CONSTELLATIONS OF PRIMES GENERATED BY TWIN PRIMES

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Abstract. We study a special set of constellations of primes generated by twin primes.

1. Introduction and main results

Let A be an infinite sequence of increasing positive integers with the infinite complementary sequence $\overline{A}$. Let H be lexicographically first strictly increasing sequence starting, say, $H(2) = h_0$ with the property that $H(n) \in A$ if and only if $n \in A$. Shevelev [1] studied several different H-sequences which possess an interesting property: let numbers a and b are arbitrary in A; let H_a and H_b are defined in the same way as sequence H but with the initial term a and b respectively; then there exists a position $n = n(a, b)$ in which H_a and H_b merge. Most likely, it is a difficult problem to find a test for A, when this property holds. Here is only one conjecture in case when $A = P$ is the sequence of all primes.

Conjecture 1. *Consider lexicographically first strictly increasing sequence starting $H(2) = 3$ with the property that $H(n)$ is prime if and only if n is prime. Then for arbitrary primes $a > b \geq 3$, H_a and H_b merge.*

Note that Conjecture 1 easily follows from each of the following weaker forms: α) $b = 3$; β) $a = Nextprime(b), b \geq 3$. If Conjecture 1 is true, there appear many natural questions. For example,

1) For a given primes $a > b \geq 3$, what is the position when H_a and H_b merge?

For the first such pairs $(5, 3); (7, 3), (7, 5); (11, 3), (11, 5), (11, 7); ...$ we get the following positions:

$$(1) \qquad 11;\ \ 47, 47;\ \ 47, 47, 11;\ \ 47, 47, 17, 17;\ \ 683, 683, 683, 683, 683; ...$$

(cf. A276676[2]).

2) What is the maximal difference between the corresponding terms of H_a and H_b?

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For example, in case when a and b are twin primes: $a = A001359(n-1)+2,\ \ b = A001359(n-1),\ \ n \geq 2$, we get

$$(2) \qquad 4, 14, 6, 6, 6, 12, 6, 8, 14, 14, 18, 36, 24, 65, 18, 6, 10, 6, 84, 14, 162, ...$$

(cf. A276826[2]).

Let us consider those lesser of twin primes $\{A001359(l)\},\ \ l \geq 1$, which correspond to terms of the sequence (2) which not exceeding 6. We have the sequence $A276848 = \{c(n)\}[2]$:

$$(3) \qquad 3, 11, 17, 29, 59, 227, 269, 1277, 1289, 1607, 2129, 2789, 3527, 3917, ...$$

We call "nearest" twin primes the corresponding pairs of twin primes $(3, 5)$, $(11, 13), (17, 19), ...$ A numerical research of sequence (3) shows that we have an interesting phenomenon: although terms c(n) == 7 or 9 (mod 10) occur often, the first terms c(n)==1 (mod 10) are 11, 165701, ... This phenomenon is explained by the following.

Theorem 1. 1) $c(n) = p \equiv 7 \pmod{10}$ *if and only if, for some* $t \geq 0$, $p = 30t + 17$ *and we have the following constellation of five consecutive primes:* $p, p+2, p+6, p+12, p+14$;
2) $c(n) = p \equiv 9 \pmod{10}$ *if and only if, for some* $t \geq 0$, $p = 30t + 29$ *and we have the following constellation of five consecutive primes:* $p, p+2, p+8, p+12, p+14$;
3) $c(n) = p \equiv 1 \pmod{10}$ *if and only if, for some* $t \geq 0$, $p = 30t + 11$ *and we have the following constellation of seven consecutive primes:* $p, p+2, p+6, p+8, p+12, p+18, p+20$.

