ times (whether left or right has equal probability), then $\mathrm{P}\big(\omega_p^{(N)}(s)\big) = (p/2)^m (1-p)^{N-m}$, independently of s. Note that there are $2^m \binom{N}{m}$ possible outcomes for an N-step walk with $m \leq N$ moves, and indeed $\sum_{m=0}^{N} 2^m \binom{N}{m} (p/2)^m (1-p)^{N-m} = (1-p+p)^N = 1$.

[4]If $I \subset \mathbb{R}$ is any closed interval, then $\int_I \delta_{\omega_k}(d\omega) = 1$ if $\omega_k \in I$ and $\int_I \delta_{\omega_k}(d\omega) = 0$ if $\omega_k \notin I$.

[5]We temporarily suppress the suffix "ζ" so as not to overload the notation.

[6]It may in principle happen for certain discrete values of s (but not of p) that different N-step paths lead to the same outcome $\omega_p^{(N)}(s)$. However, since $s > \frac{1}{2}$ is a continuous parameter, this degenerate situation is not generic. Note though that it may well happen that we humans "inadvertendly" pick precisely those non-generic cases, for instance if degeneracy occurs when $s \in \mathbb{N}$!

Let's look at two examples. After 1 step with $p \in (0,1)$ there are 3 possible positions, and the distribution (25) with $N=1$ reads

$$\varrho^{(1)}_{p;s}(d\omega) = \Big(\tfrac{1}{2}p\delta_{-1} + (1-p)\delta_0 + \tfrac{1}{2}p\delta_1\Big)(d\omega). \tag{29}$$

After 2 steps with $p \in (0,1)$ we have 9 possible positions, and (25) with $N=2$ reads

$$\begin{aligned}\varrho^{(2)}_{p;s}(d\omega) = & \Big(\Big[\tfrac{1}{2}p\delta_{-1} + (1-p)\delta_0 + \tfrac{1}{2}p\delta_1\Big] * \Big[\tfrac{1}{2}p\delta_{-\frac{1}{2^s}} + (1-p)\delta_0 + \tfrac{1}{2}p\delta_{\frac{1}{2^s}}\Big]\Big)(d\omega) \\ = & \Big(\tfrac{1}{4}p^2\delta_{-1-\frac{1}{2^s}} + \tfrac{1}{2}p(1-p)\delta_{-1+0} + \tfrac{1}{4}p^2\delta_{-1+\frac{1}{2^s}}\Big)(d\omega) + \\ & \Big(\tfrac{1}{2}p(1-p)\delta_{0-\frac{1}{2^s}} + (1-p)^2\delta_{0+0} + \tfrac{1}{2}p(1-p)\delta_{0+\frac{1}{2^s}}\Big)(d\omega) + \\ & \Big(\tfrac{1}{4}p^2\delta_{1-\frac{1}{2^s}} + \tfrac{1}{2}p(1-p)\delta_{1+0} + \tfrac{1}{4}p^2\delta_{1+\frac{1}{2^s}}\Big)(d\omega),\end{aligned} \tag{30}$$

which is precisely (28) with $N=2$; we have facilitated the comparison by writing all two-step walks which lead to the locations of the point masses explicitly, including the "non-moves." Similarly one can compute the N-th partial convolution product, although this soon gets cumbersome and does not illuminate the process any further.

The theory of convergence of probability measures (e.g. ref.[1] in [Sch03]) shows that the sequence of partial products (25) does converge to a probability measure (24) if $s > \frac{1}{2}$. Unfortunately, the expression (24) does not readily give up its secrets.

In particular, each measure (25) is singular with respect to (w.r.t.) Lebesgue measure $d\omega$, so could it be that the $N \to \infty$ limit (24) is singular as well — e.g., supported by a fractal? And if not, when $\varrho^{\zeta}_{p;s}(d\omega)$ is absolutely continuous w.r.t. $d\omega$, is its PDF perhaps not differentiable at its peaks, as hinted at by Fig. 4?

The answers to these questions will be extracted from $\mathrm{Cl}_{p;s}(t)$ in the next section.

5 The Main Theorem

In this section we use elementary calculus techniques to prove the following result:

Theorem 5.1 *Let $p \in (0, \frac{1}{2})$ & $s > \frac{1}{2}$. Then*

$$\forall\, t \in \mathbb{R}: \quad \mathrm{Cl}_{p;s}(t) = \exp\left(-C_{p;s}\,|t|^{1/s} + \varepsilon_{p;s}(|t|)\right), \tag{31}$$

where $|\varepsilon_{p;s}(|t|)| \leq K_{p;s}|t|^{1/(s+1)}$ for some constant $K_{p;s} > 0$, and where

$$C_{p;s} := -\frac{1}{s}\int_0^\infty \ln(1-p+p\cos\xi)\frac{1}{\xi^{1+1/s}}\mathrm{d}\xi = A_s B_{p;s}, \tag{32}$$

with

$$A_s := \int_0^\infty \sin\xi \frac{1}{\xi^{1/s}} \mathrm{d}\xi = \Gamma\left(1 - \tfrac{1}{s}\right)\cos\left(\tfrac{\pi}{2s}\right) \tag{33}$$

(where it is understood that $A_1 = \lim_{s\to 1}\Gamma\big(1-\frac{1}{s}\big)\cos\big(\frac{\pi}{2s}\big)$ $[= \frac{\pi}{2}]$*), and*

$$B_{p;s} := \sum_{n=0}^{\infty}(-1)^n\left(\frac{p}{1-p}\right)^{n+1}\frac{1}{2^n}\sum_{k=0}^{\lceil\frac{n-1}{2}\rceil}\binom{n}{k}\frac{(1+n-2k)^{1/s}}{1+n-k} \qquad . \tag{34}$$

