ON SOME CLASS OF SUMS

ANDREI K. SVININ

ABSTRACT. We consider some class of the sums which naturally include the sums of powers of integers. We suggest a number of conjectures concerning a representation of these sums.

1. Introduction

It is common of knowledge that the sum of powers of integers

$$S_m(n) := \sum_{q=1}^{n} q^m \tag{1.1}$$

is a polynomial $\sigma_m(n)$ in n of degree $m+1$. This proposition can be showed by using Pascal's elementary proof (see, for example [1]). Polynomials $\sigma_m(n)$ for any integer $m \geq 1$ are defined in terms of the Bernoulli numbers:

$$\sigma_m(n) = \frac{1}{m+1}\sum_{q=0}^{m}(-1)^q\binom{m+1}{q}B_q n^{m-q+1}.$$

The numbers B_k, in turn, are defined by a recursion

$$\sum_{q=0}^{k}\binom{k+1}{q}B_q = 0, \quad B_0 = 1.$$

They were discovered by Bernoulli in 1713 but as follows from [4], they were known by Faulhaber before this time.

Faulhaber's theorem says that for any odd $m \geq 3$, sum (1.1) is expressed as a polynomial of $S_1(n)$. It is common of knowledge that $S_1(n) = n(n+1)/2$. Let $t := n(n+1)$. Then Faulhaber's theorem asserts that for any $k \geq 1$, $S_{2k+1}(n)$ can be expressed as a polynomial in $\mathbb{Q}[t]$. These polynomials are referred as Faulhaber's ones. Jacobi showed in [10] that $S_{2k}(n)$ is expressed as a polynomial in $(2n+1)\mathbb{Q}[t]$.

There exists a variety of different modifications and generalizations of Faulhaber's theorem in the literature. Faulhaber itself considered r-fold sums $S_m^r(n)$ which are successively defined by

$$S_m^r(n) = \sum_{q=1}^{n} S_m^{r-1}(q), \quad r \geq 1$$

beginning from $S_m^0(n) := n^m$. He observed that $S_m^r(n)$ can be expressed as a polynomial in $t := n(n+r)$ if $m-r$ is even [11].

Date: October 19, 2016.

In this paper we consider some class of sums $S_{k,j}(n)$ which include classical sums of powers of integers (1.1) with odd $m = 2k+1$. These sums will be introduced in section 2 and 3. Particular case, namely $S_{k,j}(1)$, presents some class of binomial sums studied in [15]. Our crucial idea comes from this work. We briefly formulate the results of this article in section 4. In section 5, we formulate our main conjecture concerning a representation of the sums $S_{k,j}(n)$ in terms of some polynomials whose coefficients rationally depend on n. In section 6, we discuss the relationship of this representation with a possible generalization of Faulhaber's theorem. In section 7, we show explicit form of some n-dependent coefficients of these polynomials and suppose the relationship of these coefficients to the number of q-points in simple symmetric $(2k+1)$-Venn diagram.

2. The sums $s_{k,j}(n)$

Let us define the numbers $\{C_{j,r}(n) : j \geq 1, r = 0, \ldots, j(n-1)\}$ in the following way. Namely, let

$$\left(\sum_{q=0}^{n-1} a^{q+1}\right)^j = \sum_{q=0}^{j(n-1)} C_{j,q}(n) a^{q+j}, \tag{2.1}$$

where a is an arbitrary auxiliary nonzero positive number, taking into account that $a^l a^m = a^{l+m}$. For example, in the case $n = 1$, we have only $C_{j,0}(n) = 1$. Clearly, in the case $n = 2$, we get

$$\left(a + a^2\right)^j = \sum_{q=0}^{j} \binom{j}{q} a^{j+q},$$

that is, $C_{j,q}(2)$ is a binomial coefficient. From

$$\begin{aligned}\left(\sum_{q=0}^{n-1} a^{q+1}\right)^j &= \left(\sum_{q=0}^{(j-1)(n-1)} C_{j-1,q}(n) a^{q+j-1}\right)\left(\sum_{q=0}^{n-1} a^{q+1}\right) \\ &= C_{j-1,0}(n)a^j + \left(C_{j-1,0}(n) + C_{j-1,1}(n)\right) a^{j+1} + \cdots + \left(C_{j-1,0}(n) + \cdots\right. \\ &\quad \left.+ C_{j-1,n-1}(n)\right) a^{j+n-1} \\ &\quad + \left(C_{j-1,1}(n) + \cdots + C_{j-1,n)}(n)\right) a^{j+n} + \cdots + C_{j-1,(j-1)(n-1)}(n) a^{jn}\end{aligned}$$

