Conditions Equivalent to the Descartes-Frenicle-Sorli Conjecture on Odd Perfect Numbers

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Abstract: The Descartes-Frenicle-Sorli conjecture predicts that $k = 1$ if $q^k n^2$ is an odd perfect number with Euler prime q. In this note, we present some conditions equivalent to this conjecture.
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1 Introduction

If N is a positive integer, then we write $\sigma(N)$ for the sum of the divisors of N. A number N is *perfect* if $\sigma(N) = 2N$. We denote the abundancy index I of the positive integer w as $I(w) = \dfrac{\sigma(w)}{w}$. We also denote the deficiency D of the positive integer x as $D(x) = 2x - \sigma(x)$ [11].

Euclid and Euler showed that that an even perfect number E must have the form

$$E = (2^p - 1)\, 2^{p-1}$$

where $2^p - 1$ is a *Mersenne prime*. On the other hand, Euler showed that an odd perfect number O must have the form

$$O = q^k n^2$$

where q is an *Euler prime* (i.e., $q \equiv k \equiv 1 \pmod 4$ and $\gcd(q, n) = 1$).

It is currently unknown whether there are any odd perfect numbers. On the other hand, only 49 even perfect numbers have been found, a couple of which were discovered by the Great Internet Mersenne Prime Search [9]. It is conjectured that there are infinitely many even perfect numbers, and that there are no odd perfect numbers.

Descartes, Frenicle and subsequently Sorli conjectured that $k = 1$ [1]. Sorli conjectured $k = 1$ after testing large numbers with eight distinct prime factors for perfection [14].

Holdener presented some conditions equivalent to the existence of odd perfect numbers in [10]. In this paper, we prove the following results:

Lemma 1.1. *If $N = q^k n^2$ is an odd perfect number with Euler prime q, then $k = 1$ if and only if*

$$\frac{\sigma(n^2)}{q} \mid n^2.$$

Lemma 1.2. *If $N = q^k n^2$ is an odd perfect number with Euler prime q, then*

$$I(n^2) \leq 2 - \frac{5}{3q}.$$

Lemma 1.3. *If $N = q^k n^2$ is an odd perfect number with Euler prime q, then $k = 1$ if and only if*

$$D(n^2) \mid n^2.$$

Theorem 1.1. *If $N = q^k n^2$ is an odd perfect number with Euler prime q, then*

$$I(n^2) = 2 - \frac{5}{3q}$$

holds if and only if $k = 1$ and $q = 5$.

All of the proofs given in this note are elementary.

2 Preliminaries

Let $N = q^k n^2$ be an odd perfect number with Euler prime q.

First, we show that the following equations hold.

Lemma 2.1. *If $N = q^k n^2$ is an odd perfect number with Euler prime q, then*

$$\gcd\left(n^2, \sigma(n^2)\right) = \frac{D(n^2)}{\sigma(q^{k-1})} = \frac{\sigma(N/q^k)}{q^k}.$$

Proof. Since $N = q^k n^2$ is an odd perfect number, we have

$$\sigma(q^k)\sigma(n^2) = \sigma(N) = 2N = 2q^k n^2,$$

from which it follows that $q^k \mid \sigma(n^2)$ (because $\gcd\left(q^k, \sigma(q^k)\right) = 1$). Hence,

$$\frac{\sigma(n^2)}{q^k} = \frac{\sigma(N/q^k)}{q^k}$$

is an integer.

First, we prove that

$$\frac{D(n^2)}{\sigma(q^{k-1})} = \frac{\sigma(N/q^k)}{q^k}.$$

We rewrite the equation

$$\sigma(q^k)\sigma(n^2) = 2q^k n^2$$

as

$$\left(q^k + \sigma(q^{k-1})\right)\sigma(n^2) = 2q^k n^2$$

$$\sigma(q^{k-1})\sigma(n^2) = q^k \left(2n^2 - \sigma(n^2)\right) = q^k \cdot D(n^2)$$
$$\frac{\sigma(n^2)}{q^k} = \frac{D(n^2)}{\sigma(q^{k-1})},$$

and we are done.