Cloitre's surmise follows from Theorem 5.1. Indeed, we have the stronger result:

Corollary 5.2 *For* $p = 1/3$ *and* $s = 2$*, Theorem 5.1 reduces to*

$$\forall\, t \in \mathbb{R}: \quad \mathrm{P}_{\mathrm{CL}}(t) = e^{-C\sqrt{|t|+\varepsilon(|t|)}}, \tag{35}$$

with correction term bounded as $|\varepsilon(|t|)| \leq K|t|^{1/3}$ *for some* $K > 0$*, and with*

$$C = \int_{\mathbb{R}} \frac{\sin\xi^2}{2+\cos\xi^2}\mathrm{d}\xi = \sqrt{\frac{\pi}{2}}\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^{2n+1}}\sum_{k=0}^{\lceil\frac{n-1}{2}\rceil}\binom{n}{k}\frac{\sqrt{1+n-2k}}{1+n-k}; \tag{36}$$

numerically, $C = 0.319905585...\sqrt{\pi} \approx 0.32\sqrt{\pi}$.

Remark 5.3 *By a simple change of variables, and the fact that* ξ^2 *is even, we have*

$$C = \int_{\mathbb{R}} \frac{\sin\xi^2}{2+\cos\xi^2}\mathrm{d}\xi = 2\int_0^\infty \frac{\sin\xi^2}{2+\cos\xi^2}\mathrm{d}\xi = \int_0^\infty \frac{\sin\xi}{2+\cos\xi}\frac{1}{\xi^{1/2}}\mathrm{d}\xi = C_{\frac{1}{3};2} \tag{37}$$

Theorem 5.1 implies that $\Omega_p^\zeta(s)$ is a very regular random variable when $p \in (0, \frac{1}{2})$. Namely, by (31) the integral of $|t|^m \mathrm{Cl}_{p;s}(t)$ exists for any $m \in \{0, 1, 2, ...\}$, so by general Fourier theory the Fourier transform of $\mathrm{Cl}_{p;s}(t)$ is an infinitely often differentiable function (see, e.g., [Sch03]). Also, $\mathrm{Cl}_{p;s}(0) = 1$, so its Fourier transform has integral 1. We already know that its Fourier transform is positive. Thus we have

Corollary 5.4 *The random variable* $\Omega_p^\zeta(s)$ *defined by its characteristic function (3) with* $p \in (0, \frac{1}{2})$ *and* $s > \frac{1}{2}$ *converges with probability 1. It is continuous, having an arbitrarily differentiable (but generally not real analytic) probability density* $f_{\Omega_p^\zeta(s)}(\omega)$.

Corollary 5.4 settles our questions concerning the distribution of $\Omega^{\zeta}_{1/3}(2)$. Despite the seemingly self-similar structure hinted at in Fig. 4, $\Omega^{\zeta}_{1/3}(2)$ cannot be supported on a fractal subset of the ω line. Also, despite the appearance of singular peaks hinted at in Fig. 4, the PDF of $\Omega^{\zeta}_{1/3}(2)$ is infinitely often continuously differentiable.

It remains to prove Theorem 5.1. To offer some guidance we explain our *strategy*.

First of all, since $p \in (0, \frac{1}{2})$, we have $\mathrm{Cl}_{p;s}(t) > 0$, so we can take its logarithm and obtain the infinite series

$$\ln \mathrm{Cl}_{p;s}(t) = \sum_{n\in\mathbb{N}} \ln\left[1 - p + p\cos\left(n^{-s}t\right)\right], \ t \in \mathbb{R}, \tag{38}$$

with $p \in (0, \frac{1}{2})$ & $s > \frac{1}{2}$. We then follow the proof of Theorem 1 of [Kie13] which establishes that if $s > 1$, then for all $t \in \mathbb{R}$ one has $\sum_{n\in\mathbb{N}} \sin\left(n^{-s}t\right) = \alpha_s \mathrm{sign}(t)|t|^{1/s} + \varepsilon(|t|)$, with $\alpha_s = \Gamma\left(1 - \frac{1}{s}\right)\sin\left(\frac{\pi}{2s}\right)$ and $|\varepsilon(|t|)| \le K_s |t|^{1/(s+1)}$ for some $K_s > 0$.

Secondly, by the reflection symmetry about $t = 0$ of $\mathrm{Cl}_{p;s}(t)$ it suffices to consider $t > 0$. Yet we need to distinguish $0 \le t \le t_s$ and $t \ge t_s$ for some $t_s > 0$.

The *near side* $0 \le t \le t_s$ will be estimated with the help of a Maclaurin expansion and turn out to be subdominant.

The *far side* $t \ge t_s$ will be handled by splitting the infinite series into two parts,

$$\sum_{n\in\mathbb{N}} (\cdots_n) = \sum_{n=1}^{N_s(t/\tau)} (\cdots_n) + \sum_{n=N_s(t/\tau)+1}^{\infty} (\cdots_n), \tag{39}$$

where $(\cdots_n) = \ln\left[1 - p + p\cos\left(n^{-s}t\right)\right]$, and where $N_s(t/\tau) := \lceil (st/\tau)^{1/(s+1)} \rceil$, with $\tau < \pi/2$. The first (finite) sum in (39) will be shown to yield only a subdominant error bound. The second (infinite) sum in (39) will be interpreted as a Riemann sum approximation to an integral over the real line, the trend function, plus a subdominant error bound. We now outline this argument.

