Sums of products of binomial coefficients mod 2 and run length transforms of sequences

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Abstract

We look at some properties of functions of binomial coefficients mod 2. In particular, we derive a set of recurrence relations for sums of products of binomial coefficients mod 2 and show that they result in sequences that are the run length transforms of basic sequences. In particular, we show that the sequence $a(n) = \sum_{k=0}^{n} \binom{n-k}{2k}\binom{n}{k}$ mod 2 is the run length transform of the Fibonacci numbers and that the sequence $a(n) = \sum_{k=0}^{n} \binom{n+k}{n-k}\binom{n}{k}$ mod 2 is the run length transform of the positive integers.

1 Introduction

When is the binomial coefficient even or odd, i.e. what is $\binom{n}{k}$ mod 2? It is well known that when Pascal's triangle of binomial coefficients is taken mod 2, the result has a fractal structure in the limit and is a Sierpinski's triangle (also known as Sierpinski's gasket or Sierpinski's sieve) [1, 2].

Lucas' theorem [3] provides a simple way to determine the binomial coefficients modulo a prime. It states that for integers k, n and prime p, the following relationship holds:

$$\binom{n}{k} \equiv \prod_{i=0}^{m} \binom{n_i}{k_i} \mod p$$

where n_i and k_i are the digits of n and k in base p respectively.

When $p = 2$, n_i and k_i are the bits in the binary expansion of n and k and $\binom{n_i}{k_i}$ is 0 if and only if $n_i < k_i$. This implies that $\binom{n}{k}$ is even if and only if $n_i < k_i$ for some i.

The truth table of $n_i < k_i$ is:

n_i	k_i	$n_i < k_i$
0	0	0
0	1	1
1	0	0
1	1	0

and it is logically equivalent to $k_i \wedge (\neg n_i)$. If we think of $\wedge$, $\vee$ and $\neg$ as bitwise operations on the binary representation of numbers, then we have shown the following well-known fact [1]:

Theorem 1. $\binom{n}{k} \equiv 0 \mod 2$ *if and only if* $k \wedge (\neg n) \neq 0$.

Incidentally, for bits n_i and k_i, $n_i < k_i$ is logically equivalent to $\neg(k_i \Rightarrow n_i)$. Consider $\binom{n}{k}\binom{m}{r} \mod 2$. Clearly this is equivalent to

$$\left(\binom{n}{k} \mod 2\right)\left(\binom{m}{r} \mod 2\right)$$

Thus $\binom{n}{k}\binom{m}{r} \equiv 0 \mod 2$ if and only if $k \wedge (\neg n) \neq 0$ or $r \wedge (\neg m) \neq 0$. This in turns implies the following:

Theorem 2.

$$\binom{n}{k}\binom{m}{r} \equiv 0 \mod 2 \Leftrightarrow (k \wedge (\neg n)) \vee (r \wedge (\neg m)) \neq 0$$

Analogously we have the following result for $\left(\prod_{a=1}^{a=T} \binom{n_a}{k_a}\right) \mod 2$. Here n_a and k_a denote different integers indexed by a, not the bits of n and k.

Theorem 3.

$$\prod_{a=1}^{a=T} \binom{n_a}{k_a} \equiv 0 \mod 2 \Leftrightarrow (k_1 \wedge (\neg k_1)) \vee (k_2 \wedge (\neg k_2)) \vee \cdots \vee (k_T \wedge (\neg k_T)) \neq 0 \quad (1)$$

These equivalences will be useful in deriving properties of binomial coefficients mod 2.

2 Run length transform

The run length transform on sequences of numbers is defined as follows [4]:

Definition 1. *The run length transform of $\{S_n, n \geq 0\}$ is given by $\{T_n, n \geq 0\}$, where T_n is the product of S_i's, with i denoting the run of 1's in the binary representation of n with the convention that $T_0 = 1$.*

For instance, suppose $n = 463$, which is 111001111 in binary. It has a run of 3 1's and a run of 4 1's, and thus $T_n = S(3) \cdot S(4)$. Some fixed points of the run length transform include the sequences $\{1, 0, 0, \ldots, \}$ and $\{1, 1, 1 \cdots, \}$. In [4], the following result is proved about the run length transform:

Theorem 4. *Let $\{S_n, n \geq 0\}$ be defined by the recurrence $S_{n+1} = c_2 S_n + c_3 S_{n-1}$ with initial conditions $S_0 = 1$, $S_1 = c_1$, then the run length transform of $\{S_n\}$ is given by $\{T_n, n \geq 0\}$ satisfying $T_0 = 1$ and $T_{2n} = T_n$, $T_{4n+1} = c_1 T_n$, $T_{4n+3} = c_2 T_{2n+1} + c_3 T_n$.*

Note that the sequence S_n may not uniquely define the values of c_2 and c_3 in Theorem 4. For instance, for the sequence $S_n = \{1, 2, 4, 8, \cdots, \}$, c_2 and c_3 can be chosen to be any integers such that $2c_2 + c_3 = 4$.