we deduce that

$$C_{j,r}(n) = \sum_{q=r-n+1}^{r} C_{j-1,q}(n), \tag{2.2}$$

assuming that $C_{j-1,q}(n) = 0$ for $q < 0$ and $q > (j-1)(n-1)$. In the case $n = 2$, relation (2.2) becomes well known property for binomial coefficients:

$$\binom{j}{r} = \binom{j-1}{r-1} + \binom{j-1}{r}.$$

Clearly, the coefficient $C_{j,r}(n)$ can be presented as a proper sum of multinomial coefficients. Putting $a = 1$ into (2.1), we get the property

$$\sum_{q=0}^{j(n-1)} C_{j,q}(n) = n^j$$

for these numbers which evidently generalize well-known property of binomial coefficients. Also it is easy to get

$$\sum_{q=0}^{j(n-1)} qC_{j,q}(n) = \frac{j(n-1)}{2} n^j.$$

With the coefficients $C_{j,q}(n)$ we define the sum $s_{k,j}(n)$ as

$$s_{k,j}(n) := \sum_{q=0}^{j(n-1)} C_{j,q}(n) x_{j+q},$$

where $x_r := r^{2k+1}$. Remark that $s_{k,1}(n) = S_{2k+1}(n)$. For example, it is evident that

$$C_{2,q}(n) = \begin{cases} q+1, & q = 0, \ldots, n-2, \\ 2n-q-1, & q = n-1, \ldots, 2n-2. \end{cases}$$

Then

$$\begin{aligned} s_{k,2}(n) &:= \sum_{q=0}^{2n-2} C_{2,q}(n) x_{q+2} \\ &= \sum_{q=0}^{n-2} (q+1) x_{q+2} + \sum_{q=n-1}^{2n-2} (2n-q-1) x_{q+2} \end{aligned}$$

Shifting $q \to q-2$, we get

$$s_{k,2}(n) = \sum_{q=1}^{n} (q-1) x_q + \sum_{q=n+1}^{2n} (2n-q+1) x_q.$$

Since $q - 1 = (2n - q + 1) - (2n - 2q + 2)$, then

$$s_{k,2}(n) = -\sum_{q=1}^{n} (2n-2q+2) x_q + \sum_{q=1}^{2n} (2n-q+1) x_q. \tag{2.3}$$

3. The sums $S_{k,j}(n)$ and $\tilde{S}_{k,j}(n)$

Let

$$\tilde{S}_{k,j}(n) := \sum_{\{\lambda\} \in B_{j,jn}} \left\{ \lambda_1^{2k+1} + (\lambda_2 - n)^{2k+1} + \cdots + (\lambda_j - jn + n)^{2k+1} \right\}$$

with $B_{j,jn} := \{\lambda_k : 1 \le \lambda_1 \le \cdots \le \lambda_j \le jn\}$. Let us define

$$S_{k,j}(n) = \sum_{q=0}^{j-1} \binom{j(n+1)}{q} s_{k,j-q}(n). \tag{3.1}$$

It is obvious that

$$\tilde{S}_{k,1}(n) = S_{k,1}(n) = s_{k,1}(n) = S_{2k+1}(n).$$

Conjecture 3.1.

$$\tilde{S}_{k,j}(n) = S_{k,j}(n). \tag{3.2}$$

Using simple arguments we deduce that in the case $j = 2$

$$\begin{aligned} \tilde{S}_{k,2}(n) &= \sum_{\{\lambda\}\in B_{2,2n}} \left\{\lambda_1^{2k+1} + (\lambda_2 - n)^{2k+1}\right\} \\ &= \sum_{q=1}^{2n}(2n-q+1)x_q + \sum_{q=1}^{2n}(2n-q+1)x_{n-q+1}. \end{aligned} \tag{3.3}$$

