ASYMPTOTICS OF THE NUMBER OF STANDARD YOUNG TABLEAUX OF SKEW SHAPE

ALEJANDRO H. MORALES⋆, IGOR PAK⋆ AND GRETA PANOVA†

ABSTRACT. We give new bounds and asymptotic estimates on the number of standard Young tableaux of skew shape in a variety of special cases. Our approach is based on Naruse's hook-length formula. We also compare our bounds with the existing bounds on the numbers of linear extensions of the corresponding posets.

1. INTRODUCTION

The classical *hook-length formula* (HLF) allows one to compute the number $f^\lambda = |\mathrm{SYT}(\lambda)|$ of standard Young tableaux of a given shape [FRT]. This formula had profound applications in Enumerative and Algebraic Combinatorics, Discrete Probability, Representation Theory and other fields (see e.g. [Rom, Sag, S3]). Specifically, the HLF allows to derive asymptotics for f^λ for various families of "large" partitions λ. This was famously used to compute the diagram of a random representations of S_n with respect to the Plancherel measure $(f^\lambda)^2/n!$, see [Bia, VK] (see also [S4]).

For skew shapes λ/μ, little is known about the asymptotics, since there is no multiplicative formula for $f^{\lambda/\mu} = |\mathrm{SYT}(\lambda/\mu)|$. For large $n = |\lambda/\mu|$, the asymptotics are known for few special families of skew shapes (see [AR]), and for fixed μ (see [OO, RV, S2]). In this paper we show that the "naive HLF" give good approximations for $f^{\lambda/\mu}$ in many special cases of interest.

Formally, let λ be a partition of n. Denote by $f^\lambda = |\mathrm{SYT}(\lambda)|$ the number of standard Young tableaux of shape λ. We have:

$$\text{(HLF)} \qquad f^\lambda = \frac{n!}{\prod_{u\in\lambda} h(u)},$$

where $h(u) = \lambda_i - i + \lambda'_j - j + 1$ is the *hook-length* of the square $u = (i,j)$. Now, let λ/μ be a skew shape, $n = |\lambda/\mu|$. By analogy with the HLF, define

$$\text{(naive HLF)} \qquad F(\lambda/\mu) := \frac{n!}{\prod_{u\in\lambda/\mu} h(u)},$$

where $h(u)$ are the usual hook-lengths in λ.[1]

Our main technical tool is Theorem 3.3, which gives

$$(*) \qquad F(\lambda/\mu) \le f^{\lambda/\mu} \le \xi(\lambda/\mu)\, F(\lambda/\mu),$$

where $\xi(\lambda/\mu)$ is defined in Section 3. These bounds turn out to be sharp enough in many cases to obtain explicit good estimates for $f^{\lambda/\mu}$. In fact, they seem to be superior in many cases to standard bounds on the number $e(\mathcal{P})$ of linear extensions for general posets. We also give several examples when the lower bound is sharp but not the upper bound, and vice versa (see e.g. §10.1 and §11.2).

Let us emphasize an important special case of *thick ribbons* δ_{k+r}/δ_k, where $\delta_k = (k-1, k-2, \ldots, 2, 1)$ denotes the *staircase shape* (see Figure 4). The following result illustrates the strength our bounds (cf. §3.3).

---

[1]Note that $F(\lambda/\mu)$ is not necessarily an integer.

Theorem 1.1. *Let* $\upsilon_k = (\delta_{2k}/\delta_k)$, *where* $\delta_k = (k-1, k-2, \dots, 2, 1)$. *Then*

$$\frac{1}{6} - \frac{3}{2}\log 2 + \frac{1}{2}\log 3 + o(1) \le \frac{1}{n}\left(\log f^{\upsilon_k} - \frac{1}{2}n\log n\right) \le \frac{1}{6} - \frac{7}{2}\log 2 + 2\log 3 + o(1),$$

where $n = |\upsilon_k| = k(3k-1)/2$.

Here the LHS ≈ -0.3237, and the RHS ≈ -0.0621. Note that the number $f^{\delta_{k+r}/\delta_k}$ of standard Young tableaux for thick ribbons have been previously considered in [BR], but the tools in that paper apply only for $r \to \infty$.

We should mention that in the theorem and in many other special cases, the leading terms of the asymptotics are easy to find. Thus most of our effort is over the lower order terms which are much harder to determine (see Section 13).

The rest of the paper are structured as follows. We start with general results on linear extensions (Section 2), standard Young tableaux of skew shape (Section 3), and excited diagrams (Section 4). We then proceed to our main results concerning asymptotics for $f^{\lambda/\mu}$ in the following cases:

(1) when both λ, μ have the *Thoma–Vershik–Kerov limit* (Section 5). Here the Frobenius coordinates scale linearly and $f^{\lambda/\mu}$ grow exponentially.

(2) when both λ, μ have the *stable shape limit* (Section 6). Here the row and column lengths scale as $\sqrt{n}$, and $f^{\lambda/\mu} \approx \sqrt{n!}$ up to an exponential factor.

(3) when λ/μ have *subpolynomial depth* (Section 7). Here both the row and column lengths of λ/μ grow as $n^{o(1)}$, and $f^{\lambda/\mu} \approx n!$ up to a factor of *intermediate growth* (i.e. super-exponential and subfactorial), which can be determined by the depth growth function.

(4) when λ/μ is a large ribbon hook (Section 10). Here λ/μ scale linearly along fixed curve, and $f^{\lambda/\mu} \approx n!$ up to an exponential factor.

(5) when λ/μ is a *slim shape* (Section 11). Here μ is fixed and $\ell, \lambda_\ell/\ell \to \infty$, where $\ell = \ell(\lambda)$. Here $f^{\lambda/\mu} \sim f^\lambda f^\mu/|\mu|!$.

We illustrate these cases with various examples. Further examples and more specialized applications are given in sections 8 and 12. We conclude with final remarks in Section 13.

2. Linear extensions of posets

2.1. **Notation.** We assume the reader is familiar with standard definitions and notation of Young diagrams, Young tableaux, ranked posets, linear extensions, chains, antichains, etc. In case of confusion, we refer the reader to [S3], and will try to clarify the notation throughout the paper.

