Generalized K-Shift Forbidden Substrings in Permutations

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In this note we continue the analysis started in [2] and generalize propositions regarding permutations that avoid substrings $12, 23, \ldots, (n-1)n$, (and others) to permutations that for k fixed, $k < n$, avoid substrings $j(j+k)$, $1 \leq j \leq n-k$, (*ie.* k-shifts in general, as defined in Section 2). We count the number of such permutations and relate them to generalized derangement numbers.

Keywords: Generalized derangements, permutations, linear arrangements, forbidden substrings, fixed points, k-shifts, bijections.

1. Introduction and Previous Results

In this section we summarize some results obtained in [2] and we recall the following definitions[2]:

d_n := the number of permutations on $[n]$ that avoid substrings $12, 23, \ldots, (n-1)n$.

D_n := the number of permutations on $[n]$ that avoid substrings $12, 23, \ldots, (n-1)n, n1$.

Der_n := the nth derangement number, *ie.*

$$Der_n = n! \sum_{k=0}^{n} \frac{(-1)^n}{k!}. \tag{1.1}$$

In [2] we discussed the existing result

$$d_n = \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} (n-j)!. \tag{1.2}$$

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[2]Note: In [2], the term "linear arrangement" was used instead of "permutation", and "pattern" instead of "substring". Here we use the more conventional terminology. Permutations are meant to be in one-line notation.

We also proved (Equation 2.1)

$$D_n = n! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!}, \tag{1.3}$$

which is equivalent to

$$D_n = \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} (n-j)!. \tag{1.4}$$

Finally in Proposition 2.4, we proved that $D_n = Der_n + (-1)^{n-1}$, $n \geq 1$, which we called the "alternating derangement sequence" since these numbers alternate plus or minus one from the derangement sequence itself. This is sequence A000240 in OEIS [3].

Now we extend the results to forbidden substrings that are not one space apart but k spacings apart (what we call "k-shifts" in the following section).

2. Main Lemmas and Propositions

2.1 Results for $\{d_n\}$ and its k-shifts $\{d_n^k\}$

For the sake of compactness, we define $\{d_n\}$ as the set of permutations on $[n]$ that avoid substrings $12, 23, \ldots, (n1)n, n1$, with d_n being the number of such permutations.

We generalize to k-shifts $\{d_n^k\}$, $k \leq n$, as the set of permutations on $[n]$ that for fixed k, avoid substrings $j(j+k)$, $1 \leq j \leq n-k$. We let d_n^k be the number of such permutations (the reason for power notation will become apparent in the next section).

The forbidden substrings in these permutations can be pictured as a diagonal running k places to the right of the main diagonal of an $n \times n$ chessboard (hence the term "k-shifts"). The permutations that avoid these substrings are not too difficult to handle, and in fact we can count them for any k, as we show in the following proposition.

Proposition 2.1. *For k fixed, $k \leq n$, if d_n^k denotes the number of permutations that avoid substrings $j(j+k)$, $1 \leq j \leq n-k$, then*

$$d_n^k = \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} (n-j)!. \tag{2.1}$$

Proof. For any n and k fixed, there are a total of

$$\binom{n-k}{j}(n-j)!$$

forbidden substrings of length j since the combinatorial term counts the number of ways to get such substrings while the term $(n-j)!$ counts the permutations of the substrings and the remaining elements (note that substrings are not necessarily disjoint but may overlap). Using inclusion-exclusion we get the result. □

We note that the case $k = 1$ is just the result we had for d_n in Equation 1.2.

Corollary 2.2. *The following relation holds for* d_n^k*:*

$$d_n^{k+1} = d_n^k + d_{n-1}^k. \tag{2.2}$$

Proof. By Equation 2.1 and elementary methods. □

Now we define $d_n^0 := Der_n$, which makes sense since in a chessboard of forbidden positions, a derangement is represented by an X in the position (j, j) *ie.* a 0-shift.