Notice that we have several x_r's with negative subscript r in (3.3). Clearly, we must put $x_r = -x_{-r}$. Taking into account this rule, we rewrite the second sum in (3.3) as

$$\sum_{q=1}^{n}(n+q)x_q - \sum_{q=1}^{n}(n-q)x_q = 2\sum_{q=1}^{n} qx_q.$$

Adding to (3.3)

$$-\sum_{q=1}^{n}(2n-2q+2)x_q + \sum_{q=1}^{n}(2n-2q+2)x_q$$

and taking into account (2.3), we finally get

$$\begin{aligned} \tilde{S}_{k,2}(n) &= s_{k,2}(n) + (2n+2)\sum_{q=1}^{n} x_q \\ &= s_{k,2}(n) + \binom{2(n+1)}{1} s_{k,1}(n) \\ &= S_{k,2}(n). \end{aligned}$$

4. Special case of the sums (3.1)

In the case $n = 1$, the sum (3.1) becomes

$$S_{k,j}(1) = \sum_{q=0}^{j-1}\binom{2j}{q}(j-q)^{2k+1}. \tag{4.1}$$

This type of the sums was studied in [2], [14], [15]. More exactly, the authors of these works considered binomial sums of the form

$$\mathcal{S}_m(j) = \sum_{q=0}^{2j}\binom{2j}{q}|j-q|^m.$$

It is evident that $S_{k,j}(1) = \mathcal{S}_{2k+1}(j)/2$.

Some bibliographical remarks are as follows. Bruckman in [2] asked to prove that $\mathcal{S}_3(j) = j^2\left(\begin{array}{c}2j\\ j\end{array}\right)$. Strazdins in [14] solved this problem and conjectured that $\mathcal{S}_{2k+1}(j) = \tilde{P}_k(x)|_{x=j}\binom{2j}{j}$ with some monic polynomial $\tilde{P}_k(x)$ for any $k \geq 0$. Tuenter showed in [15] that it is almost true. More exactly, he proved that there exists a sequence of polynomials $P_k(x)$ sush that

$$\mathcal{S}_{2k+1}(j) = P_k(x)|_{x=j} j\binom{2j}{j} = P_k(x)|_{x=j} \frac{(2j)!}{(j-1)!j!}.$$

One can see that polynomial $\tilde{P}_k(x)$ is monic only for $k = 0, 1$. Next, we will describe these polynomials. The following proposition [15] is verified by direct computations.

Proposition 4.1. *The sums $S_{k,j}(1)$ enjoy recurrence relation*

$$S_{k,j}(1) = j^2 S_{k-1,j}(1) - 2j(2j-1)S_{k-1,j-1}(1). \tag{4.2}$$

Remark 4.2. It was in fact proved for the sums $\mathcal{S}_{2k+1}(j)$ in [15].

In what follows we introduce the sequence of positive integer numbers:

$$g_1 = 1, \quad g_j = \frac{j}{(j-1)!}\prod_{q=1}^{j-1}(j+q), \quad j \geq 2.$$

It should be noted that the sequence $\{g_j : j \geq 1\}$ satisfy the recurrent relation

$$g_{j+1} = 2\frac{2j+1}{j} g_j. \tag{4.3}$$

The numbers g_j are given in the following table:

j	1	2	3	4	5	6	7	8	9
g_j	1	6	30	140	630	2772	12012	51480	218790

They constitute A002457 integer sequence in [13].

Let us write

$$S_{k,j}(1) = P_{k,j} g_j$$

with some numbers $P_{k,j}$ to be calculated. Taking into account (4.3), it can be easily seen that (4.2) is valid if the relation

$$P_{k+1,j} = j^2 \left(P_{k,j} - P_{k,j-1}\right) + jP_{k,j-1} \tag{4.4}$$

does.

Now we need in polynomials studied in [15] which are defined by a recurrent relation

$$P_{k+1}(x) = x^2 \left(P_k(x) - P_k(x-1)\right) + xP_k(x-1) \tag{4.5}$$

with initial condition $P_0(x) = 1$. The first eight polynomials yielded by (4.5) are as follows.

$$P_0(x) = 1 \quad P_1(x) = x, \quad P_2(x) = x(2x-1),$$

$$P_3(x) = x(6x^2 - 8x + 3),$$

$$P_4(x) = x(24x^3 - 60x^2 + 54x - 17),$$

$$P_5(x) = x(120x^4 - 480x^3 + 762x^2 - 556x + 155),$$

$$P_6(x) = x(720x^5 - 4200x^4 + 10248x^3 - 12840x^2 + 8146x - 2073).$$

$$\begin{aligned} P_7(x) &= x\left(5040x^6 - 40320x^5 + 139440x^4 - 263040x^3\right. \\ &\quad \left. +282078x^2 - 161424x + 38227\right). \end{aligned}$$

