ON WITHIN-PERFECTNESS AND NEAR-PERFECTNESS

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ABSTRACT. The analytic aspect of within-perfectness and near-perfectness was considered by Erdös, Pomerance, Harman, Wolke, Pollack and Shevelev. We generalize these concepts by introducing a threshold function k, which is positive and increasing on $[1,\infty)$. Let $\ell \geq 1$. A natural number n is an $(\ell;k)$-within-perfect number if $|\sigma(n) - \ell n| < k(n)$. A natural number n is a k-near-perfect number if n can be written as a sum of all but at most $k(n)$ of its divisors. We study the asymptotic densities and bounds for our new notions as k varies. We denote the number of k-near-perfect numbers up to x by $\#N(k;x)$. For k-near-perfectness in which k is a constant, we improve the previous result of Pollack and Shevelev considerably by establishing for $k \geq 4$,

$$\#N(k;x) \ll_k \frac{x}{\log x}(\log\log x)^{j_0(k)},$$

where $j_0(k)$ is the smallest integer such that

$$j_0(k) > \frac{\log(k+1)}{\log 2} - \frac{\log 5}{\log 2},$$

and unconditionally for a large class of positive integers $k \geq 4$ we have

$$\#N(k;x) \asymp_k \frac{x}{\log x}(\log\log x)^{f(k)},$$

where

$$f(k) = \left\lfloor \frac{\log(k+4)}{\log 2} \right\rfloor - 3,$$

For $4 \leq k \leq 9$, we determine asymptotic formulae for $\#N(k;x)$.

CONTENTS

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1. INTRODUCTION

Let $\sigma(n)$ be the sum of all positive divisors of n. A natural number n is *perfect* if $\sigma(n) = 2n$, is *ℓ-perfect* if $\sigma(n) = \ell n$ and is *multiply perfect* if $n \mid \sigma(n)$. Perfect numbers have played a prominent role in classical number theory for millennia. Euclid and Euler proved that n is an even perfect number if and only if n is of the form $2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are both prime. A well-known conjecture claims that there are infinitely many even, but no odd, perfect numbers. Despite the fact that these conjectures remain unproven, there has been significant progress on studying the distribution of perfect numbers during the 20th century [Vo, HoWi, Ka, Er]. The sharpest known result is due to Hornfeck and Wirsing, who established that the number of multiply perfect numbers up to x is at most $x^{o(1)}$ as $x \to \infty$.

Pomerance [Po] studied a closely related notion. Let $\ell \geq 2$ and k be integers. We call a natural number n *(ℓ, k)-almost-perfect* if $\sigma(n) = \ell n + k$. By estimating the count of *sporadic* solutions of the congruence $\sigma(n) \equiv k \pmod{n}$, he proved that as $x \to \infty$, the number of (ℓ, k)-almost-perfect numbers up to x is at most $x/\log x$.

We can further generalize the notion of (ℓ, k)-almost perfect number by replacing the constant integer k above by a threshold function $k(y)$ and ℓ is a real number at least 1. We call a natural number n $(\ell; k)$-*within-perfect* if $|\sigma(n) - \ell n| < k(n)$. This was first studied by Wolke [Wo] and Harman [Ha] in terms of Diophantine approximation.[1] They showed that for any real $\ell \geq 1$ and for any $c \in (0.525, 1)$, there exists infinitely many natural numbers that are $(\ell; y^c)$-within-perfect.

We describe the phase-transition behaviour of the densities of within-perfect numbers in terms of the distribution function of $\sigma(n)/n$, where Davenport [Da] proved that this distribution function exists. Our result is as follows.

Theorem 1.1. *Let $D(\cdot)$ denote the distribution function of $\sigma(n)/n$. We may extend the definition of $D(\cdot)$ to $\mathbb{R}$ by defining $D(u) = 0$ for $u < 1$. Let $W(\ell; k)$ the set of all $(\ell; k)$-within-perfect numbers.*

- *(a) If $k(n) = o(n)$, then $W(\ell; k)$ has asymptotic density 0.*
- *(b) If $k(n) \sim cn$ for some $c > 0$, then $W(\ell; k)$ has asymptotic density $D(\ell + c) - D(\ell - c)$.*
- *(c) If $k(n) \asymp n$, then $W(\ell; k)$ has positive lower density and upper density strictly less than 1.*
- *(d) If $n = o(k(n))$, then $W(\ell; k)$ has asymptotic density 1.*

By refining the techniques of Pomerance, we have the following results which describe the distribution of within-perfect numbers in the sublinear regime more precisely.

Theorem 1.2. *Let $\ell \geq 2$ be an integer and k be a positive constant. Let $W(\ell; k; x) = W(\ell; k) \cap [1, x]$.*

- *(a) Suppose there are ℓ-perfect numbers. Then there exists a constant $c_1 = c_1(\ell)$ such that if $k > c_1$, then*
$$c_2 \ = \ c_2(k, \ell) \ := \sum_{\substack{m < k/\ell \\ \sigma(m) = \ell m}} \frac{1}{m} \ > \ 0 \tag{1.1}$$
and as $x \to \infty$,
$$\#W(\ell; k; x) \ \sim \ c_2 \frac{x}{\log x}. \tag{1.2}$$
For $k \leq c_1$, as $x \to \infty$,
$$\#W(\ell; k; x) \ \leq \ 2kx^{1/2 + o(1)}, \tag{1.3}$$
where $o(1)$ does not depend on k and ℓ.

[1] Analogous problems were also considered by Erdős, Schinzel [Sc], Harman [Ha], and Alkan-Ford-Zaharescu [AlFoZa1, AlFoZa2].

(b) *Suppose there is no ℓ-perfect number. Then for all $k > 0$, as $x \to \infty$,*

$$\#W(\ell; k; x) \ \leq \ 2kx^{1/2+o(1)}, \tag{1.4}$$

where $o(1)$ does not depend on k and ℓ.

Theorem 1.3. *Suppose $k(y) \leq y^{\epsilon}$ for large y and k is a positive increasing unbounded function. Consider the following set*

$$\Sigma := \left\{ \frac{\sigma(m)}{m} : m \geq 1 \right\} \subset \mathbb{Q}. \tag{1.5}$$

(a) *If $\ell \in \Sigma$, then we have*

$$\lim_{x \to \infty} \frac{\#W(\ell; k; x)}{x/\log x} \ = \sum_{\sigma(m) \, = \, \ell m} \frac{1}{m} \tag{1.6}$$

unconditionally for $\epsilon \in (0, 1/3)$ and if we assume Conjecture 2.7, then we have (1.6) *for $\epsilon \in (0, 1)$.*

(b) *If $\ell \in (\mathbb{Q} \cap [1, \infty)) \setminus \Sigma$, $\ell = a/b$, $a > b \geq 1$, a, b are coprime integers and $\epsilon \in (0, 1/3)$, then we have the following upper bound*

$$\#W(\ell; k; x) \ = \ O(\max\{a, b^3\} x^{\min\{3/4, \ \epsilon + 2/3\} + o(1)}), \tag{1.7}$$

as $x \to \infty$. Assume Conjecture 2.7. Now for $\epsilon \in (0, 1)$, we have as $x \to \infty$

$$\#W(\ell; k; x) \ = \ O(\max\{a, b^3\} x^{\epsilon} (\log x)^{O(1)}). \tag{1.8}$$

From Theorem 1.3, we can see that a more natural, informative distribution function for within-perfect numbers in the sublinear regime is the following

$$D'_{\epsilon}(r) \ := \ \lim_{x \to \infty} \frac{\#W(r; y^{\epsilon}; x)}{x/\log x} \tag{1.9}$$

for $r \in [1, \infty)$. In terms of this new distribution function, we have the following simple result.

Corollary 1.4. *For $\epsilon \in (0, 1/3)$, D'_{ϵ} is discontinuous on a dense subset of $[1, \infty)$.*

Another line of generalization of perfect numbers was initiated by Sierpiński [Si] in which a natural number is *pseudoperfect* if it is a sum of some subset of its proper divisors. Pseudoperfect numbers are clearly abundant (i.e., $\sigma(n) > 2n$). The asymptotic density of abundant numbers is between 0.24761 and 0.24765 [De, Kob]. Therefore a substantial proportion of natural numbers are not pseudoperfect. Nonetheless, Erdös and Benkoski [Erd, BeEr] proved that the asymptotic density for pseudoperfect numbers, as well as that of abundant numbers that are not pseudoperfect (or *weird numbers* in [BeEr]), exist and are positive.

Pollack and Shevelev [PoSh] studied a subclass of pseudoperfect numbers. A natural number is said to be *k-near-perfect* if it is a sum of all of its proper divisors with *at most* k exceptions. Those exceptions are said to be *redundant divisors*. It turns out restricting the number of exceptional divisors would lead to asymptotic density 0. More precisely, they showed that the number of 1-near-perfect numbers up to x is at most $x^{3/4+o(1)}$ [2] and in general for $k \geq 1$ the number of k-near-perfect numbers up to x is at most $\frac{x}{\log x}(\log\log x)^{O_k(1)}$, where $O_k(1)$ can be taken to be $k-1$ and is at least $\lfloor \frac{\log(k+4)}{\log 2} \rfloor - 3$.

By allowing k to increase with n – in other words, we let larger natural numbers n have more exceptional divisors – we explore the possibility of a positive density $k(n)$-near-perfect number set. If such a set exists, we look for its critical order of magnitude and at the phase-transition behavior. We have the following theorem.

[2] This is a result stated in [AnPoPo]. In the original paper of Pollack and Shevelev [PoSh], the upper bound was given by $x^{5/6+o(1)}$.

Theorem 1.5. *Denote by $N(k)$ the set of all k-near-perfect numbers. Let $N(k;x) = N(k) \cap [1,x]$. If the asymptotic density of $N(\log y)^{\log 2+\epsilon})$ is $c \in [0,1]$ for some $\epsilon > 0$, then for any positive strictly increasing function k such that $k(y) \geq (\log y)^{\log 2+\epsilon}$ for large y, the asymptotic density of $N(k)$ is also c.*

Before stating our next theorem, we introduce the following notion. Let k be a natural number. We say a finite subset B of $N(k)$ is *k-admissible* if for any $m_1, m_2 \in B$ with $m_1 \neq m_2$, we have one of the following

(1) At least one of the natural numbers $\text{lcm}[m_1,m_2]/m_1$, $\text{lcm}[m_1,m_2]/m_2$ is not square-free.
(2) If both of the natural numbers $\text{lcm}[m_1,m_2]/m_1$, $\text{lcm}[m_1,m_2]/m_2$ are square-free, then $\gcd(\text{lcm}[m_1,m_2]/m_1, m_1)$ and $\gcd(\text{lcm}[m_1,m_2]/m_2, m_2)$ are strictly greater than 1.

We let $\mathscr{C}(k)$ be the set of all k-admissible subsets and M be the constant

$$M := \frac{6}{\pi^2} \sup_{k\in\mathbb{N}} \sup_{B\in\mathscr{C}(k)} \sum_{m\in B} \frac{\phi(m)}{m^2}, \tag{1.10}$$

where $\phi(\cdot)$ is the Euler's totient function.

Theorem 1.6. *Let k be a positive strictly increasing function.*

(a) If $k(y) > (\log y)^{\log 2+\epsilon}$ for some $\epsilon \in (0,1)$, then $N(k)$ has positive lower density of at least M and

$$0.0715251 \approx \frac{4981}{7056\pi^2} \leq M \leq 0.24765. \tag{1.11}$$

(b) If $k(y) < (\log y)^{\epsilon}$ for some $\epsilon \in (0, \log 2)$. Then $N(k)$ has asymptotic density 0. In fact, we have

$$\#N(k;x) \ll_\epsilon \frac{x}{(\log x)^{r(\epsilon)}}, \tag{1.12}$$

where

$$r(\epsilon) := 1 - \frac{\epsilon(1+\log_2 2 - \log\epsilon)}{\log 2} \in (0,1). \tag{1.13}$$

For a more precise upper bound, see the discussion in Section 3.1.

On the other hand, by modifying the method of [PoSh], we improve the their result by proving asymptotic formulae of $\#N(k;x)$ for $4 \leq k \leq 9$ and determining exact orders of $\#N(k;x)$ for a large portion of integers $k \geq 4$. We conject that we can replace 'liminf' by 'lim' and '$\geq$' by '$=$' in (1.20) and (1.22) respectively.

Theorem 1.7. *For $4 \leq k \leq 9$, we have*

$$\#N(k;x) \sim c_k \frac{x}{\log x} \tag{1.14}$$

as $x \to \infty$, where

$$c_4 = c_5 = \frac{1}{6},\ c_6 = \frac{17}{84},\ c_7 = c_8 = \frac{493}{1260}, c_9 = \frac{179017}{360360}. \tag{1.15}$$

Theorem 1.8. *For $k \geq 4$, as $x \to \infty$*

$$\#N(k;x) \ll_k \frac{x}{\log x}(\log\log x)^{j_0(k)}, \tag{1.16}$$

where $j_0(k)$ is the smallest integer such that

$$j_0(k) > \frac{\log(k+1)}{\log 2} - \frac{\log 5}{\log 2}. \tag{1.17}$$

Let f be the following function defined for integers $k \geq 4$.

$$f(k) = \left\lfloor \frac{\log(k+4)}{\log 2} \right\rfloor - 3. \tag{1.18}$$

For integer $k \in [4,\infty)_\mathbb{Z} \setminus (\{10,11\} \cup \{2^{s+2} - i : s \geq 3, i = 5,6\})$, we have

$$\#N(k;x) \asymp_k \frac{x}{\log x}(\log\log x)^{f(k)}. \tag{1.19}$$

Moreover,

$$\liminf_{x\to\infty} \frac{\#N(k;x)}{\frac{x}{\log x}(\log\log x)^{r-2}} \ \geq \ \frac{c_k}{(r-2)!}, \tag{1.20}$$

where $r \geq 2$ *and*

$$c_k = \begin{cases} 17/84 \ \text{ if } \ k = 3\cdot 2^r - 6 \\ 493/1260 \ \text{ if } \ k \in [3\cdot 2^r - 5, 4\cdot 2^r - 8]_{\mathbb{Z}} \\ 179017/360360 \ \text{ if } \ k = 4\cdot 2^r - 7. \end{cases} \tag{1.21}$$

If $k \in [4\cdot 2^r - 4, 6\cdot 2^r - 7]_{\mathbb{Z}}$ *for some* $r \geq 1$*, then*

$$\liminf_{x\to\infty} \frac{\#N(k;x)}{\frac{x}{\log x}(\log\log x)^{r-1}} \ \geq \ \frac{1}{6(r-1)!}. \tag{1.22}$$

Our last theorem is motivated by the following question raised by Erdös and Benkoski in [BeEr]. They asked if $\sigma(n)/n$ can be arbitrarily large for weird n. They suggested that the answer should be negative but this remains to be an open problem. We ask for an analogue to k-exactly-perfect n, where a natural number is said to be k-*exactly-perfect* if it is a sum of all of its proper divisors with *exactly* k exceptions. We have the following weaker result.

Theorem 1.9. *Denote the set of all* k*-exactly-perfect numbers by* $E(k)$ *and we write* $E(k;x) := E(k)\cap[1,x]$*. Let* M *be the set of all natural numbers of the form* $2q$*, where* q *is a prime such that* $2^q - 1$ *is also a prime. Let* $E_\epsilon(k;x) = \{n \leq x : n \in E(k)$ *and* $\sigma(n) \geq 2n + n^\epsilon\}$*, where* $\epsilon \in (0, 1/3)$*. Assume that there is no odd perfect number. For large* k *and* $k \notin M$*, we have*

$$\lim_{x\to\infty} \frac{\#E_\epsilon(k;x)}{\#E(k;x)} \ = \ 1. \tag{1.23}$$

Moreover, we have the following unconditional results. Equation (1.23) *holds for* $k = 8$, $2^{s+2} - 4$ *(*$2 \leq s \leq 8$*)* , $3\cdot 2^s - 5$ *(*$2 \leq s \leq 8$*)* , $3\cdot 2^s - 6$ *(*$3 \leq s \leq 8$*)* , $2^{s+2} - 7$ *(*$2 \leq s \leq 8$*) and for* $k = 4, 6$*, we have*

$$\lim_{x\to\infty} \frac{\#E_\epsilon(k;x)}{\#E(k;x)} \ = \ 0. \tag{1.24}$$

We use the following notations throughout this article.

- We write $f(x) \asymp g(x)$ if there exist positive constants c_1, c_2 such that $c_1 g(x) < f(x) < c_2 g(x)$ for sufficiently large x.
- We write $f(x) \sim g(x)$ if $\lim_{x\to\infty} f(x)/g(x) = 1$.
- We write $f(x) = O(g(x))$ or $f(x) \ll g(x)$ if there exists a positive constant C such that $f(x) < Cg(x)$ for sufficiently large x.
- We write $f(x) = o(g(x))$ if $\lim_{x\to\infty} f(x)/g(x) = 0$.
- In all cases, subscripts indicate dependence of implied constants on other parameters.
- Denote by $[a,b]_{\mathbb{Z}}$ the collection of all *integers* n such that $a \leq n \leq b$.
- Denote by $\log_k x$ the k-th iterate of logarithm. For example, $\log_1 x = \log x$, $\log_2 x = \log\log x$.