To further simplify the notation, we use the same letter to denote the partition and the corresponding Young diagram. To avoid the ambiguity, unless explicitly stated otherwise, we always assume that skew partitions are connected. To describe disconnected shapes, we use $\lambda/\mu \circ \pi/\tau$ notation.

We make heavy use of Stirling's formula $\log n! = n\log n - n + O(\log n)$. Here and everywhere below log denotes natural logarithm. There is similar formula for the *double factorial* $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$, the *superfactorial* $\Phi(n) = 1! \cdot 2! \cdots n!$, the *double superfactorial* $\Psi(n) = 1! \cdot 3! \cdot 5! \cdots (2n-1)!$, and the *super doublefactorial* $\Lambda(n) = 1!! \cdot 3!! \cdot 5!! \cdots (2n-1)!!$:

$$\log(2n-1)!! = n\log n + (\log 2 - 1)n + O(1),$$

$$\log \Phi(n) = \frac{1}{2}n^2\log n - \frac{3}{4}n^2 + 2n\log n + O(n),$$

$$\log \Psi(n) = n^2\log n + \left(\log 2 - \frac{3}{2}\right)n^2 + \frac{5}{2}n\log n + O(n),$$

$$\log \Lambda(n) = \frac{1}{2}n^2\log n + \left(\frac{\log 2}{2} - \frac{3}{4}\right)n^2 + \frac{1}{2}n\log n + O(n)$$

(see [OEIS, A001147], [OEIS, A008793], [OEIS, A168467]), and [OEIS, A057863]).

2.2. **Ranked posets.** Let $\mathcal{P}$ be a ranked poset on a finite set X with linear ordering denoted by $\prec$. Unless stated otherwise, we assume that $|X| = n$. Let $e(\mathcal{P})$ be the number of *linear extensions* of $\mathcal{P}$.

Denote by $\ell = \mathrm{L}(\mathcal{P})$ and $m = \mathrm{M}(\mathcal{P})$ the length of the *longest chain* and the *longest antichain*, respectively. Let $r_1, \ldots, r_k$ denote the number of of elements in X of each rank, so $k \geq \ell$ and $n = r_1 + \ldots + r_k$. Similarly, let $\mathcal{P}$ have a decomposition into chains $C_1, \ldots, C_m$, and denote $\ell_i = |C_i|$, so $\ell_1 + \ldots + \ell_m = n$. Recall that such decompositions exists by the *Dilworth theorem* (see e.g. [Tro]).

Theorem 2.1. *For every ranked poset $\mathcal{P}$ as above, we have:*

$$ r_1! \cdots r_k! \leq e(\mathcal{P}) \leq \frac{n!}{\ell_1! \cdots \ell_m!} \, . $$

These bounds are probably folklore; for the lower bound see e.g. [Bri]. They are easy to derive but surprisingly powerful. We include a quick proof for completeness.

Proof. For the lower bound, label elements of rank 1 with numbers $1, \ldots, r_1$ in any order, elements of rank 2 with numbers $r_1+1, \ldots, r_1+r_2$ in any order, etc. All these labelings are clearly linear extensions and the bound follows. For the upper bound, observe that every linear extension of $\mathcal{P}$ when restricted to chains, defines a set partition of $\{1, \ldots, n\}$ into m subsets of sizes $|C_1|, \ldots, |C_m|$. $\square$

Remark 2.2. In Theorem 2.1, one can replace the LHS with the product $|A_1|! \cdots |A_k|!$ for any antichain decomposition, and the RHS with $n!/|C_1|! \cdots |C_k|!$ for any chain decomposition, not necessarily minimal.[2] It is also worth noting that the upper bound in Theorem 2.1 is always better than the easy to use upper bound $e(\mathcal{P}) \leq m^n$ (cf. [Bri, BT]).

Let us also mention the following unusual bound for the number of linear extensions of general posets. Denote also by $\mathrm{B}(x) = \#\{y \in \mathcal{P}, y \succcurlyeq x\}$ the size of the upper ideal in $\mathcal{P}$ spanned by x.

Theorem 2.3 ([HP]). *For every poset $\mathcal{P}$, in the notation above, we have:*

$$ e(\mathcal{P}) \geq \frac{n!}{\prod_{x \in \mathcal{P}} \mathrm{B}(x)} \, . $$

The lower bound was proposed by Stanley [S3, Exc. 3.57] and proved by Hammett and Pittel [HP]. It is tight for forests, i.e. disjoint unions of tree posets (see e.g. [Sag]). Note that this bound can be different for the poset $\mathcal{P}$ and the *dual poset* $\mathcal{P}^*$. We refer to §13.1 further references on the number of linear extensions.

2.3. **Square shape.** The following is the motivating example for this paper. Let $\lambda = (k^k)$, $\mu = \varnothing$, $n = k^2$. Clearly, $e(k^k) = |\mathrm{SYT}(\lambda)|$ (see [OEIS, A039622]).

Observe that $(r_1, r_2, \ldots) = (1, 2, \ldots, k-1, k, k-1, \ldots, 1)$ in this case and $m = \mathrm{M}(k^k) = k$. Theorem 2.1 gives $e(k^k) \geq \Phi(k) \, \Phi(k-1)$, which implies

$$ (\boxplus) \qquad \log e(k^k) \geq \log \Phi(k) + \log \Phi(k-1) = \frac{1}{2} n \log n - \frac{3}{2} n + O(\sqrt{n} \log n) \, . $$

For the upper bound, observe that $\mathcal{Q}_n = (k^k)$ can be decomposed into chains of lengths $2k-1, \ldots, 3, 1$, each involving two adjacent diagonals. We have then:

$$
\begin{aligned}
\log e(k^k) &\leq \log \binom{n}{2k-1, \, 2k-3, \, \ldots, \, 3, \, 1} = \log n! - \log \Psi(k) \\
(\oplus) \qquad &\leq \frac{1}{2} n \log n + \left(\frac{1}{2} - \log 2 \right) n + O(\sqrt{n} \log n) \, .
\end{aligned}
$$