3 Recurrence relations of product of binomial coefficients mod 2

Definition 2. *Consider integers $a_i \geq 0$, $i = 1, \cdots, 4$, $a_1 \in \{0, 1\}$, $a_3 \in \{0, 1\}$, $0 \leq a_1 + a_2$, and $0 \leq a_3 + a_4$. Let $F(n, k) = \begin{pmatrix} a_1 n + a_2 k \\ a_3 n + a_4 k \end{pmatrix} \begin{pmatrix} n \\ k \end{pmatrix} \mod 2$ and $g(n, k) = ((a_3 n + a_4 k) \wedge \neg(a_1 n + a_2 k)) \vee (k \wedge \neg n)$*

By Theorem 2, $F(n, k) = 1$ if and only if $g(n, k) = 0$. We show that F satisfies various recurrence relations.

Theorem 5. *The following relations hold for the function F:*

- *$F(n, k) = 0$ if $k > n$,*
- *$F(4n, 4k) = F(2n, 2k) = F(n, k)$,*
- *$F(2n, 2k+1) = F(4n+1, 4k+2) = F(4n+1, 4k+3) = F(4n+2, 4k+1) = F(4n+2, 4k+3) = F(4n, 4k+1) = F(4n, 4k+2) = F(4n, 4k+3) = 0$,*
- *$F(4n+1, 4k) = F(n, k)$ if $a_1 = 1$ or $a_3 = 0$ and $F(4n+1, 4k) = 0$ otherwise.*
- *$F(4n+3, 4k) = F(n, k)$ if $a_1 = 1$ or $a_3 = 0$ and $F(4n+3, 4k) = 0$ otherwise.*
- *$F(2n+1, 2k) = F(n, k)$ if $a_1 = 1$ or $a_3 = 0$ and $F(2n+1, 2k) = 0$ otherwise.*

Proof. Since $\begin{pmatrix} n \\ k \end{pmatrix} = 0$ if $k > n$, $F(n,k) = 0$ if $k > n$.

$g(2n, 2k) = (2(a_3 n + a_4 k) \wedge \neg(2(a_1 n + a_2 k))) \vee (2k \wedge \neg 2n) = 2g(n,k)$ since the lsb (least significant bit) is 0, i.e. $F(2n, 2k) = F(n,k)$.

Next $F(n,k) = 0$ if $\begin{pmatrix} n \\ k \end{pmatrix} \equiv 0 \mod 2$, i.e. if $(k \wedge \neg n) > 0$. Then it is easy to see that $F(2n, 2k+1) = F(4n+1, 4k+2) = F(4n+1, 4k+3) = F(4n+2, 4k+1) = F(4n+2, 4k+3) = 0$ and $F(4n, 4k+i) = 0$ for $1 \le i \le 3$.

$g(4n+1, 4k) = (4(a_3 n + a_4 k) + a_3 \wedge \neg 4(a_1 n + a_2 k) + a_1) \vee (4k \wedge \neg 4n+1)$. The least significant 2 bits is equal to $a_3 \wedge \neg a_1 \mod 4$, so $g(4n+1, 4k) = g(n,k)$ and $F(4n+1, 4k) = F(n,k)$ if $a_3 \wedge \neg a_1 \equiv 0 \mod 4$ and $F(4n+1, 4k) = 0$ otherwise.

$g(4n+3, 4k) = (4(a_3 n + a_4 k) + 3a_3 \wedge \neg(4(a_1 n + a_2 k) + 3a_1)) \vee (4k \wedge \neg 4n + 3) = 4g(n,k)$ if $a_1 = 1$ or $a_3 = 0$., i.e. $F(4n+3, 4k) = F(n,k)$ if $a_1 = 1$ or $a_3 = 0$ and $F(4n+3, 4k) = 0$ otherwise.