2. $(\ell;k)$-WITHIN-PERFECT NUMBERS

In this section, we prove our results on $(\ell;k)$-within-perfect numbers, namely Theorems 1.1, 1.2 and 1.3. In Theorem 1.1, we interpret the within-perfect condition in terms of the Davenport distribution function $D(\cdot)$ and then use its continuity. In Theorem 1.2 and 1.3, we apply the results concerning the solutions of the congruence $\sigma(n) \ \equiv \ k \pmod n$.

2.1. **Phase-transition behavior of asymptotic densities of $W(\ell;k)$.** Distribution function is a crucial notion in this section. We state its definition as follows.

Definition 2.1 (Distribution function). *Let $-\infty \le a < b \le \infty$. A function $F : (a,b) \to \mathbb{R}$ is a distribution function if F is increasing, right continuous, $F(a+) = 0$, and $F(b-) = 1$. An arithmetic function $f : \mathbb{N} \to \mathbb{R}$ has a distribution function if there exists a distribution function F such that*

$$\lim_{x\to\infty} \frac{1}{x}\#\{n \le x : f(n) \le u\} = F(u)$$

at all points of continuity of F.

It is a theorem of Davenport [Da] that $\sigma(n)/n$ has a continuous and strictly increasing distribution function on $[1,\infty)$. Denote by $D(\cdot)$ the distribution function of $\sigma(n)/n$ and extend the definition of $D(\cdot)$ to $\mathbb{R}$ by defining $D(u) = 0$ for $u < 1$. The problem concerning the existence of a distribution function for an additive arithmetic function is completely resolved by the *Erdős-Wintner Theorem* [ErWi]. For details, see [Te].

Proof of Theorem 1.1. For part (a), label all of the $(\ell;k)$-within-perfect numbers by n_j in increasing order. Then for any $j \in \mathbb{N}$,

$$\left|\frac{\sigma(n_j)}{n_j} - \ell\right| \ < \ \frac{k(n_j)}{n_j}. \tag{2.1}$$

Fix $\epsilon > 0$. Since

$$\lim_{j\to\infty} \frac{k(n_j)}{n_j} \ = \ 0, \tag{2.2}$$

there exists $L \in \mathbb{N}$ such that for any $j \ge L$, we have

$$\left|\frac{k(n_j)}{n_j}\right| \ < \ \epsilon. \tag{2.3}$$

Hence we have

$$\begin{aligned} \frac{1}{x}\#\{n \le x : |\sigma(n) - \ell n| < k(n)\} \ &\le \ \frac{1}{x}\#\left\{j \ge L : n_j \le x, \left|\frac{\sigma(n_j)}{n_j} - \ell\right| < \frac{k(n_j)}{n_j}\right\} + \frac{L}{x} \\ &\le \ \frac{1}{x}\#\left\{j \ge L : n_j \le x, \left|\frac{\sigma(n_j)}{n_j} - \ell\right| < \epsilon\right\} + \frac{L}{x} \\ &\le \ \frac{1}{x}\#\left\{n \le x : \left|\frac{\sigma(n)}{n} - \ell\right| < \epsilon\right\} + \frac{L}{x}. \end{aligned} \tag{2.4}$$

Now,

$$\limsup_{x\to\infty} \frac{1}{x}\#\{n \le x : |\sigma(n) - \ell n| < k(n)\} \ \le \ \limsup_{x\to\infty} \frac{1}{x}\#\left\{n \le x : \left|\frac{\sigma(n)}{n} - \ell\right| < \epsilon\right\} = D(\ell+\epsilon) - D(\ell-\epsilon). \tag{2.5}$$

By Davenport's theorem, $D(\cdot)$ is continuous. Letting $\epsilon \to 0$, we have

$$\limsup_{x\to\infty} \frac{1}{x}\#\{n \le x : |\sigma(n) - \ell n| < k(n)\} \ = \ 0. \tag{2.6}$$

This completes the proof of part (a).

For part (b), fix $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that for any $n \ge N$,

$$c - \epsilon \ < \ \frac{k(n)}{n} < c + \epsilon. \tag{2.7}$$

For $x \geq N$, observe that

$$\begin{aligned} &\frac{1}{x}\#\left\{n \leq x : \left|\frac{\sigma(n)}{n} - \ell\right| < c - \epsilon\right\} - \frac{1}{x}\#\left\{n \leq N : \left|\frac{\sigma(n)}{n} - \ell\right| < c - \epsilon\right\} \\ \leq\; &\frac{1}{x}\#\{n \leq x : |\sigma(n) - \ell n| < k(n)\} - \frac{1}{x}\#\{n \leq N : |\sigma(n) - \ell n| < k(n)\} \\ \leq\; &\frac{1}{x}\#\left\{n \leq x : \left|\frac{\sigma(n)}{n} - \ell\right| < c + \epsilon\right\} - \frac{1}{x}\#\left\{n \leq N : \left|\frac{\sigma(n)}{n} - \ell\right| < c + \epsilon\right\}, \end{aligned} \tag{2.8}$$

which implies, by Davenport's Theorem,

$$\lim_{x\to\infty} \frac{1}{x}\#\{n \leq x : |\sigma(n) - \ell n| < k(n)\} \;=\; D(\ell + c) - D(\ell - c). \tag{2.9}$$

The proof of part (c) is essentially the same as that of part (b), so we omit the details here.
For part (d), for any $j \in \mathbb{N}$ there exists $n_j \in \mathbb{N}$ such that for any $n \geq n_j$,

$$\frac{n}{k(n)} \;<\; \frac{1}{j}. \tag{2.10}$$

For $x \geq n_j$, we have

$$\begin{aligned} \frac{1}{x}\#\{n \leq x : |\sigma(n) - \ell n| < jn\} \;&\leq\; \frac{n_j}{x} + \frac{1}{x}\#\{n_j \leq n \leq x : |\sigma(n) - \ell n| < jn\} \\ &\leq\; \frac{n_j}{x} + \frac{1}{x}\#\{n \leq x : |\sigma(n) - \ell n| < k(n)\} \end{aligned} \tag{2.11}$$

and

$$D(\ell + j) \;=\; \liminf_{x\to\infty} \frac{1}{x}\#\{n \leq x : |\sigma(n) - \ell n| < jn\} \;\leq\; \liminf_{x\to\infty} \frac{1}{x}\#\{n \leq x : |\sigma(n) - \ell n| < k(n)\} \;\leq\; 1. \tag{2.12}$$

Letting $j \to \infty$ and by Davenport's theorem, we have the conclusion for part (d). □

2.2. **Explicit bounds for $W(\ell; k; x)$ for k being constant.** In this section, $\ell \geq 2$ and k are integers. Denote by $S(\ell, k)$ the set of all (ℓ, k)-almost-perfect numbers and $S(\ell, k; x) = S(\ell, k) \cap [1, x]$. Following Anavi, Pollack, Pomerance and Shevelev [AnPoPo, Po, PoPo, PoSh], we use the following definitions regarding the solutions of a special type of congruence involving the arithmetic function $\sigma(n)$.

Definition 2.2. *Let k be an integer. Consider the congruence in natural numbers*

$$\sigma(n) \;\equiv\; k \pmod{n}. \tag{2.13}$$

A natural number n is a regular solution *of (2.1) if n is of the form*

$$n = pm \text{ where } p \text{ is prime, } p \nmid m,\; m \mid \sigma(m), \text{ and } \sigma(m) = k. \tag{2.14}$$

Other solutions of (2.13) *are known as* sporadic solutions.

It was first observed in [Po] that the sporadic solutions occur much less frequently than the regular solutions. The following are the known results on this theme.

Lemma 2.3 (Pomerance [Po])**.** *For each fixed integer k, the number of sporadic solutions up to x is at most $x\exp(-(1/\sqrt{2} + o(1))\sqrt{\log x \log\log x})$ as $x \to \infty$.*

Lemma 2.4 (Pollack-Shevelev [PoSh])**.** *Let $x \geq 3$. Uniformly for integers k with $|k| < x^{2/3}$, the number of sporadic solutions up to x is at most $x^{2/3+o(1)}$.* [3]

Lemma 2.5 (Anavi-Pollack-Pomerance [AnPoPo])**.** *Uniformly for integers k with $|k| \leq x^{1/4}$, the number of sporadic solutions up to x is at most $x^{1/2+o(1)}$ as $x \to \infty$.* [4]

[3] In fact, $o(1)$ can be taken to be $C/\sqrt{\log\log x}$ for some absolute constant $C > 0$. The explicit choice of $o(1)$ follows from the estimate of Pollack: $\sum_{n\leq x} \gcd(\sigma(n), n) \leq x^{1+C/\sqrt{\log\log x}}$ for $x \geq 3$. See [PoSh].

[4] The choice of $o(1)$ here can also be made explicit, see [AnPoPo].

Lemma 2.6 (Pollack-Pomerance [PoPo]). *Uniformly for integers k with $0 < |k| \leq x^{1/4}$, the number of solutions up to x of the congruence* (2.13) *for which $\sigma(n)$ is odd is at most $|k|x^{1/4+o(1)}$ as $x \to \infty$.* [5]

However, the above lemmas should be far from best possible according to Remark 3 of [AnPoPo]. In fact, Anavi, Pollack and Pomerance conjected the following based on a heuristic regarding the average number of sporadic solutions.

Conjecture 2.7 (Anavi-Pollack-Pomerance [AnPoPo]). *The number of sporadic solutions to* (2.13) *less than or equal to x is at most $(\log x)^{O(1)}$ uniformly for $x \geq 3$ and $|k| \leq x/2$.*

We first settle the distribution of $W(\ell; k)$ for the case k being a constant by establishing the following lemma. This lemma refines the original result due to Pomerance (see Corollary 3 of [Po]).

Lemma 2.8. *For fixed integers k, ℓ with $\ell \geq 2$, as $x \to \infty$, we have*

(a) If k/ℓ is an ℓ-perfect number, then

$$\#S(\ell, k; x) \ \sim \ \frac{\ell}{k}\frac{x}{\log x}. \tag{2.15}$$

(b) If k/ℓ is not an ℓ-perfect number, then

$$\#S(\ell, k; x) \ \leq \ x^{1/2+o(1)}. \tag{2.16}$$

In the case of ℓ is even and k is odd, the upper bound can be replaced by $|k|x^{1/4+o(1)}$.

Proof of Lemma 2.8. If $n \in S(\ell, k)$, then $\sigma(n) \equiv k \pmod n$. Consider n of the form (2.14). Then

$$(1+p)k = \sigma(p)\sigma(m) = \sigma(n) = \ell pm + k.$$

This implies $\sigma(m) = k = \ell m$. So, m is an ℓ-perfect number.

(a) If k/ℓ is ℓ-perfect, then obviously $\{n \in \mathbb{N} : n = p(k/\ell),\, p \nmid (k/\ell)\}$ is the set of all regular solutions of $\sigma(n) \equiv k \pmod n$ and it is a subset of $S(\ell, k)$. Then by Lemma 2.5, for large x we have

$$\#\{n \leq x : n = p(k/\ell),\, p \nmid (k/\ell)\} \ \leq \ \#S(\ell, k; x) \ \leq \ \#\{n \leq x : n = p(k/\ell),\, p \nmid (k/\ell)\} + x^{1/2+o(1)}. \tag{2.17}$$

By the Prime Number Theorem, as $x \to \infty$, we have

$$\#\{n \leq x : n = p(k/\ell),\, p \nmid (k/\ell)\} \ \sim \ \frac{\ell}{k}\frac{x}{\log x}. \tag{2.18}$$

The results for part (a) follow from equations (2.17) and (2.18).

(b) If k/ℓ is not an ℓ-perfect number, then the congruence (2.13) has no regular solution. Then $\#S(\ell, k; x) \leq x^{1/2+o(1)}$ follows directly from Lemma 2.5.

It is an elementary fact that $\sigma(n)$ is odd if and only if n is a perfect square or two times a perfect square. So it is trivial that if ℓ is even and k is odd, $\#S(\ell, k; x) = O(x^{1/2})$, which surpasses the upper bound $x^{1/2+o(1)}$. In this case we use Lemma 2.6.

□

By assuming Conjecture 2.7, Lemma 2.8 can be strengthened to say that $\#S(\ell, k; x)$ is at most $(\log x)^{O(1)}$ if k/ℓ is not ℓ-perfect. Conjecture 2.7 is best possible from the simple observation that powers of 2 are in $S(2, -1)$. However, $(\log x)^{O(1)}$ should not always be the correct order of magnitude of $\#S(\ell, k; x)$ when k/ℓ is not perfect. For example: it is widely conjectured that there are no quasiperfect numbers, and the number of perfect numbers [6] up to x is asymptotic to

$$\frac{e^\gamma}{\log 2}\log\log x, \tag{2.19}$$

[5] We can take $x^{o(1)}$ to be $\exp(O(\log x/\log\log x))$.

[6] A heuristic argument from Pomerance (which can be found in [Pol]) suggests that there are no odd perfect numbers.

where γ is the Euler-Mascheroni constant.

Remark 2.9. *The results of this lemma are illustrated in Figure 1.*

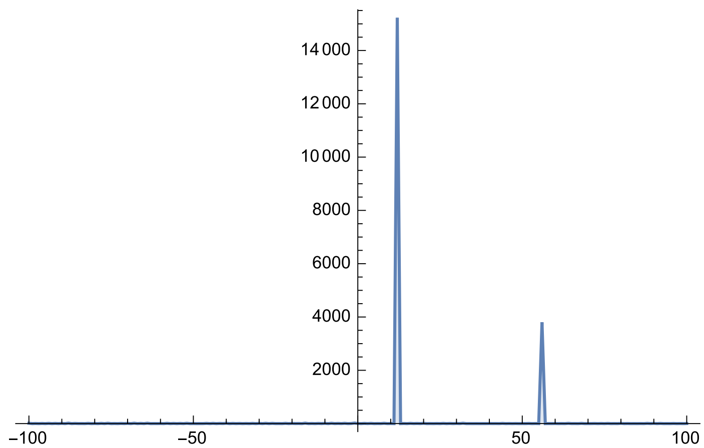

FIGURE 1. The x-axis is k and the y-axis is the number of $(2,k)$-almost-perfect numbers up to 10^6. There are spikes at $k = 12$ and $k = 56$, illustrating the results of Lemma 2.8.