Next, we show that

$$\gcd\left(n^2, \sigma(n^2)\right) = \frac{D(n^2)}{\sigma(q^{k-1})}.$$

We already know that

$$\sigma(n^2) = q^k \cdot \left(\frac{D(n^2)}{\sigma(q^{k-1})}\right).$$

Since $\sigma(q^k)\sigma(n^2) = 2q^k n^2$, we also obtain

$$\frac{2n^2}{\sigma(q^k)} = \frac{\sigma(n^2)}{q^k} = \frac{D(n^2)}{\sigma(q^{k-1})}.$$

This implies that

$$n^2 = \frac{\sigma(q^k)}{2} \cdot \left(\frac{D(n^2)}{\sigma(q^{k-1})}\right).$$

It follows that

$$\gcd\left(n^2, \sigma(n^2)\right) = \frac{D(n^2)}{\sigma(q^{k-1})}$$

since

$$\gcd\left(q^k, \frac{\sigma(q^k)}{2}\right) = \gcd(q^k, \sigma(q^k)) = 1.$$

This concludes the proof. □

Remark 2.1. *Dris obtained the lower bound 3 for $\sigma(N/q^k)/q^k$ in [6] and [7]. The following papers obtain (ever-increasing) lower bounds for this quantity: [8], [4], [2], [5].*

Remark 2.2. *Notice that*

$$\frac{\sigma(n^2)}{q^k} = \frac{2n^2}{\sigma(q^k)} > \frac{8}{5} \cdot \left(\frac{n^2}{q^k}\right)$$

since $I(q^k) < 5/4$ holds unconditionally (i.e., for $k \geq 1$). Additionally, note that

$$\frac{8}{5} \cdot \left(\frac{n^2}{q^k}\right) > \frac{8n}{5}$$

is true if $q^k < n$.

Dris conjectured in [6] that $q^k < n$. Recently, Brown has announced a proof for $q < n$, and that $q^k < n$ holds "in many cases" [3].

Remark 2.3. *It is an easy exercise to prove that $q^k < n$ implies the biconditional*

$$q^k < n \Leftrightarrow \sigma(q^k) < \sigma(n) \Leftrightarrow \frac{\sigma(q^k)}{n} < \frac{\sigma(n)}{q^k}.$$

We refer the interested reader to MSE (`http://math.stackexchange.com/q/713035`) for an expository proof.

Next, we prove the following lemmas.

Lemma 2.2. *If $N = q^k n^2$ is an odd perfect number with Euler prime q, then*

$$I(n^2) \geq 2 - \frac{5}{3q} \Rightarrow (k = 1 \land q = 5).$$

Proof. Note that

$$I(n^2) = \frac{2}{I(q^k)} = \frac{2q^k(q-1)}{q^{k+1}-1} = 2 - 2 \cdot \left(\frac{q^k - 1}{q^{k+1}-1}\right).$$

If

$$I(n^2) \geq 2 - \frac{5}{3q}$$

then we obtain

$$2 - 2 \cdot \left(\frac{q^k - 1}{q^{k+1}-1}\right) \geq 2 - \frac{5}{3q}$$

$$\frac{5}{3q} \geq 2 \cdot \left(\frac{q^k - 1}{q^{k+1}-1}\right)$$

$$5q^{k+1} - 5 \geq 6q^{k+1} - 6q$$

$$0 \geq q^{k+1} - 6q + 5,$$

which then implies that $k = 1$. (Otherwise, if $k > 1$ we have

$$0 \geq q^{k+1} - 6q + 5 \geq q^6 - 6q + 5$$

since $k \equiv 1 \pmod 4$, contradicting $q \geq 5$.) Now, since $k = 1$, we get

$$0 \geq q^2 - 6q + 5 = (q-5)(q-1)$$

which implies that $1 \leq q \leq 5$. Together with $q \geq 5$, this means that $q = 5$. This concludes the proof. $\square$

Lemma 2.3. *If $N = q^k n^2$ is an odd perfect number with Euler prime q, then $k = 1$ implies*

$$I(n^2) \leq 2 - \frac{5}{3q}.$$

Proof. Suppose that $k = 1$. By Lemma 1.1, we have $\sigma(n^2)/q \mid n^2$. This implies that there exists an (odd) integer d such that

$$n^2 = d \cdot \left(\frac{\sigma(n^2)}{q}\right).$$

Note that, from the equation $\sigma(N) = 2N$, we obtain (upon setting $k = 1$)

$$(q+1)\sigma(n^2) = \sigma(q)\sigma(n^2) = 2qn^2$$

from which we get

$$d = \frac{n^2}{\sigma(n^2)/q} = \frac{q+1}{2}.$$

Notice that, when $k = 1$, we can derive
$$\frac{5}{3} \leq I(n^2) = \frac{2}{I(q)} = \frac{2q}{q+1} < 2$$
so that we have
$$\frac{q}{2} < d = \frac{q}{I(n^2)} \leq \frac{3q}{5}.$$
Additionally, note that, when $k = 1$, we have
$$I(n^2) = \frac{2}{I(q)} = \frac{2q}{q+1} = \frac{2q+2}{q+1} - \frac{2}{q+1} = 2 - \frac{1}{\frac{q+1}{2}} = 2 - \frac{1}{d}.$$
Consequently, we obtain
$$\frac{q}{2} < d \leq \frac{3q}{5}$$
$$\frac{5}{3q} \leq \frac{1}{d} < \frac{2}{q}$$
$$2 - \frac{2}{q} < 2 - \frac{1}{d} = I(n^2) \leq 2 - \frac{5}{3q},$$
and we are done. □

3 The proof of Lemma 1.1

Let $N = q^k n^2$ be an odd perfect number with Euler prime q.