Note that Equation 2.2 generalizes the relation in Lemma 2.3 in [2], and we have the following equations starting at $n = 1$:

$$\begin{aligned} d_n = &d_n^1 = Der_n + Der_{n-1} \\ &d_n^2 = d_n + d_{n-1} \\ &d_n^3 = d_n^2 + d_{n-1}^2 \quad \cdots \end{aligned}$$

Using the inital condition condition $d_2^1 = Der_2$, Equation 2.2 defines a binomial-type relation, which, upon iteration, gives us the triangle in Table 1 in the Appendix[3].

It is interesting to note from the triangle that we may get d_n^k starting only from derangement numbers. For example, to get, d_8^5 *ie.* the number of permutations of length 8 with forbidden substrings $\{16, 27, 38\}$, we can start from the upper-left corner of the table and by successive addition along the triangle we can reach the cell $d_8^5 = 27,240$ (or we can obviously use Equation 2.1). Note in particular that for $k = n-1, d_n^k = n! - k!$.

2.2 Results for $\{D_n\}$ and its k-shifts $\{D_n^k\}$

Now we define $\{D_n\}$ as the set of permutations on $[n]$ that avoid substrings $12, 23, \ldots, (n-1)n, n1$, with D_n being the number of such permutations.

[3] This triangle follows the same recurrence as the so-called Euler's Difference Table, which originally had no combinatorial interpretation. Euler's Table also has $n!$ terms at the beginning of each column, which don't apply in our context of k-shifts.

We generalize to k-shifts $\{D_n^k\}$ as the set of permutations on $[n]$ that for fixed k, $k \leq n$, avoid substrings $j(j+k)$, $1 \leq j \leq n-k$, and for $j > n-k$, avoid substrings $(n-k+j)j$, $1 \leq j \leq k$. We let D_n^k be the number of such permutations. These forbidden substrings are easily seen along an $n \times n$ chessboard, where for $j > n-k$, the forbidden positions start again from the first column along a diagonal $(n-k)$ places below the main diagonal.

It turns out that the numbers D_n^k are more difficult to get. They depend on whether n is prime, and more generally, on whether n and k are relatively prime.

Proposition 2.3. *For permutations $\{D_n^k\}$ with k relative prime to n, $n \geq 3$, we can form a forbidden substring of length $j = n-1$ for any k, $k = 1, 2, \ldots, n-1$.*

Proof. Start with forbidden substrings $12, 23, \ldots, (n-1)n$ in $\{D_n\}$ and form the permutation $(12\ldots n)$ in cycle notation. Since any k-shift corresponds to a k-power of the permutation, we see that the longest cycle will have length $n/(n,k)$ for any k, where (n,k) stands for the greatest common divisor. Hence the longest cycle length will be achieved for $(n,k) = 1$, and in this case we will have a cycle of length n, which represents a forbidden substring of length $j = n-1$. □

Note that the proof of the proposition justifies the power notation in D_n^k (and in k-shifts in general).

Note also that the proposition is not true if k is not relative prime to n, for example in the case $n = 6$ and $k = 2$. In this case, the forbidden substrings are $\{13, 24, 35, 46, 51, 62\}$, and we cannot form a cycle of length $n = 6$ (and hence a substring of length 5) using these substrings.

Corollary 2.4. *For all permutations in $\{D_n^k\}$ with k relative prime to n, $n \geq 3$, there exist forbidden substrings of any length j, $j = 1, 2, \ldots, n-1$.*

Proof. By the previous proposition, for $(n,k) = 1$ we can get the longest forbidden substring of length $j = n-1$. Note that it can be considered either a single substring of length $n-1$ or $n-1$ overlapping substrings of length 2. Hence once this substring is obtained, we can split it to get any number of forbidden substrings $j, j = 1, \ldots, n-1$. □

Proposition 2.5. *The number of permutations in $\{D_n^k\}$ with k relative prime to n, $n \geq 3$, $k < n$, is the same as the number of permutations in $\{D_n\}$.*

Proof. By the previous corollary and proposition, since for k's such that $(n,k) = 1$, we can have any number j of forbidden substrings, $j = 1, \ldots, n-1$. It is easy to count that there are exactly $\binom{n}{j}$ ways to get j forbidden substrings (either disjoint or overlapping), and $(n-j)!$ permutations of these substrings and the remaining elements. Then by inclusion-exclusion we get that

$$D_n^k = \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} (n-j)!.$$

But this is the same as Equation 1.4, which counts the number of permutations in $\{D_n\}$. □