Proof of Theorem 1.2. Suppose ℓ-perfect numbers exist. Let $m_0 = m_0(\ell)$ be the smallest one. Take $c_1 = \ell m_0$. Hence for a constant $k > c_1$,

$$c_2 := \sum_{\substack{m<k/\ell \\ \sigma(m)=\ell m}} \frac{1}{m} > 0. \tag{2.20}$$

Then

$$\frac{\#W(\ell;k;x)}{x/\log x} = \frac{\log x}{x} \sum_{|r|<k} \#S(\ell,r;x) = \frac{\log x}{x} \Bigg(\sum_{\substack{|r|<k \\ r/\ell \text{ is } \ell\text{-perfect}}} \#S(\ell,r;x) + \sum_{\substack{|r|<k \\ r/\ell \text{ is not } \ell\text{-perfect}}} \#S(\ell,r;x) \Bigg). \tag{2.21}$$

By Lemmas 2.5 and 2.8, there exists an absolute constant $C > 0$ such that for $x \geq \max\{k^4, C\}$,

$$\sum_{\substack{|r|<k \\ r/\ell \text{ is not } \ell\text{-perfect}}} \#S(\ell,r;x) \leq 2kx^{1/2+o(1)}. \tag{2.22}$$

So

$$\frac{\log x}{x} \sum_{\substack{|r|<k \\ r/\ell \text{ is not } \ell\text{-perfect}}} \#S(\ell,r;x) \leq \frac{2k\log x}{x^{1/2+o(1)}} \to 0 \text{ as } x \to \infty. \tag{2.23}$$

By Lemma 2.8, as $x \to \infty$, we have

$$\frac{\log x}{x} \sum_{\substack{|r|<k \\ r/\ell \text{ is } \ell\text{-perfect}}} \#S(\ell,r;x) \to \sum_{\substack{|r|<k \\ r/\ell \text{ is } \ell\text{-perfect}}} \frac{\ell}{k} = c_2. \tag{2.24}$$

Therefore for $k > c_1$, we have as $x \to \infty$,

$$\#W(\ell;k;x) \sim c_2 \frac{x}{\log x}. \tag{2.25}$$

The rest of the cases, i.e., (1.3), (1.4), are trivial. □

2.3. **Within-Perfectness for sublinear function k.**

Proof of Theorem 1.3. We first prove the lower bound. Fix any natural number r. Since k is increasing and unbounded, there exists $y_0 = y_0(k,r)$ such that for $y \geq y_0$, we have $k(y) \geq r$. Then

$$\#\{y_0 \leq n \leq x : |\sigma(n) - \ell n| < r\} \leq \#\{y_0 \leq n \leq x : |\sigma(n) - \ell n| < k(n)\}. \tag{2.26}$$

From this we have

$$\#W(\ell;r;x) + O(y_0) \leq \#W(\ell;k;x) \tag{2.27}$$

and by Theorem 1.2, we have

$$\liminf_{x\to\infty} \frac{\#W(\ell;k;x)}{x/\log x} \;\geq\; \liminf_{x\to\infty} \frac{\#W(\ell;r;x)}{x/\log x} \;=\; \sum_{\substack{m<r/\ell \\ \sigma(m)=\ell m}} \frac{1}{m}. \tag{2.28}$$

Letting $r \to \infty$, we find

$$\liminf_{x\to\infty} \frac{\#W(\ell;k;x)}{x/\log x} \;\geq\; \sum_{\sigma(m)=\ell m} \frac{1}{m}. \tag{2.29}$$

For the upper bound, assume $k(y) \leq y^\epsilon$ for large y and $\epsilon \in (0, 1/3)$. Let $W'(\ell;k;x) = \{n \leq x : |\sigma(n) - \ell n| < k(x)\}$. Clearly since k is increasing, $\#W(\ell;k;x) \leq \#W'(\ell;k;x) \leq \#W'(\ell;y^\epsilon;x)$. We rewrite the Diophantine inequality described in $W'(\ell;y^\epsilon;x)$ as a collection of Diophantine equations over certain range, i.e.,

$$\sigma(n) - \ell n \;=\; k, \text{ where } k \in \mathbb{Z},\ |k| < x^\epsilon. \tag{2.30}$$

In particular, we have a collection of congruences of the form (2.13):

$$\sigma(n) \;\equiv\; k \pmod{n}, \text{ where } k \in \mathbb{Z},\ |k| < x^\epsilon. \tag{2.31}$$

By Lemma 2.4, the number of $n \in W'(\ell;y^\epsilon;x)$ not of the form (2.14) is

$$\leq 2x^\epsilon x^{2/3+o(1)} = 2x^{2/3+\epsilon+o(1)}, \tag{2.32}$$

which is negligible. So we may assume n is of the form (2.14).
Next by the Prime Number Theorem and the Hornfeck-Wirsing Theorem, we have

$$\#\{n \leq x : n \text{ is of the form (2.14) with } p \leq x^\epsilon\} \;\ll\; \frac{x^\epsilon}{\log x^\epsilon} x^{o(1)} \;\ll_\epsilon\; \frac{x^{\epsilon+o(1)}}{\log x}, \tag{2.33}$$

which is again negligible. Hence, we may assume n is of the form (2.14) with $p > x^\epsilon$.
Now suppose that $\sigma(m) = rm$ for some $r \geq \ell + 1$ and $p > x^\epsilon$. Then

$$\begin{aligned}\sigma(n) - \ell n \;&=\; \sigma(p)\sigma(m) - \ell pm \;=\; (1+p)(rm) - \ell pm \;=\; m(r + p(r-\ell)) \\ &\geq\; p > x^\epsilon. \end{aligned} \tag{2.34}$$

We have n does not belong to $W'(\ell;y^\epsilon;x)$, which is a contradiction.

On the other hand, consider the case where $\sigma(m) = rm$ with $2 \leq r \leq \ell - 1$ and $p > x^\epsilon$. Note that $r + p(r-\ell) \geq 0$ implies $p < r \leq \ell - 1$. For $x > (2\ell)^{1/\epsilon}$, we have a contradiction. Now suppose that $r + p(r - \ell) < 0$. Then $|\sigma(n) - \ell n| < x^\epsilon$ if and only if $m[(\ell - r)p - r] < x^\epsilon$. By Merten's estimate, the number of such n is

$$\begin{aligned}&\leq \sum_{2\leq r\leq \ell-1} \sum_{x^\epsilon<p\leq x} \frac{x^\epsilon}{(\ell-r)p - r} \;\leq\; (\ell-2)x^\epsilon \sum_{x^\epsilon<p\leq x} \frac{1}{p-\ell+1} \;\leq\; 2(\ell-2)x^\epsilon \sum_{x^\epsilon<p\leq x} \frac{1}{p} \\ &\ll\; (\ell-2)x^\epsilon \log\log x. \end{aligned} \tag{2.35}$$

Therefore, we may assume n is of the form (2.14) with $p > x^\epsilon$ and $\sigma(m) = \ell m$.

By partial summation and Hornfeck-Wirsing Theorem, we have for any $z \geq 1$,

$$\sum_{\substack{m \leq z \\ \sigma(m) = \ell m}} \frac{\log m}{m} = \int_1^z \frac{\log t}{t} dP(t) = \frac{\log z}{z^{1-o(1)}} + \int_1^z \frac{\log t}{t^{2-o(1)}}\, dt \ll 1, \tag{2.36}$$

where $P(z) = \#\{m \leq z : \sigma(m) = \ell m\}$. From these we can see that both of the series

$$\sum_{\sigma(m) = \ell m} \frac{\log m}{m}, \quad \sum_{\sigma(m) = \ell m} \frac{1}{m} \tag{2.37}$$

converge. We have $x \geq n = pm > x^{\epsilon} m$ and so $m < x^{1-\epsilon}$.
For $m \leq x^{1-\epsilon}$, since

$$0 < \frac{\log m}{\log x} \leq 1 - \epsilon < 1, \tag{2.38}$$

we have

$$\left(1 - \frac{\log m}{\log x}\right)^{-1} = 1 + O_{\epsilon}\left(\frac{\log m}{\log x}\right). \tag{2.39}$$

Let c be any constant greater than 1. By the Prime Number Theorem, there exists $x_0 = x_0(c) > 0$ such that for $x \geq x_0$, we have

$$\pi(x) < c\, \frac{x}{\log x}. \tag{2.40}$$

Then for $x \geq x_0^{1/\epsilon}$, we have the number of n of the form (2.14), with $p > x^{\epsilon}$ and $\sigma(m) = \ell m$, is bounded above by

$$\begin{aligned}\sum_{\substack{m \leq x^{1-\epsilon} \\ \sigma(m)=\ell m}} \pi\left(\frac{x}{m}\right) &< c \sum_{\substack{m \leq x^{1-\epsilon} \\ \sigma(m)=\ell m}} \frac{x/m}{\log(x/m)} = c\, \frac{x}{\log x} \sum_{\substack{m \leq x^{1-\epsilon} \\ \sigma(m)=\ell m}} \frac{1}{m} + O_{\epsilon}\left(\frac{cx}{(\log x)^2} \sum_{\substack{m \leq x^{1-\epsilon} \\ \sigma(m)=\ell m}} \frac{\log m}{m}\right) \\ &< c\, \frac{x}{\log x} \sum_{\sigma(m)=\ell m} \frac{1}{m} + O_{\epsilon}\left(\frac{cx}{(\log x)^2}\right). \end{aligned} \tag{2.41}$$

Therefore,

$$\limsup_{x \to \infty} \frac{\#W(\ell; k; x)}{x/\log x} \leq c \sum_{\sigma(m)=\ell m} \frac{1}{m}. \tag{2.42}$$

Since the choice of constant $c > 1$ is arbitrary, we have

$$\limsup_{x \to \infty} \frac{\#W(\ell; k; x)}{x/\log x} \leq \sum_{\sigma(m)=\ell m} \frac{1}{m}. \tag{2.43}$$

Combining with (2.29), we have

$$\lim_{x \to \infty} \frac{\#W(\ell; k; x)}{x/\log x} \tag{2.44}$$

exists and is equal to

$$\sum_{\sigma(m)=\ell m} \frac{1}{m}. \tag{2.45}$$

Now suppose $\ell \geq 2$ is an integer such that there is no ℓ-perfect number. A similar calculation can be done to positive increasing function k with $k(y) \leq y^{1/4}$ with the bounds in (2.32), (2.33) and (2.35) being replaced by $2x^{3/4+o(1)}$, $x^{1/4+o(1)}/\log x$ and $x^{1/4} \log\log x$ respectively.

Then by the above argument, we have $\#W(\ell; k; x) \ll \ell x^{\min\{3/4, \epsilon+2/3\}+o(1)}$. The conclusions under Conjecture 2.7 can be proven similarly.

For the rational case, its proof is very similar to that of the integral case, except one has to revise the definitions of regular and sporadic solutions of a suitable congruence in terms of $\sigma(n)$. Let $a > b \geq 1$ are integers and $\gcd(a,b) = 1$. Suppose we would like to count

$$\#W\left(\frac{a}{b}; y^\epsilon, x\right) := \#\left\{n \leq x : \left|\sigma(n) - \frac{a}{b}n\right| < n^\epsilon\right\}. \tag{2.46}$$

We are led to a slightly more general congruence

$$b\,\sigma(n) \ \equiv\ k \ (\mathrm{mod}\, n), \tag{2.47}$$

for integers k satisfying $|k| < bx^{2/3}$. If $b \mid k$, then we say n is a *regular solution* to the congruence (2.47) if

$$n = pm, \text{ where } p \text{ is a prime not dividing } m\ ,\ m \mid b\,\sigma(m), \text{ and } \sigma(m) = \frac{k}{b}. \tag{2.48}$$

It is easy to check that regular solutions are indeed solution of (2.47). We say solutions that are not regular *sporadic*. If $b \nmid k$, then we declare that the congruence (2.47) has no regular solution (or all of its solutions are sporadic). The following result is a direct adaptation of the corresponding results found in [AnPoPo], [PoSh], [Po1] or [Po2]. We shall not repeat the argument here.

Theorem 2.10. *Let $x \geq b$ and let k be an integer with $|k| < bx^{2/3}$. Then the number of sporadic solutions to congruence* (2.47) *is at most $b^2 x^{2/3+o(1)}$ as $x \to \infty$, where $o(1)$ is uniform in k.*

This completes the proof. □

Proof of Corollary 1.4. It follows from a theorem of Anderson (see [Pol] P. 270) that $(\mathbb{Q} \cap [1,\infty)) \setminus \Sigma$ is dense in $[1,\infty)$. Observe that D_ϵ' takes the value 0 on $(\mathbb{Q} \cap [1,\infty)) \setminus \Sigma$ but it takes positive values on Σ by Theorem 1.3. So D_ϵ' is discontinuous on Σ. It is a well-known theorem that Σ is again dense in $[1,\infty)$ (see [Pol] P. 275). This completes the proof. □

From the table and the graph below, we can see that the rate of convergence of $\lim_{x\to\infty} \frac{\#W(2;k;x)}{x/\log x}$, where $k(y) = y^\epsilon$ and ϵ is close to 1, is quite slow (in fact, $\sum_{\sigma(m)=2m} \frac{1}{m} \approx 0.2045$). We calculate $\frac{\#W(2;k;x)}{x/\log x}$ for various $k(y)$ at $x = 1,000,000$, $x = 10,000,000$, and $x = 20,000,000$.

$k(y)$	$x = 1,000,000$	$x = 10,000,000$	$x = 20,000,000$
$y^{0.9}$	3.661860	3.305180	3.196040
$y^{0.8}$	1.141480	0.945623	0.908751
$y^{0.7}$	0.494278	0.435395	0.426470
$y^{0.6}$	0.311567	0.274586	0.267904
$y^{0.5}$	0.276559	0.259482	0.255962
$y^{0.4}$	0.264968	0.252956	0.250063
$y^{0.3}$	0.225980	0.247837	0.247299
$y^{0.2}$	0.151238	0.195911	0.197430

TABLE 1. $\frac{\#W(2;k;x)}{x/\log x}$ for various values of x and $k(y)$.

Our method gives no conclusion for the cases $\ell \notin \mathbb{Q}$ or k is a positive increasing unbounded function satisfying $y^\epsilon = o(k(y))$ for any $\epsilon \in (0,1)$. The situations remain unchanged even if we assume Conjecture 2.7. Therefore, we list these as open problems for further investigations.

Problem 2.11. *What is the order of magnitude of $\#W(\ell;k;x)$ for sublinear k such that $y^\epsilon = o(k(y))$ for any $\epsilon \in (0,1)$?*

Problem 2.12. *Suppose k is a sublinear positive increasing function. What is the order of magnitude of $\#W(\ell;k;x)$ for irrational ℓ? We conject that it is bounded above by x^δ for some $\delta > 0$.*

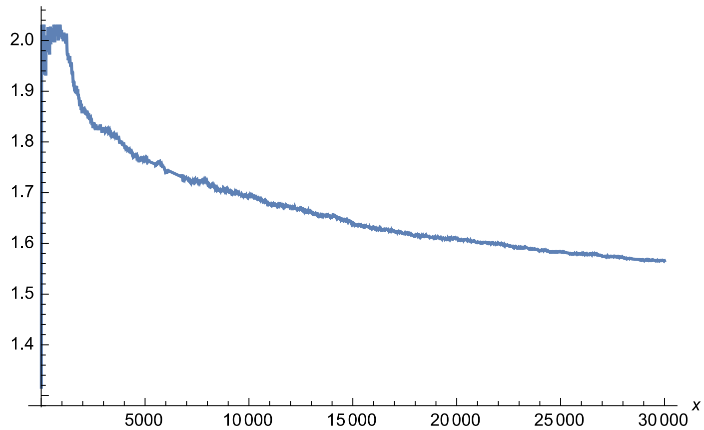

FIGURE 2. This plot shows the quantity

$$\frac{\#W(2;k;x)}{x/\log x}$$

with $k(y) = y^{0.8}$ for x up to $30,000$.

Problem 2.13. *What is the set of all points of continuity of our new distribution function D'_ϵ ?*

For example: we consider $\#W(2; y/\log y; x)/(x/\log x)$. The plot from $x = 2$ to $x = 10,000$ is given in Figure 3.

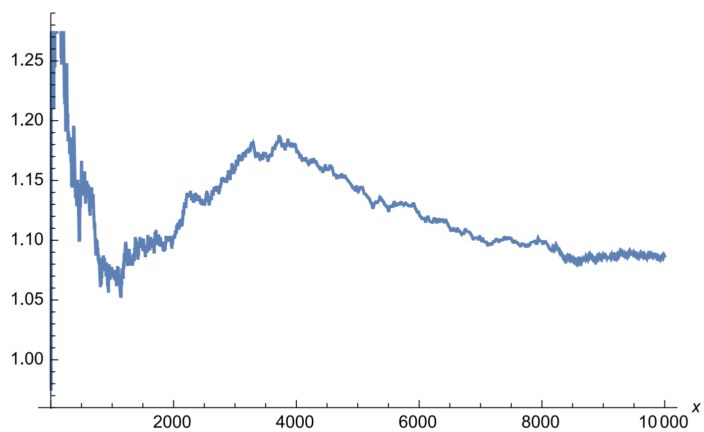

FIGURE 3. This plot shows the quantity $\#W(2;k;x)/(x/\log x)$ with $k(y) = y/\log y$ for $x = 2$ to $10,000$.

3. k-NEAR-PERFECT NUMBERS

3.1. **Near-Perfectness with k being non-constant.** The range of our positive increasing function k under consideration is

$$k(y) \ < \ \exp\left(C\frac{\log y}{\log\log y}\right), \tag{3.1}$$

where C is any constant greater than $\log 2$. In fact, the divisor function, $\tau(n)$, has the following well-known property for its extremal order [HaWr]:

$$\limsup_{n\to\infty} \frac{\log \tau(n)}{\log n/\log\log n} \ = \ \log 2. \tag{3.2}$$

Next we introduce the notion of smooth numbers as follows.

Definition 3.1 (Smooth Number)**.** *Let $y \geq 2$. Then a natural number n is said to be y-smooth if all of its prime factors is at most y. Let $x \geq y \geq 2$. Denote by $\Phi(x, y)$ the set of all y-smooth numbers up to x. We also denote the largest prime factor of n by $P^+(n)$. Hence,*

$$\#\Phi(x, y) = \#\{n \leq x : P^+(n) \leq y\}. \tag{3.3}$$

We have the following well-known trivial estimate.

Lemma 3.2. *Let*

$$u \ = \ \frac{\log x}{\log y}. \tag{3.4}$$

Uniformly for $x \geq y \geq 2$, we have

$$\#\Phi(x, y) \ \ll \ x \exp(-u/2). \tag{3.5}$$

Denote by $\Omega(n)$ the number of prime factors of n *counting multiplicities* and let

$$\Omega(r; x) \ := \ \{n \leq x : \Omega(n) = r\}. \tag{3.6}$$

The size of $\Omega(k; x)$ can be estimated by the following results due to Landau, Hardy and Ramanujan (see [HaWr], [HaRa] or Chapter III.3 of [Te]). These results also hold for $\omega(n)$, the number of *distinct* primes of n.