$g(2n+1, 2k) = (2(a_3 n + a_4 k) + a_3 \wedge \neg(2(a_1 n + a_2 k) + a_1)) \vee (2k \wedge \neg 2n + 1)$. thus $F(2n+1, 2k) = 0$ if $a_3 \wedge \neg a_1 \not\equiv 0 \mod 2$. Otherwise, $g(2n+1, 2k) = 2g(n,k)$ and $F(2n+1, 2k) = F(n,k)$.

□

4 Sums of products of binomial coefficients mod 2

In this section, we show that for various values of a_i's, the sequence $a(n) = \sum_{k=0}^{n} F(n,k)$ corresponds to the run length transforms of well-known sequences. It is clear that $a(n) \le \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \mod 2$ with equality when $a_1 = a_4 = 1$, $a_2 = a_3 = 0$ or when $a_1 = a_2 = a_3 = a_4 = 1$. The sequence $g(n) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \mod 2$ is known as Gould's sequence or Dress's sequence and is the run length transform of the positive powers of 2: $\{1, 2, 4, 8, 16, 32, \cdots\}$ (see OEIS [5] sequence A001316).

Lemma 1. *The sequence $a(n)$ satisfies the following properties:*

- $a(0) = 1$,
- $a(2n) = a(n)$,
- *If $a_1 = 1$ or $a_3 = 0$, then $a(4n+1) = a(n) + \sum_{k=0}^{n} F(4n+1, 4k+1)$, otherwise $a(4n+1) = \sum_{k=0}^{n} F(4n+1, 4k+1)$,*
- *If $a_1 = 1$ or $a_3 = 0$, then $a(4n+3) = a(n) + \sum_{m=1}^{3} \sum_{k=0}^{n} F(4n+3, 4k+m)$, otherwise $a(4n+3) = \sum_{m=1}^{3} \sum_{k=0}^{n} F(4n+3, 4k+m)$*
- *If $a_1 = 1$ or $a_3 = 0$, then $a(2n+1) = a(n) + \sum_{k=0}^{n} F(2n+1, 2k+1)$, otherwise $a(2n+1) = \sum_{k=0}^{n} F(2n+1, 2k+1)$.*

Proof. $a(0)$ is trivially equal to 1.

$a(2n) = \sum_{k=0}^{2n} F(2n,k) = \sum_{k=0}^{n} F(2n,2k) + \sum_{k=0}^{n-1} F(2n,2k+1)$ which is equal to $a(n)$ by Theorem 5.

Suppose $a_1 = 1$ or $a_3 = 0$. By Theorem 5, the following relations hold: $a(4n+1) = \sum_{m=0}^{3}\sum_{k=0}^{n} F(4n+1,4k+m) - F(4n+1,4n+2) - F(4n+1,4n+3) = \sum_{m=0}^{3}\sum_{k=0}^{n} F(4n+1,4k+m) = \sum_{k=0}^{n} F(n,k) + \sum_{k=0}^{n} F(4n+1,4k+1)$; $a(4n+3) = \sum_{m=0}^{3}\sum_{k=0}^{n} F(4n+3,4k+m) = \sum_{k=0}^{n} F(n,k) + \sum_{m=1}^{3}\sum_{k=0}^{n} F(4n+3,4k+m)$; $a(2n+1) = \sum_{k=0}^{2n+1} F(2n+1,k) = \sum_{k=0}^{n} F(2n+1,2k) + \sum_{k=0}^{n} F(2n+1,2k+1) = \sum_{k=0}^{n} F(n,k) + \sum_{k=0}^{n} F(2n+1,2k+1)$. □

In particular, if $a_1 = 1$ or $a_3 = 0$ then $\sum_{k=0}^{n} F(2n+1,2k+1) = a(2n+1) - a(n)$, an equation which we will use often in the sequel.

4.1 Run length transform of the Fibonacci sequence

First, consider the case $a_1 = 1$, $a_2 = -1$, $a_3 = 0$, $a_4 = 2$.

Lemma 2. *For $a_1 = 1$, $a_2 = -1$, $a_3 = 0$, $a_4 = 2$, the following relations hold for the function F:*

- $F(4n+1,4k+1) = F(4n+3,4k+3) = 0,$
- $F(4n+3,4k+1) = F(n,k),$
- $F(4n+3,4k+2) = F(2n+1,2k+1).$

Proof. $g(4n+1,4k+1) = (8k+2 \wedge \neg(4(n-k)) \vee (4k+1 \wedge \neg 4n+1) \neq 0$, i.e $F(4n+1,4k+1) = 0$.