By Lemma 2.1, we have
$$\frac{D(n^2)}{\sigma(q^{k-1})} = \frac{\sigma(N/q^k)}{q^k}.$$
This equation can be rewritten as
$$D(n^2) = \frac{\sigma(n^2)}{q} \cdot I(q^{k-1}).$$
Suppose that $\sigma(n^2)/q \mid n^2$. Trivially, we know that $\sigma(n^2)/q \mid \sigma(n^2)$. Thus, we have
$$\frac{\sigma(n^2)}{q} \mid \left(2n^2 - \sigma(n^2)\right) = D(n^2),$$
giving
$$\frac{\sigma(n^2)}{q} \mid \frac{\sigma(n^2)}{q} \cdot I(q^{k-1}).$$
This implies that $I(q^{k-1})$ is an integer. Since $1 \leq I(q^{k-1}) < 5/4$, we obtain $k = 1$.

Now assume that $k = 1$. We obtain
$$2n^2 - \sigma(n^2) = D(n^2) = \frac{\sigma(n^2)}{q}.$$
Again, since $\sigma(n^2)/q \mid \sigma(n^2)$, this implies
$$\frac{\sigma(n^2)}{q} \mid n^2$$
since $\sigma(n^2)/q$ is odd.

This concludes the proof of Lemma 1.1.

4 The proof of Lemma 1.2

Let $N = q^k n^2$ be an odd perfect number with Euler prime q.

Assume to the contrary that
$$I(n^2) > 2 - \frac{5}{3q}.$$
Following the proof of Lemma 2.2, we get
$$0 > q^{k+1} - 6q + 5.$$
Since $k \equiv 1 \pmod 4$, then $k \geq 1$, which implies that
$$0 > q^{k+1} - 6q + 5 \geq q^2 - 6q + 5 = (q-5)(q-1).$$
This then finally gives $1 < q < 5$, contradicting $q \geq 5$.

We therefore conclude that
$$I(n^2) \leq 2 - \frac{5}{3q},$$
and this finishes the proof of Lemma 1.2.

5 The proof of Lemma 1.3

Let $N = q^k n^2$ be an odd perfect number with Euler prime q.

By Lemma 2.1, we have
$$\frac{D(n^2)}{\sigma(q^{k-1})} = \frac{2n^2}{\sigma(q^k)}.$$
Multiplying throughout the last equation by $\sigma(q^{k-1})\sigma(q^k)$, we get
$$D(n^2)\sigma(q^k) = 2n^2\sigma(q^{k-1}).$$
If $k = 1$, then it is evident that $D(n^2) \mid 2n^2$, from which it follows that $D(n^2) \mid n^2$ since $D(n^2)$ is odd.

Now, assume that $D(n^2) \mid n^2$. Then we have
$$\frac{\sigma(q^k)}{2\sigma(q^{k-1})} = \frac{n^2}{D(n^2)}$$
is an integer. Since $\gcd\left(\sigma(q^{k-1}), \sigma(q^k)\right) = 1$, the previous equation then implies that $k = 1$.

This concludes the proof of Lemma 1.3. In particular, we have shown that the Descartes-Frenicle-Sorli conjecture for odd perfect numbers $q^k n^2$ is true if and only if the non-Euler part n^2 is *deficient-perfect* [12].

6 The proof of Theorem 1.1

Let $N = q^k n^2$ be an odd perfect number with Euler prime q.

We want to prove that the equation

$$I(n^2) = 2 - \frac{5}{3q}$$

holds if and only if $k = 1$ and $q = 5$.

Suppose that

$$I(n^2) = 2 - \frac{5}{3q}.$$

Following the proof of Lemma 2.1, we get

$$0 = q^{k+1} - 6q + 5.$$

Assume to the contrary that $k > 1$. Since $k \equiv 1 \pmod 4$, we obtain

$$0 = q^{k+1} - 6q + 5 \geq q^6 - 6q + 5.$$

This contradicts $q \geq 5$. Thus, we have established that $k = 1$.

Substituting $k = 1$ into $0 = q^{k+1} - 6q + 5$, we have

$$0 = q^2 - 6q + 5 = (q - 5)(q - 1)$$

which implies that $q = 5$ since $q \geq 5$. This takes care of one direction of Theorem 1.1.