Again in case when a and b are twin primes: $a = A001359(n-1)+2,\ \ b = A001359(n-1),\ \ n \geq 2$, consider the smallest $m = m(n)$ such that $H_a(m) - H_b(m) > 6$ or 0 if there is no such m. We get the sequence

$$(4) \qquad 0, 13, 0, 0, 0, 9, 0, 11, 11, 5, 3, 15, 3, 7, 3, 0, 3, 0, 3, 5, 7, 3, 11, ...$$

Theorem 2. *The all distinct numbers in the sequence* (4) *are* 0, 3, 5, 7, 9, 11, 13, 15 *and* 17 *only.*

2. Proof of Theorem 1 (sufficiency)

1) Let $p = 30t + 17$, set $a = 30t + 19, b = p$. Let $\{b, a\}$ are twin primes. If also $30t + 23, 30t + 29, 30t + 31$ are prime, then we have

$$H_a(2) = 30t + 19,\ \ H_b(2) = 30t + 17;$$
$$H_a(3) = 30t + 23,\ \ H_b(3) = 30t + 19;$$
$$H_a(4) = 30t + 24,\ \ H_b(4) = 30t + 20;$$

$$H_a(5) = 30t + 29, \;\; H_b(5) = 30t + 23;$$
$$H_a(6) = 30t + 30, \;\; H_b(6) = 30t + 24;$$
$$H_a(7) = 30t + 31, \;\; H_b(7) = 30t + 29;$$
$$H_a(8) = 30t + 32, \;\; H_b(8) = 30t + 30;$$
$$H_a(9) = 30t + 33, \;\; H_b(9) = 30t + 32;$$
$$H_a(10) = 30t + 34, \;\; H_b(10) = 30t + 33;$$
$$H_a(11) = Nextprime(30t + 34), \;\; H_b(11) = Nextprime(30t + 33).$$

Since $Nextprime(30t+34) = Nextprime(30t+33)$ then $H_a(n) = H_b(n)$ for $n >= 11$ and, for $n >= 2$, $\max(H_a(n) - H_b(n) = 6$ which holds for $n = 5, 6$.

2) Let $p = 30t + 29$, set $a = 30t + 31, b = p$. Let $\{b, a\}$ are twin primes. If also $30t + 37, 30t + 41, 30t + 43$ are prime, then we have

$$H_a(2) = 30t + 31, \;\; H_b(2) = 30t + 29;$$
$$H_a(3) = 30t + 37, \;\; H_b(3) = 30t + 31;$$
$$H_a(4) = 30t + 38, \;\; H_b(4) = 30t + 32;$$
$$H_a(5) = 30t + 41, \;\; H_b(5) = 30t + 37;$$
$$H_a(6) = 30t + 42, \;\; H_b(6) = 30t + 38;$$
$$H_a(7) = 30t + 43, \;\; H_b(7) = 30t + 41;$$
$$H_a(8) = 30t + 44, \;\; H_b(8) = 30t + 42;$$
$$H_a(9) = 30t + 45, \;\; H_b(9) = 30t + 44;$$
$$H_a(10) = 30t + 46, \;\; H_b(10) = 30t + 45;$$
$$H_a(11) = Nextprime(30t + 46), \;\; H_b(11) = Nextprime(30t + 45).$$

Since $Nextprime(30t+46) = Nextprime(30t+45)$ then $H_a(n) = H_b(n)$ for $n >= 11$ and, for $n >= 2$, $\max(H_a(n) - H_b(n) = 6$ which holds for $n = 3, 4$.

3) Let $p = 30t + 11$, set $a = 30t + 13, b = p$. Let $\{b, a\}$ are twin primes. If also $30t+17, 30t+19, 30t+23, 30t+29, 30t+31$ are all prime, then we have

$$H_a(2) = 30t + 13, \;\; H_b(2) = 30t + 11;$$
$$H_a(3) = 30t + 17, \;\; H_b(3) = 30t + 13;$$
$$H_a(4) = 30t + 18, \;\; H_b(4) = 30t + 14;$$
$$H_a(5) = 30t + 19, \;\; H_b(5) = 30t + 17;$$
$$H_a(6) = 30t + 20, \;\; H_b(6) = 30t + 18;$$
$$H_a(7) = 30t + 23, \;\; H_b(7) = 30t + 19;$$