In other words, the lower and upper bounds agree in the leading term of the asymptotics but not in the second term. Let us compare this with an exact value of $e(k^k)$. The HLF gives:

$$ e(k^k) = \frac{n! \, \Phi(k-1)^2}{\Phi(2k-1)} \, , $$

---

[2]In fact, in some cases a larger chain decomposition gives a better bound, see e.g. http://mathoverflow.net/q/251773.

which implies

$$\log e(k^k) \;=\; \frac{1}{2}\, n \log n \;+\; \left(\frac{1}{2} - 2\log 2\right) n \;+\; O\big(\sqrt{n}\, \log n\big).$$

Since $\left(\frac{1}{2} - 2\log 2\right) \approx -0.8863$ and $\left(\frac{1}{2} - \log 2\right) \approx -0.1931$, we conclude that the true constant -0.8863 of the second asymptotic term lies roughly halfway between the lower bound -1.5 in ($\boxplus$) and the upper bound -0.1931 in ($\oplus$). Let us mention also that the lower bound $e(k^k) \geq n!/(k!)^{2k}$ in Theorem 2.3 is much too weak.

2.4. **Skew shapes.** Let λ/μ be a skew shape Young diagram (see Figure 1). To simplify the notation, we use λ/μ to also denote the corresponding posets of squares increasing downward and to right. The main object of this paper is the asymptotic analysis of

$$e(\lambda/\mu) \;=\; f^{\lambda/\mu} \;=\; |\operatorname{SYT}(\lambda/\mu)|,$$

the number of standard Young tableaux of shape λ/μ. Note that both are standard notation in different areas; we use them interchangeably throughout the paper.

The following determinant formula due to Feit [Feit] is a standard result in the area, often referred to as the *Jacobi–Trudi identity* (see e.g. [Sag, S3]):

$$(\triangledown) \qquad f^{\lambda/\mu} \;=\; n! \, \det \left(\frac{1}{\lambda_i - \mu_j - i + j}\right)_{i,j=1}^{\ell(\lambda)}.$$

Unfortunately, due to the alternating sign nature of the determinant, this formula is difficult to use in the asymptotic context. Here is the only (quite weak) general bound we found in the literature.

Proposition 2.4. *For every skew shape λ/μ, we have:*

$$f^{\lambda/\mu} \;\leq\; \frac{|\lambda|!\, f^\mu}{|\mu|!\, f^\lambda} \;=\; \frac{\prod_{u \in \lambda} h(u)}{\prod_{v \in \mu} h(v)}.$$

Proof. Recall the standard equalities for the *Littlewood–Richardson* (LR–) *coefficients* $c^\lambda_{\mu\nu}$, where $|\lambda| = |\mu| + |\nu|$. We have:

$$f^\mu f^\nu \binom{|\mu| + |\nu|}{|\mu|} \;=\; \sum_{\lambda \vdash |\mu| + |\nu|} c^\lambda_{\mu\nu} f^\lambda \quad \text{and} \quad f^{\lambda/\mu} \;=\; \sum_{\nu \vdash |\lambda| - |\mu|} c^\lambda_{\mu\nu} f^\nu.$$

From here and the Burnside identity [Sag, S3], we have:

$$f^{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} f^\nu \leq \sum_\nu \binom{|\lambda|}{|\mu|} \frac{f^\mu f^\nu}{f^\lambda} f^\nu = \frac{f^\mu}{f^\lambda} \binom{|\lambda|}{|\mu|} \sum_\nu (f^\nu)^2 = \frac{f^\mu \, |\lambda|!}{f^\lambda \, |\mu|!\, |\nu|!} \, |\nu|! = \frac{|\lambda|!\, f^\mu}{|\mu|!\, f^\lambda},$$

proving the first inequality. The equality follows from the HLF. $\square$

Example 2.5. Let $\lambda/\mu = (4^2 32/21)$ be a skew shape of size $n = 10$. The Jacobi–Trudi formula ($\triangledown$) gives $f^{\lambda/\mu} = e(\lambda/\mu) = 3060$.

For the poset λ/μ, the length of the longest antichain is $m = 4$, and the number of elements of each rank are $3, 4, 3$. Similarly, the poset can be decomposed into four chains of sizes $3, 3, 3, 1$ (see Figure 1). Theorem 2.1 gives:

$$864 \;=\; 3!\,4!\,3! \;\leq\; e(\lambda/\mu) \;\leq\; \frac{10!}{3!\,3!\,3!\,1!} \;=\; 16800.$$

The sizes $\textsc{b}(x)$ are given in Figure 1. Then Theorem 2.3 gives a slightly weaker bound:

$$e(\lambda/\mu) \;\geq\; \frac{10!}{2^2 3^2 5^2 6} \;=\; 672.$$

Finally, Proposition 2.4 and the HLF gives the following very weak upper bound:

$$e(\lambda/\mu) \;\leq\; \frac{13!\, e(21)}{3!\, e(4^2 32)} \;=\; \frac{13! \cdot 2}{3! \cdot 8580} \;=\; 241920.$$

FIGURE 1. The shape $\lambda/\mu = (4^2 32/21)$, its hook-lengths, the Hasse diagram of poset $\mathcal{P}$, and the sizes of the upper ideals spanned by each element of the poset.

Remark 2.6. Calculations similar to the square shape can be done for various other geometric shapes with known nice product formulas. Beside the usual Young diagrams these include *shifted diagrams* and various ad hoc shapes as in Figure 2 (see [AR, Pan] and the last example in [MPP3]). In all these cases,

$$(\heartsuit) \qquad \log e(\mathcal{P}) = \frac{1}{2} n \log n + O(n).$$

While the bounds in Theorem 2.1 again give the leading term correctly, they are all off in the second asymptotic term. This observation is the key starting point for this work. Roughly speaking, in many cases, the inequalities $(*)$ in the introduction make the gap between the second asymptotic term smaller.
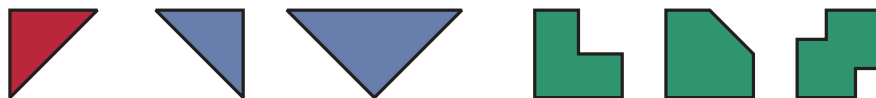

FIGURE 2. Staircase shape, two shifted shapes and three ad hoc shapes with product formulas.