Introducing a recursion operator $R := x^2\left(1-\Lambda^{-1}\right)+x\Lambda^{-1}$, where Λ is a shift operator acting as $\Lambda(f(x)) = f(x+1)$, one can write $P_k(x) = R^k(1)$. One could notice that the polynomial $P_k(x)$ has $k!$ as a coefficient at x^k. In addition, as was noticed in [15], the constant terms of the polynomials $P_k(x)/x$ constitute a sequence of the Genocchi numbers with opposite sign, that is,

$$-G_2 = 1,\quad -G_4 = -1,\quad -G_6 = 3,\quad -G_8 = -17,\quad -G_{10} = 155,\quad -G_{12} = -2073, \ldots$$

Recall that the Genocchi numbers are defined with the help of generating function

$$\frac{2x}{e^x+1} = \sum_{q\geq 1} G_q \frac{x^q}{q!}$$

and are related to the Bernoulli numbers as $G_{2k} = 2\left(1-2^{2k}\right)B_{2k}$.

Comparing (4.5) with (4.4) we conclude that $P_{k,j} = P_k(x)|_{x=j}$.

Remark 4.3. As was noticed in [15], polynomials $P_k(x)$ can be obtained as a special case of Dumont-Foata polynomials of three variables [3].

5. General case

Let $t := n(n+1)$. Our main conjecture is as follows.

Conjecture 5.1. *There exists some polynomials $P_k(t,x)$ in x of degree k whose coefficients rationally depend on t such that*

$$S_{k,j}(n) = \frac{t^{k+1}}{2^{k+1}} P_k(j) g_j(n),$$

where

$$g_1(n) = 1,\quad g_j(n) := \frac{j}{(j-1)!}\prod_{q=1}^{j-1}(jn+q).$$

The first eight polynomials $P_k(t,x)$ are

$$P_0(t,x) = 1,\quad P_1(t,x) = x,\quad P_2(t,x) = x\left(2x - \frac{2(t+1)}{3t}\right),$$

$$P_3(t,x) = x\left(6x^2 - \frac{16(t+1)}{3t}x + \frac{4(t+1)^2}{3t^2}\right),$$

$$P_4(t,x) = x\left(24x^3 - \frac{40(t+1)}{t}x^2 + \frac{24(t+1)^2}{t^2}x - \frac{24(t+1)^3+8t^2}{5t^3}\right),$$

$$\begin{aligned} P_5(t,x) &= x\left(120x^4 - \frac{320(t+1)}{t}x^3 + \frac{1016(t+1)^2}{3t^2}x^2\right.\\ &\left.\quad - \frac{160(t+1)^3+32t^2}{t^3}x + \frac{80\left((t+1)^4+t^2(t+1)\right)}{3t^4}\right), \end{aligned}$$

$$\begin{aligned}
P_6(t,x) &= x\left(720x^5 - \frac{2800(t+1)}{t}x^4 + \frac{13664(t+1)^2}{3t^2}x^3\right.\\
&\quad - \frac{55936(t+1)^3 + 7632t^2}{15t^3}x^2 + \frac{22112(t+1)^4 + 13664t^2(t+1)}{15t^4}x\\
&\quad \left. - \frac{22112(t+1)^5 + 44224t^2(t+1)^2}{105t^5}\right),
\end{aligned}$$

$$\begin{aligned}
P_7(t,x) &= x\left(5040x^6 - \frac{26880(t+1)}{t}x^5 + \frac{185920(t+1)^2}{3t^2}x^4\right.\\
&\quad - \frac{76800(t+1)^3 + 7680t^2}{t^3}x^3\\
&\quad + \frac{157088(t+1)^4 + 67968t^2(t+1)}{3t^4}x^2\\
&\quad - \frac{17920(t+1)^5 + 22528t^2(t+1)^2}{t^5}x\\
&\quad \left. + \frac{6720(t+1)^6 + 22400t^2(t+1)^3 + 1344t^4}{3t^6}\right).
\end{aligned}$$

Unfortunately, in general case we do not know a recursion relation for these polynomials except for the case $t = 2$. One can check that $P_k(t,x)|_{t=2} = P_k(x)$, where $P_k(x)$ are Tuenter's polynomials introduced above.