$$H_a(8) = 30t + 24, \;\; H_b(8) = 30t + 20;$$
$$H_a(9) = 30t + 25, \;\; H_b(9) = 30t + 21;$$
$$H_a(10) = 30t + 26, \;\; H_b(10) = 30t + 22;$$
$$H_a(11) = 30t + 29, \;\; H_b(11) = 30t + 23;$$
$$H_a(12) = 30t + 30, \;\; H_b(12) = 30t + 24;$$
$$H_a(13) = 30t + 31, \;\; H_b(13) = 30t + 29;$$
$$H_a(14) = 30t + 32, \;\; H_b(14) = 30t + 30;$$
$$H_a(15) = 30t + 33, \;\; H_b(15) = 30t + 32;$$
$$H_a(16) = 30t + 34, \;\; H_b(16) = 30t + 33;$$
$$H_a(17) = Nextprime(30t + 34), \;\; H_b(17) = Nextprime(30t + 33).$$

So $H_a(n) = H_b(n), \;\; n \geq 17$, and and, for $n >= 2$, $\max(H_a(n) - H_b(n) = 6$ which holds for $n = 11, 12.$■

3. Proof of Theorem 1 (necessity) and of Theorem 2

1) Let $c(n) = p \equiv 7 \pmod{10}$. By Section 2, if we have five consecutive primes of the form

$$(5) \qquad 30t + 17 = p, 30t + 19, 30t + 23, 30t + 29, 30t + 31, \;\; t \geq 0,$$

then, for all $n \geq 2$, $(H_a(n) - H_b(n)) \leq 6$. Let us show that, if the condition (5) does not hold, then $(H_a(n) - H_b(n)) > 6$. Denote by m the smallest n when $(H_a(n) - H_b(n)) > 6$.

1a) Let $30t + 17, 30t + 19$ be prime, but $30t + 23$ be composite. Then we have

$$H_a(2) = 30t + 19, \;\; H_b(2) = 30t + 17;$$
$$H_a(3) = Nextprime(30t + 23) \geq 30t + 29, \;\; H_b(3) = 30t + 19.$$

So $(H_a(3) - H_b(3)) \geq 10$, i.e. $m = 3$.

1b) Let $30t + 17, 30t + 19, 30t + 23$ be prime, but $30t + 29$ be composite. We have

$$H_a(2) = 30t + 19, \;\; H_b(2) = 30t + 17;$$
$$H_a(3) = 30t + 23, \;\; H_b(3) = 30t + 19;$$
$$H_a(4) = 30t + 24, \;\; H_b(4) = 30t + 20;$$
$$H_a(5) = Nextprime(30t + 29) \geq 30t + 31, \;\; H_b(5) = 30t + 23.$$

So $(H_a(5) - H_b(5)) \geq 8$, i.e. $m = 5$.

1c) Let $30t+17, 30t+19, 30t+23, 30t+29$ be prime, but $30t+31$ be composite. We have

$$H_a(4) = 30t+24, \;\; H_b(4) = 30t+20;$$
$$H_a(5) = 30t+29, \;\; H_b(5) = 30t+23;$$
$$H_a(6) = 30t+30, \;\; H_b(6) = 30t+24;$$
$$H_a(7) = Nextprime(30t+31) \geq 30t+37, \;\; H_b(7) = 30t+29.$$

So $(H_a(7) - H_b(7)) \geq 8$, i.e. $m = 7$. This proves the case $c(n) \equiv 7 \pmod{10}$ with $m = 3, 5, \;\; or \;\; 7$

2) Now let $c(n) = p \equiv 9 \pmod{10}$. By Section 2, if we have five consecutive primes of the form

$$(6) \qquad 30t+29 = p, 30t+31, 30t+37, 30t+41, 30t+43, \;\; t \geq 0,$$

then, for all $n \geq 2$, $(H_a(n) - H_b(n)) \leq 6$. Let us show that, if the condition (6) does not hold, then $(H_a(n) - H_b(n)) > 6$ and $3 \leq m \leq 17$.