3. HOOKS FORMULAS FOR SKEW SHAPES

3.1. **Definition of excited diagrams.** Let λ/μ be a skew partition and D be a subset of the Young diagram of λ. A cell $u = (i, j) \in D$ is called *active* if $(i + 1, j)$, $(i, j + 1)$ and $(i + 1, j + 1)$ are all in $\lambda \setminus D$. Let u be an active cell of D, define $\alpha_u(D)$ to be the set obtained by replacing (i, j) in D by $(i + 1, j + 1)$. We call this replacement an *excited move*. An *excited diagram* of λ/μ is a subset of squares in λ obtained from the Young diagram of μ after a sequence of excited moves on active cells. Let $\mathcal{E}(\lambda/\mu)$ be the set of excited diagrams of λ/μ, and let $\xi(\lambda/\mu) = |\mathcal{E}(\lambda/\mu)|$.

The following explicit characterization in [MPP1] is also very helpful. Let D be a subset of squares in λ with the same number of squares in each diagonal as μ. Define an order relation on squares $(i, j) \preccurlyeq (i', j')$ if and only if $i \leq i'$ and $j \leq j'$. Then D is an excited diagram if and only if the relation $\preccurlyeq$ on squares of μ hold for D.

We conclude with an explicit formula for $\xi(\lambda/\mu)$. For the diagonal Δ that passes through the cell (i, μ_i), denote by ϑ_i the row in which Δ intersects the boundary of λ.

Theorem 3.1 ([MPP1]). *Let λ/μ be skew partition and let $\ell = \ell(\mu)$. In the notation above, we have:*

$$\xi(\lambda/\mu) = \det\left[\binom{\vartheta_i + \mu_i - i + j - 1}{\vartheta_i - 1}\right]_{i,j=1}^{\ell}$$

This formula follows from a characterization of the excited diagrams as certain *flagged tableaux* of shape μ with entries in row i at most ϑ_i, a different border strip decomposition, and the Lindström–Gessel–Viennot lemma. We refer to [MPP1, MPP2] for details and the references.

3.2. **NHLF and its implications.** The following recent result is the crucial advance which led to our study (cf. [MPP1, MPP2]).

Theorem 3.2 (Naruse [N2]). *Let λ, μ be partitions, such that $\mu \subset \lambda$. We have:*

(NHLF) $$e(\lambda/\mu) = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in \lambda \backslash D} \frac{1}{h(u)} \, .$$

We can now present a corollary of the NHLF, which is the main technical tool of this paper. For a general bound of this type, it is quite powerful in applications (see below). It is also surprisingly easy to prove.

Theorem 3.3. *For every skew shape λ/μ, $|\lambda/\mu| = n$, we have:*

(∗) $$F(\lambda/\mu) \leq e(\lambda/\mu) \leq \xi(\lambda/\mu) \, F(\lambda/\mu) \,,$$

where

$$F(\lambda/\mu) = n! \prod_{u \in \lambda/\mu} \frac{1}{h(u)}$$

is defined as in the introduction.

Note that when restricted to skew shapes, the lower bound in Theorem 2.3 is clearly weaker than the lower bound (∗), and coincides with it exactly for the *ribbon hooks* (shapes with at most one square in every diagonal).

Proof. The lower bound follows from the NHLF, since μ is an excited diagram in $\mathcal{E}(\lambda/\mu)$. For the upper bound, note that under the excited move the product $\prod_{u \in \lambda \backslash D} h(u)$ increases. Thus this product is minimal for $D = \mu$, and the upper bound follows from the NHLF. $\square$

Remark 3.4. Note also that both bounds in Theorem 2.1 are symmetric with respect to taking a dual poset. On the other hand, one can apply Theorem 3.3 to either (λ/μ) or the dual shape $(\lambda/\mu)^*$ obtained by a 180 degrees rotation.

Example 3.5. As in Example 2.5, let $\lambda/\mu = (4^2 32/21)$ and $e(\lambda/\mu) = 3060$. The hook-lengths in this case are given in Figure 1. We have

$$F(\lambda/\mu) = \frac{10!}{5 \, 4^2 \, 3^2 \, 2^2} = 1260 \,.$$

In this case we have $\vartheta_1 = 2$, $\vartheta_2 = 3$, and by Theorem 3.1

$$\xi(\lambda/\mu) = \det \begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix} = 5 \,.$$

In this case, Theorem 3.3 gives:

$$1260 \leq e(\lambda/\mu) \leq 1260 \cdot 5 = 6300 \,.$$

Both bounds are better than the bounds in Example 2.5.

3.3. **Comparison of bounds.** To continue the theme of this section, we make many comparisons between the bounds on $e(\lambda/\mu)$ throughout the paper, both asymptotically and in special cases. Here is the cleanest comparison, albeit under certain restrictions.

Fix a skew shape $\upsilon = (\lambda/\mu)$ with $\ell = \mathrm{L}(\lambda)$. Let $A_1, \ldots, A_\ell$ be an antichain decomposition with elements in A_k lying in the antidiagonal $A_k = \{(i, j) \in \upsilon \mid i + j = k + s - 1\}$, where $s = \min\{i + j \mid (i, j) \in \upsilon\}$. Denote $r_k = |A_k|$.

Theorem 3.6. *In the notation above, suppose $r_1 \leq r_2 \leq \ldots \leq r_k$. Then we have:*

$$r_1! \cdots r_k! \leq F(\lambda/\mu) \,.$$

For example, the theorem applies to the thick ribbon shapes (see §8).