$g(4n+3,4k+1) = (8k+2 \wedge \neg(4(n-k)+2)) \vee (4k+1 \wedge \neg 4n+3) = (4(2k \vee n-k)) \vee 4(k \wedge \neg n) = g(n,k)$.

Note that $(4k+2 \wedge \neg 4n+3) = (4k \wedge \neg 4n) = 2(2k \wedge \neg 2n)$ and $(2k+1 \wedge \neg 2n+1) = (2k \wedge \neg 2n)$. Similarly $(4k+3 \wedge \neg 4n+3) = (4k \wedge \neg 4n) = 2(2k+1 \wedge \neg 2n+1)$.

$g(4n+3,4k+2) = (8k+4 \wedge \neg(4(n-k)+1)) \vee (4k+2 \wedge \neg 4n+3) = 2[4k+2 \wedge \neg 2(n-k)) \vee (2k+1 \wedge \neg 2n+1)]$ where we have use the fact that $(8k+4 \wedge \neg(4(n-k)+1)) = (8k+4 \wedge \neg(4(n-k)))$. This implies that $F(4n+3,4k+2) = F(2n+1,2k+1)$.

$g(4n+3,4k+3) = (8k+6 \wedge \neg(4(n-k)) \vee (4k+3 \wedge \neg 4n+3)$. Since $8k+6 \wedge \neg 4(n-k) \neq 0$ This implies that $F(4n+3,4k+3) = 0$. □

Theorem 6. *Let $a(n) = \sum_{k=0}^{n} \begin{pmatrix} n-k \\ 2k \end{pmatrix} \begin{pmatrix} n \\ k \end{pmatrix} \mod 2$. Then $a(n)$ satisfies the equations $a(0) = 1$, $a(2n) = a(n)$, $a(4n+1) = a(n)$ and $a(4n+3) = a(2n+1) + a(n)$. In particular, $a(n)$ is the run length transform of the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, \cdots$.*

Proof. By Lemma 1,$a(0) = 1$ and $a(2n) = a(n)$. Next, by Lemma 1 and Lemma 2, $a(4n+1) = a(n)$. Similarly $a(4n+3) = a(n) + \sum_{m=1}^{3}\sum_{k=0}^{n} F(4n+3,4k+m) = a(n) + \sum_{k=0}^{n} F(n,k) + F(2n+1,2k+1) = a(2n+1) + a(n)$. By Theorem 4, $a(n)$ is the run length transform of the Fibonacci sequence 1,1,2,3,5,8,13,... □

Note that in this case $a(n)$ corresponds to OEIS sequence A246028. Other values of a_i can also generate the same sequence. For instance, it can be shown that $\sum_{k=0}^{n} \binom{2k}{n-k}\binom{n}{k}$ mod 2 , $\sum_{k=0}^{n} \binom{n+3k}{2k}\binom{n}{k}$ mod 2, and $\sum_{k=0}^{n} \binom{n+3k}{n+k}\binom{n}{k}$ mod 2 all correspond to the run length transform of the Fibonacci sequence as well.

4.2 Run length transform of the truncated Fibonacci sequence

Next, consider the case $a_1 = a_3 = 0$, $a_2 = 3$, $a_4 = 1$.

Lemma 3. *For $a_1 = a_3 = 0$, $a_2 = 3$, $a_4 = 1$, the following relations hold for the function F:*

- $F(4n+1, 4k+1) = F(4n+3, 4k+1) = F(n, k)$,
- $F(4n+3, 4k+2) = F(2n+1, 2k+1)$,
- $F(4n+3, 4k+3) = 0$.

Proof. $g(4n+1, 4k+1) = (4k+1 \wedge \neg(12k+3) \vee (4k+1 \wedge \neg 4n+1) = (4k \wedge \neg 12k) \vee (4k \wedge \neg 4n)$, i.e $F(4n+1, 4k+1) = F(n, k)$.

$g(4n+3, 4k+1) = (4k+1 \wedge \neg(12k+3) \vee (4k+1 \wedge \neg 4n+3) = (4k \wedge \neg 12k) \vee (4k \wedge \neg 4n)$ and $F(4n+3, 4n+1) = F(n, k)$.