Since $\tau < \pi/2$, when t gets large any two consecutive arguments t/n^s and $t/(n+1)^s$ of the cosine functions will come to lie within one quarter period of cosine whenever $n > \lceil (st/\tau)^{1/(s+1)} \rceil$. Moreover, with increasing n, for fixed t/τ, the consecutive arguments t/n^s and $t/(n+1)^s$ will be more and more closely spaced. And so, when τ is sufficiently small, with increasing t the part of the sum of $\ln \mathrm{Cl}_{p;s}(t)$ with $n > N_s(t/\tau)$ becomes an increasingly better Riemann sum approximation to

$$\int_{N_s(t/\tau)+1}^{\infty} \ln\left[1 - p + p\cos\left(\nu^{-s}t\right)\right] \mathrm{d}\nu.$$

More precisely, using the variable substitution $\nu^{-s}t = \xi$, we have (informally)

$$\sum_{n=N_s(t/\tau)+1}^{\infty} \ln\left[1-p+p\cos\left(n^{-s}t\right)\right] \approx t^{1/s}\tfrac{1}{s}\int_0^{t/(N_s(t/\tau)+1)^s} \ln\left[1-p+p\cos\xi\right]\frac{1}{\xi^{1+1/s}}\mathrm{d}\xi. \quad (40)$$

Since $s > 1/2$, the upper limit of integration at r.h.s.(40) goes to ∞ like $Kt^{1/(s+1)}$ when $t \to \infty$. The limiting integral is an improper Riemann integral which converges absolutely for all $s > 1/2$, yielding

$$\tfrac{1}{s}\int_0^{\infty} \ln\left[1-p+p\cos\xi\right]\frac{1}{\xi^{1+1/s}}\mathrm{d}\xi \equiv -C_{p;s}. \quad (41)$$

This heuristic argument will be made rigorous by supplying the subdominant error bounds, using only senior level undergraduate mathematics.

Lastly, the integral (41) will be evaluated with the help of a rapidly converging geometric series expansion and a recursion which involves the Catalan numbers.

Proof of Theorem 5.1:

First of all, if $t_s > 0$ is sufficiently small, then for the *near side* $0 \leq t \leq t_s$ we have the Maclaurin expansion $\ln \mathrm{Cl}_{p;s}(t) = -\frac{1}{2}p\,\zeta(2s)t^2 + O(t^4)$. It follows that $|\ln \mathrm{Cl}_{p;s}(t) + C_{p;s}t^{1/s}| \leq Kt^{1/(s+1)}$ for some $K > 0$ when $0 \leq t \leq t_s$. Here and in all estimates below, K is a generic positive constant which may depend on p, s, τ, t_s.

As for the *far side* $t \geq t_s$, the first sum at r.h.s.(39) is estimated by

$$\Bigg|\sum_{n=1}^{N_s(t/\tau)} \ln\left[1-p+p\cos\left(n^{-s}t\right)\right]\Bigg| \;\leq\; |\ln(1-2p)|\lceil(st/\tau)^{1/(s+1)}\rceil \;\leq\; Kt^{1/(s+1)}, \quad (42)$$

where we used the triangle inequality and

$$\left|\ln\left[1-p+p\cos\left(n^{-s}t\right)\right]\right| \leq |\ln(1-2p)|.$$

For the second sum at r.h.s.(39) we find, for some $\nu_n \in [n, n+1]$,

$$\Big| \sum_{n=N_s(t/\tau)+1}^{\infty} \ln\big[1-p+p\cos\big(n^{-s}t\big)\big] - \int_{N_s(t/\tau)+1}^{\infty} \ln\big[1-p+p\cos\big(\nu^{-s}t\big)\big]\, d\nu \Big| = \tag{43}$$

$$\Big| \sum_{n=N_s(t/\tau)+1}^{\infty} \Big(\ln\big[1-p+p\cos\big(n^{-s}t\big)\big] - \int_{n}^{n+1} \ln\big[1-p+p\cos\big(\nu^{-s}t\big)\big]\, d\nu \Big) \Big| = \tag{44}$$

$$\Big| \sum_{n=N_s(t/\tau)+1}^{\infty} \Big(\ln\big[1-p+p\cos\big(n^{-s}t\big)\big] - \ln\left[1-p+p\cos\left(\nu_n^{-s}t\right)\right] \Big) \Big| = \tag{45}$$

$$\Big| \sum_{n=N_s(t/\tau)+1}^{\infty} \int_{t/\nu_n^s}^{t/n^s} \frac{p\sin(\xi)}{1-p+p\cos(\xi)} \mathrm{d}\xi \Big| \leq \tag{46}$$

$$\frac{p}{1-2p} \sum_{n=N_s(t/\tau)+1}^{\infty} \int_{t/\nu_n^s}^{t/n^s} \big|\sin\xi\big| \mathrm{d}\xi \leq \tag{47}$$

$$\frac{p}{1-2p} \sum_{n=N_s(t/\tau)+1}^{\infty} t\big(\tfrac{1}{n^s} - \tfrac{1}{\nu_n^s}\big) \leq \tag{48}$$

$$\frac{p}{1-2p} \sum_{n=N_s(t/\tau)+1}^{\infty} t\big(\tfrac{1}{n^s} - \tfrac{1}{(n+1)^s}\big) = \tag{49}$$

$$\frac{p}{1-2p} t \lceil (st/\tau)^{1/(s+1)} + 1 \rceil^{-s} \leq \tag{50}$$

$$K t^{\frac{1}{s+1}};$$

here, (43) is obviously true, whereas (44) expresses the mean value theorem for some $\nu_n \in [n, n+1]$, and (45) holds by the fundamental theorem of calculus; inequality (46) holds by the triangle inequality in concert with $\cos\xi \geq -1$, (47) holds since $|\sin\xi| \leq 1$, followed by elementary integration, while (48) is due to the monotonic decrease of $\nu \mapsto \nu^{-s}$ for $s > 1/2$, with $\nu_n \in [n, n+1]$; equality (49) holds because the sum at l.h.s.(49) is telescoping; lastly, inequality (50) holds because $\lceil x \rceil$ differs from x by at most 1, and for large x one can basically ignore the $+1$ in its argument.