It could be noticed that coefficients of the polynomials $P_k(t,x)$ have a special form. Namely, let

$$P_k(t,x) = x\left(p_{k,0}(t)x^{k-1} - p_{k,1}(t)x^{k-2} + \cdots + (-1)^{k-1}p_{k,k-1}(t)\right). \tag{5.1}$$

Based on actual calculations, it can be supposed the following.

Conjecture 5.2. *The coefficients of t-dependent polynomial (5.1) are given by*

$$p_{k,j}(t) = \frac{r_{k,j}(t)}{t^j},$$

where the polynomials $r_{k,j}(t)$ are of the form

$$r_{k,j}(t) = \sum_{q=0}^{m} \alpha_{k,j,q} t^{2q}(t+1)^{j-3q}$$

with rational positive nonzero numbers $\alpha_{k,j,q}$. Here $m \geq 0$, by definition, is the result of division of the number j by 3 with some remainder l, that is, $j = 3m + l$.

It should be noticed that the last conjecture is quite strong. With this conjecture the number N_k of parameters which entirely define polynomial $P_k(t,x)$ is presented in the following table:

k	1	2	3	4	5	6	7	8	9
N_k	1	2	3	5	7	9	12	15	18

It is the A001840 integer sequence in [13]. We have in fact verified conjecture 5.2 up to $k = 11$ and calculated all corresponding coefficients $\alpha_{k,j,q}$.

It is evident that for all polynomials $P_k(t,x)$ we have $p_{k,0}(t) = k!$. Actual calculations show that

$$\alpha_{k,1,0} = \frac{(k-1)(k+1)}{9}k!, \quad k \geq 2,$$
$$\alpha_{k,2,0} = \frac{(k-2)(k+1)(5k^2+k-3)}{810}k!, \quad k \geq 3,$$
$$\alpha_{k,3,0} = \frac{(k-3)(k+1)(175k^4-70k^3-724k^2+643k-690)}{765450}k!, \quad k \geq 4,$$
$$\alpha_{k,3,1} = \frac{(k-3)(k+1)(2k^2-4k+5)}{1575}k!, \quad k \geq 4, \ldots$$

Looking at these patterns it can be supposed that

$$\alpha_{k,j,0} = (k-j)(k+1)p_j(k)k!, \quad k \geq j+1$$

with some polynomial $p_j(k)$ of degree $2j-2$.

6. Faulhaber's theorem

With the polynomials $P_k(t,x)$ we are able, for example, to calculate

$$S_{0,j}(n) = \frac{t}{2}\frac{j}{(j-1)!}\prod_{q=1}^{j-1}(jn+q), \quad S_{1,j}(n) = \frac{t^2}{4}\frac{j^2}{(j-1)!}\prod_{q=1}^{j-1}(jn+q),$$

$$S_{2,j}(n) = \frac{t^2}{12}\left\{(3j-1)t-1\right\}\frac{j^2}{(j-1)!}\prod_{q=1}^{j-1}(jn+q),$$

$$S_{3,j}(n) = \frac{t^2}{24}\left\{(9j^2-8j+2)t^2-(8j-4)t+2\right\}\frac{j^2}{(j-1)!}\prod_{q=1}^{j-1}(jn+q),$$

$$\begin{aligned} S_{4,j}(n) &= \frac{t^2}{20}\{(15j^3-25j^2+15j-3)t^3-(25j^2-30j+10)t^2 \\ &\quad +(15j-9)t-3\}\frac{j^2}{(j-1)!}\prod_{q=1}^{j-1}(jn+q), \end{aligned}$$

$$\begin{aligned} S_{5,j}(n) &= \frac{t^2}{24}\{(45j^4-120j^3+127j^2-60j+10)t^4 \\ &\quad -(120j^3-254j^2+192j-50)t^3+(127j^2-180j+70)t^2 \\ &\quad -(60j-40)t+10\}\frac{j^2}{(j-1)!}\prod_{q=1}^{j-1}(jn+q), \end{aligned}$$

$$\begin{aligned} S_{6,j}(n) &= \frac{t^2}{840}\{(4725j^5-18375j^4+29890j^3-24472j^2+9674j-1382)t^5 \\ &\quad -(18375j^4-59780j^3+76755j^2-44674j+9674)t^4 \\ &\quad +(29890j^3-73416j^2+64022j-19348)t^3-(24472j^2+38696j-16584)t^2 \\ &\quad +(9674j-6910)t-1382\}\frac{j^2}{(j-1)!}\prod_{q=1}^{j-1}(jn+q). \end{aligned}$$