Note that $(4k+2 \wedge \neg 4n+3) = (4k \wedge \neg 4n) = 2(2k \wedge \neg 2n)$ and $(2k+1 \wedge \neg 2n+1) = (2k \wedge \neg 2n)$. Similarly $(4k+3 \wedge \neg 4n+3) = (4k \wedge \neg 4n) = 2(2k+1 \wedge \neg 2n+1)$.

$g(4n+3, 4k+2) = (4k+2 \wedge \neg(12k+6)) \vee (4k+2 \wedge \neg 4n+3) = 2[(2k+1 \wedge \neg 6k+3) \vee (2k+1 \wedge \neg 2n+1)]$. This implies that $F(4n+3, 4k+2) = F(2n+1, 2k+1)$.

$g(4n+3, 4k+3) = (4k+3 \wedge \neg(12k+6) \vee (4k+3 \wedge \neg 4n+3)$. Since $4k+3 \wedge \neg 12k+6 \neq 0$ This implies that $F(4n+3, 4k+3) = 0$ □

Theorem 7. *Let $a(n) = \sum_{k=0}^{n} \binom{3k}{k}\binom{n}{k}$ mod 2. Then $a(n)$ satisfies the equations $a(0) = 1$, $a(2n) = a(n)$, $a(4n+1) = 2a(n)$ and $a(4n+3) = a(2n+1) + a(n)$. In particular, $a(n)$ is the run length transform of the truncated Fibonacci sequence $1, 2, 3, 5, 8, 13, \cdots$.*

Proof. By Lemma 1, $a(0) = 1$ and $a(2n) = a(n)$. Next, by Lemma 1 and Lemma 3, $a(4k+1) = 2a(n)$. Similarly, $a(4n+3) = a(n) + \sum_{m=1}^{3}\sum_{k=0}^{n} F(4n+3, 4k+m) = a(n) + \sum_{k=0}^{n} F(n,k) + F(2n+1, 2k+1) = a(2n+1) + a(n)$. By Theorem 4, $a(n)$ is the run length transform of the truncated Fibonacci sequence $1, 2, 3, 5, 8, 13, \cdots$. □

Note that in this case $a(n)$ corresponds to OEIS sequence A245564. This sequence is also equal to $\sum_{k=0}^{n} \binom{3k2^m}{k2^m}\binom{n}{k}$ mod 2 and $\sum_{k=0}^{n} \binom{3k2^m}{2k2^m}\binom{n}{k}$ mod 2 for all integers $m \geq 0$.

4.3 Run length transform of $\{1, 1, 2, 4, 8, 16, 32, \cdots\}$

Next, consider the case $a_1 = 1$, $a_2 = a_3 = 0$, $a_4 = 2$.

Lemma 4. *For $a_1 = 1$, $a_2 = a_3 = 0$, $a_4 = 2$, the following relations hold for the function F:*

- $F(4n + 1, 4k + 1) = 0$
- $F(4n + 3, 4k + 1) = F(n, k)$,
- $F(4n + 3, 4k + 2) = F(4n + 3, 4k + 3) = F(2n + 1, 2k + 1)$.

Proof. $g(4n+1, 4k+1) = (8k+2\wedge\neg(4n+1)\vee(4k+1\wedge\neg 4n+1) \neq 0$, i.e $F(4n+1, 4k+1) = 0$.

$g(4n+3, 4k+1) = (8k+2\wedge\neg(4n+3)\vee(4k+1\wedge\neg 4n+3) = 4[(2k\wedge\neg n)\vee(k\wedge\neg n)$ where we use the fact that $(8k + 2 \wedge \neg(4n + 3)(8k \wedge \neg(4n)$ and thus $F(4n + 3, 4n + 1) = F(n, k)$.

Note that $(4k+2\wedge\neg 4n+3) = (4k\wedge\neg 4n) = 2(2k\wedge\neg 2n)$ and $(2k+1\wedge\neg 2n+1) = (2k\wedge\neg 2n)$. Similarly $(4k + 3 \wedge \neg 4n + 3) = (4k \wedge \neg 4n) = 2(2k + 1 \wedge \neg 2n + 1)$.

$g(4n + 3, 4k + 2) = (8k + 4 \wedge \neg(4n + 3)) \vee (4k + 2 \wedge \neg 4n + 3) = 2[(4k + 2 \wedge \neg 2n + 1) \vee (2k + 1 \wedge \neg 2n + 1)]$, where we use the fact that $(8k + 4 \wedge \neg(4n + 3)) = (8k + 4 \wedge \neg(4n + 2))$. This implies that $F(4n + 3, 4k + 2) = F(2n + 1, 2k + 1)$.