As an example of the previous proposition, consider again $n = 8$, $k = 5$. In this case, forbidden substrings in $\{D_8^5\}$ are $\{16, 27, 38, 41, 52, 63, 74, 85\}$. It is easy to count that there are $\binom{8}{4}$ forbidden substrings of length $j = 4$ and $(8-4)!$ permutations of these substrings and the remaining elements. For example, a substring of length 4 (alternatively, four substrings of length 2) is given by 1638 74 and we count $(8-4)! = 4!$ permutations of the four blocks 1638 74 2 5.

Corollary 2.6. *For all permutations in $\{D_p^k\}$ with p prime, $p \geq 3$, we can form a substring of length $j = p - 1$ for any k-shift, $k = 1, 2, \ldots, p - 1$.*

Proof. $(p, k) = 1$, $k = 1, 2, \ldots, p - 1$. □

Corollary 2.7. *The number of permutations in $\{D_p^k\}$ for any k-shift, $k = 1, 2, \ldots, p-1$, is the same as the number of permutations in $\{D_p\}$, p prime, $p \geq 3$.*

Proof. Same proof as in the previous corollary. □

The maximum cycle length achieved for a particular n and k is a very important statistic. In fact, for any fixed n, k-shifts that have the same maximum cycle length also have the same number of permutations, as can be seen in Table 2 in the Appendix[4].

3. Relationships with Generalized Derangements

3.1 For $\{d_n\}$, we need k-permutations

For $\{d_n\}$, we can refer to generalized derangements as discussed in [1]. In this case, if $D(n, k, r)$ denotes the number of k-permutations of n elements that have r fixed points, then we have from [1] that

$$D(n, k, r) = \frac{\binom{k}{r}}{(n-k)!} \sum_{j=0}^{k-r} (-1)^j \binom{k-r}{j} (n-r-j)!. \tag{3.1}$$

From this we have our first proposition:

Proposition 3.1. *If $D(n, k, r)$ denotes the number of k-permutations of n elements that have r fixed points, then d_n can be written as:*

$$d_n = D(n, n-1, 0). \tag{3.2}$$

[4]No similar table appears in other references to our knowledge.

Proof. If we let in $k = n-1$ and $r = 0$ in Equation 3.1, we have that

$$D(n, n-1, 0) = \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} (n-j)!, \tag{3.3}$$

which is just Equation 1.2 for d_n. □

Hence we may interpret from Equation 3.2 that d_n counts the number of derangements of an $(n-1)$-permutation from an n-element set.

This seems intuitive since $\{d_n\}$ consists of the set of permutations that avoid the $n-1$ substrings $12, 23, \ldots, (n-1)n$, while of course, for $k = n$, we have that $D(n, n, 0)$ is just the derangement number Der_n, which counts the derangements of an n-element set.

As we will see, to get a similar expression for D_n, we will not only need to consider k-permutations, but also the number of fixed points. Before doing so we have the following generalization.

3.2 Generalization to subsets of forbidden substrings

Note that by definition $\{d_n\}$ is the set of permutations of $[n]$ that avoid substrings $12, 23, \ldots, (n-1)n$. In this definition we consider the $n-1$ forbidden substrings taken all at the time, but note that we can take subsets of them. If in $\{d_n\}$ we define P^k as a subset of k forbidden substrings, $k \leq n-1$, then we have the following theorem.

Theorem 3.2. *The number of permutations in $\{d_n\}$ that avoid subsets of k forbidden substrings is given by d_n^{n-k}.*

Proof. We see from Equation 3.1 that if $D(n, k, r)$ denotes the number of k-permutations of n elements that have r fixed points, then letting $r = 0$ and multiplying by $(n-k)!$ ways to permute allowed substrings yields

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} (n-j)!, \tag{3.4}$$

which is just d_n^{n-k} by Proposition 2.1. □

What Theorem 3.2 means is that permutations in $\{d_n\}$ taking subsets of k forbidden substrings are k-derangements on n-element sets. Hence when $k = n-1$, we can see from Equation 3.4 that $D(n, n-1, 0)$ is just d_n as before, and when $k = n$, $D(n, n, 0)$ is just Der_n.