Proof. Consider the hooks of the squares on a given antidiagonal A_i. Let the number of squares of rank $\geq i$ (i.e. on the antidiagonal or below/right of it) be N_i. Every such square belongs to at most 2 hooks h_u and h_v with $u, v \in A_i$, and that happens only when it is contained in the "subtriangle" with antidiagonal A_i (everything left after erasing rows and columns which have no boxes in A_i). There are at most $\binom{r_i}{2}$ boxes in such a rectangle, and cannot be more than all boxes below diagonal i, which is $N_i - r_i$. In fact, since $r_{i+1} \geq r_i$, there is at least one box below the diagonal which is not counted twice, so the bound is $N_i - r_i - 1$. Hence by counting the squares covered by a hook, we have:

$$\sum_{u \in A_i} h_u \ \leq \ N_i \ + \ \min\left\{N_i - r_i - 1, \binom{r_i}{2}\right\}.$$

Noting that for the last diagonal we have $h_u = 1$ and $N_k = r_k$, we obtain the following

$$
(\maltese) \qquad \prod_{i=1}^{k} r_i! \prod_i \prod_{u \in A_i} h_u \ \leq \ \prod_{i=1}^{k} r_i! \prod_{i=1}^{k-1} \left(\frac{\sum_{u \in A_i} h_u}{r_i}\right)^{r_i}
$$

$$
\leq \ r_k! \prod_{i=1}^{k-1} \frac{r_i!\left(N_i + \min\{N_i - r_i - 1, \binom{r_i}{2}\}\right)^{r_i}}{r_i^{r_i}}.
$$

For the first inequality is the AM–GM inequality for the product of hooks on a given diagonal, and for the second – the estimate for their sum.

Next we will need the following inequality for binomial coefficients.

Lemma 3.7. *Let $t \geq r \geq 3$. Then*

$$\binom{t+r}{r} \ \geq \ \left(\frac{2t+r-1}{r}\right)^r.$$

We use the lemma to estimate the RHS of inequality ($\maltese$). For $r_i \geq 3$, take $t = N_i - r_i \geq r_i$ for $i < k$. We have:

$$\frac{(2N_i - 1 - r_i)^{r_i}}{r_i^{r_i}} \ \leq \ \binom{N_i}{r_i}.$$

For $r_i = 2$, we have $\min(N_i - 2, 1) \leq 1$ and the inequality becomes

$$\frac{(N_i + \min\{N_i - 2, 1\})^2}{2^2} \ \leq \ \frac{(N_i + 1)^2}{4} \ \leq \ \binom{N_i}{2},$$

which holds trivially whenever $N_i \geq 5$. For $N_i \leq 4$ and $r_i = 2$, under the assumptions, we must have a partition of size at most 5; such cases are checked by direct computation. Finally, for $r_i = 1$, we have $\min(N_i - 1, 0) = 0$, and we have the equality. Putting all these together in the RHS of ($\maltese$), we have

$$\prod_{i=1}^{k} r_i! \prod_u h_u \ \leq \ \prod_{i=1}^{k} r_i! \prod_{i=1}^{k} \binom{N_i}{r_i} \ = \ N_1!,$$

since $N_i = r_i + N_{i+1}$ and $N_1 = N$. Dividing both sides by the product of hooks gives the desired inequality. $\qquad\square$

Remark 3.8. The antichain decomposition in the theorem can be generalized from rank antichains to all ordered antichain decompositions $(A_1, \ldots, A_\ell)$, such that $x \in A_i$, $y \in A_j$, $i < j$, then $x \prec y$. The proof extends verbatim; we omit the details. Note also that the theorem cannot be extended to general skew shapes. The examples include $(2^2/1)$, $(3^3/321)$ and $(4^4/2^2)$.

Prof of Lemma 3.7. First, let $t = r \geq 3$, the inequality becomes

$$(3.1) \qquad\qquad \binom{2r}{r} \ \geq \ (3 - 1/r)^r.$$

A direct calculation shows that it holds for $r = 3, 4, 5$ (and not for $r = 2$). Now assume $r \geq 6$, and take the logarithm of the LHS and apply the lower and upper bounds from Stirling's approximation:

$$\log(2r)! - 2\log r! \geq 1 + (2r + 1/2)\log 2 - \frac{1}{2}\log r - 2\log \sqrt{2\pi} \geq r \log 3$$

since the difference is an increasing function of r for $r \geq (4\log 2 - 2\log 3)^{-1} \approx 1.738$, and the difference is positive (≈ 0.338) when $r = 6$. This gives (3.1) for $r \geq 6$.

Next, let $t > r$, and replace $\binom{t+r}{r}$ by the smaller $\binom{t-1+r}{r}$. Taking logarithms, we have

$$\log \binom{t - 1 + r}{r} - r \log \left(\frac{2t + r - 1}{r} \right) = \sum_{i=0}^{r-1} \log(t + i) - r \log(2t + r - 1) + r \log r - \log r! \,.$$

The derivative of the last function satisfies

$$\sum_{i=0}^{r-1} \frac{1}{t + i} - \frac{r}{t + \frac{r-1}{2}} \geq 0,$$

by the arithmetic mean vs. harmonic mean inequality. So this is an increasing function in t and its minimal value for $t \geq r + 1$ is achieved when $t = r + 1$, when by inequality (3.1) is already positive, so the claim holds. $\qquad\square$

4. Bounds on the number of excited diagrams

4.1. **Non-intersecting paths.** We recall that the excited diagrams of λ/μ are in bijection with families of certain non-intersecting grid paths $\gamma_1, \ldots, \gamma_k$ with a fixed set of start and end points, which depend only on λ/μ. This was proved in [MPP1, MPP2], and based on the earlier works by Kreiman [Kre], Lascoux and Schützenberger [LasS], and Wachs [Wac], on flagged tableaux.

Formally, given a connected skew shape λ/μ, there is unique family of border-strips (i.e. non-intersecting paths) $\gamma_1^*, \ldots, \gamma_k^*$ in λ with support λ/μ, where each border strip γ_i^* begins at the southern box (a_i, b_i) of a column and ends at the eastern box (c_i, d_i) of a row [Kre, Lemma 5.3]. Moreover, all non-intersecting paths $(\gamma_1, \ldots, \gamma_k)$ contained in λ with $\gamma_i : (a_i, b_i) \to (c_i, d_i)$ are in correspondence with excited diagrams of the shape λ/μ [Kre, §5.5].