Lemma 3.3 (Landau)**.** *Fix an integer $r \geq 1$. As $x \to \infty$, we have*

$$\#\Omega(r; x) \ \sim \ \frac{1}{(r-1)!} \frac{x}{\log x} (\log\log x)^{r-1}. \tag{3.7}$$

Lemma 3.4 (Hardy-Ramanujan)**.** *Uniformly for $x \geq 1$ and integers $r \geq 1$, we have*

$$\#\Omega(r; x) \ll \frac{x}{\log x} \frac{(\log\log x + O(1))^{k-1}}{(k-1)!}. \tag{3.8}$$

Parallel to within-perfect numbers, we study the phase-transition behaviour of k-near-perfect numbers. We need the notion of 'normal order' of arithmetic functions.

Definition 3.5 (Normal order)**.** *Let f and g be positive arithmetic functions. We say f has normal order g if for any $\epsilon > 0$, we have*

$$\lim_{x \to \infty} \frac{1}{x} \# \left\{ n \leq x : \left| \frac{f(n)}{g(n)} - 1 \right| \geq \epsilon \right\} \ = \ 0. \tag{3.9}$$

We have the following classical theorem known as Hardy-Ramanujan Theorem [HaRa].

Lemma 3.6. *$\log \tau(n)$ has normal order $\log 2 \log\log n$. $\omega(n)$ and $\Omega(n)$ have normal order $\log\log n$.*

Theorem 1.5 follows from the definition of k-near-perfect numbers and the normal order of $\log \tau(n)$.

Proof of Theorem 1.5. Suppose the asymptotic density of $N((\log y)^{\log 2 + \epsilon})$ exists for some $\epsilon > 0$ and is equal to c. Let k be any positive increasing function on $[1, \infty)$ such that $k(y) > (\log y)^{\log 2 + \epsilon}$ for large $y \geq 1$. Clearly $N((\log y)^{\log 2 + \epsilon}) \subset N(k)$ and we have

$$c \ \leq \ \liminf_{x \to \infty} \frac{1}{x} N(k; x). \tag{3.10}$$

On the other hand,

$$\begin{aligned} \frac{1}{x} N(k; x) \ &= \ \frac{1}{x} N((\log y)^{\log 2 + \epsilon}) + \frac{1}{x} \#((N(k) \setminus N((\log y)^{\log 2 + \epsilon}) \cap [1, x]) \\ &\leq \ \frac{1}{x} N((\log y)^{\log 2 + \epsilon}) + \frac{1}{x} \#\{n \leq x : \tau(n) \ \geq \ (\log n)^{\log 2 + \epsilon}\}. \end{aligned} \tag{3.11}$$

By Lemma 3.6,

$$\limsup_{x \to \infty} \frac{1}{x} \#\{n \leq x : \tau(n) \ \geq \ (\log n)^{\log 2 + \epsilon}\} \ = \ 0. \tag{3.12}$$

Thus we have

$$\limsup_{x\to\infty} \frac{1}{x} N(k;x) \;\le\; c, \tag{3.13}$$

which proves Theorem 1.5. □

If $\log k(y) \le \epsilon \log\log y$ for some $\epsilon \in (0, \log 2)$, then from the definition of normal order we have $\#\{n \le x : \tau(n) \le 2k(n)\} = o(x)$. This is the non-trivial estimate we need for our adaptation of [PoSh]. In fact, one can have a better estimate than $\#\{n \le x : \tau(n) \le 2k(n)\} = o(x)$, such as having an explicit upper bound. This is done by Rankin's method, jointly with a lemma due to Hall, Halberstam and Richert [HaRi].

Lemma 3.7. *For* $y \in (0,1)$,

$$\sum_{n\le x} y^{\log \tau(n)} \;\ll\; \frac{x}{\log x} \sum_{n\le x} \frac{y^{\log \tau(n)}}{n}. \tag{3.14}$$

Proof. See Chapter III.3 of [Te]. This is a fairly general theorem for multiplicative functions and now we specialize to our case. □

Lemma 3.8. *Uniformly for* $\alpha \in (0,1)$,

$$\#\{n \le x : \log \tau(n) \le \alpha \log 2 \log\log x\} \;\ll\; x(\log x)^{-B(\alpha)} \tag{3.15}$$

where $B(\alpha) = \alpha \log \alpha - \alpha + 1$.

Proof. First observe that for $y < 1$, we have

$$\left(\frac{3}{2}\right)^{\log y} \frac{1}{p} + \left(\frac{4}{2}\right)^{\log y} \frac{1}{p^2} + \cdots \le \left(\frac{3}{2}\right)^{\log y} \frac{1}{p}\left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) \;=\; \left(\frac{3}{2}\right)^{\log y} \frac{1}{p-1} \;\ll\; \frac{1}{p}. \tag{3.16}$$

By Rankin's method, we have

$$\begin{aligned}
\sum_{n\le x} \frac{y^{\log \tau(n)}}{n} \;&=\; \sum_{n\le x} \frac{\tau(n)^{\log y}}{n} \\
&\le\; \prod_{p\le x} \left(1 + \frac{2^{\log y}}{p} + \frac{3^{\log y}}{p^2} + \cdots\right) \\
&=\; \prod_{p\le x} \left(1 + \frac{2^{\log y}}{p}\left(1 + \left(\frac{3}{2}\right)^{\log y} \frac{1}{p} + \left(\frac{4}{2}\right)^{\log y} \frac{1}{p^2} + \cdots\right)\right) \\
&=\; \prod_{p\le x} \left(1 + \frac{2^{\log y}}{p} + O\left(\frac{1}{p^2}\right)\right) \\
&=\; \exp\left(\sum_{p\le x} \log\left(1 + \frac{2^{\log y}}{p} + O\left(\frac{1}{p^2}\right)\right)\right) \\
&=\; \exp\left(\sum_{p\le x} \left(\frac{2^{\log y}}{p} + O\left(\frac{1}{p^2}\right)\right)\right) \\
&=\; \exp\left(2^{\log y} \log\log x + O(1) + O\left(\frac{1}{x}\right)\right) \\
&\ll\; (\log x)^{2^{\log y}}.
\end{aligned} \tag{3.17}$$

We also have

$$\sum_{n\le x} y^{\log \tau(n)} \;\ll\; \frac{x}{\log x} \sum_{n\le x} \frac{y^{\log \tau(n)}}{n} \;\ll\; x(\log x)^{2^{\log y}-1}, \tag{3.18}$$

which yields

$$\begin{aligned}\#\{n \leq x : \log\tau(n) \leq \alpha\log 2\log\log x\} &\leq \sum_{n\leq x} y^{\log\tau(n)-\alpha\log 2\log\log x}\\ &\ll xy^{-\alpha\log 2\log\log x}(\log x)^{2^{\log y}-1}\\ &= (\log x)^{-\alpha\log 2\log y+2^{\log y}-1}. \end{aligned}\tag{3.19}$$

Let $f_\alpha(y) = -\alpha\log 2\log y + 2^{\log y} - 1$. It is easy to see that $f_\alpha(y)$ has a minimum point at $y = \alpha^{1/\log 2}$. Plugging this into (3.19), the result follows. □

Corollary 3.9. *Uniformly for a positive increasing function k with*

$$k(y) < (\log y)^\epsilon \tag{3.20}$$

for large y, where $\epsilon \in (0, \log 2)$, we have

$$\#\{n \leq x : \tau(n) \leq 2k(n)\} \ll \frac{x}{\log x}\, k(x)^{(1+\log_2 2)/\log 2}\exp\left(\left(1+\frac{\log k(x)}{\log 2}\right)(\log_3 x - \log_2 2k(x))\right) \tag{3.21}$$

Moreover, this estimate is non-trivial, i.e., the right-hand side of (3.21) *is $o(x)$ as $x \to \infty$. It is also easy to see that the right-hand side of* (3.21) *is greater than $x/(\log x)^2$.*

Proof. Observe that

$$\frac{\log 2k(x)}{\log 2\log_2 x} \in (0,1).$$

Now by Lemma 3.8, we have

$$\begin{aligned}\#\{n \leq x : \tau(n) \leq 2k(n)\} &\leq \#\{n \leq x : \tau(n) \leq 2k(x)\}\\ &\ll x(\log x)^{-B((\log 2k(x))/(\log 2\log_2 x))}\\ &\ll \frac{x}{\log x}\exp\left(\frac{\log 2k(x)}{\log 2}\log\frac{(e\log 2)\log_2 x}{\log 2k(x)}\right)\\ &\ll \frac{x}{\log x}\, k(x)^{(1+\log_2 2)/\log 2}\exp\left(\left(1+\frac{\log k(x)}{\log 2}\right)(\log_3 x - \log_2 2k(x))\right)\end{aligned}$$

□

Proof of Theorem 1.6. We first prove part (a) of Theorem 1.6. Suppose $\epsilon > 0$ is given and k is a natural number. For $m \in N(k)$, define

$$A(m) := \{n \in \mathbb{N} : n = mm', m' \in Q, (m, m') = 1\}, \tag{3.22}$$

where Q is the set of all positive square-free numbers with $1 \in Q$. The number of proper divisors of $n \in A(m)$ is $\tau(m)\cdot 2^s - 1$, where $\Omega(m') = s$. Suppose $m = d_1 + \cdots + d_j$, where $1 \leq d_1 < \cdots < d_j < n$ are proper divisors of m and $j + k \geq \tau(m) - 1$. Then $n = d_1m' + \cdots + d_jm'$ is a sum of j of its proper divisors. Then the number of redundant divisors of n is $\tau(m)\cdot 2^s - 1 - j \leq \tau(m)(2^s - 1) + k$, i.e., $n \in N(\tau(m)(2^s - 1) + k)$.

We can see that $B \in \mathscr{C}(k)$ contains at most one square-free number and $A(m_1) \cap A(m_2) = \emptyset$ for any $m_1, m_2 \in B$ with $m_1 \neq m_2$. Let $r = \max\{\tau(m) : m \in B\}$.

For $\exp((r+k)^{2/\epsilon}) \leq m'$, $m \in B$ and $s \leq (1 + \frac{\epsilon}{2\log 2})\log\log m'$, we have

$$\log\left(\tau(m) + \frac{k}{2^s}\right) \leq \log(r+k) \leq \frac{\epsilon}{2}\log\log m' \tag{3.23}$$

and

$$s + \frac{1}{\log 2} \log\left(\tau(m) + \frac{k}{2^s}\right) \;\le\; \left(1 + \frac{\epsilon}{\log 2}\right) \log\log m'. \tag{3.24}$$

Therefore,

$$2^s \cdot \tau(m) + k \;\le\; (\log m')^{\log 2+\epsilon} \;\le\; (\log mm')^{\log 2+\epsilon}. \tag{3.25}$$

Denote by $\mu(\cdot)$ the Möbius function. From the classical estimate (see [HaWr])

$$\#(Q \cap [1,x]) = \sum_{n\le x} |\mu(n)| = \frac{6x}{\pi^2} + O(\sqrt{x}), \tag{3.26}$$

and the Hardy-Ramanujan Theorem for $\Omega(\cdot)$, we have

$$\lim_{x\to\infty} \frac{1}{x} \#\{n \le x : \Omega(n) < (1+\epsilon)\log\log n, n \in Q\} \;=\; \frac{6}{\pi^2}. \tag{3.27}$$

By using inclusion-exclusion principle and the above observations, for $x \ge \max B$, we have

$$\begin{aligned}
&\#N((\log y)^{\log 2+\epsilon}; x)\\
\ge& \sum_{m\in B} \#\left\{m' \le \frac{x}{m} : (m,m') = 1, m' \in Q, \tau(m)\cdot 2^s + k \le (\log mm')^{\log 2+\epsilon}\right\}\\
\ge& \sum_{m\in B} \#\left\{\exp((r+k)^{2/\epsilon}) \le m' \le \frac{x}{m} : (m,m') = 1, m' \in Q, \Omega(m') \le (1 + \frac{\epsilon}{2\log 2}) \log\log m'\right\}\\
=& \sum_{m\in B} \#\left\{m' \le \frac{x}{m} : (m,m') = 1, m' \in Q, \Omega(m') \le (1 + \frac{\epsilon}{2\log 2}) \log\log m'\right\} + O_{r,k,\epsilon}(\#B)\\
=& \left(\frac{6}{\pi^2}(1+o(1)) \sum_{m\in B} \frac{\phi(m)}{m^2}\right) x + o_{r,k,\epsilon}(x),
\end{aligned} \tag{3.28}$$

as $x \to \infty$, where $\phi(\cdot)$ is the Euler's totient function.

In other words,

$$\liminf_{x\to\infty} \frac{\#N((\log y)^{\log 2+\epsilon}; x)}{x} \;\ge\; \frac{6}{\pi^2} \sum_{m\in B} \frac{\phi(m)}{m^2} \tag{3.29}$$

for any $\epsilon > 0$, $k \in \mathbb{N}$ and $B \in \mathscr{C}(k)$.

Recall that M is defined by

$$M \;:=\; \frac{6}{\pi^2} \sup_{k\in\mathbb{N}} \sup_{B\in\mathscr{C}(k)} \sum_{m\in B} \frac{\phi(m)}{m^2}.$$

We have for any $\epsilon > 0$

$$\liminf_{x\to\infty} \frac{\#N((\log y)^{\log 2+\epsilon}; x)}{x} \;\ge\; M, \tag{3.30}$$

and from [Kob],

$$M \;\le\; 1 - D(2) \in (0.24761, 0.24765). \tag{3.31}$$

Now we take $k = 1$ and from the sequence A181595 of [OEIS]

$$N(1) \;=\; \{6, 12, 18, 20, 24, 28, 40, 88, 104, 196, 224, 234, \ldots\} \tag{3.32}$$

We pick our admissible subset B of $N(1)$ inductively, starting with $6 \in B$. In this way from the list of $N(1)$ above, we have the following admissible set

$$B \;=\; \{6, 12, 18, 24, 224\} \tag{3.33}$$

and we have

$$M \;\geq\; \frac{4981}{7056\pi^2} \approx 0.0715251. \tag{3.34}$$

This lower bound for the constant M is clearly far from the best. It would be interesting to pursue further on its computational aspect.

Now we prove part (b) of Theorem 1.6. It is an adaptation of the argument in [PoSh]. Let $y = x^{1/4\log\log x}$. Consider the following three sets form a partition of $N(k; x)$.

$$\begin{aligned} N_1(k;x) \;&:=\; \{n \in N(k;x):\; n \text{ is } y\text{-smooth}\} \\ N_2(k;x) \;&:=\; \{n \in N(k;x) : P^+(n) > y \text{ and } P^+(n)^2 | n\} \\ N_3(k;x) \;&:=\; \{n \in N(k;x) : P^+(n) > y \text{ and } P^+(n) \mid\mid n\}. \end{aligned} \tag{3.35}$$

By Lemma 3.2, we have

$$\#N_1(k;x) \;\leq\; \#\Phi(x,y) \;=\; x\exp(-2\log\log x) \;=\; \frac{x}{(\log x)^2}. \tag{3.36}$$

We have the following trivial estimate:

$$\#N_2(k;x) \;\leq\; \sum_{p>y} \frac{x}{p^2} \;\ll\; \frac{x}{y} \;=\; x^{1-1/\log\log x} \;=\; x\exp(-\log x/\log\log x) \ll \frac{x}{(\log x)^2}. \tag{3.37}$$

For $n \in N_3(k; x)$, we can write

$$n = pm, \;\; \text{where} \;\; p = P^+(n) > \max\{y, P^+(m)\}. \tag{3.38}$$

Further partition $N_3(k; x)$ into $N_3'(k; x)$ and $N_3''(k; x)$, where $N_3'(k; x)$ consists of $n \in N_3(k; x)$ such that $\tau(m) \leq k$ and $N_3''(k; x) := N_3(k; x) \setminus N_3'(k; x)$.