For the other direction, assume that $k = 1$ and $q = 5$. We want to show that

$$I(n^2) = 2 - \frac{5}{3q}.$$

Note that, when $k = 1$ and $q = 5$, we obtain

$$I(n^2) = \frac{2}{I(q)} = \frac{2q}{q+1} = \frac{5}{3}.$$

We also get

$$2 - \frac{5}{3q} = 2 - \frac{1}{3} = \frac{5}{3}$$

so that we have

$$I(n^2) = 2 - \frac{5}{3q},$$

as desired.

7 Concluding Remarks

We end with some remarks related to the biconditional

$$k = 1 \Longleftrightarrow \Bigg(D(n^2) \mid n^2\Bigg).$$

Suppose that $k = 1$. By Lemma 1.3 and Lemma 2.1, we obtain

$$D(n^2) = \gcd\left(n^2, \sigma(n^2)\right) = \frac{\sigma(n^2)}{q} = \frac{n^2}{(q+1)/2}.$$

Multiplying throughout the equations by $q(q+1)/2$, we have

$$D(n^2) \cdot \left(\frac{q(q+1)}{2}\right) = \left(\frac{q+1}{2}\right) \cdot \sigma(n^2) = qn^2 = N.$$

In fact, as shown by Slowak [13], every odd perfect number N has the form

$$N = q^k \cdot \frac{\sigma(q^k)}{2} \cdot d$$

for some $d > 1$. We give a quick proof of this fact here.

By Lemma 2.1, we obtain

$$\frac{D(n^2)}{\sigma(q^{k-1})} = \gcd\left(n^2, \sigma(n^2)\right) = \frac{\sigma(n^2)}{q^k} = \frac{n^2}{\sigma(q^k)/2}.$$

Multiplying throughout the equations by $q^k\sigma(q^k)/2$, we get

$$\frac{q^k\sigma(q^k)}{2} \cdot \frac{D(n^2)}{\sigma(q^{k-1})} = \frac{q^k\sigma(q^k)}{2} \cdot \gcd\left(n^2, \sigma(n^2)\right) = q^k n^2 = N,$$

where

$$d = \frac{D(n^2)}{\sigma(q^{k-1})} = \gcd(n^2, \sigma(n^2)) > 1$$

by Remark 2.1.

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References

[1] B. D. Beasley, Euler and the ongoing search for odd perfect numbers, ACMS 19th Biennial Conference Proceedings, Bethel University, May 29 to Jun. 1, 2013.

[2] K. A. Broughan, D. Delbourgo, Q. Zhou, Improving the Chen and Chen result for odd perfect numbers, *Integers*, **13** (2013), #A39.

[3] P. A. Brown, A partial proof of a conjecture of Dris, preprint (2016), `http://arxiv.org/abs/1602.01591`.

[4] F-J. Chen and Y-G. Chen, On odd perfect numbers, *Bull. Aust. Math. Soc.*, **86** (2012), 510-514.

[5] F-J. Chen and Y-G. Chen, On the index of an odd perfect number, *Colloq. Math.*, **136** (2014), 41-49.

[6] J. A. B. Dris, Solving the Odd Perfect Number Problem: Some Old and New Approaches, M. Sc. Math thesis, De La Salle University, Manila, Philippines, 2008.

[7] J. A. B. Dris, The abundancy index of divisors of odd perfect numbers, *J. Integer Seq.*, **15** (2012), Article 12.4.4.

[8] J. A. B. Dris and F. Luca, A note on odd perfect numbers, *Fibonacci Quart.*, **54** (2016), no. 4, 291-295.

[9] G. Woltman and S. Kurowski, The Great Internet Mersenne Prime Search, `http://www.mersenne.org/primes/`. Last viewed: September 9, 2016.

[10] J. A. Holdener, Conditions equivalent to the existence of odd perfect numbers, *Math. Mag.*, **79** (2006), 389-391.

[11] N. J. A. Sloane, OEIS sequence A033879 - Deficiency of n, or $2n - \sigma(n)$, `http://oeis.org/A033879`.

[12] C. F. E. Adajar, OEIS sequence A271816 - Deficient-perfect numbers: Deficient numbers n such that $n/(2n - \sigma(n))$ is an integer, `http://oeis.org/A271816`.

[13] J. Slowak, Odd perfect numbers, *Math. Slovaca*, **49** (1999), 253-254.

[14] R. M. Sorli, Algorithms in the Study of Multiperfect and Odd Perfect Numbers, Ph. D. Thesis, University of Technology, Sydney, 2003, `http://hdl.handle.net/10453/20034`.