Proposition 4.1 (Kreiman [Kre], see also [MPP2]). *The non-intersecting paths $(\gamma_1, \ldots, \gamma_k)$ in λ where $\gamma_i : (a_i, b_i) \to (c_i, d_i)$ are uniquely determined by their support and moreover these supports are in bijection with complements of excited diagrams of λ/μ.*

Corollary 4.2. *In the notation above,*

$$\xi(\lambda/\mu) = \#\{non\text{-}intersecting\ paths\ (\gamma_1, \ldots, \gamma_k) \mid \gamma_i \subseteq \lambda,\ \gamma_i : (a_i, b_i) \to (c_i, d_i)\}.$$

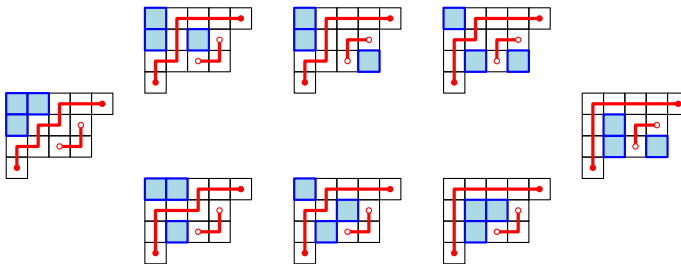See Figure 3 for an example of the proposition and the corollary.

FIGURE 3. The eight non-intersecting paths (γ_1, γ_2) whose support are the complement of the excited diagrams of the shape $(54^2 1/21)$.

4.2. **Bounds on** $\xi(\lambda/\mu)$**.** We prove the following two general bounds. They are elementary but surprisingly powerful in applications.

Lemma 4.3. *Let* λ/μ *be a skew shape,* $n = |\lambda/\mu|$*. Then* $\xi(\lambda/\mu) \leq 2^n$*.*

Proof. Each path is determined by its vertical and horizontal steps. We conclude:

$$\xi(\lambda/\mu) \leq \prod_{i=1}^{k} 2^{|\gamma_i|-1} = 2^{n-k} \leq 2^n \,,$$

where k is the number of paths γ_i, as above. $\square$

The *Durfee square* in Young diagram λ is the maximal square which fits λ. Let $d(\lambda)$ be the size of the Durfee square.

Lemma 4.4. *Let* λ/μ *be a skew shape and let* $d = d(\lambda)$*. Then* $\xi(\lambda/\mu) \leq n^{2d^2}$*.*

Proof. Since λ/μ is connected, the path traced by the border of λ is a ribbon of length at most n and height at least d, so we must have that $\lambda_1, \ell(\lambda) \leq n - d$. Then $\lambda \subset \nu$, where $\nu = \left((n-d)^d, d^{n-2d}\right)$. This implies $\xi(\lambda/\mu) \leq \xi(\nu/\mu)$.

As before, we estimate $\xi(\nu/\mu)$ by the number of non-intersecting lattice paths inside that shape. Consider the lattice paths above the diagonal $i = j$. Since the shape ν/μ within the top d rows is the complement of a straight shape, the lattice paths there all start at the bottom row and follow the shape μ, ending at the rightmost column, which is of length d. So the paths above the diagonal are at most d and their endpoints are no longer fixed. Now each such path has length at most n. Those above diagonal, have at most $(d-1)$ vertical steps and those below diagonal have at most $(d-1)$ horizontal steps. Therefore,

$$\xi(\lambda/\mu) \leq \xi(\nu/\mu) \leq \binom{n-1}{d-1}^{2d} \leq n^{2d^2} \,,$$

as desired. $\square$

5. The Thoma–Vershik–Kerov limit shape

Denote by $a(\lambda) = (a_1, a_2, \ldots)$, $b(\lambda) = (b_1, b_2, \ldots)$ the Frobenius coordinates of λ. Let $\overline{\alpha} = (\alpha_1, \ldots, \alpha_k)$, $\overline{\beta} = (\beta_1, \ldots, \beta_k)$ be fixed sequences in $\mathbb{R}_+^k$. We say that a sequence of partitions $\{\lambda^{(n)}\}$ has a *Thoma–Vershik–Kerov* (TVK) *limit* $(\overline{\alpha}, \overline{\beta})$, write $\lambda^{(n)} \to (\overline{\alpha}, \overline{\beta})$, if $a_i/n \to \alpha_i$ and $b_i/n \to \beta_i$ as $n \to \infty$, for all $1 \leq i \leq k$.

Theorem 5.1. *Let* $\{\upsilon_n = \lambda^{(n)}/\mu^{(n)}\}$ *be a sequence of skew shapes with a TVK limit. Formally, in the notation above, suppose* $\lambda^{(n)} \to (\overline{\alpha}, \overline{\beta})$*, where* $\alpha_1, \beta_1 > 0$*, and* $\mu^{(n)} \to (\overline{\pi}, \overline{\tau})$ *for some* $\overline{\alpha}, \overline{\beta}, \overline{\pi}, \overline{\tau} \in \mathbb{R}_+^k$*. Then*

$$\log e(\upsilon_n) = cn + o(n) \quad as \quad n \to \infty \,,$$

where

$$c = c(\overline{\alpha}, \overline{\beta}, \overline{\pi}, \overline{\tau}) = \gamma \log \gamma - \sum_{i=1}^{k} (\alpha_i - \pi_i) \log(\alpha_i - \pi_i) - \sum_{i=1}^{k} (\beta_i - \tau_i) \log(\beta_i - \tau_i)$$

and

$$\gamma = \sum_{i=1}^{k} (\alpha_i + \beta_i - \pi_i - \tau_i) \,.$$

Note the $d(\lambda) \leq k$ condition is essential in the theorem, as otherwise $\xi(\lambda/\mu)$ can be superexponential (see Section 11).