For $n \in N_3(k; x)$, we count the number of possible p and m in (3.38). Clearly, the number of possible m is at most x/y. Since n is k-near-perfect, there exists a set of proper divisors D_n of n with $\#D_n \leq k(n)$ such that

$$\sigma(n) \;=\; 2n + \sum_{d \in D_n} d. \tag{3.39}$$

Consider

$$\begin{aligned} D_n^{(1)} \;&:=\; \{d \in D_n : p \nmid d\}, \\ D_n^{(2)} \;&:=\; \{d/p : d \in D_n, p \mid d\}. \end{aligned}$$

Then

$$(1+p)\sigma(m) \;=\; \sigma(pm) \;=\; 2pm + \sum_{d \in D_n^{(1)}} d + p \sum_{d \in D_n^{(2)}} d. \tag{3.40}$$

Reducing both sides mod p yields

$$p \;\Bigg|\; \left(\sigma(m) - \sum_{d \in D_n^{(1)}} d\right). \tag{3.41}$$

For $d \in D_n^{(1)}$, we have $d \mid m$ and

$$\left(\sigma(m) - \sum_{d \in D_n^{(1)}} d\right) \;\; \text{is a sum of divisors of } m. \tag{3.42}$$

For $n \in N_3''(k;x)$,

$$\sigma(m) - \sum_{d \in D_n^{(1)}} d > 0. \tag{3.43}$$

Therefore,

$$\begin{aligned}\left(\sigma(m) - \sum_{d \in D_n^{(1)}} d\right) &\le \sigma(m) \\ &\ll m \log\log m \\ &\ll x^{1-1/\log\log x} \log((1-1/\log\log x)\log x) \\ &\ll (x \log\log x) \exp(-\log x/\log\log x). \end{aligned} \tag{3.44}$$

Thus for each possible value of $\left(\sigma(m) - \sum_{d\in D_n^{(1)}} d\right)$, there are $\ll \log x$ prime factors.
We may also assume $\tau(m) \le (\log x)^3$. Indeed by the well-known estimate $\sum_{n \le x} \tau(n) \ll x \log x$ and $2\tau(m) = \tau(n)$,

$$\#\{n \le x : \tau(m) > (\log x)^3\} \ll \frac{x}{(\log x)^2}. \tag{3.45}$$

Under this assumption, the number of possible values for $\left(\sigma(m) - \sum_{d\in D_n^{(1)}} d\right)$ is $\le (1+\tau(m))^k \le (1+(\log x)^3)^{k(x)}$ and hence the number of possible p is $\ll (\log x)(1+(\log x)^3)^{k(x)}$.

Suppose $k(y) < (\log y)^\epsilon$ for some $\epsilon \in (0, \log 2)$.

$$\begin{aligned}\#N_3''(k;x) &\ll \frac{x}{y}(\log x)(1+(\log x)^3)^{k(x)} \ll x\log x \exp\left(-\frac{\log x}{\log\log x}\right)\exp(k(x)\log(1+(\log x)^3)) \\ &= x \log x \exp\left(-\frac{\log x}{\log\log x}\right)\exp\left(3k(x)\log\log x + O\left(\frac{k(x)}{(\log x)^3}\right)\right) \\ &\ll x\log x \exp\left(-\frac{\log x}{\log\log x} + 3(\log x)^{\log 2}\log\log x\right) \\ &\ll x \log x \exp\left(-\frac{\log x}{2\log\log x}\right) \ll \frac{x}{(\log x)^2}. \end{aligned} \tag{3.46}$$

By Corollary 3.9, we have

$$\begin{aligned}\#N(k;x) &\le \#N((\log y)^\epsilon; x) \\ &\ll \frac{x}{\log x}(\log x)^{\epsilon(1+\log_2 2)/\log 2} \exp\left(\left(1+\frac{\epsilon \log_2 x}{\log 2}\right)(\log_3 x - \log_2 2(\log x)^\epsilon)\right) \\ &\ll_\epsilon \frac{x}{\log x}(\log x)^{\epsilon(1+\log_2 2)/\log 2}\exp\left(-\log\epsilon\left(1+\frac{\epsilon\log_2 x}{\log 2}\right)\right) \\ &\ll_\epsilon \frac{x}{(\log x)^{r(\epsilon)}}, \end{aligned} \tag{3.47}$$

where

$$r(\epsilon) := 1 - \frac{\epsilon(1+\log_2 2 - \log\epsilon)}{\log 2} \in (0,1). \tag{3.48}$$

This completes the proof of Theorem 1.6. □

We end this section with the remark that we can improve the bound (3.21) (hence that of $\#N(k;x)$) if $k(y) < \exp\left(\sqrt{\frac{\log 2}{2}\log_3 y}\right)$ by establishing the following.

Lemma 3.10. *Uniformly for positive increasing function k with*

$$k(y) < \exp\left(\epsilon \frac{\log_2 y}{\log_3 y}\right) \tag{3.49}$$

for large y, where $\epsilon \in (0, \log 2)$, we have

$$\#\{n \leq x : \tau(n) \leq 2k(n)\} \ll \frac{x}{\log x} \exp\left(\frac{1}{\log 2} \log\ k(x) \log_3 x + O\left(\frac{\log\ k(x)}{\log_2 x}\right)\right). \tag{3.50}$$

This estimate is non-trivial and it is trivial that the right-hand side of (3.50) *is larger than* $x/(\log x)^2$.

Proof. Since $2^{\omega(n)} \leq \tau(n)$ and $k(y) < (\log y)^\epsilon$, we have

$$\begin{aligned}\#\{n \leq x : \tau(n) \leq 2k(n)\} &\leq \#\left\{n \leq x : \omega(n) \leq 1 + \frac{\log\ k(n)}{\log 2}\right\} \\ &\leq \sum_{r \leq 1 + \frac{\log k(x)}{\log 2}} \#\{n \leq x : \omega(n) = r\} \\ &\ll \sum_{r \leq 1 + \frac{\log k(x)}{\log 2}} \frac{x}{\log x} \frac{(\log_2 x + O(1))^{r-1}}{(r-1)!} \\ &\ll \frac{x}{\log x} (\log_2 x + O(1))^{\frac{\log k(x)}{\log 2}} \\ &= \frac{x}{\log x} \exp\left(\frac{1}{\log 2} \log\ k(x) \log_3 x + O\left(\frac{\log\ k(x)}{\log_2 x}\right)\right),\end{aligned} \tag{3.51}$$

and this is a non-trivial estimate if

$$k(x) < \exp\left(\epsilon \frac{\log_2 x}{\log_3 x}\right) \tag{3.52}$$

for some $\epsilon \in (0, \log 2)$. □

Now suppose $k(y) < \exp\left(\sqrt{\frac{\log 2}{2} \log_3 y}\right)$. Then

$$\log_2 2k(x)\left(1 + \frac{\log k(x)}{\log 2}\right) \ < \ \frac{2}{\log 2} (\log k(x))^2 \ < \ \log_3 x. \tag{3.53}$$

We have

$$\frac{1}{\log 2} \log k(x) \log_3 x + O\left(\frac{\log k(x)}{\log_2 x}\right) < \frac{1 + \log_2 2}{\log 2} \log k(x) + \left(1 + \frac{\log k(x)}{\log 2}\right)(\log_3 x - \log_2 2k(x)), \tag{3.54}$$

hence improving the bound (3.21).

3.2. **Near-Perfectness with k being constant: improving previous results.** Throughout this section, k is a fixed natural number. From the remark at the end of the last section, we have

$$\#N(k; x) \ \ll \ \frac{x}{\log x} (\log \log x)^{\lfloor \frac{\log k}{\log 2} \rfloor}. \tag{3.55}$$

Now we know that the exponent of $\log \log x$ is between $\lfloor \frac{\log(k+4)}{\log 2} \rfloor - 3$ and $\lfloor \frac{\log k}{\log 2} \rfloor$ inclusively. In order to have a precise determination of the exponent, we have to refine the counting done in [PoSh]. In the proof of Theorem 1.6, observe that $N_3'(k; x)$ contributes the most to $N(k; x)$, but the restriction on m, i.e., $\tau(m) \leq k$, merely provides a very crude upper bound. There should be more arithmetic information on m. Moreover, we remark that the assumption $\tau(m) > k$ is more than needed to do the counting. Hence, we partition $N_3(k;x)$ differently from [PoSh] as follows:

$$N_3^{(1)}(k;x) \;:=\; \{n \in N_3(k;x) : \text{all of the positive divisors of } m \text{ are redundant divisors of } n\},$$
$$N_3^{(2)}(k;x) \;:=\; N_3(k;x) \setminus N_3^{(1)}(k;x). \tag{3.56}$$

We have the following key lemma which allows us to count $\#N(k;x)$ precisely.

Lemma 3.11. *Suppose n is of the form* (3.38). *Then $n \in N_3^{(1)}(k;x)$ if and only if $\tau(m) \le k$ and m is an $(k-\tau(m))$-near-perfect number. In particular, if $n \in N_3^{(1)}(k;x)$, then m is a $\frac{k-1}{2}$-near-perfect number.*

Proof of Lemma 3.11. Suppose $n \in N_3^{(1)}(k;x)$. There exists a set of proper divisors D_n of n with $\#D_n \le k$ such that

$$\sigma(n) \;=\; 2n + \sum_{d\in D_n} d. \tag{3.57}$$

Partition D_n into two subsets according to whether $d \in D_n$ is divisible by p or not. More precisely, define

$$D_n^{(1)} \;:=\; \{d \in D_n : p \nmid d\},$$
$$D_n^{(2)} \;:=\; \{d/p : d \in D_n,\ p \mid d\}. \tag{3.58}$$

Then

$$(1+p)\sigma(m) \;=\; \sigma(pm) \;=\; 2pm + \sum_{d\in D_n^{(1)}} d + p \sum_{d\in D_n^{(2)}} d. \tag{3.59}$$

By the definition of $N_3^{(1)}(k;x)$ and the fact that $p \nmid m$, $D_n^{(1)}$ is the set of all positive divisors of m. Hence

$$\sigma(m) = \sum_{d\in D_n^{(1)}} d. \tag{3.60}$$

We have

$$\sigma(m) \;=\; 2m + \sum_{d\in D_n^{(2)}} d. \tag{3.61}$$

Since $\#D_n^{(1)} = \tau(m)$ and $\#D_n^{(1)} + \#D_n^{(2)} = \#D_n \le k$, we have $\#D_n^{(2)} \le k - \tau(m)$. Note that $D_n^{(2)}$ consists of proper divisors of m. This proves m is an $(k-\tau(m))$-near-perfect number. The converse is trivial.
By observing $\#D_n^{(2)} \le \min\{k-\tau(m), \tau(m)-1\}$, if $n \in N_3^{(1)}(k;x)$, then m is $\frac{k-1}{2}$ -near-perfect. This completes the proof of Lemma 3.11. □

Here we explain the role of Lemma 3.11 in our modification. We have finitely many possible values for $\tau(m)$. For each possible value of $\tau(m)$, we can determine all possible forms of m in terms of prime factorizations. Then by the criterion that m has to be an $(k-\tau(m))$-near-perfect number, we have a finite collection of polynomial Diophantine equations in primes (See the proof of Lemma 3.15). This gives all of the possible values of m. Note that in Lemma 3.11, there is no restriction on prime p. Therefore for each such m, there corresponds to $\asymp_m x/\log x$ natural numbers $n \in N_3^{(1)}(k;x)$. For smaller $k \ge 4$, there are only finitely many such m; this explains why $\#N(k;x)$ has order $x/\log x$.

We prove the following lemmata that can reduce the amount of calculations.

Lemma 3.12. *Prime powers cannot be k-near-perfect for any natural number k.*

Proof. Suppose $m = q^\ell$ is a k-near-perfect number for some k. Then from

$$\sigma(m) \;=\; 2m + \sum_{d\in D_m} d, \tag{3.62}$$

where D_m is a set of proper divisors of m with $\#D_m \le k$, we have

$$q^\ell = \sum_{a \in A} q^a, \tag{3.63}$$

where A is a subset of $\{0, \dots, \ell - 1\}$; however, this contradicts the uniqueness of q-ary representation. This completes the proof. □

Corollary 3.13. *m cannot be k-near-perfect for any integers $k \ge 0$ if $\tau(m)$ is prime. Hence if m is a k-near-perfect number for some $k \ge 0$, then $\tau(m) \ge 4$.*

The following is a result of [ReCh], which is a complete classification of 1-near-perfect numbers with two distinct prime factors. It is not strictly necessary for our method, but it reduces the amount of calculations considerably.

Lemma 3.14. *A 1-near-perfect number which is not perfect and has two distinct prime factors is of the form*

(1) $2^{t-1}(2^t - 2^k - 1)$, *where* $2^t - 2^k - 1$ *is prime,*
(2) $2^{2p-1}(2^p - 1)$, *where p is a prime such that $2^p - 1$ is also a prime.*
(3) $2^{p-1}(2^p - 1)^2$, *where p is a prime such that $2^p - 1$ is also a prime.*
(4) 40.

Lemma 3.15. *If m is a k-near-perfect number for some $k \ge 0$ and $\tau(m) = 4$ or $\tau(m) = 6$, then $m \in \{6, 12, 18, 20, 28\}$.*

Proof. Suppose m is a k-near-perfect number for some $k \ge 0$. If $\tau(m) = 4$, then by Lemma 3.12, m is of the form qr, where q, r are distinct primes and we have one of the following cases:

$$\begin{aligned}
(1+q)(1+r) &= 2qr \\
(1+q)(1+r) &= 2qr + 1 \\
(1+q)(1+r) &= 2qr + q \\
(1+q)(1+r) &= 2qr + 1 + q \\
(1+q)(1+r) &= 2qr + q + r \\
(1+q)(1+r) &= 2qr + 1 + q + r.
\end{aligned} \tag{3.64}$$

From these equations, we have $m = 6$. The case for $\tau(m) = 6$ is similar but with more equations to be considered. For $\tau(m) = 6$, m is of the form $q^2 r$, where q, r are distinct primes. In fact, for any $k \ge 3$, there is no k-near-perfect number with 6 positive divisors. Moreover, all of the 2-near-perfect numbers with 6 positive divisors are indeed 1-near-perfect.

□

Proof of Theorem 1.7. Let $y = x^{1/\log_2 x}$ as before. From the proof of Theorem 1.6, we have the following estimates:

$$\#N_1(k;x),\ \#N_2(k;x),\ \#N_3^{(2)}(k;x) \ll_k \frac{x}{(\log x)^2}. \tag{3.65}$$

Now consider $n \in N_3^{(1)}(k;x)$. Then $n = pm$, where p is a prime $> \max\{y, P^+(m)\}$ and $m \in N(k - \tau(m))$. The following is a case-by-case analysis.

For $k = 4, 5$, by Corollary 3.13, $\tau(m) = 4$ and $m \in N(1)$. By Lemma 3.15, we have $m = 6$.
By the Prime Number Theorem, we have

$$\begin{aligned}
\#N_3^{(1)}(4;x) = \pi(x/6) - \pi(x^{1/\log\log x}) &= \frac{1}{6}\frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right) + O\left(\frac{x^{1/\log\log x}\log\log x}{\log x}\right) \\
&= \frac{1}{6}\frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right).
\end{aligned} \tag{3.66}$$

Therefore,

$$\#N(4;x)=\#N_1(4;x)+\#N_2(4;x)+\#N_3^{(1)}(4;x)+\#N_3^{(2)}(4;x)=\frac{1}{6}\frac{x}{\log x}+O\left(\frac{x}{(\log x)^2}\right). \quad (3.67)$$

The same result holds for $\#N(5;x)$.

For $k=6$, we have $\tau(m)\in\{4,6\}$.

- If $\tau(m)=4$, then $m\in N(2)$. We have $m=6$.
- If $\tau(m)=6$, then $m\in N(0)$. We have $m=28$.

Therefore, we have

$$\#N(6;x)\ \sim\ \frac{17}{84}\frac{x}{\log x}. \quad (3.68)$$

For $k=7$, we have $\tau(m)\in\{4,6\}$. For $k=8$, $\tau(m)\in\{4,6,8\}$.

- If $\tau(m)=4$, then $m\in N(3)$. We have $m=6$.
- If $\tau(m)=6$, then $m\in N(1)$. We have $m\in\{12,18,20,28\}$.
- If $k=8$ and $\tau(m)=8$, then $m\in N(0)$. m has at most 3 prime factors. It is an elementary fact that m cannot be an odd perfect number. By Euclid-Euler Theorem, m is of the form $2^{p-1}(2^p-1)$ for some prime p such that 2^p-1 is also a prime. Then $8=\tau(m)=2p$, which is a contradiction. Hence, there is no such m.

Therefore, we have

$$\#N(7;x),\#N(8;x)\ \sim\ \frac{493}{1260}\frac{x}{\log x}. \quad (3.69)$$

For $k=9$, $\tau(m)\in\{4,6,8,9\}$. Again if $\tau(m)=4$ or 6, $m\in\{6,12,18,20,28\}$.

- If $\tau(m)=8$, then $m\in N(1)$. By the discussion in the case $k=8$, m cannot be perfect. By Lemma 3.12, we have m is of the form q^3r or qrs, where q,r,s are distinct primes. For the first case we have $m\in\{24,40,56,88,104\}$ by using Lemma 3.14. For the second case, we consider the following set of equations
$$\begin{aligned}(1+q)(1+r)(1+s)&=2qrs+1,\\(1+q)(1+r)(1+s)&=2qrs+q,\\(1+q)(1+r)(1+s)&=2qrs+qr,\end{aligned} \quad (3.70)$$
in which it is easy to check all of them have no solution.
- If $\tau(m)=9$, then $m\in N(0)$. By similar discussion in the case of $k=8$, there is no such m.