Corollary 3.3. *For k fixed, the number of permutations in $\{d_n\}$ produced by any subset of k forbidden substrings, P^k, is invariant.*

Proof. From Theorem 3.2, the number of permutations in $\{d_n\}$ produced by any of the $\binom{n-1}{k}$ subsets P^k is d_n^{n-k}. □

Corollary 3.3 means that if we take $k \leq n-1$ subsets of forbidden substrings, this would mean permutations in $\{d_n\}$ avoiding subsets of $P^k \subseteq P^{n-1}$, $P^{n-1} := P = \{12, 23, \ldots, (n-1)n\}$, and the number of such permutations is invariant for k fixed. For example, for $n = 4$ and $k = 2$, we have the following sets of forbidden substrings: $P_1^2 = \{12, 23\}$, $P_2^2 = \{23, 34\}$, $P_3^2 = \{12, 34\}$, and the number of permutations of 4 elements that avoid these substrings is given by $d_4^2 = 14$ in all three cases. Similarly, for $k = 1$, the sets of forbidden substrings are $P_1^1 = \{12\}$, $P_2^1 = \{23\}$, $P_3^1 = \{34\}$, and the number of permutations that avoid these substrings is $d_4^3 = 18$ in all three cases. Note that in the case $k = 3$ we have $d_4^1 = d_4$, and the case $k = 4$ is $d_4^0 = Der_4$.

3.3 For $\{D_n\}$, need fixed points

To analyze further the permutations in $\{D_n\}$, we have that by adding the substring $n1$ to the set $P = 12, 23, \ldots, (n-1)n$ of forbidden substrings, we now have a total of n invalid substrings, but we cannot describe $\{D_n\}$ just in terms of k-permutations. We need to consider fixed points.

We have the following result:

Proposition 3.4. *If $D(n, k, r)$ denotes the number of k-permutations of n elements that have r fixed points, then D_n can be written as:*

$$D_n = \frac{r}{\binom{n+r-1}{r-1}} D(n+r-1, n+r-1, r). \tag{3.5}$$

Proof. Let $h(n, r)$ denote a permutation on an n-element set that leaves r elements fixed. Using Proposition 6.1 from [2] we have that $h(n, 1) = D_n$. Then, since $h(n, r) = \binom{n}{r} Der(n-r)$, using Lemma 3.1 in [2], *ie.* $D_n = nDer_{n-1}$, we have that $h(n, 2) = \binom{n}{2} D_{n-1}/(n-1)$, and in general that

$$h(n, r) = \frac{\binom{n}{r-1}}{r} D_{n-r+1}. \tag{3.6}$$

Substituting n by $n + r - 1$ and rearranging we get the result. □

To check Equation 3.5 for the case $r = 1$, we see that in this case the equation reduces to $D_n = D(n, n, 1)$, so we recover the result from Proposition 6.1 in [2]. This result shows that D_n not only counts the number of permutations of $[n]$ that avoid substrings $12, 23, \ldots, (n-1)n, n1$, but also the number of n-permutations of an n-element set with exactly 1 fixed point.

References

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APPENDIX

n	Der_n	d_n	d_n^2	d_n^3	d_n^4	d_n^5
1	0					
2	1	1				
3	2	3	4			
4	9	11	14	18		
5	44	53	64	78	96	
6	265	309	362	426	504	600
7	1.854	2.119	2.428	2.790	3.216	3.720
8	14.833	16.687	18.806	21.234	24.024	27.240

Table 1: Some values of d_n^k

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$n=2$	0					
$n=3$	3	3				
$n=4$	8	8	8			
$n=5$	45	45	45	45		
$n=6$	264	270	240	270	264	
$n=7$	1.855	1.855	1.855	1.855	1.855	1.855
$n=8$	14.832	14.816	14.832	13.824	14.832	14.816
$n=9$	133.497	133.497	134.298	133.497	133.497	134.298

Table 2: Some values of D_n^k