Proof. First, note that $d(v_n) \leq k$, so $\xi(v_n) < n^{2k^2}$. Thus, by Theorem 3.3, it suffices to compute the lead term in the asymptotics of $F(\lambda^{(n)}/\mu^{(n)})$. Observe that $|v_n| \sim \gamma n$. Next, observe that in the i-th row, up to an additive constant $\leq d(\lambda) = k$, all the hooks are $1, \ldots, (\alpha_i - \pi_i + o(1))n$, and in the i-th column the product of hooks is $1, \ldots, (\beta_i - \tau_i + o(1))n$. Therefore, by the Stirling formula, we have:

$$\log F(v_n) \sim \log(\gamma n)! - \sum_i \log\big[(\alpha_i - \pi_i)n\big]! - \sum_i \log\big[(\beta_i - \tau_i)n\big]!$$

$$\sim n\left(\gamma \log \gamma - \sum_i (\alpha_i - \pi_i)\log(\alpha_i - \pi_i) - \sum_i (\beta_i - \tau_i)\log(\beta_i - \tau_i)\right),$$

as $n \to \infty$. We omit the easy details. $\qquad\square$

Remark 5.2. When $\mu = \varnothing$, the exponential nature of f^λ was recently studied in [GM]; one can view our results as a asymptotic version in the skew shape case. For a fixed partition μ, Okounkov and Olshanski [OO] give an explicit formula for the ratio $f^{\lambda/\mu}/f^\lambda$ which can be computed explicitly in this case (see also [CGS, S2]). Note also that the notion of TVK limit is rather weak, as it is oblivious to adding $s = o(n)$ to the rows (columns), which can affect $f^{\lambda/\mu}$ by a factor of $2^{\Theta(s)}$. Thus the error term in the theorem cannot be sharpened.

6. The stable shape

6.1. **The usual stable shape.** Let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing continuous function. Consider a sequence of partitions $\{\lambda^{(n)}\}$, such that the rescaled diagrams $\frac{1}{\sqrt{n}}\big[\lambda^{(n)}\big]$ converge uniformly to ω, where $[\lambda]$ denotes the curve giving the boundary of Young diagram λ. In this case we say that a sequence of partitions $\{\lambda^{(n)}\}$ has a *stable shape* ω, and write $\lambda^{(n)} \to \omega$.

Suppose we are given two stable shapes $\omega, \pi : \mathbb{R}_+ \to \mathbb{R}_+$, such that $\pi(x) \leq \omega(x)$ for all $x \geq 0$. To simplify the notation, denote by $\mathcal{C} = \mathcal{C}(\omega/\pi) \subset \mathbb{R}_+^2$ the region between the curves. One can view $\mathcal{C}$ as the stable shape of skew diagrams, and denote by $\text{area}(\omega/\pi)$ the area of $\mathcal{C}$.

Theorem 6.1. *Let $\omega, \pi : [0, a] \to [0, b]$ be continuous non-increasing functions, and suppose that $\text{area}(\omega/\pi) = 1$. Let $\{v_n = \lambda^{(n)}/\mu^{(n)}\}$ be a sequence of skew shapes with the stable shape ω/π, i.e. $\lambda^{(n)} \to \omega$, $\mu^{(n)} \to \pi$. Then*

$$\log e(v_n) \sim \frac{1}{2} n \log n \quad \text{as} \quad n \to \infty.$$

We prove the theorem below. Let us first state a stronger result for a more restrictive notion of stable shape limit, which shows how the stable shape ω/π appears in the second term of the asymptotic expansion.

Remark 6.2. Note that as stated, the theorem applies to nice geometric shapes with $\sqrt{n}$ scaling and the Plancherel shape ω, see [VK] (see also [Rom, S4]), but not the Erdős–Szekeres limit shape (see e.g. [DVZ]), since random partitions have $\Theta(\sqrt{n}\log n)$ parts. The proof, however, can be adapted to work in this case (see also §13.6).
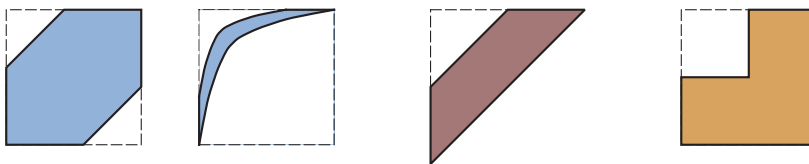

FIGURE 4. Two stable shapes which fit a square, a stable shape of thick ribbons and inverted thick hooks.

6.2. **Strongly stable shape.** Let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing continuous function. Suppose sequence of partitions $\{\lambda^{(n)}\}$ satisfies the following property

$$(\sqrt{n} - L)\omega \subset \left[\lambda^{(n)}\right] \subset (\sqrt{n} + L)\omega, \quad \text{for some } L > 0,$$

where we write $[\lambda]$ to denote a function giving the boundary of Young diagram λ. In this case we say that a sequence of partitions $\{\lambda^{(n)}\}$ has a *strongly stable shape* ω, and write $\lambda^{(n)} \mapsto \omega$.

In the notation above, define the *hook function* $\hbar : \mathcal{C} \to \mathbb{R}_+$ to be the scaled function of the hooks: $\hbar(x, y) := h\big(\lfloor x\sqrt{n}\rfloor, \lfloor y\sqrt{n}\rfloor\big)$.

Theorem 6.3. *Let* $\omega, \pi : [0, a] \to [0, b]$ *be continuous non-increasing functions, and suppose that* $\mathrm{area}(\omega/\pi) = 1$. *Let* $\{v_n = \lambda^{(n)}/\mu^{(n)}\}$ *be a sequence of skew shapes with the strongly stable shape* ω/π, *i.e.* $\lambda^{(n)} \mapsto \omega$ *and* $\mu^{(n)} \mapsto \pi$. *Then*

$$-\big(1 + c(\omega/\pi)\big)n + o(n) \;\le\; \log e(v_n) - \frac{1}{2}n \log n \;\le\; -\big(1 + c(\omega/\pi)\big)n + \log\xi(v_n) + o(n),$$

as $n \to \infty$, *where*

$$c(\omega/\pi) = \iint_{\mathcal{C}} \log \hbar(x, y)\, dx\, dy.$$

Note that by Lemma 4.3, we always have $\log\xi(v_n) \le (\log 2)n$.