For the integral in (43) the variable substitution $\nu^{-s}t = \xi$ yields

$$t^{1/s} \tfrac{1}{s} \int_0^{t/(N_s(t/\tau)+1)^s} \ln\left[1-p+p\cos\xi\right] \frac{\mathrm{d}\xi}{\xi^{1+1/s}} = t^{1/s} \left[-C_{p;s} - \tfrac{1}{s} \int_{t/(N_s(t/\tau)+1)^s}^{\infty} \ln\left[1-p+p\cos\xi\right] \frac{\mathrm{d}\xi}{\xi^{1+1/s}} \right]. \tag{51}$$

Using again the estimate $|\ln\left[1-p+p\cos\xi\right]| \leq -\ln(1-2p)$, we find (for $t \geq 1$):

$$t^{1/s} \left| \int_{t/(N_s(t/\tau)+1)^s}^{\infty} \ln\left[1-p+p\cos\xi\right] \frac{1}{s\xi^{1+1/s}} \mathrm{d}\xi \right| \leq |\ln(1-2p)| \lceil (st/\tau)^{1/(s+1)} + 1 \rceil \tag{52}$$

$$\leq K t^{1/(s+1)}. \tag{53}$$

This completes the proof of (31).

It remains to prove (32), (33), (34). Integration by parts yields

$$C_{p;s} \equiv -\tfrac{1}{s}\int_0^\infty \ln\left[1-p+p\cos\xi\right]\frac{1}{\xi^{1+1/s}}\mathrm{d}\xi = \int_0^\infty \frac{p\sin\xi}{1-p+p\cos\xi}\frac{1}{\xi^{1/s}}\mathrm{d}\xi, \tag{54}$$

where the integral at r.h.s.(54) converges absolutely when $s \in (1/2, 1)$, but only conditionally when $s \geq 1$. With the help of the geometric series r.h.s.(54) becomes

$$\frac{p}{1-p}\int_0^\infty \frac{\sin\xi}{1+\frac{p}{1-p}\cos\xi}\frac{1}{\xi^{1/s}}\mathrm{d}\xi = \sum_{n=0}^\infty (-1)^n\left(\frac{p}{1-p}\right)^{n+1}\int_0^\infty \sin\xi\cos^n\xi\frac{1}{\xi^{1/s}}\mathrm{d}\xi; \tag{55}$$

the exchange of summation and integration is justified for $s \in (1/2, 1)$ by Fubini's theorem, and for $s \geq 1$ by a more careful limiting argument involving the definition of the conditional convergent integrals as limit $R \to \infty$ of integrals from 0 to R. Repeatedly using the trigonometric identity $2\sin(\alpha)\cos(\beta) = \sin(\alpha+\beta) - \sin(\alpha-\beta)$, eventually followed by a simple rescaling of the integration variable, now yields

$$\int_0^\infty \sin\xi\cos^n\xi\frac{1}{\xi^{1/s}}\mathrm{d}\xi = \int_0^\infty \sin\xi\frac{1}{\xi^{1/s}}\mathrm{d}\xi\ \frac{1}{2^n}\sum_{k=0}^{\lceil\frac{n-1}{2}\rceil}\left[\binom{n}{k} - \binom{n}{k-1}\right](1+n-2k)^{\frac{1}{s}-1}, \tag{56}$$

where it is understood that when $k = 0$ one has $\binom{n}{-1} = 0$. The integral at r.h.s.(56) is A_s given in (33). Inserting (56) into (55) and pulling out A_s yields r.h.s.(32) with

$$B_{p;s} := \sum_{n=0}^\infty (-1)^n\left(\frac{p}{1-p}\right)^{n+1}\frac{1}{2^n}\sum_{k=0}^{\lceil\frac{n-1}{2}\rceil}\left[\binom{n}{k} - \binom{n}{k-1}\right](1+n-2k)^{\frac{1}{s}-1}, \tag{57}$$

and a simple manipulation of r.h.s.(57) now yields (34).

This completes the entirely elementary proof of Theorem 5.1. QED

6 Lévy Trends and Fluctuations

In this section we display the PDFs for a small selection of random Riemann-ζ walks $\Omega_p^\zeta(s)$, obtained by numerical Fourier transform of their characteristic functions $\mathrm{Cl}_{p;s}(t)$. We compare them with the Fourier transform of their trend functions $\exp\left(-C_{p;s}\,|t|^{1/s}\right)$, which are known as Lévy-stable distributions with *stability parameter* $\alpha = 1/s$, *skewness parameter* $\beta = 0$, *scale parameter* $c = C_{p;s}^s$, and *median* $\mu = 0$; see [PrRo69]. The comparison will highlight the importance of the fluctating factors $\exp\left(\varepsilon_{p;s}(|t|)\right)$ in the characteristic functions $\mathrm{Cl}_{p;s}(t)$.