$$\begin{aligned} S_{7,j}(n) &= \frac{t^2}{48}\{(945j^6-5040j^5+11620j^4-14400j^3+9818j^2-3360j+420)t^6 \\ &-(5040j^5-23240j^4+44640j^3-43520j^2+21024j-3920)t^5 \\ &+(11620j^4-43200j^3+63156j^2-42048j+10584)t^4 \\ &-(14400j^3-39272j^2+37824j-12600)t^3 \\ &+(9818j^2-16800j+7700)t^2 \\ &-(3360j-2520)t+420\}\frac{j^2}{(j-1)!}\prod_{q=1}^{j-1}(jn+q). \end{aligned}$$

Looking at these patterns we could suggest the following.

Conjecture 6.1. *$S_{k,j}(n)$ is expressed as a polynomial in $\prod_{q=1}^{j-1}(jn+q)\mathbb{Q}[t]$.*

For $j=1$, conjecture 6.1 becomes well known Faulhaber's theorem [6] which was, in fact, proved by Jacobi in [10]. The first eight Faulhaber's polynomials are as follows:

$$S_{0,1}(n)=\frac{t}{2},\ S_{1,1}(n)=\frac{t^2}{4},\ S_{2,1}(n)=\frac{t^2}{12}(2t-1),\ S_{3,1}(n)=\frac{t^2}{24}\left(3t^2-4t+2\right),$$

$$S_{4,1}(n)=\frac{t^2}{20}\left(2t^3-5t^2+6t-3\right),\ S_{5,1}(n)=\frac{t^2}{24}\left(2t^4-8t^3+17t^2-20t+10\right),$$

$$S_{6,1}(n)=\frac{t^2}{840}\left(60t^5-350t^4+1148t^3-46584t^2+2764t-1382\right),$$

$$S_{7,1}(n)=\frac{t^2}{48}t^2(3t^6-24t^5+112t^4-352t^3+718t^2-840t+420)$$

In general, one usually write

$$S_{k,1}(n)=\frac{1}{2(k+1)}\sum_{q=0}^{k}A_q^{(k+1)}t^{k-q+1}.$$

where $A_0^{(k)}=1$ and $A_{k-1}^{(k)}=0$. One knows quite a lot about the coefficients $A_q^{(k)}$. Jacobi proved that the coefficients $A_q^{(k)}$ enjoy the recurrence relation

$$(2k+2)(2k+1)A_q^{(k)}=2(k-q+1)(2k-2q+1)A_q^{(k+1)}+(k-q+1)(k-q+2)A_{q-1}^{(k+1)}$$

and tabulated some of them. It was shown by Knuth in [11] that these coefficients satisfy quite simple implicit recurrence relation

$$\sum_{q=0}^{r}\binom{k-q}{2r+1-2q}A_q^{(k)}=0,\quad r>0 \tag{6.1}$$

which yields an infinite triangle system of equations from which one easily obtains

$$A_1^{(k)}=-\frac{(k-2)k}{6},\quad A_2^{(k)}=\frac{(k-3)(k-1)k(7k-8)}{360},$$

$$A_3^{(k)}=-\frac{(k-4)(k-2)(k-1)k(31k^2-89k+48)}{15120},$$

$$A_4^{(k)}=\frac{(k-5)(k-3)(k-2)(k-1)k(127k^3-691k^2+1038k-384)}{604800},\ldots$$

Gessel and Viennot showed in [9] that a solution of system (6.1) can be presented as a $k \times k$ determinant

$$A_q^{(k)} = \frac{1}{(1-k)\cdots(q-k)} \begin{vmatrix} \binom{k-q+1}{3} & \binom{k-q+1}{1} & 0 & \cdots & 0 \\ \binom{k-q+2}{5} & \binom{k-q+2}{3} & \binom{k-q+2}{1} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \binom{k-1}{2k-1} & \binom{k-1}{2k-3} & \binom{k-1}{2k-5} & \cdots & \binom{k-1}{1} \\ \binom{k}{2k+1} & \binom{k-1}{2k-1} & \binom{k-1}{2k-3} & \cdots & \binom{k}{3} \end{vmatrix}$$

(see also [5]). A bivariate generating function for the coefficients $A_q^{(k)}$ was obtained in [8].