2a) Let in (6) $30t+37$ is composite. Then we have

$$H_a(2) = 30t+31, \;\; H_b(2) = 30t+29;$$
$$H_a(3) = Nextprime(30t+37) \geq 30t+41, \;\; H_b(3) = 30t+31.$$

So $(H_a(3) - H_b(3)) \geq 10$, i.e. $m = 3$.

2b) Let $30t+37$ be prime, but $30t+41$ be composite.

2ba) Let $30t+37$ be prime, but $30t+41, 30+49$ be composite.
We have

$$H_a(2) = 30t+31, \;\; H_b(2) = 30t+29;$$
$$H_a(3) = 30t+37, \;\; H_b(3) = 30t+31;$$
$$H_a(4) = 30t+38, \;\; H_b(4) = 30t+32;$$
$$H_a(5) = Nextprime(30t+43) \geq 30t+47, \;\; H_b(5) = 30t+37,$$

if $30t+43$ is composite, $m = 5$.

$$H_a(5) = 30t+43, \;\; H_b(5) = 30t+37,$$

if $30t+43$ is prime;

$$H_a(6) = 30t+44, \;\; H_b(6) = 30t+38;$$
$$H_a(7) = Nextpime(30t+47) \geq 30t+53, \;\; H_b(7) = 30t+43,$$

if $30+47$ is composite, $m=7$.

$$H_a(7)=30t+47,\;\; H_b(7)=30t+43,$$

if $30t+47$ is prime;

$$H_a(8)=30t+48,\;\; H_b(8)=30t+44;$$

$$H_a(9)=30t+49,\;\; H_b(9)=30t+45;$$

$$H_a(10)=30t+50,\;\; H_b(10)=30t+46;$$

$$H_a(11)=Nextprime(30t+53)\geq 30t+59,\;\; H_b(11)=30t+47;$$

if $30t+53$ is composite, $m=11$.

$$H_a(11)=30t+53,\;\; H_b(11)=30t+47,$$

if $30t+53$ is prime;

$$H_a(12)=30t+54,\;\; H_b(12)=30t+48;$$

$$H_a(13)=Nextprime(30t+54)\geq 30t+61,\;\; H_b(13)=30t+53,$$

if $30+59$ is composite, $m=13$.

$$H_a(13)=30t+59,\;\; H_b(13)=30t+53,$$

if $30t+59$ is prime;

$$H_a(14)=30t+60,\;\; H_b(14)=30t+54;$$

$$H_a(15)=30t+62,\;\; H_b(15)=30t+55,$$

if $30t+61$ is prime, $m=15$.

$$H_a(15)=30t+61,\;\; H_b(15)=30t+55,$$

if $30t+61$ is composite;

$$H_a(16)=30t+62,\;\; H_b(16)=30t+56;$$

$$H_a(17)=Nextprime(30t+62)\geq 30t+67,\;\; H_b(17)=30t+59,$$

and $m=17$, which completes proof of the case 2ba)

2bb) Let $30t+37, 30t+49$ be prime, but $30t+41$, $30t+47$ be composite. Now the rows $H_a(2),...,H_a(6)$ are as in 2ba). But here we have $30t+43$ and $30t+49$ are prime. So,

$$H_a(7)=30t+49,\;\; H_b(7)=30t+43,$$

$$H_a(8)=30t+50,\;\; H_b(8)=30t+44;$$

$$H_a(9)=30t+51,\;\; H_b(9)=30t+46;$$

$$H_a(10)=30t+52,\;\; H_b(10)=30t+47;$$

$$H_a(11) = Nextprime(30t+53) \geq 30t+59, \;\; H_b(11) = 30t+49,$$

if $30t + 53$ is composite, $m = 11$.