The first figure shows the PDF $f_{\Omega_p^\zeta(s)}(\omega)$ for Cloitre's parameter values $p = 1/3$ and $s = 2$, together with the pertinent Lévy PDF (here $\mathcal{C}$ and $\mathcal{S}$ are Fresnel integrals)

$$f_{\Omega^{\text{Lévy}}_{1/2;0;C^2;0}}(\omega) = 2\pi u^3 \Big(\sin\big(\tfrac{\pi}{2}u^2\big)\big[\tfrac{1}{2} - \mathcal{S}(u)\big] + \cos\big(\tfrac{\pi}{2}u^2\big)\big[\tfrac{1}{2} - \mathcal{C}(u)\big] \Big), \tag{58}$$

where $u = C/\sqrt{2\pi|\omega|}$ and $C = C_{1/3;2}$; cf. the histogram Fig. 4.

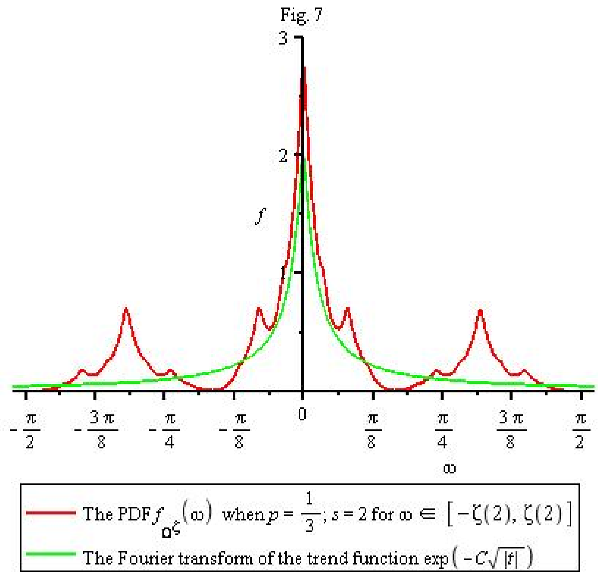
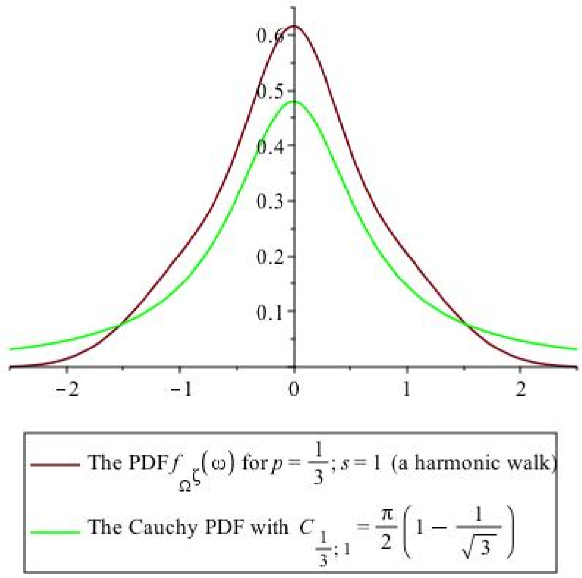


Fig. 7 reveals that the stable distribution (58) obtained by Fourier transform of the Lévy trend factor $\exp(-C\sqrt{|t|})$, which captures the "large scale" behavior of $\mathrm{P}_{\text{CL}}(t)$ asymptotically exactly but misses all of its "small scale" details (recall Fig. 1 and Fig. 2), only very crudely resembles the distribution obtained by the Fourier transform of $\mathrm{P}_{\text{CL}}(t)$. Also, we recall that the random variable $\Omega^\zeta_{1/3}(2)$ takes its values in the interval $[-\zeta(2), \zeta(2)]$, so $f_{\Omega^\zeta_{1/3}(2)}(\omega)$ vanishes identically outside this interval. By contrast, Lévy-stable PDF are "heavy-tailed" (except when $\alpha = 2$, i.e. $s = 1/2$, which is excluded here); in particular, it follows from (58) (see also [PrRo69]) that

$$f_{\Omega^{\text{Lévy}}_{\frac{1}{2};0;C;0}}(\omega) \sim C^2_{1/3;2}\sqrt{\pi}/(2|\omega|)^{3/2} \quad (\omega \to \infty). \tag{59}$$

Next we turn to the borderline case $s = 1$, which is particularly interesting. When $p \neq 1$ this random walk is a generalization of the harmonic random walk ($p = 1$) studied by Kac [Kac59], Morrison [Mor95], and Schmuland [Sch03]. Furthermore, when $p \in (0, \frac{1}{2})$ the "trend factor" of the characteristic function for $\Omega^\zeta_p(1)$ becomes $e^{-C_{p;1}|t|}$: the characteristic function of a Cauchy random variable with "theoretical spread" $C_{p;1}$ (which is explicitly computable; see below). The next Figure displays the PDF $f_{\Omega^{\zeta}_{1/3}(1)}(\omega)$ for the harmonic random walk with $p = 1/3$, together with the Cauchy distribution of theoretical spread $C_{1/3;1}$ about 0; cf. the histogram in Fig. 5.