$g(4n + 3, 4k + 3) = (8k + 6 \wedge \neg(4n + 3)) \vee (4k + 3 \wedge \neg 4n + 3) = 2[(4k + 2 \wedge \neg(2n + 1)) \vee (2k + 1 \wedge \neg 2n + 1)$.This implies that $F(4n + 3, 4k + 3) = F(2n + 1, 2k + 1)$.

□

Theorem 8. *Let $a(n) = \sum_{k=0}^{n} \binom{n}{2k}\binom{n}{k} \mod 2$. Then $a(n)$ satisfies the equations $a(0) = 1$, $a(2n) = a(n)$, $a(4n + 1) = a(n)$ and $a(4n + 3) = 2a(2n + 1)$. In particular, $a(n) = \{1, 1, 1, 2, 1, 1, 2, 4, 1, 1, 1, 2, \cdots\}$ is the run length transform of the $1, 1, 2, 4, 8, 16, 32, \cdots$, i.e. 1 plus the positive powers of 2.*

Proof. By Lemma 1, $a(0) = 1$ and $a(2n) = a(n)$. Next, by Lemma 1 and Lemma 4, $a(4n + 1) = a(n)$. Similarly, $a(4n + 3) = a(n) + \sum_{m=1}^{3}\sum_{k=0}^{n} F(4n + 3, 4k + m) = 2a(n) + 2\sum_{k=0}^{n} F(2n + 1, 2k + 1) = 2a(2n + 1)$. By Theorem 4, $a(n)$ is the run length transform of the sequence $1, 1, 2, 4, 8, 16, 32, \cdots$. □

4.4 Run length transform of $\{1, 2, 2, 2, 2, 2, \cdots\}$

Next, consider the case $a_1 = 1$, $a_2 = a_4 = 2$, $a_3 = 0$.

Lemma 5. *For $a_1 = 1$, $a_2 = a_4 = 2$, $a_3 = 0$, the following relations hold for the function F:*

- $F(4n + 1, 4k + 1) = F(n, k)$
- $F(4n + 3, 4k + 1) = F(4n + 3, 4k + 3) = 0$,
- $F(4n + 3, 4k + 2) = F(2n + 1, 2k + 1)$,

Proof. $g(4n+1, 4k+1) = (8k+2 \wedge \neg(4n+8k+3) \vee (4k+1 \wedge \neg 4n+1) = 4[(2k \wedge \neg(n+2k) \vee (k \wedge \neg n)]$, i.e $F(4n+1, 4k+1) = F(n, k)$.

$g(4n+3, 4k+1) = (8k+2 \wedge \neg(4n+8k+5) \vee (4k+1 \wedge \neg 4n+3) \neq 0$, i.e. $F(4n+3, 4k+1) = 0$.

Note that $(4k+2 \wedge \neg 4n+3) = (4k \wedge \neg 4n) = 2(2k \wedge \neg 2n)$ and $(2k+1 \wedge \neg 2n+1) = (2k \wedge \neg 2n)$. Similarly $(4k+3 \wedge \neg 4n+3) = (4k \wedge \neg 4n) = 2(2k+1 \wedge \neg 2n+1)$.

$g(4n+3, 4k+2) = (8k+4 \wedge \neg(4n+8k+7)) \vee (4k+2 \wedge \neg 4n+3) = 2[(4k+2 \wedge \neg 2n+4k+3) \vee (2k+1 \wedge \neg 2n+1)]$ where we use $(8k+4 \wedge \neg(4n+8k+7)) = (8k+4 \wedge \neg(4n+8k+6))$. This implies that $F(4n+3, 4k+2) = F(2n+1, 2k+1)$.

$g(4n+3, 4k+3) = (8k+6 \wedge \neg(4n+8k+9)) \vee (4k+3 \wedge \neg 4n+3) \neq 0$, i.e. $F(4n+3, 4k+3) = 0$. □

Theorem 9. *Let* $a(n) = \sum_{k=0}^{n} \binom{n+2k}{2k} \binom{n}{k} \mod 2$. *Then* $a(n)$ *satisfies the equations* $a(0) = 1$, $a(2n) = a(n)$, $a(4n+1) = 2a(n)$ *and* $a(4n+3) = a(2n+1)$. *In particular,* $a(n) = \{1, 2, 2, 2, 2, 4, 2, 2, 2, 4, \cdots\}$ *is the run length transform of the* $1, 2, 2, 2, 2, 2, 2, \cdots$.