Therefore, we have

$$\#N(9;x)\ \sim\ \frac{179017}{360360}\frac{x}{\log x}. \quad (3.71)$$

□

Remark 3.16. *It was established in* [PoSh] *that* $\#N(k;x)\ll x\exp(-(c_k+o(1))\sqrt{\log x\log\log x})$, *where* $c_2=\sqrt{6}/6\approx 0.4082$ *and* $c_3=\sqrt{2}/4\approx 0.3535$. *By our modification, we recover this result with improved constants and replacement of* $o(1)$ *by* $O(\log_3 x/\log_2 x)$.
We first introduce the following standard, more precise estimate for $\#\Phi(x,y)$ *which can be found in Chapter 9 of* [DeLu].

Let $u = \log x/\log y$. Then uniformly for $(\log x)^3 \le y \le x$, we have

$$\#\Phi(x,y) \ = \ x\exp\left(-u\log u + O(u\log\log u)\right). \tag{3.72}$$

Since $\#N(2;x) \le \#N(3;x)$, it suffices to consider the case $k=3$ only. By Corollary 3.13, $N_3^{(1)}(3;x)$ is an empty set. We remark that the choice of y is different from before and it is important for the quality of the upper bound. Hence,

$$\begin{aligned}\#N(3;x) &= \#N_1(3;x) + \#N_2(3;x) + \#N_3^{(2)}(3;x)\\ &\ll x\exp(-u\log u + O(u\log\log u)) + \frac{x}{y} + \frac{x}{y}(\log x)^{10}\\ &\ll x\exp(-u\log u + O(u\log\log u)) + \frac{x}{y}(\log x)^{10}.\end{aligned} \tag{3.73}$$

We should choose y such that

$$\exp(-u\log u + O(u\log\log u)) = \frac{(\log x)^{10}}{y}, \tag{3.74}$$

or

$$u\log u + O(u\log\log u) = \log y - 10\log\log x. \tag{3.75}$$

This suggests us to choose $\log y = \sqrt{\log x \log\log x}$, which is clearly admissible. From this we can see that

$$u = \sqrt{\frac{\log x}{\log\log x}} \quad \textit{and} \quad u\log u = \frac{1}{2}\sqrt{\frac{\log x}{\log\log x}}(\log\log x - \log\log\log x) \asymp \log y. \tag{3.76}$$

Therefore,

$$\begin{aligned}\#N(3;x) &\ll x\exp\left(-\frac{1}{2}\sqrt{\log x\log\log x}\left(1 + O\left(\frac{\log\log\log x}{\log\log x}\right)\right)\right)\\ &\quad + x\exp(-\sqrt{\log x\log\log x} + 10\log\log x)\\ &\ll x\exp\left(-\frac{1}{2}\sqrt{\log x\log\log x}\left(1 + O\left(\frac{\log\log\log x}{\log\log x}\right)\right)\right).\end{aligned} \tag{3.77}$$

This upper bound is in fact the best we can do by using the partition described before in terms of smooth numbers. We need a more refine counting to handle the cases $k=2$ and $k=3$. However, for the cases $k \ge 4$, this is the right partition that leads us to the sharp results. We are going to discuss in the following.

Remark 3.17. *With the assumption that the set $\{m \in N(2) : \tau(m) = 8\}$ is finite, we have*

$$\#N(10;x) \ \sim \ c_{10}\frac{x}{\log x}, \tag{3.78}$$

for some constant c_{10} satisfying

$$c_{10} \ge \frac{78806633}{156396240}. \tag{3.79}$$

With the assumptions that the sets $\{m \in N(3) : \tau(m) = 8\}$ and $\{m \in N(2) : \tau(m) = 9\}$ are finite, we have

$$\#N(11;x) \ \sim \ c_{11}\frac{x}{\log x}, \tag{3.80}$$

for some constant c_{11} satisfying

$$c_{11} \ge \frac{53072311991}{104316292080}. \tag{3.81}$$

The exact values for c_{10} and c_{11} can be found as above, but the computations become tedious.

The amount of calculations increases significantly as k grows in the above method. Moreover, it is not easy to solve those Diophantine equations in primes systematically in general. It is of interest to ask for better ways to handle the general cases. The key idea is to apply Lemma 3.11 and our partition repeatedly.

First it is essential to estimate the size of following set for $j \geq 1$ and $x \geq y \geq 2$:

$$\Phi_j(x,y) \ := \ \{n \leq x : n = p_1 \cdots p_j m_j, P^+(m_j) \leq y < p_j < \cdots < p_1\}. \tag{3.82}$$

Obtaining a lower bound for $P_j(x)$ is easy. It is simply an observation of the fact that

$$\{n \leq x : n = p_1 \cdots p_j m_j, m_j \leq y < p_j < \cdots < p_1\} \subset \Phi_j(x,y) \tag{3.83}$$

and the following lemma. The idea is that in the set $\Omega(r;x)$, the numbers that are *square-free* contribute the most. Then the rest follows from Landau's Theorem.

Lemma 3.18.

$$\#\{n \leq x : n = p_1 \cdots p_s, p_1 > \cdots > p_s\} \ \sim \ \frac{1}{(s-1)!}\frac{x}{\log x}(\log\log x)^{s-1}. \tag{3.84}$$

Proof of Lemma 3.18. First observe that

$$\{n \leq x : \Omega(n) = s\} = \bigcup_{\substack{a_1+\cdots+a_r=s \\ a_1,\ldots,a_r \geq 1 \\ r \geq 1}} \{n \leq x : n = p_1^{a_1}\cdots p_r^{a_r}, p_1 > \cdots > p_r\}. \tag{3.85}$$

Consider one of the sets $\{n \leq x : n = p_1^{a_1}\cdots p_r^{a_r}, p_1 > \cdots > p_r\}$ forming the partition above with $a_j \geq 2$ for some $1 \leq j \leq r$, $a_1, \ldots, a_r \geq 1$ and $a_1 + \cdots + a_r = s$.

By partial summation and Landau's Theorem, we have

$$\begin{aligned}\sum_{\substack{m \leq x \\ \Omega(m)=s-a_j}} \left(\frac{1}{m}\right)^{1/a_j} &= \int_2^x \frac{d\,\#\Omega(s-a_j;t)}{t^{1/a_j}} = \frac{\#\Omega(s-a_j;x)}{x^{1/a_j}} + \frac{1}{a_j}\int_2^x \frac{\#\Omega(s-a_j;t)}{t^{1+1/a_j}}\,dt \\ &\ll \frac{1}{x^{1/a_j}}\frac{x}{\log x}(\log\log x)^{s-a_j-1} + \int_2^x \frac{1}{t^{1/a_j}\log t}(\log\log t)^{s-a_j-1}\,dt. \end{aligned} \tag{3.86}$$

We claim that

$$\int_2^x \frac{1}{t^{1/a_j}\log t}(\log\log t)^{s-a_j-1}\,dt = o\left(x^{1-\frac{1}{a_j}}\frac{(\log\log x)^{s-1}}{\log x}\right). \tag{3.87}$$

First note that

$$\frac{d}{dx}x^{1-\frac{1}{a_j}}\frac{(\log\log x)^{s-1}}{\log x} = \left(1-\frac{1}{a_j}\right)(1+o(1))x^{-\frac{1}{a_j}}\frac{(\log\log x)^{s-1}}{\log x}. \tag{3.88}$$

Then by this and the Fundamental Theorem of Calculus,

$$\begin{aligned}\frac{\frac{d}{dx}\int_2^x \frac{1}{t^{1/a_j}\log t}(\log\log t)^{s-a_j-1}\,dt}{\frac{d}{dx}x^{1-\frac{1}{a_j}}\frac{(\log\log x)^{s-1}}{\log x}} &= \frac{\frac{1}{x^{1/a_j}\log x}(\log\log x)^{s-a_j-1}}{\left(1-\frac{1}{a_j}\right)(1+o(1))x^{-\frac{1}{a_j}}\frac{(\log\log x)^{s-1}}{\log x}} \\ &= \frac{1}{\left(1-\frac{1}{a_j}\right)(1+o(1))(\log\log x)^{a_j}} = o(1)\end{aligned} \tag{3.89}$$

as $x \to \infty$. From L'Hôpital's Rule, the claim follows. Hence

$$\sum_{\substack{m \leq x \\ \Omega(m)=s-a_j}} \left(\frac{1}{m}\right)^{1/a_j} = o\left(x^{1-\frac{1}{a_j}}\frac{(\log\log x)^{s-1}}{\log x}\right) \tag{3.90}$$

and

$$\begin{aligned}\#\{n \leq x : n = p_1^{a_1}\cdots p_r^{a_r}, p_1 > \cdots > p_r\} &\leq \sum_{\substack{m\leq x \\ \Omega(m)=s-a_j}} \sum_{p_j^{a_j}\leq \frac{x}{m}} 1 \ll \sum_{\substack{m\leq x \\ \Omega(m)=s-a_j}} \left(\frac{x}{m}\right)^{1/a_j} \\ &= x^{1/a_j} \sum_{\substack{m\leq x \\ \Omega(m)=s-a_j}} \left(\frac{1}{m}\right)^{1/a_j} \\ &= o\left(\frac{x}{\log x}(\log\log x)^{s-1}\right). \end{aligned} \tag{3.91}$$

From the fact that

$$\#\Omega(s;x) = \#\{n \leq x : n = p_1\cdots p_s, p_1 > \cdots > p_s\} + \sum_{\substack{a_1+\cdots+a_r=s \\ a_1,\ldots,a_r\geq 1 \\ \exists j\in\{1,\ldots,r\}: a_j\geq 2}} \#\{n\leq x : n = p_1^{a_1}\cdots p_r^{a_r}, p_1 > \cdots > p_r\} \tag{3.92}$$

and Landau's Theorem, we have

$$\frac{1}{(s-1)!}(1+o(1))\frac{x}{\log x}(\log\log x)^{s-1} = \#\{n \leq x : n = p_1\cdots p_s, p_1 > \cdots > p_s\} + o\left(\frac{x}{\log x}(\log\log x)^{s-1}\right). \tag{3.93}$$

Hence,

$$\#\{n \leq x : n = p_1\cdots p_s, p_1 > \cdots > p_s\} \sim \frac{1}{(s-1)!}\frac{x}{\log x}(\log\log x)^{s-1}. \tag{3.94}$$

This completes the proof of Lemma 3.18. □

Therefore, we have

$$\begin{aligned}\#\Phi_j(x,y) &\geq \sum_{m_j\leq y} \sum_{\substack{n_j\leq \frac{x}{m_j} \\ n_j = p_1\cdots p_j \\ \text{for some } p_1>\cdots p_j>y}} 1 \\ &\gg \sum_{m_j\leq y} \frac{x/m_j}{\log(x/m_j)}\left(\log\log\frac{x}{m_j}\right)^{j-1} \\ &\geq \frac{x}{\log x}\left(\log\log\frac{x}{y}\right)^{j-1}\sum_{m_j\leq y}\frac{1}{m_j} \\ &\gg \frac{x\log y}{\log x}\left(\log\log\frac{x}{y}\right)^{j-1}. \end{aligned} \tag{3.95}$$

For the upper bound of $\#\Phi_j(x,y)$, we use the smooth number bound (3.5) and the following standard upper bound sieve estimate (see [FoHa]).

Lemma 3.19. *Suppose A is a finite subset of natural number, P is a subset of primes and $z > 0$. Let*

$$P(z) = \prod_{\substack{p\in P \\ p\leq z}} p. \tag{3.96}$$

Denote by $S(A,P,z)$ the set

$$\{n \in A : (n, P(z)) = 1\} \tag{3.97}$$

and by A_d the set

$$\{a \in A : d \mid a\}. \tag{3.98}$$

Suppose g is a multiplicative function satisfying

$$0 \le g(p) < 1 \text{ for } p \in P \text{ and } g(p) = 0 \text{ for } p \notin P \tag{3.99}$$

and there exists some constants $B > 0$ and $\kappa \ge 0$ such that

$$\prod_{y \le p \le w} (1 - g(p))^{-1} \le \left(\frac{\log w}{\log y}\right)^{\kappa} \exp\left(\frac{B}{\log y}\right) \tag{3.100}$$

for $2 \le y < w$.

Let $X > 0$. For d which is a product of distinct primes from P, define

$$r_d := \#A_d - Xg(d). \tag{3.101}$$

Suppose for some $\theta > 0$, we have

$$\sum_{\substack{d \mid P(z) \\ d \le X^{\theta}}} |r_d| \le C \frac{x}{(\log x)^{\kappa}}. \tag{3.102}$$

Then for $2 \le z \le X$, we have

$$\#S(A, P, z) \ll_{\kappa, \theta, C, B} XV(z), \tag{3.103}$$

where

$$V(z) := \prod_{\substack{p \le z \\ p \in P}} (1 - g(p)). \tag{3.104}$$

Lemma 3.20. *Suppose $x \ge y \ge 2$ and $y \le x^{o(1)}$. For every $j \ge 1$, we have*

$$\#\Phi_j(x, y) \ll \frac{x \log y}{\log x} (\log \log x)^{j-1}. \tag{3.105}$$

Proof. With the notation as in Lemma 3.19. Let A be the set of all natural numbers up to x, P be the set of primes in $(y, x^{1/(j+1)}]$, $z := x^{1/(j+1)}$, $X := x$ and $g(d) := 1/d$.
$S(A, P, z)$ consists of all natural numbers up to x whose prime factors are $\le y$ or $> x^{\frac{1}{j+1}}$. (Note that there are at most j prime factors $> x^{\frac{1}{j+1}}$.) By Merten's estimates, we can see that all of the assumptions of Lemma 3.19 are satisfied and hence we have

$$\#S(A, P, z) \ll \frac{x \log y}{\log x}. \tag{3.106}$$

Therefore,

$$\#Q^{(j)}(x) := \#\{n \le x : n = p_1 \cdots p_j m_j, P^+(m_j) \le y < x^{\frac{1}{j+1}} < p_j < \cdots < p_1\} \ll \frac{x \log y}{\log x}. \tag{3.107}$$

For $1 \le i \le j-1$, denote by $Q^{(i)}(x)$ the set

$$Q^{(i)}(x) := \{n \le x : n = p_1 \cdots p_j m_j, P^+(m_j) \le y < p_j < \cdots < p_{i+1} \le x^{\frac{1}{j+1}} < p_i < \cdots < p_1\} \tag{3.108}$$

and by $Q^{(0)}(x)$ the set

$$Q^{(0)}(x) := \{n \le x : n = p_1 \cdots p_j m_j, P^+(m_j) \le y < p_j < \cdots < p_1 \le x^{\frac{1}{j+1}}\}. \tag{3.109}$$

For $1 \le i \le j-1$, we use the same kind of estimate of $S(A, P, z)$ with the same choices of parameters above, except this time we choose

$$X := \frac{x}{p_{i+1} \cdots p_j} \tag{3.110}$$

and A be the set of all natural numbers up to X, for some fixed choices of primes $p_{i+1},\dots p_j$.

$$\begin{aligned}
\#Q^{(i)}(x) &= \sum_{y<p_j<\dots<p_{i+1}\le x^{\frac{1}{j+1}}} \sum_{\substack{P^+(m_j)\le y \\ p_1>\dots>p_i>x^{\frac{1}{j+1}} \\ p_1\cdots p_i m_j\le x/(p_{i+1}\cdots p_j)}} 1 \\
&\ll \sum_{y<p_j<\dots<p_{i+1}\le x^{\frac{1}{j+1}}} \frac{x}{p_{i+1}\cdots p_j}\frac{\log y}{\log x} \\
&\le \frac{x\log y}{\log x}\Bigg(\sum_{p\le x^{\frac{1}{j+1}}}\frac{1}{p}\Bigg)^{j-i} \ll \frac{x\log y}{\log x}(\log\log x)^{j-i}. \qquad (3.111)
\end{aligned}$$

$$\begin{aligned}
\#Q^{(0)}(x) &= \sum_{y<p_j<\dots<p_1\le x^{\frac{1}{j+1}}} \sum_{\substack{P^+(m_j)\le y \\ m_j\le x/(p_1\cdots p_j)}} 1 \\
&\ll \sum_{y<p_j<\dots<p_1\le x^{\frac{1}{j+1}}} \frac{x}{p_1\cdots p_j}\exp\left(-\frac{\log(x/p_1\cdots p_j)}{2\log y}\right) \\
&\le \sum_{y<p_j<\dots<p_1\le x^{\frac{1}{j+1}}} \frac{x}{p_1\cdots p_j}\exp\left(-\frac{1}{2(j+1)}\frac{\log x}{\log y}\right) \\
&\le x\exp\left(-\frac{1}{2(j+1)}\frac{\log x}{\log y}\right)\Bigg(\sum_{p\le x^{\frac{1}{j+1}}}\frac{1}{p}\Bigg)^{j} \\
&\ll x(\log\log x)^j\exp\left(-\frac{1}{2(j+1)}\frac{\log x}{\log y}\right) \qquad (3.112)
\end{aligned}$$

As a result,

$$\#\Phi_j(x,y) = \sum_{i=0}^{j}\#Q^{(i)}(x) \ll \frac{x\log y}{\log x}(\log\log x)^{j-1}. \qquad (3.113)$$

This completes the proof of Lemma 3.20. □

We are now ready to give the proof of Theorem 1.8. We note that it is immaterial to choose $y = x^{1/\log\log x}$ in the proof of Theorem 1.7 (for the case $k \ge 4$). It is simply a usual, convenient choice as in [PoSh]. But at least we must have $y \ge (\log x)^\alpha$, $\alpha > 3k+2$ (refer to the estimation of $\#N_3^{(2)}(k;x)$ in Theorem 1.7). In Theorem 1.8 with the consideration of Lemma 3.20, it is the best to choose y of the form $(\log x)^\alpha$. We also note that for $j \ge 1$, $\#\Phi_j(x,y)$ decays much slower than $\#\Phi(x,y)$.