The discrepancy between the PDF $f_{\Omega^{\zeta}_{1/3}(1)}(\omega)$ for the harmonic random walk with $p = \frac{1}{3}$ and the Cauchy distribution of theoretical spread $C_{1/3;1}$ about 0 visible in Fig. 8 is not quite as flagrant as the corresponding discrepancy in Fig. 7. Not so outside the shown interval, though: the Cauchy distribution is heavy-tailed, while $f_{\Omega^{\zeta}_{1/3}(1)}(\omega)$ has moments of all order. This can be shown by adaptation of the estimates for $f_{\Omega^{\zeta}_{1/2}(1)}(\omega)$ given by Schmuland [Sch03], or by noticing that $\mathrm{Cl}_{1/3;1}(t)$ has infinitely many derivatives at $t = 0$ and invoke Fourier theory (as also explained in [Sch03]).

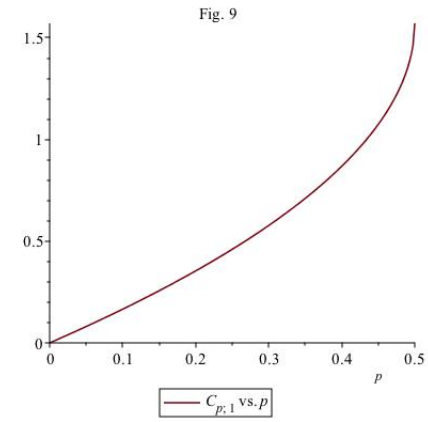
We also vindicate our claim that one can compute $C_{p;1}$ explicitly. First of all,

$$\sum_{k=0}^{\lceil (n-1)/2 \rceil} \binom{n}{k} \frac{1+n-2k}{1+n-k} = \binom{n}{\lfloor n/2 \rfloor}, \tag{60}$$

which is A001405 in Sloane's OEIS. Now $\frac{1}{2^0}\binom{0}{\lfloor 0/2 \rfloor} = 1$ while $\frac{1}{2^n}\binom{n}{\lfloor n/2 \rfloor} = \frac{1}{2^{n-1}}\binom{n-1}{\lfloor n-1/2 \rfloor}$ when $n = 2m$ with $m \in \mathbb{N}$, and using that $\sum_{m=0}^{\infty} \frac{1}{2^{2m}}\binom{2m}{m} x^{2m} = \frac{1}{\sqrt{1-x^2}}$ we compute

$$B_{p;1} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{p}{1-p}\right)^{n+1} \frac{1}{2^n} \binom{n}{\lfloor n/2 \rfloor} = 1 - \sqrt{1-2p} \quad \text{for} \quad p \in (0, \tfrac{1}{2}); \tag{61}$$

so with $A_1 = \frac{\pi}{2}$ we obtain $C_{p;1} = A_1 B_{p;1}$ in closed form, displayed in Fig. 9.



7 Open Problems

The following problems seem to be particularly worthy of further pursuit.

7.1 The interval $\frac{1}{2} \leq p \leq 1$

Theorem 5.1 cannot be true for $p \in [1/2, 1]$ because $\mathrm{Cl}_{p;s}(t)$ has zeros when $p \geq 1/2$, and for $p > 1/2$ will even be sign-changing. Indeed, when $p \in (1/2, 1]$, then (31) cannot hold with real $\varepsilon_{p;s}(|t|)$. When $p = 1/2$, then $\mathrm{Cl}_{p;s}(t)$ is non-negative and (31) could seem to still hold, but the function $\varepsilon_{p;s}(|t|)$ must diverge to $-\infty$ whenever $\mathrm{Cl}_{p;s}(t) = 0$; so the *absolute bound on the error term* $\varepsilon_{p;s}(|t|)$ stated in Theorem 5.1 must fail. A modified version of Theorem 5.1 may well hold for all $p \in (0, 1]$, though.

Namely, the integral in (32) is continuous as function of $p \in (0, \frac{1}{2}]$ for all $s > 1/2$. *Assuming* that also $p \mapsto \mathrm{Cl}_{p;s}(t)$ is continuous at $p = 1/2$ we can conclude that the limit $\lim_{p \nearrow 1/2} \exp\left(\varepsilon_{p;s}(|t|)\right) =: F_{1/2;s}(|t|)$ exists for $s > 1/2$, too, and then our Theorem 5.1 has an analog for $p = 1/2$, stating that $\mathrm{Cl}_{1/2;s}(t) = \exp\bigl(-C_{1/2;s}\,|t|^{1/s}\bigr) F_{1/2;s}(|t|)$, where $F_{1/2;s}$ is a subdominant, fluctuating, non-negative function with zeros whenever $t \in \{\pm n^s(2k-1)\pi | k \in \mathbb{N}, n \in \mathbb{N}\}$.

Moreover it is easy to show (see our excercise at the end of the Appendix) that $\mathrm{Cl}_{1/2;s}(t) = \mathrm{Cl}_{1;s}^2(t/2)$, which implies that also $\mathrm{Cl}_{1;s}(t) = \exp\left(-C_{1;s}\,|t|^{1/s}\right) F_{1;s}(|t|)$, where $F_{1;s}$ is a subdominant, fluctuating and *sign-changing* function, having zeros whenever $t \in \{\pm n^s(2k-1)\pi/2 | k \in \mathbb{N}, n \in \mathbb{N}\}$, satisfying $F_{1;s}(|t|)^2 = F_{1/2;s}(2|t|)$, and where $C_{1;s} = 2^{-1+1/s} C_{1/2;s}$. So Theorem 5.1 also has an analog for $p = 1$.