Proof. By Lemma 1, $a(0) = 1$, $a(2n) = a(n)$. Next, by Lemma 1 and Lemma 5, $a(4n+1) = a(n) + \sum_{k=0}^{n} F(n, k) = 2a(n)$. Similarly, $a(4n+3) = a(n) + \sum_{m=1}^{3} \sum_{k=0}^{n} F(4n+3, 4k+m) = a(n) + \sum_{k=0}^{n} F(2n+1, 2k+1) = a(2n+1)$. By Theorem 4, $a(n)$ is the run length transform of the Fibonacci sequence $1, 2, 2, 2, 2, \cdots$. □

This sequence is also generated by $\sum_{k=0}^{n} \binom{n+2k}{n} \binom{n}{k} \mod 2$.

4.5 Run length transform of the positive integers

OEIS sequence A106737 is defined as $a(n) = \sum_{k=0}^{n} \binom{n+k}{n-k} \binom{n}{k} \mod 2$. It was noted that the following recursive relationships appear to hold: $a(2n) = a(n)$, $a(4n+1) = 2a(n)$ and $a(4n+3) = 2a(2n+1) - a(n)$. In this section we show that this is indeed the case.

Let $a_1, a_2, a_3 = 1$ and $a_4 = -1$, i.e. $F(n, k) = \binom{n+k}{n-k} \binom{n}{k} \mod 2$ and $g(n, k) = ((n-k) \wedge \neg(n+k)) \vee (k \wedge \neg n)$.

Lemma 6. *For* $a_1, a_2, a_3 = 1$ *and* $a_4 = -1$, *the following relations hold for the function* F *and* g*:*

- $F(4n+1, 4k+1) = F(n, k)$,
- $F(4n+3, 4k+1) = 0$,
- $F(4n+3, 4k+2) = F(4n+3, 4k+3) = F(2n+1, 2k+1)$.

Proof. $g(4n+1, 4k+1) = (4(n-k) \wedge \neg(4(n+k)+2)) \vee (4k+1 \wedge \neg 4n+1) = 4((n-k) \wedge \neg(n+k)) \vee 4(k \wedge \neg n)$, i.e. $F(4n+1, 4k+1) = F(n, k)$.

$g(4n+3,4k+1) = (4(n-k)+2 \wedge \neg(4(n+k+1))) \vee (4k+1 \wedge \neg 4n+3) > 0$ since $(4(n-k)+2 \wedge \neg(4(n+k+1))) > 0$, i.e. $F(4n+3,4k+1) = 0$.

Note that $(4k+2 \wedge \neg 4n+3) = (4k \wedge \neg 4n) = 2(2k \wedge \neg 2n)$ and $(2k+1 \wedge \neg 2n+1) = (2k \wedge \neg 2n)$. Similarly $(4k+3 \wedge \neg 4n+3) = (4k \wedge \neg 4n) = 2(2k+1 \wedge \neg 2n+1)$.

$g(4n+3,4k+2) = (4(n-k)+1 \wedge \neg(4(n+k+1)+1)) \vee (4k+2 \wedge \neg 4n+3) = 2[(2(n-k) \wedge \neg 2(n+k+1)) \vee (2k+1 \wedge \neg 2n+1)]$ where we use $(4(n-k)+1 \wedge \neg(4(n+k+1)+1)) = (4(n-k) \wedge \neg(4(n+k+1)))$. This implies that $F(4n+3,4k+2) = F(2n+1,2k+1)$.

$g(4n+3,4k+3) = (4(n-k) \wedge \neg(4(n+k+1)+2)) \vee (4k+3 \wedge \neg 4n+3) = 2[(2(n-k) \wedge \neg 2(n+k+1)) \vee (2k+1 \wedge \neg 2n+1)]$ where we use $(4(n-k) \wedge \neg(4(n+k+1)+2)) = (4(n-k) \wedge \neg(4(n+k+1)))$, and thus $F(4n+3,4k+3) = F(2n+1,2k+1)$. □

Theorem 10. *For OEIS sequence A106737, $a(0) = 1$, $a(2n) = a(n)$, $a(4n+1) = 2a(n)$ and $a(4n+3) = 2a(2n+1) - a(n)$. Furthermore, $a(n)$ is the run length transform of the positive integers.*