Proof of Theorem 1.8. By Lemma 3.11 and the proof of Theorem 1.7, the major contribution to $\#N(k;x)$ comes from numbers of the form $n = p_1m_1$ with $p_1 > y_1 := (\log x)^{3k+10}$ being a prime, $p_1 > P^+(m_1)$ and $m_1 \in N(\frac{k-1}{2})$. Then we use our new partition on $N(\frac{k-1}{2};\frac{x}{y})$ and repeat the similar estimations done in Theorem 1.7.

In general for $j \geq 1$, suppose we repeat this process for j times, we would like to show that

$$\#\Big\{n \leq x : n = p_1 \cdots p_j m_j, p_1 > \cdots > p_j > \max\{y_1, P^+(m_j)\}, m_j \in N_1\Big(\frac{k-(2^j-1)}{2^j}\Big) \cup N_2\Big(\frac{k-(2^j-1)}{2^j}\Big) \cup N_3^{(2)}\Big(\frac{k-(2^j-1)}{2^j}\Big)\Big\} \ll_k \frac{x}{\log x}(\log\log x)^j. \tag{3.114}$$

Firstly by Lemma 3.20, we have

$$\#\Big\{n \leq x : n = p_1 \cdots p_j m_j, p_1 > \cdots > p_j > y_1 \geq P^+(m_j), m_j \in N_1\Big(\frac{k-(2^j-1)}{2^j}\Big)\Big\} \leq \#\Phi^{(j)}(x, y_1) \ll \frac{x \log y_1}{\log x}(\log\log x)^{j-1} \ll_k \frac{x}{\log x}(\log\log x)^j. \tag{3.115}$$

Secondly, observe that

$$\sum_{\substack{m_j < \frac{x}{y_1^j} \\ P^+(m_j)^2 || m_j \\ P^+(m_j) > y_1}} \frac{1}{m_j} \leq \sum_{\substack{p_{j+1} > y_1 \\ p_{j+1}^2 r < \frac{x}{y_1^j}}} \frac{1}{p_{j+1}^2 r} = \sum_{p_{j+1} > y_1} \frac{1}{p_{j+1}^2} \sum_{r < \frac{x}{y_1^j p_{j+1}^2}} \frac{1}{r} \ll \sum_{p_{j+1} > y_1} \frac{1}{p_{j+1}^2} \log \frac{x}{y_1^j p_{j+1}^2} \ll \frac{1}{y_1} \log \frac{x}{y_1^{j+2}} \tag{3.116}$$

and

$$\sum_{\substack{\sqrt{x} < m_j \leq \frac{x}{p_1 \cdots p_j} \\ P^+(m_j)^2 | m_j \\ P^+(m_j) > y_1}} 1 \leq \sum_{\substack{p_{j+1}^2 r \leq \frac{x}{p_1 \cdots p_j} \\ p_{j+1} > y_1}} 1 \leq \sum_{p_{j+1} > y_1} \sum_{r \leq \frac{x}{p_1 \cdots p_j p_{j+1}^2}} 1 \leq \frac{x}{y_1} \frac{1}{p_1 \cdots p_j}. \tag{3.117}$$

By using (3.116), (3.117) and Lemma 3.18, we have

$$\begin{aligned}
&\#\Big\{n \leq x : n = p_1 \cdots p_j m_j, p_1 > \cdots > p_j > P^+(m_j) > y_1, m_j \in N_2\Big(\frac{k-(2^j-1)}{2^j}\Big)\Big\} \\
&\leq \#\Big\{n \leq x : n = p_1 \cdots p_j m_j, p_1 > \cdots > p_j > P^+(m_j) > y_1, P^+(m_j)^2 | m_j\Big\} \\
&\leq \sum_{\substack{m_j \leq \sqrt{x} \\ P^+(m_j)^2 || m_j \\ P^+(m_j) > y_1}} \sum_{\substack{p_1 > \cdots > p_j > y_1 \\ p_1 \cdots p_j \leq \frac{x}{m_j}}} 1 + \sum_{\substack{p_1 > \cdots > p_j > y_1 \\ p_1 \cdots p_j \leq \sqrt{x}}} \sum_{\substack{\sqrt{x} < m_j \leq \frac{x}{p_1 \cdots p_j} \\ P^+(m_j)^2 | m_j \\ P^+(m_j) > y_1}} 1 \\
&\ll \sum_{\substack{m_j < \sqrt{x} \\ P^+(m_j)^2 || m_j \\ P^+(m_j) > y_1}} \frac{x/m_j}{\log(x/m_j)} \Big(\log\log \frac{x}{m_j}\Big)^{j-1} + \frac{x}{y_1}\Big(\sum_{p \leq \sqrt{x}} \frac{1}{p}\Big)^j \\
&\ll \frac{x}{\log x}(\log\log x)^{j-1} \sum_{\substack{m_j < \frac{x}{y_1^j} \\ P^+(m_j)^2 || m_j \\ P^+(m_j) > y_1}} \frac{1}{m_j} + \frac{x}{y_1}(\log\log x)^j \\
&\ll \frac{x}{\log x}(\log\log x)^{j-1} \frac{1}{y_1} \log \frac{x}{y_1^{j+2}} + \frac{x}{y_1}(\log\log x)^j \ll \frac{x}{(\log x)^{3k+10}}(\log\log x)^j.
\end{aligned} \tag{3.118}$$

Denote by $M(k)$ the set of all natural numbers n with the properties that $n \in N(k)$, n can be written of the form $n = pm$ with $p > P^+(m)$ and there exists a set D_n consists of proper divisors of n such that

$$\sigma(m) - \sum_{d \in D_n^{(1)}} d > 0, \tag{3.119}$$

where $D_n^{(1)}$ is defined to be the set $\{d \in D_n : p \nmid d\}$ as before.
Denote by $M(k;x)$ the set of all elements of $M(k)$ up to x. The estimation of the size of $M(k;x)$ is very similar to that in Theorem 1.7. However note that here we take $y_1 = (\log x)^{3k+10}$,

$$u = \frac{\log x}{\log y_1} = \frac{\log x}{(3k+10)\log\log x} \tag{3.120}$$

and hence

$$\#\Phi(x, y_1) \ll x \exp\left(-\frac{1}{2}\frac{\log x}{(3k+10)\log\log x}\right) \ll_k \frac{x}{(\log x)^2}. \tag{3.121}$$

$$\#\{n \in M(k;x) : P^+(n) > y_1\} \ll_k \min\left\{\frac{x}{(\log x)^2}, \frac{x}{y_1}(\log x)^{3k+1}\right\} \ll \frac{x}{(\log x)^2}. \tag{3.122}$$

Therefore,

$$\#M(k;x) = \#\{n \in M(k;x) : P^+(n) \le y_1\} + \#\{n \in M(k;x) : P^+(n) > y_1\} \ll_k \frac{x}{(\log x)^2} \tag{3.123}$$

and by partial summation, we have

$$\sum_{n \in M(k)} \frac{1}{n} < \infty. \tag{3.124}$$

We have

$$\begin{aligned}
&\#\left\{n \le x : n = p_1 \cdots p_j m_j, p_1 > \cdots > p_j > P^+(m_j) > y_1, m_j \in N_3^{(2)}\left(\frac{k-(2^j-1)}{2^j}\right)\right\} \\
&\le \#\left\{n \le x : n = p_1 \cdots p_j m_j, p_1 > \cdots > p_j > P^+(m_j) > y_1, m_j \in M\left(\frac{k-(2^j-1)}{2^j}\right)\right\} \\
&\le \sum_{\substack{p_1 > \cdots > p_j > y_1 \\ p_1 \cdots p_j \le \sqrt{x}}} \sum_{\substack{m_j \le \frac{x}{p_1 \cdots p_j} \\ m_j \in M(\frac{k-(2^j-1)}{2^j})}} 1 + \sum_{\substack{m_j \le \sqrt{x} \\ m_j \in M(\frac{k-(2^j-1)}{2^j})}} \sum_{\substack{p_1 > \cdots > p_j > y_1 \\ p_1 \cdots p_j \le \frac{x}{m_j}}} 1 \\
&\ll_k \sum_{\substack{p_1 > \cdots > p_j > y_1 \\ p_1 \cdots p_j \le \sqrt{x}}} \frac{\frac{x}{p_1 \cdots p_j}}{(\log \frac{x}{p_1 \cdots p_j})^2} + \sum_{\substack{m_j \le \sqrt{x} \\ m_j \in M(\frac{k-(2^j-1)}{2^j})}} \frac{\frac{x}{m_j}}{\log \frac{x}{m_j}} \left(\log\log \frac{x}{m_j}\right)^{j-1} \\
&\ll \frac{x}{(\log x)^2} \sum_{\substack{p_1 > \cdots > p_j > y_1 \\ p_1 \cdots p_j \le \sqrt{x}}} \frac{1}{p_1 \cdots p_j} + \frac{x}{\log x}(\log\log x)^{j-1} \sum_{\substack{m_j \le \sqrt{x} \\ m_j \in M(\frac{k-(2^j-1)}{2^j})}} \frac{1}{m_j} \\
&\ll \frac{x}{(\log x)^2}\left(\sum_{p \le \sqrt{x}} \frac{1}{p}\right)^j + \frac{x}{\log x}(\log\log x)^{j-1} \ll \frac{x}{\log x}(\log\log x)^{j-1}.
\end{aligned} \tag{3.125}$$

By Lemma 3.11, we have

$$\begin{aligned}&\left\{n \le x : n = p_1 \cdots p_j m_j, p_1 > \cdots > p_j > P^+(m_j) > y_1, m_j \in N_3^{(1)}\left(\frac{k-(2^j-1)}{2^j}\right)\right\}\\ = &\left\{n \le x : n = p_1 \cdots p_j p_{j+1} m_{j+1}, p_1 > \cdots > p_j > p_{j+1} > \max\{y_1, P^+(m_{j+1})\},\right.\\ &\left. m_{j+1} \in N\left(\frac{k-(2^{j+1}-1)}{2^{j+1}}\right)\right\} \end{aligned} \tag{3.126}$$

and the process repeats. Pick the smallest integer $j_0 = j_0(k)$ such that

$$\frac{k-(2^{j_0}-1)}{2^{j_0}} < 4. \tag{3.127}$$

i.e.,

$$j_0 > \frac{\log(k+1)}{\log 2} - \frac{\log 5}{\log 2}. \tag{3.128}$$

By using partial summation and the upper bound (3.77) for $\#N(3;x)$, we have

$$\sum_{m \in N(3)} \frac{1}{m} < \infty. \tag{3.129}$$

Therefore, we have

$$\begin{aligned}&\#\left\{n \le x : n = p_1 \cdots p_{j_0} m_{j_0}, p_1 > \cdots > p_{j_0} > P^+(m_{j_0}), m_j \in N_3^{(1)}\left(\frac{k-(2^{j_0}-1)}{2^{j_0}}\right)\right\}\\ \le\ &\#\left\{n \le x : n = p_1 \cdots p_{j_0} m_{j_0}, p_1 > \cdots > p_{j_0} > P^+(m_{j_0}), m_{j_0} \in N(3)\right\}\\ \le\ &\sum_{\substack{m_{j_0} \le \sqrt{x}\\ m_{j_0} \in N(3)}} \sum_{\substack{p_1 > \cdots > p_{j_0}\\ p_1 \cdots p_{j_0} \le \frac{x}{m_{j_0}}}} 1 + \sum_{\substack{p_1 > \cdots > p_{j_0}\\ p_1 \cdots p_{j_0} \le \sqrt{x}}} \sum_{\substack{m_{j_0} \le \frac{x}{p_1 \cdots p_{j_0}}\\ m_{j_0} \in N(3)}} 1\\ \ll\ &\sum_{\substack{m_{j_0} \le \sqrt{x}\\ m_{j_0} \in N(3)}} \frac{\frac{x}{m_{j_0}}}{\log \frac{x}{m_{j_0}}}\left(\log\log \frac{x}{m_{j_0}}\right)^{j_0-1} + \sum_{\substack{p_1 > \cdots > p_{j_0}\\ p_1 \cdots p_{j_0} \le \sqrt{x}}} \frac{\frac{x}{p_1 \cdots p_{j_0}}}{(\log \frac{x}{p_1 \cdots p_{j_0}})^2}\\ \ll\ &\frac{x}{\log x}(\log\log x)^{j_0-1} \sum_{\substack{m_{j_0} \le \sqrt{x}\\ m_{j_0} \in N(3)}} \frac{1}{m_{j_0}} + \frac{x}{(\log x)^2} \sum_{\substack{p_1 > \cdots > p_{j_0}\\ p_1 \cdots p_{j_0} \le \sqrt{x}}} \frac{1}{p_1 \cdots p_{j_0}}\\ \ll\ &\frac{x}{\log x}(\log\log x)^{j_0-1}. \end{aligned} \tag{3.130}$$

Taking stock, we have

$$\#N(k;x) \ \ll_k\ \frac{x}{\log x}(\log\log x)^{j_0}. \tag{3.131}$$

We have not used Lemma 3.11 fully as we aim at obtaining an improved upper bound *for all* $k \ge 4$ conveniently and unconditionally while in the following we only handle *a large portion* of integers $k \ge 4$ and the treatment is more delicate. At the last step of the above process we only conclude that $m_{j_0} \in N(3)$ and use the fact $\sum_{m \in N(3)} \frac{1}{m} < \infty$. Indeed it is possible to obtain more information on m_j by using another inductive process as follows.

By the same kind of estimates we have done in the proof of Theorem 1.8, for $j \geq 2$, we have

$$\#\Big\{n \leq x : n = p_1 \cdots p_{j-1} m_{j-1}, p_1 > \cdots > p_{j-1} > \max\{y_1, P^+(m_{j-1})\},$$
$$m_{j-1} \in N_1(k-(2^{j-1}-1)\tau(m_{j-1})) \cup N_2(k-(2^{j-1}-1)\tau(m_{j-1})) \cup N_3^{(2)}(k-(2^{j-1}-1)\tau(m_{j-1}))\Big\}$$
$$\ll_k \frac{x}{\log x}(\log\log x)^{j-1} \tag{3.132}$$

and

$$\#\Big\{n \leq x : n = p_1 \cdots p_{j-1} m_{j-1}, p_1 > \cdots > p_{j-1} > P^+(m_{j-1}) > y_1, m_{j-1} \in N_3^{(1)}(k-(2^{j-1}-1)\tau(m_{j-1}))\Big\}$$
$$= \#\Big\{n \leq x : n = p_1 \cdots p_{j-1} p_j m_j, p_1 > \cdots > p_{j-1} > p_j > \max\{y_1, P^+(m_j)\},$$
$$m_j \in N(k-(2^j-1)\tau(m_j))\Big\} \tag{3.133}$$

and the process continues. However it is different from the situation of Theorem 1.8, now we are allowed us to solve out *finitely many* possible m_j such that $(2^j-1)\tau(m_j) \leq k$ and $m_j \in N(k-(2^j-1)\tau(m_j))$ for suitably chosen j. In this case, by Lemma 3.18, we have

$$\#N(k;x) \ \ll_k \ \frac{x}{\log x}(\log\log x)^{j-1}. \tag{3.134}$$

Moreover by Lemma 3.11, we have

$$\{n \leq x : n = m_j p_1 \cdots p_j, p_1 > \cdots > p_j > P^+(m_j), m_j \in N(k-(2^j-1)\tau(m_j))\} \subset N(k;x). \tag{3.135}$$

Therefore,

$$\liminf_{x\to\infty} \frac{\#N(k;x)}{\frac{x}{\log x}(\log\log x)^{j-1}} \ \geq \ \sum_{m_j \in N(k-(2^j-1)\tau(m_j))} \frac{1}{m_j}. \tag{3.136}$$

By Lemma 3.12, we have $\tau(m_j) \geq 4$. Also,

$$\tau(m_j) \ \leq \ \frac{k}{2^j-1}. \tag{3.137}$$

Therefore we have

$$j \ \leq \ \frac{\log(k+4)}{\log 2} - 2. \tag{3.138}$$

(1) We consider k is of the form $2^{s+2}+\ell$ for $\ell \geq -4$. For

$$s \ > \ s_0(\ell) \ := \ \frac{\log(\ell+6)}{\log 2} - 1, \tag{3.139}$$

we have

$$s \ < \ \frac{\log(k+4)}{\log 2} - 2 \ < \ s+1 \tag{3.140}$$

and hence we choose $j = s$. *For each* $\ell \geq -4$, we have not covered every single integer $s \geq 1$. (In fact it is even worse that $s_0(\ell) \to \infty$ as $\ell \to \infty$.)
For $\ell \geq -4$, define the following set:

$$T_\ell^{(1)} \ := \ \left\{2^{s+2}+\ell : s > \frac{\log(\ell+6)}{\log 2} - 1\right\}. \tag{3.141}$$

Part of the integers omitted by a single $T_\ell^{(1)}$ can be covered by the other $T_{\ell'}^{(1)}$, but the totality of $T_\ell^{(1)}$ ($\ell \geq -4$) still does *not* cover every single natural number k. We need more coverings of this type.