$$H_a(11) = 30t+53, \;\; H_b(11) = 30t+49,$$

if $30t + 53$ is prime;

$$H_a(12) = 30t+54, \;\; H_b(12) = 30t+50;$$

$$H_a(13) = Nextprime(30t+59) \geq 30t+61, \;\; H_b(13) = 30t+53,$$

if $30t + 59$ is composite, $m = 13$.

$$H_a(13) = 30t+59, \;\; H_b(13) = 30t+53,$$

if $30t + 59$ is prime;

$$H_a(14) = 30t+60, \;\; H_b(14) = 30t+54;$$

$$H_a(15) = 30t+62, \;\; H_b(15) = 30t+55,$$

if $30t + 61$ is prime, $m = 15$.

$$H_a(15) = 30t+61, \;\; H_b(15) = 30t+55,$$

if $30t + 61$ is composite;

$$H_a(16) = 30t+62, \;\; H_b(16) = 30t+56;$$

$$H_a(17) = Nextprime(30t+62) \geq 30t+67, \;\; H_b(17) = 30t+59$$

and $m = 17$, which completes proof of the case 2bb)

2bc) Let $30t+37, 30t+47, 30t+49$ be prime, but $30t+41$, be composite. Now again the rows $H_a(2), ..., H_a(6)$ are as in 2ba). But here we have $30t + 43$, $30t + 47$ and $30t + 49$ are prime. So,

$$H_a(7) = 30t+47, \;\; H_b(7) = 30t+43,$$

$$H_a(8) = 30t+48, \;\; H_b(8) = 30t+44;$$

$$H_a(9) = 30t+50, \;\; H_b(9) = 30t+45;$$

$$H_a(10) = 30t+51, \;\; H_b(10) = 30t+46;$$

$$H_a(11) = Nextprime(30t+53) \geq 30t+59, \;\; H_b(11) = 30t+47,$$

if $30t + 53$ is composite, $m = 11$.

$$H_a(11) = 30t+53, \;\; H_b(11) = 30t+47,$$

if $30t + 53$ is prime;

$$H_a(12) = 30t+54, \;\; H_b(12) = 30t+48;$$

$$H_a(13) \geq 30t+59, \;\; H_b(12) = 30t+49$$

and $m = 13$.
2c) Let $30t + 37, 30t + 41$ be prime, but $30t + 43$ be composite.
We have

$$H_a(2) = 30t + 31, \ \ H_b(2) = 30t + 29;$$

$$H_a(3) = 30t + 37, \ \ H_b(3) = 30t + 31;$$

$$H_a(4) = 30t + 38, \ \ H_b(4) = 30t + 32;$$

$$H_a(5) = 30t + 41, \ \ H_b(5) = 30t + 37$$

$$H_a(6) = 30t + 42, \ \ H_b(6) = 30t + 38;$$

$$H_a(7) = Nextprime(30t + 47) \geq 30t + 49, \ \ H_b(7) = 30t + 41,$$

if $30t + 47$ is composite, $m = 7$.

$$H_a(7) = 30t + 47, \ \ H_b(7) = 30t + 41,$$

if $30t + 47$ is prime;

$$H_a(8) = 30t + 48, \ \ H_b(8) = 30t + 42;$$

$$H_a(9) = 30t + 50, \ \ H_b(9) = 30t + 43,$$

if $30t + 49$ is prime, $m = 9$.

$$H_a(9) = 30t + 49, \ \ H_b(9) = 30t + 43,$$

if $30t + 49$ is composite;

$$H_a(10) = 30t + 50, \ \ H_b(10) = 30t + 44;$$

$$H_a(11) = Nextprime(30t + 53) \geq 30t + 59, \ \ H_b(11) = 30t + 47,$$

if $30t + 53$ is composite, $m = 11$.

$$H_a(11) = 30t + 53, \ \ H_b(11) = 30t + 47,$$

if $30t + 53$ is prime.
The two last rows coincide with the corresponding rows of 2ba). Therefore, 2c) has the same end as 2ba) with $m = 13, 15$ and $m = 17$.
This proves the case 2c) and together with 2a), 2ba), 2bb) and 2bc) completes the proof of the case $c(n) \equiv 9 \pmod{10}$ with $m = 3, 5, 7, 9, 11, 13, 15$ or 17.