Furthermore, a half-angle identity and an obvious substitution of variables yields

$$\int_0^\infty \ln(\tfrac{1}{2} + \tfrac{1}{2}\cos\xi)\frac{1}{\xi^{1+1/s}}\mathrm{d}\xi = \int_0^\infty \ln(\cos^2\tfrac{\xi}{2})\frac{1}{\xi^{1+1/s}}\mathrm{d}\xi = 2^{1-\frac{1}{s}}\int_0^\infty \ln|\cos\xi|\frac{1}{\xi^{1+1/s}}\mathrm{d}\xi; \quad (62)$$

l.h.s.(62)$=-sC_{1/2;s}$, and with $C_{1;s} = 2^{-1+1/s} C_{1/2;s}$ now r.h.s.(62)$=-s2^{1-\frac{1}{s}} C_{1;s}$.

With the boundary values of the interval $[\frac{1}{2}, 1]$ covered, it is reasonable to suspect

Conjecture 7.1 *Let $p \in [\frac{1}{2}, 1]$ & $s > \frac{1}{2}$. Then $\exists K_{p;s} > 0$ such that*

$$\forall\, t \in \mathbb{R}: \quad \mathrm{Cl}_{p;s}(t) = \exp\left(-C_{p;s}\, |t|^{1/s}\right) F_{p;s}(|t|), \tag{63}$$

where $C_{p;s}$ is given by the absolutely convergent, multiply improper Riemann integral

$$C_{p;s} := -\frac{1}{s}\int_0^\infty \ln|1 - p + p\cos\xi|\, \frac{1}{\xi^{1+1/s}} \mathrm{d}\xi, \tag{64}$$

and $F_{p;s}$ is a generally sign-changing function, bounded by $|F_{p;s}(|t|)| \leq \exp(K_{p;s}|t|^{\frac{1}{s+1}})$.

It should be straightforward to prove this, but the proof, and the evaluation of $C_{p;s}$, will not anymore be undergraduate business. For $p = \frac{1}{2}$ and $s = 1$ one has[7] $C_{1/2;1} = \mathrm{PV}\int_0^\infty \frac{\sin\xi}{1+\cos\xi}\frac{1}{\xi}\mathrm{d}\xi = \mathrm{PV}\int_0^\infty \frac{\tan\xi}{\xi}\mathrm{d}\xi = C_{1;1} (= \frac{\pi}{2}$; see [BeGl77]), where "PV" means *principal value*, so presumably $C_{p;s} = \mathrm{PV}\int_0^\infty \frac{p\sin\xi}{1-p+p\cos\xi}\frac{1}{\xi^{1/s}}\mathrm{d}\xi$ for $p \in [\frac{1}{2}, 1]$ and $s > \frac{1}{2}$.

7.2 Why Lévy trends?

What is the probabilistic reason for the occurrence of the symmetric Lévy $\frac{1}{s}$-stable distributions associated with the trend factors when $p \in (0, 1/2)$ (or $p \in (0, 1]$)? We recall that X is a *Lévy-stable* random variable if and only if $X = c_1X_1 + c_2X_2$, where X_1 and X_2 are i.i.d. copies of X and c_1 and c_2 are suitable positive constants; see also [GaFr03]. Where is this "Lévy stability" hiding in the random Riemann-ζ walks?

7.3 Are there "perfectly typical" random Riemann-ζ walks?

If the (possibly uncountable) intersection of all typical subsets of the set of random Riemann-ζ walks for given $p \in (0, 1]$ and $s > 0$ is non-empty, then the answer is "Yes!" — in that case it would be very interesting to exhibit a perfectly typical walk explicitly, if at all possible. It is also conceivable that the intersection set is empty.

7.4 Complex random Riemann-ζ walks

What happens if one extends $\Omega_p^\zeta(s)$ to complex s? The Riemann hypothesis implies for $\zeta(s)$ itself that its extremal walks with $\mathrm{Im}(s) \neq 0$ converge to the origin if and only if $\mathrm{Re}(s) = 1/2$ and $\mathrm{Im}(s)$ is the imaginary part of a nontrivial zero of $\zeta(s)$. Does $\mathrm{Re}(s) = 1/2$ play a special role also for the random Riemann-ζ walks?

[7] Personal communications by Larry Glasser and Norm Frankel, Dec. 2016

Appendix on Power Walks

If instead of a step size which decreases by the power law $n \mapsto n^{-s}$ one uses an exponentially decreasing step size $n \mapsto s^{-n}$ with $s > 1$, the outcome is a "random geometric series" (a sum over powers of $1/s$ with random coefficients $R_p(n) \in \{-1, 0, 1\}$),

$$\Omega_p^{\text{pow}}(s) := \sum_{n\in\mathbb{N}} R_p(n) \frac{1}{s^n}, \quad s > 1, \quad p \in (0, 1]; \tag{65}$$

the pertinent walks are called "geometric walks." With more general random coefficients one simply speaks of "random power series" and their "power walks."

All these random variables $\Omega_p^{\text{pow}}(s)$ have characteristic functions with infinite trigonometric product representations obtainable from our (22) by replacing ${}^{\zeta} \to {}^{\text{pow}}$ and $n^{-s} \to s^{-n}$. Some of these can be evaluated in terms of elementary functions. We register a few special cases, beginning with three geometric walks and ending with a countable family of more general (but simple) power walks.