From (3.139), we have

$$4 \le \tau(m_s) \le \frac{2^{s+2}+\ell}{2^s-1} < 6. \tag{3.142}$$

Again by Lemma 3.12, $\tau(m_s) = 4$ and $m_s \in N(\ell+4)$. By Lemma 3.15, we have $m_s = 6$. Then we have

$$\#N(k;x) \ll_k \frac{x}{\log x}(\log\log x)^{s-1} \tag{3.143}$$

and

$$\liminf_{x\to\infty} \frac{\#N(k;x)}{\frac{x}{\log x}(\log\log x)^{s-1}} \ge \frac{1}{6(s-1)!} \tag{3.144}$$

if $k = 2^{s+2}+\ell$, $\ell \ge -4$ and $s > \frac{\log(\ell+6)}{\log 2} - 1$.

It is easy to see that the sets $T_\ell^{(1)}$'s are pairwise disjoint for $\ell \ge -4$ and

$$\bigcup_{\ell\ge -4} T_\ell^{(1)} = \bigcup_{r\ge 1}[4\cdot 2^r-4, 6\cdot 2^r-7]_{\mathbb{Z}}. \tag{3.145}$$

(2) For k of the form $2^{s+2}-\ell$ with $\ell > 8$ and $s \ge \frac{\log(\ell-4)}{\log 2} - 1$, we choose $j = s-1$. Then

$$4 \le \tau(m_{s-1}) \le \frac{2^{s+2}-\ell}{2^{s-1}-1} = 8 - \frac{\ell-8}{2^{s-1}-1} < 8. \tag{3.146}$$

By Lemma 3.12, $\tau(m_{s-1}) = 4$ or 6. If $\tau(m_{s-1}) = 4$, then $m_{s-1} = 6$. Now suppose $\tau(m_{s-1}) = 6$. By Lemma 3.15, it suffices to consider $m_{s-1} \in N(1)$ and hence $m_{s-1} \in N(\min\{1, 2^s-\ell+6\})$.

We consider s in the range

$$s \ge \frac{\log(\ell-5)}{\log 2} \tag{3.147}$$

so that

$$\min\{1, 2^s-\ell+6\} = 1. \tag{3.148}$$

In this case, $m_{s-1} \in \{6, 12, 18, 20, 28\}$. Hence, we have

$$\#N(k;x) \ll_k \frac{x}{\log x}(\log\log x)^{s-2}. \tag{3.149}$$

Moreover,

$$\liminf_{x\to\infty} \frac{\#N(k;x)}{\frac{x}{\log x}(\log\log x)^{s-2}} \ge \frac{493}{1260(s-2)!}. \tag{3.150}$$

The sets

$$T_\ell^{(2)} := \left\{2^{s+2}-\ell : s \ge \frac{\log(\ell-5)}{\log 2}\right\} \tag{3.151}$$

are pairwise disjoint for $\ell > 8$ and

$$\bigcup_{\ell>8} T_\ell^{(2)} = \bigcup_{r\ge 2}[3\cdot 2^r-5, 4\cdot 2^r-9]_{\mathbb{Z}}. \tag{3.152}$$

On the other hand, if $2^s-\ell+6 = 0$ and $\tau(m_{s-1}) = 6$, then $m_{s-1} \in N(0)$ and $m_{s-1} = 28$. Hence for $k = 3\cdot 2^s - 6$, we have

$$\#N(k;x) \ll_k \frac{x}{\log x}(\log\log x)^{s-2} \tag{3.153}$$

and

$$\liminf_{x\to\infty} \frac{\#N(k;x)}{\frac{x}{\log x}(\log\log x)^{s-2}} \;\geq\; \frac{17}{84(s-2)!}. \tag{3.154}$$

(3) For k of the form $2^{s+2}-8$ and $s \geq 2$ (i.e., $\ell = 8$), we have $4 \leq \tau(m_{s-1}) \leq 8$ and $m_{s-1} \in N(2^{s+2}-8-(2^{s-1}-1)\tau(m_{s-1}))$. This is settled as in the case $k = 8$ of Theorem 1.7. Therefore,

$$\#N(k;x) \;\ll_k\; \frac{x}{\log x}(\log\log x)^{s-2} \tag{3.155}$$

and

$$\liminf_{x\to\infty} \frac{\#N(k;x)}{\frac{x}{\log x}(\log\log x)^{s-2}} \;\geq\; \frac{493}{1260(s-2)!}. \tag{3.156}$$

(4) For k of the form $2^{s+2}-7$ and $s \geq 3$ (i.e., $\ell = 7$), we have $4 \leq \tau(m_{s-1}) \leq 8$ and $m_{s-1} \in N(2^{s+2}-7-(2^{s-1}-1)\tau(m_{s-1}))$. This is settled as in the case $k = 9$ of Theorem 1.7. Therefore,

$$\#N(k;x) \;\ll_k\; \frac{x}{\log x}(\log\log x)^{s-2}. \tag{3.157}$$

and

$$\liminf_{x\to\infty} \frac{\#N(k;x)}{\frac{x}{\log x}(\log\log x)^{s-2}} \;\geq\; \frac{179017}{360360(s-2)!}. \tag{3.158}$$

Similar to above define

$$\begin{aligned} T^{(3)} &:= \{3\cdot 2^s - 6 : s \geq 2\}, \\ T^{(4)} &:= \{2^{s+2} - 8 : s \geq 2\}, \\ T^{(5)} &:= \{2^{s+2} - 7 : s \geq 3\}, \\ T^{(6)} &:= \{2^{s+2} - 6 : s \geq 3\}, \\ T^{(7)} &:= \{2^{s+2} - 5 : s \geq 4\}. \end{aligned} \tag{3.159}$$

We have $T_\ell^{(1)}, T_\ell^{(2)}, T^{(3)}, \ldots, T^{(7)}$ all pairwise disjoint and

$$\bigcup_{\ell\geq -4} T_\ell^{(1)} \cup \bigcup_{\ell>8} T_\ell^{(2)} \cup \bigcup_{3\leq i\leq 5} T^{(i)} = [4,\infty)_{\mathbb{Z}} \setminus \left(\{9,10,11,27\} \cup \bigcup_{6\leq i\leq 7} T^{(i)}\right). \tag{3.160}$$

This completes the proof of Theorem 1.8. □

Remark 3.21. *For k of the form $2^{s+2}-6$ ($s \geq 3$) and $2^{s+2}-5$ ($s \geq 4$), we have*

$$\#N(k;x) \;\ll_k\; \frac{x}{\log x}(\log\log x)^{s-2}, \tag{3.161}$$

$$\liminf_{x\to\infty} \frac{\#N(k;x)}{\frac{x}{\log x}(\log\log x)^{s-2}} \;\geq\; \frac{179017}{360360}\frac{1}{(s-2)!}, \tag{3.162}$$

provided that $\{m \in N(3) : \tau(m) = 8\}$ is a finite set.

Remark 3.22. *Heuristically, one expects that natural numbers of the form* $n = p_1 \cdots p_{j-1} p_j m_j$ *with* $p_1 > \cdots > p_{j-1} > p_j > P^+(m_j)$ *and* $m_j \in N(k-(2^j-1)\tau(m_j))$ *(*) (with the choice of* j *made in the proof of Theorem 1.8) contributes the most to* $\#N(k;x)$ *(*$k \geq 4$*). This would lead to asymptotic formulae of* $\#N(k;x)$*.*
However we fail to do so. The sizes of the sets $\Phi_{j-1}(x,y)$ *(*$j \geq 2$*) are much larger than that of* $\Phi(x,y)$ *and it is already the best that we choose* y *of the form* $(\log x)^\alpha$*. Even so, the size of upper bound of* $\#\Phi_{j-1}(x,y)$ *is the same as that given by (*). Therefore, we fail to locate exactly the major contributions of* $\#N(k;x)$*. A possible solution for this is to repeat our process by one time less. At the same time the computations would become more tedious.*

Remark 3.23. *Our method of studying near-perfectness can be carried over to exact-perfectness for some special cases. We state our results here without proof.*

Theorem 3.24. *Denote by* $E(k)$ *the set of all* k*-exactly-perfect numbers. Let* $E(k;x) = E(k) \cap [1,x]$*. Then as* $x \to \infty$*,*

$$\#E(k;x) \sim c_k \frac{x}{\log x}, \tag{3.163}$$

where

$$c_4 = \frac{1}{6},\ c_6 = \frac{1}{28},\ c_7 = \frac{17}{90},\ c_8 = \frac{5}{36},\ c_9 = \frac{12673}{120120}. \tag{3.164}$$

Moreover, we have

$$\#E(5;x) \ll x \exp\left(-\frac{1}{2}\sqrt{\log x \log\log x}\left(1 + O\left(\frac{\log\log\log x}{\log\log x}\right)\right)\right), \tag{3.165}$$

$$\#E(3\cdot 2^s - 5\,;x),\ \#E(3\cdot 2^s - 6\,;x),\ \#E(2^{s+2} - 7\,;x) \asymp_s \frac{x}{\log x}(\log\log x)^{s-2}, \tag{3.166}$$

$$\#E(2^{s+2} - 4\,;x) \asymp_s \frac{x}{\log x}(\log\log x)^{s-1}. \tag{3.167}$$

We suggest to investigate the distribution of exact-perfect numbers further.
Note that $E(k_1)$ *and* $E(k_2)$ *are not necessarily disjoint. For example:* $12, 18 \in E(1) \cap E(2)$*. Hence, we also suggest investigating the size of* $E_{k_1,k_2}(x) := E(k_1) \cap E(k_2) \cap [1,x]$*. Table 2 compares values of* $E_{1,2}(x)$*,* $E_1(x)$*, and* $E_2(x)$ *for* x *up to* 10^6*.*

x	$E_{1,2}(x)$	$E_1(x)$	$E_2(x)$	$E_{1,2}(x)/E_1(x)$	$E_{1,2}(x)/E_2(x)$
10^2	5	7	14	0.714	0.357
10^3	6	15	48	0.400	0.125
10^4	8	21	143	0.381	0.056
10^5	9	33	301	0.272	0.030
10^6	11	45	571	0.244	0.019

TABLE 2. Comparison of values of $E_{1,2}(x)$, $E_1(x)$, and $E_2(x)$ for x up to 10^6.

3.3. **Concluding Remark.**

Proof of Theorem 1.9. For $k \in M$, $k = 2q$ for some prime q such that $2^q - 1$ is also a prime. Let $\epsilon \in (0,1)$ and $m = 2^{q-1}(2^q - 1)$. Since m is a perfect number, m is the sum of its proper divisors. The number of proper divisors of m is $\tau(m) - 1 = 2q - 1$. Hence, pm is a sum of $2q-1$ of its proper divisors. The number of proper divisors of pm is $\tau(pm) - 1 = 4q - 1$. So, pm is a sum of all of its proper divisors with exactly $(4q-1) - (2q-1) = 2q$ exceptions, i.e., $pm \in E(k)$. Clearly $\sigma(pm) - 2pm < (pm)^\epsilon$ if $p > (2m^{1-\epsilon})^{1/\epsilon}$ and $p \nmid m$. This proves

$$\liminf_{x\to\infty} \frac{\#(E(k;x) \setminus E_\epsilon(k;x))}{x/\log x} \geq \frac{1}{m}. \tag{3.168}$$

Now suppose $\epsilon \in (0, 1/3)$. By the same argument as in Theorem 1.3, we have

$$\#(E(k;x) \setminus E_\epsilon(k;x)) \ \leq \ \#\{n \leq x : n \in E(k), n = pm', p \nmid m', \sigma(m') = 2m'\} + O(x^{2/3+\epsilon+o(1)}). \tag{3.169}$$

For $n \in E(k)$ with $n = pm'$, $p \nmid m'$ and $\sigma(m') = 2m'$, we have

$$pm' \ = \sum_{d_1 \in D_1} d_1 + p \sum_{d_2 \in D_2} d_2, \tag{3.170}$$

where D_1 is a subset of positive divisors of m', D_2 is a subset of proper divisors of m' with $\#D_1 + \#D_2 = \tau(pm') - 1 - k = 2\tau(m') - 1 - k$.

Suppose that $D_1 \neq \emptyset$. Then

$$1 \leq \sum_{d_1 \in D_1} d_1 \leq \sigma(m') = 2m'. \tag{3.171}$$

Reducing (3.170) modulo p, we have

$$p \ \Big| \ \sum_{d_1 \in D_1} d_1. \tag{3.172}$$

The number of possible values for p is $O(\log 2m') = O(\log x)$. Hence, the number of possible values for such n is $O(x^{o(1)} \log x)$ by Hornfeck-Wirsing Theorem, which is negligible.

Now suppose that $D_1 = \emptyset$. Then $\#D_2 = 2\tau(m') - 1 - k$ and

$$m' \ = \sum_{d_2 \in D_2} d_2. \tag{3.173}$$

Since $\sigma(m') = 2m'$, we have $\#D_2 = \tau(m') - 1$. Therefore, $\tau(m') - 1 = 2\tau(m') - 1 - k$, i.e., $\tau(m') = k$. Nielsen [Ni] has recently shown that an odd perfect number has at least 10 prime factors and hence it has at least 1024 distinct positive divisors. Hence, assume $k < 1024$ or there is no odd perfect number. We have $m' = 2^{q'-1}(2^{q'} - 1)$ for some prime q' such that $2^{q'} - 1$ is also prime, by using Euclid-Euler Theorem. So $k = \tau(m') = 2q' \in M$. Hence if $k \notin M$, then we have a contradiction and

$$\#(E(k;x) \setminus E_\epsilon(k;x)) \ = \ O(x^{o(1)} \log x) + O(x^{2/3+\epsilon+o(1)}) = O(x^{2/3+\epsilon+o(1)}). \tag{3.174}$$

If $k \in M$, then $k = 2q$ for some prime q such that $2^q - 1$ is also a prime. Then $q' = q$ and so $m' = m$. Hence,

$$\#(E(k;x) \setminus E_\epsilon(k;x)) \ \leq \ \#\{n \leq x : n = pm,\ p \nmid m\} + O(x^{o(1)} \log x) + O(x^{2/3+\epsilon+o(1)}). \tag{3.175}$$

By the Prime Number Theorem, we have

$$\limsup_{x\to\infty} \frac{\#(E(k;x) \setminus E_\epsilon(k;x))}{x/\log x} \ \leq \ \frac{1}{m}. \tag{3.176}$$

As a result,

$$\lim_{x\to\infty} \frac{\#(E(k;x)\setminus E_\epsilon(k;x))}{x/\log x} \;=\; \frac{1}{m}. \tag{3.177}$$

It was shown in [PoSh], by using a form of prime number theorem of Drmota, Mauduit and Rivat, that for all large k, the number of k-exactly-perfect numbers up to x is $\gg_k x/\log x$.

Therefore,

$$\frac{\#(E(k;x)\setminus E_\epsilon(k;x))}{\#E(k;x)} \ll_k \frac{\log x}{x^{1/3-\epsilon-o(1)}} \tag{3.178}$$

for large $k \notin M$, $\epsilon \in (0,1/3)$ and with the assumption that there is no odd perfect number. In this case,

$$\lim_{x\to\infty} \frac{\#E_\epsilon(k;x)}{\#E(k;x)} \;=\; 1. \tag{3.179}$$

For $k = 8$, $2^{s+2}-4$ $(2 \le s \le 8)$, $3\cdot 2^s - 5$ $(2 \le s \le 8)$, $3 \cdot 2^s - 6$ $(3 \le s \le 8)$, $2^{s+2} - 7$ $(2 \le s \le 8)$, by the above argument and Theorem 3.24, we have unconditionally that (3.179) holds.
For $k = 4, 6$, we have unconditionally that

$$\lim_{x\to\infty} \frac{\#E_\epsilon(k;x)}{\#E(k;x)} \;=\; 0. \tag{3.180}$$

□

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