7. The conjectural relationship of the coefficients $p_{k,k-1}(t)$ to simple symmetric Venn diagrams

Let us rewrite the coefficients $p_{k,k-1}(t)$ being expressed via n, that is,

$$p_{k,k-1}(n) = p_{k,k-1}(t)|_{t=n(n+1)}.$$

For example,

$$p_{1,0}(n) = 1, \quad p_{2,1}(n) = -\frac{2}{3}\frac{n^2+n+1}{n(n+1)}, \quad p_{3,2}(n) = \frac{4}{3}\frac{n^4+2n^3+3n^2+2n+1}{n^2(n+1)^2},$$

$$p_{4,3}(n) = -\frac{24}{5}\frac{n^6+3n^5+\frac{19}{3}n^4+\frac{23}{3}n^3+\frac{19}{3}n^2+3n+1}{n^3(n+1)^3},$$

$$p_{5,4}(n) = \frac{80}{3}\frac{n^8+4n^7+11n^6+19n^5+23n^4+19n^3+11n^2+4n+1}{n^4(n+1)^4},$$

$$p_{6,5}(n) = -\frac{22112}{105}\frac{n^{10}+5n^9+17n^8+38n^7+61n^6+71n^5+61n^4+38n^3+17n^2+5n+1}{n^5(n+1)^5}, \ldots$$

Looking at these patterns we see that

$$p_{k,k-1}(n) = c_k \frac{v_k(n)}{n^{k-1}(n+1)^{k-1}}, \tag{7.1}$$

where $v_k(n)$ is a monic polynomial of degree $2k-2$. All these polynomials are invariant with respect to transformation

$$v_k(n) \mapsto n^{2k-2} v_k\left(\frac{1}{n}\right). \tag{7.2}$$

Also it worth to remark that the polynomial $v_4(n)$ unlike the others, has several fractional coefficients.

Conjecture 7.1. *Polynomials $v_k(n)$ are given by*

$$v_k(n) = \sum_{q=1}^{2k-1} \frac{\binom{2k}{q} + (-1)^{q+1}}{2k+1} n^{2k-q-1}, \tag{7.3}$$

while the coefficients c_k are expressed via Bernoulli numbers as

$$c_k = (2k+1)2^k B_{2k}. \tag{7.4}$$

Let us notice that if (7.3) is valid then the invariance of corresponding polynomial with respect to (7.4) is obvious in virtue of the invariance of binomial coefficients.

Let $p = 2k+1$. It is known that if p is simple then

$$T(p,q) = \frac{\binom{p-1}{q} + (-1)^{q+1}}{p}, \quad p \geq 5$$

is the number of q-points on the left side of a crosscut of simple symmetric p-Venn diagram [12]. This integer sequence is known as A219539 sequence in [13]. It is evident that the row sum

$$t_p := \sum_{q=1}^{p-2} T(p,q) = \frac{2^{p-1}-1}{p}$$

can be calculated as $v_k(n)|_{n=1}$. The Fermat quotients $(2^{p-1}-1)/p$ for simple p constitute integer sequence A007663 in [13]. Taking into account (7.1) and (7.4), we get

$$p_{k,k-1}(n)|_{n=1} = 2(2^{2k}-1)B_{2k} = -G_{2k}.$$

8. Discussion

In the paper we have considered some class of sums $S_{k,j}(n)$ and conjectured a representation of these sums in terms of a sequence of the polynomials $\{P_k(t,x) : k \geq 0\}$. This assumption is resulted from computational experiments and supported by a large amount of actual calculations. For $n = 1$, we get the well-known results from [15]. This also confirms our assumptions. The conjectural relationship of several coefficients of polynomials $P_k(t,x)$ being expressed via n to simple symmetric Venn diagrams is quite unexpected and requires explanation.

Acknowledgments

This work was supported in part by the Council for Grants of the President of Russian Foundation for state support of the leading scientific schools, project NSh-8081.2016.9.

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Andrei K. Svinin, Matrosov Institute for System Dynamics and Control Theory of Siberian Branch of Russian Academy of Sciences, P.O. Box 292, 664033 Irkutsk, Russia

E-mail address: `svinin@icc.ru`