(i) Setting $p = 1$ and $s = 2$ gives the chacteristic function (see formula (1) of [Mor95])

$$\Phi_{\Omega_1^{\text{pow}}(2)}(t) = \prod_{n\in\mathbb{N}} \cos\left(\frac{t}{2^n}\right) \equiv \frac{\sin t}{t}, \tag{66}$$

an infinite product[8] representation of the sinc function derived by Euler algebraically by exploiting the trigonometric angle-doubling formulas (see [Mor95]). Recall that $\operatorname{sinc}(t) = \int_{-1}^{1} \frac{1}{2} e^{it\omega} d\omega$ is the (inverse) Fourier transform of the PDF $f_{\Omega^{\text{unif}}}(\omega)$ of the uniform random variable Ω^{unif} on $[-1, 1]$, i.e. $f_{\Omega^{\text{unif}}}(\omega) = \frac{1}{2}$ if $\omega \in [-1, 1]$, and $f_{\Omega^{\text{unif}}}(\omega) = 0$ otherwise. Indeed, $\Omega_1^{\text{pow}}(2)$ is a random walk representation of Ω^{unif} equivalent to the binary representation of $[0, 1]$: recalling that any real number $x \in [0, 1]$ has a binary representation[9] $x = 0.b_1b_2b_3... \equiv \sum_{n\in\mathbb{N}} b_n/2^n$ with $b_n \in \{0, 1\}$, and noting that if $x \in [0, 1]$ then $\omega := 2x - 1 \in [-1, 1]$, it follows that any real number $\omega \in [-1, 1]$ has a binary representation $\omega = \sum_{n\in\mathbb{N}} r_2(n)/2^n$ with $r_2(n) \in \{-1, 1\}$. It is manifest that any such representation of ω is an outcome of $\Omega_1^{\text{pow}}(2)$.

(ii) $\Omega_1^{\text{pow}}(3)$ is the random variable for which the characteristic function

$$\Phi_{\Omega_1^{\text{pow}}(3)}(t) = \prod_{n\in\mathbb{N}} \cos\left(\frac{t}{3^n}\right) =: \Phi_{\Omega^{\text{Cantor}}}(t/2) \tag{67}$$

[8] By substituting $\pi/2$ for t and repeatedly using a trigonometric angle-halving identity one arrives at Viète's infinite product for $2/\pi$, allegedly the first infinite product ever proposed.

[9] Those representations are not unique and one needs to consider their equivalence classes to identify them uniquely with their real outcome on $[0, 1]$, cf. [Kac59].

is a trigonometric product discussed in [Mor95]. Morrison explains that $\Phi_{\Omega^{\text{CANTOR}}}(t)$ is the characteristic function of a random variable Ω^{CANTOR} that is uniformly distributed over the Cantor set constructed from $[-1,1]$ by removing middle thirds ad infinitum. For uniform distributions on other Cantor sets, see [DFT94].

We remark that this is a nice example of a random walk whose endpoints are distributed by a singular distribution, in the sense that the Cantor set obtained from $[-1,1]$ has Lebesgue measure 0. As pointed out to us by one of the referees, the distribution of $\Omega_1^{\text{pow}}(s)$ is singular and concentrated on some Cantor set *for all* $s>2$. The referee also pointed out that for $1<s<2$ the story is more complicated: Solomyak [Sol95] proved that the distribution of $\Omega_1^{\text{pow}}(s)$ is absolutely continuous (i.e., it is equivalent to a PDF, an integrable function) *for almost every* $s\in(1,2)$; see also [PeSo96]. However, the distribution of $\Omega_1^{\text{pow}}(s)$ is not absolutely continuous for all $s\in(1,2)$ — in 1939 Erdős found values of $s\in(1,2)$ for which the distribution is singular; these are still the only ones known. See [PSS00] for further reading.

(iii) Setting $p=\frac{2}{3}$ and $s=3$ yields

$$\Phi_{\Omega_{2/3}^{\text{pow}}(3)}(t)=\prod_{n\in\mathbb{N}}\left[\frac{1}{3}+\frac{2}{3}\cos\left(\frac{t}{3^n}\right)\right]\equiv\frac{\sin(t/2)}{t/2}, \tag{68}$$

which becomes formula (9) of [Mor95] under the rescaling $t\mapsto 2t$. (As pointed out to us by the other referee, this infinite product representation of the sinc function appears also as Exercise 3 on page 11 of [Kac59].) Recalling our discussion of example (i), we conclude that (68) is the characteristic function of the uniform random variable on the interval $[-\frac{1}{2},\frac{1}{2}]$, expressed as a random walk equivalent to the ternary representation of the real numbers in $[0,1]$, shifted to the left by $-\frac{1}{2}$.

(iv) The sinc representations (66) and (68) (after rescaling $t\mapsto 2t$) are merely the first two members of a countable family of infinite trigonometric product representations of $\sin t/t$ derived by Kent Morrison [Mor95], and given by

$$\frac{\sin t}{t}=\prod_{n\in\mathbb{N}}\sum_{m=1-s}^{s-1}\frac{1-(-1)^{s+m}}{2s}\cos\left(\frac{m}{s^n}t\right),\ 1<s\in\mathbb{N}; \tag{69}$$

s even in (69) is formula (12) in [Mor95], s odd in (69) is formula (13) in [Mor95]. These representations of the characteristic function of the uniform random variable over $[-1,1]$ are obtained by considering random walks that enter with equal likelihood into any one of s branches which "s-furkate" off of every vertex of a symmetric tree centered at 0, equivalent to the usual "s-ary" representation of the real numbers in $[0,1]$ (shifted to the left by $-\frac{1}{2}$ and scaled up by a factor 2). When $s>3$ these are no longer random geometric series, but still simple random power series.

(v) Here is our excercise: Prove that $\Phi_{\Omega_{1/2}^{\times}(s)}(t)=\Phi^2_{\Omega_1^{\times}(s)}(t/2)$, where ${}^{\times}={}^{\text{pow}}$ or ${}^{\times}={}^{\zeta}$.

Acknowledgement

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