3) Finally let $c(n) = p \equiv 1 \pmod{10}$. By Section 2, if we have seven consecutive primes of the form

(7) $30t+11 = p, 30t+13, 30t+17, 30t+19, 30t+23, 30t+29, 30t+31, \ \ t \geq 0,$

then, for all $n \geq 2$, $(H_a(n) - H_b(n)) \leq 6$. Again let us show that, if the condition (7) does not hold, then $(H_a(n) - H_b(n)) > 6$ and $3 \leq m \leq 13$.
3a) Let $30t + 11, 30t + 13$ be prime, but $30t + 17$ be composite.

We have

$$H_a(2) = 30t + 13, \;\; H_b(2) = 30t + 11;$$

$$H_a(3) = Nextprime(30t + 19) \geq 30t + 23, \;\; H_b(3) = 30t + 13,$$

if $30t + 19$ is composite, $m = 3$.

$$H_a(3) = 30t + 19, \;\; H_b(3) = 30t + 13,$$

if $30t + 19$ is prime;

$$H_a(4) = 30t + 20, \;\; H_b(4) = 30t + 14;$$

$$H_a(5) = Nextprime(30t + 23) \geq 30t + 29, \;\; H_b(5) = 30t + 19,$$

if $30t + 23$ is composite, $m = 5$.

$$H_a(5) = 30t + 23, \;\; H_b(5) = 30t + 19,$$

if $30t + 23$ is prime.

$$H_a(6) = 30t + 24, \;\; H_b(6) = 30t + 20;$$

$$H_a(7) = Nextprime(30t + 29) \geq 30t + 31, \;\; H_b(7) = 30t + 23,$$

if $30t + 29$ is composite, $m = 7$.

$$H_a(7) = 30t + 29, \;\; H_b(7) = 30t + 23,$$

if $30 + 29$ is prime;

$$H_a(8) = 30t + 30, \;\; H_b(8) = 30t + 24;$$

$$H_a(9) = 30t + 32, \;\; H_b(9) = 30t + 25,$$

if $30t + 31$ is prime and $m = 9$.

$$H_a(9) = 30t + 31, \;\; H_b(9) = 30t + 25,$$

if $30t + 31$ is composite;

$$H_a(10) = 30t + 32, \;\; H_b(10) = 30t + 26;$$

$$H_a(11) = Nextprime(30t + 32) \geq 30t + 37, \;\; H_b(11) = 30t + 29$$

and $m = 11$. This proves 3a).

3b) Let $30t + 11, 30t + 13, 30t + 17$ be prime, but $30t + 19$ be composite. We have

$$H_a(2) = 30t + 13, \;\; H_b(2) = 30t + 11;$$

$$H_a(3) = 30t + 17, \;\; H_b(3) = 30t + 13;$$

$$H_a(4) = 30t + 18, \;\; H_b(4) = 30t + 14;$$

$$H_a(5) = Nextprime(30t + 23) \geq 30t + 29, \;\; H_b(5) = 30t + 17,$$

if $30t + 23$ be composite, $m = 5$.

$$H_a(5) = 30t + 23, \ \ H_b(5) = 30t + 17,$$

if $30t + 23$ is prime;

$$H_a(6) = 30t + 24, \ \ H_b(6) = 30t + 18;$$

$$H_a(7) = Next(30t + 29) \geq 31, \ \ H_b(7) = 30t + 23,$$

if $30t + 29$ be composite, $m = 7$.

$$H_a(7) = 30t + 29, \ \ H_b(7) = 30t + 23,$$

if $30t+29$ be prime. Since the last two rows coincide with the corresponding rows $H_a(7)'s$ of the previous point 3a), then 3b) has the same end as 3a) with $m = 9$ and $m = 11$. This proves 3b).