Proof. As before, by Lemma 1, $a(0) = 1$ and $a(2n) = a(n)$. Next by Lemma 1 and Lemma 6, $a(4n+1) = a(n) + \sum_{k=0}^{n} F(n,k) = 2a(n)$. Similarly, $a(4n+3) = a(n) + \sum_{k=0}^{n} F(2n+1,2k+1) + F(2n+1,2k+1) = 2a(2n+1) - a(n)$. By Theorem 4, $a(n)$ is the run length transform of the positive integers $1,2,3,4,\cdots$ □

This sequence is also generated by each of the following expressions: $\sum_{k=0}^{n} \binom{n+k}{2k}\binom{n}{k}$ mod 2, $\sum_{k=0}^{n} \binom{n+2k}{k}\binom{n}{k}$ mod 2 and $\sum_{k=0}^{n} \binom{n+2k}{n+k}\binom{n}{k}$ mod 2.

4.6 A fixed point of the run length transform

The all ones sequence $\{1,1,1,\cdots\}$ (OEIS sequence A000012) is a fixed point of the run length transform. We next show that it is also expressible as sums of products of binomial coefficients mod 2. To prove this, we consider the case $a_1 = a_4 = 1$, $a_2 = -1$, $a_3 = 0$.

Lemma 7. *For $a_1 = a_4 = 1$, $a_2 = -1$, $a_3 = 0$, the following relations hold for the function F:*

- $F(4n+1,4k+1) = F(4n+3,4k+1) = F(4n+3,4k+2) = F(4n+3,4k+3) = 0$

Proof. $g(4n+1,4k+1) = (4k+1 \wedge \neg(4(n-k)) \vee (4k+1 \wedge \neg 4n+1) \neq 0$, i.e. $F(4n+1,4k+1) = 0$.

$g(4n+3,4k+1) = (4k+1 \wedge \neg(4(n-k)+2) \vee (4k+1 \wedge \neg 4n+3) \neq 0$, i.e. $F(4n+3,4k+1) = 0$.

Note that $(4k+2 \wedge \neg 4n+3) = (4k \wedge \neg 4n) = 2(2k \wedge \neg 2n)$ and $(2k+1 \wedge \neg 2n+1) = (2k \wedge \neg 2n)$. Similarly $(4k+3 \wedge \neg 4n+3) = (4k \wedge \neg 4n) = 2(2k+1 \wedge \neg 2n+1)$.

$g(4n+3,4k+2) = (4k+2 \wedge \neg(4(n-k)+1)) \vee (4k+2 \wedge \neg 4n+3) \neq 0$, i.e. $F(4n+3,4k+2) = 0$.
$g(4n+3,4k+3) = (4k+3 \wedge \neg(4(n-k)) \vee (4k+3 \wedge \neg 4n+3) \neq 0$, i.e. $F(4n+3,4k+3) = 0$. □

Theorem 11. *For $n,k \geq 0$, $\binom{n-k}{k}\binom{n}{k}$ is odd if and only if $k = 0$.*

Proof. Define $a(n) = \sum_{k=0}^{n} \binom{n-k}{k} \binom{n}{k} \mod 2$. By Lemma 1 and Lemma 7, $a(0) = 1$, $a(n) = a(2n)$ and $a(4n+1) = a(n)$, $a(4n+3) = a(n)$. By Theorem 4, $a(n)$ is the run length transform of the sequence $1, 1, 1, 1, \cdots$, i.e. $a(n) = 1$ for all $n \geq 0$. The conclusion then follows since $\binom{n-k}{k} \binom{n}{k} = 1$ when $k = 0$. □

Theorem 11 can also been shown by looking at the Sierpinski's triangle generated by Pascal's triangle mod 2 and paraphrasing Theorem 11 as: if starting from the left edge of the triangle moving k steps to the right reaches a point of the Sierpinski's triangle, then continuing moving diagonally k steps must necessary reach a void of the Sierpinski's triangle.

5 Conclusions

The run length transform has been useful in analyzing the number of ON cells in a cellular automata after n iterations [4]. We show here that the run length transform can also characterize sums of products of binomial coefficients mod 2. Given the fact that several cellular automata can generate the Sierpinski's triangle [1] which are equivalent to Pascal's triangle mod 2, this is not surprising and suggests that there is a close relationship between cellular automata and functions of binomial coefficients mod 2.

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