3c) Let $30t + 11, 30t + 13, 30t + 17, 30t + 19$ be prime, but $30t + 23$ be composite.
We have

$$H_a(2) = 30t + 13, \ \ H_b(2) = 30t + 11;$$

$$H_a(3) = 30t + 17, \ \ H_b(3) = 30t + 13;$$

$$H_a(4) = 30t + 18, \ \ H_b(4) = 30t + 14;$$

$$H_a(5) = 30t + 19, \ \ H_b(5) = 30t + 17;$$

$$H_a(6) = 30t + 20, \ \ H_b(6) = 30t + 18;$$

$$H_a(7) = Nextprime(30t + 23) \geq 30t + 29, \ \ H_b(7) = 30t + 19$$

and $m = 7$. This proves 3c).

3d) Let $30t+11, 30t+13, 30t+17, 30t+19, 30t+23$ be prime, but $30t+29$ be composite.
We have

$$H_a(6) = 30t + 20, \ \ H_b(6) = 30t + 18;$$

$$H_a(7) = 30t + 23, \ \ H_b(7) = 30t + 19;$$

$$H_a(8) = 30t + 24, \ \ H_b(8) = 30t + 20;$$

$$H_a(9) = 30t + 25, \ \ H_b(9) = 30t + 21;$$

$$H_a(10) = 30t + 26, \ \ H_b(10) = 30t + 22;$$

$$H_a(11) = Nextprime(30t + 29) \geq 30t + 31, \ \ H_b(11) = 30t + 23$$

and $m = 11$. This proves 3d).
Finally,
3e) Let $30t + 11, 30t + 13, 30t + 17, 30t + 19, 30t + 23, 30t + 29$ be prime, but $30t + 31$ be composite.

We have

$$H_a(10) = 30t + 26, \;\; H_b(10) = 30t + 22;$$
$$H_a(11) = 30t + 29, \;\; H_b(11) = 30t + 23;$$
$$H_a(12) = 30t + 30, \;\; H_b(12) = 30t + 24;$$
$$H_a(13) = Nextprime(30t + 31) \geq 30t + 37, \;\; H_b(13) = 30t + 29$$

and $m = 13$. This completes the proof of the case 3e) and the proof of the case $c(n) \equiv 1 \pmod{10}$ with $m = 3, 5, 7, 9, 11,$ *or* 13. The proved cases 1),2) and 3) complete the proof of both Theorem 1 and 2. ■

Corollary 1. *In sequence* (2) *there is only one term equal to* 4*; all other terms are greater than or equal to* 6.

Also from 2ba), 2bb) and 2c) we deduce the following.

Corollary 2. *In sequence* (4) $m = 17$ *appears if and only if we have seven primes: either*

$$p = 30t + 29, p + 2, p + 8, p + 12, p + 18, p + 24, p + 30$$

or

$$p, p + 2, p + 8, p + 14, p + 20, p + 24, p + 30$$

or

$$p, p + 2, p + 8, p + 14, p + 18, p + 24, p + 30,$$

but $p + 32$ *is composite.*

Corollary 3. *In sequence* (4) $m = 15$ *appears if and only if we have one of the constellations of eight primes: either*

$$p = 30t + 29, p + 2, p + 8, p + 12, p + 18, p + 24, p + 30, p + 32$$

or

$$p, p + 2, p + 8, p + 14, p + 20, p + 24, p + 30, p + 32$$

or

$$p, p + 2, p + 8, p + 14, p + 18, p + 24, p + 30, p + 32.$$

So $m = 15$ occurs asymptotically less than $m = 17$. Analogous statements one can obtain for other positive values of m among which $m = 3$ appears asymptotically most frequently.

References

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