Proof. Theorem 3.3 gives:

$$\log F(v_n) \;\le\; \log e(v_n) \;\le\; \log F(v_n) + \log\xi(v_n).$$

Now, observe that $|v_n| = n + O(\sqrt{n})$ as $n \to \infty$. Using the Stirling formula, the definition and compactness of the stable shape $\mathcal{C} \subset [a \times b]$,[3] we have:

$$\log F(v_n) = |v_n|! - \sum_{u \in v_n} \log h(u) = n\log n - n + O(\sqrt{n}\log n) -$$

$$- n \cdot \iint_{\mathcal{C}} \log\big(\sqrt{n}\,\hbar(x, y)\big)\, dx\, dy + o(n) = \frac{1}{2}n\log n - n - c(\omega/\pi)n + o(n),$$

where the $o(n)$ error term comes from approximation of the sum with the scaled integral. This implies both parts of the theorem. □

Proof of Theorem 6.1. The proof is similar, but the error terms are different. First, by uniform convergence, we have $|v_n| = n + o(n)$, which only implies

$$|v_n|! = \frac{1}{2}n\log n + o(n\log n) \quad \text{as} \quad n \to \infty.$$

The same error term $o(n\log n)$ also appears from the scaled integral bound. The details are straightforward. □

7. THE SUBPOLYNOMIAL DEPTH SHAPE

Let $g(n) : \mathbb{N} \to \mathbb{N}$ be an integer function which satisfies $1 \le g(n) \le n$, $g(n) \to \infty$ and $\log g(n) = o(\log n)$ as $n \to \infty$. The last condition is equivalent to $g(n) = n^{o(1)}$, hence the name. Function $g(n)$ is said to have *subpolynomial growth* (cf. [GP]). Examples include $e^{\sqrt{\log n}}$, $(\log n)^\pi$, $n^{1/\log\log n}$, etc.

For a skew shape λ/μ, define $\mathrm{width}(\lambda/\mu) := \min\{\lambda_1, \lambda_1'\}$, and $\mathrm{depth}(\lambda/\mu) := \max_{u \in \lambda/\mu} h(u)$. For example, for thick ribbons $v_k = \delta_{3k}/\delta_{2k}$, we have $\mathrm{width}(v_k) = 3k - 1$ and $\mathrm{depth}(v_k) = 2k - 1$.

We say that a sequence of skew partitions $\{v_n = \lambda^{(n)}/\mu^{(n)}\}$ has *subpolynomial depth shape* if

$$(\circ) \qquad \mathrm{width}(v_n) = \Theta\left(\frac{n}{g(n)}\right) \quad \text{and} \quad \mathrm{depth}(v_n) = \Theta\big(g(n)\big),$$

where $g(n)$ is a subpolynomial growth function.

---

[3]In [FeS], such shapes are called *balanced*.

Theorem 7.1. *Let $\{\nu_n = \lambda^{(n)}/\mu^{(n)}\}$ be a sequence of skew partitions with a subpolynomial depth shape associated with the function $g(n)$. Then*

$$\log e(\nu_n) = n \log n - \Theta\big(n \log g(n)\big) \quad as \quad n \to \infty.$$

Proof. First, by Lemma 4.3, we have $\xi(\nu_n) \leq 2^n$. Thus, by Theorem 3.3, we have:

$$\log e(\nu_n) = \log F(\nu_n) + O(n) = n \log n + O(n) - \sum_{u \in \nu_n} \log h(u).$$

First, observe:

$$\sum_{u \in \nu_n} \log h(u) \leq n \log\big[\mathrm{depth}(\nu_n)\big] = O\big(n \log g(n)\big).$$

In the other direction, suppose for simplicity that $\mathrm{width}(\nu_n) = \lambda_1$. By $(\circ)$ and total area count, we have: of the $\mathrm{width}(\nu_n)$ columns, at least α proportion of them have length $> \beta\, \mathrm{depth}(\nu_n)$, for some $\alpha, \beta > 0$. Here the constants α, β depend only on the constants implied by the $\Theta(\cdot)$ notation and independent on n. We conclude:

$$\sum_{u \in \nu_n} \log h(u) \geq \alpha\, \mathrm{width}(\nu_n) \cdot \log\left[\beta\, \mathrm{depth}(\nu_n)\right]!$$

$$\geq \alpha\Theta\left(\frac{n}{g(n)}\right) \cdot \beta\Theta\big(g(n)\big) \log\big[\beta\Theta\big(g(n)\big)\big] = \Omega\big(n \log g(n)\big),$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

8. Thick ribbons

8.1. **Proof of Theorem 1.1.** As in the introduction, let $\delta_k = (k-1, k-2, \ldots, 2, 1)$ and $\upsilon_k = \delta_{2k}/\delta_k$ (see [MPP2]). Note that thick ribbons υ_k have strongly stable trapezoid shape (see Figure 4). We have $n = |\upsilon_k| = 3k(k+1)/2$. For simplicity, we assume that k is even (for odd k the formulas are slightly different). In the notation of §2.1, we have:

$$F(\upsilon_k) = \frac{n!}{(2k-1)!!^k\, \Lambda(k-1)}\,.$$

This gives the lower bound in Theorem 1.1 :

$$\log e(\upsilon_k) \geq \log F(\upsilon_k) = \frac{1}{2}n \log n + \left(\frac{1}{6} - \frac{3\log 2}{2} + \frac{\log 3}{2}\right)n + o(n)$$

$$\geq \frac{1}{2}n \log n - 0.3237n + o(n).$$

For the upper bound, we obtain a closed formula for the number of excited diagrams.

Proposition 8.1. *Let $\upsilon_k = (\delta_{2k}/\delta_k)$ be a thick ribbon, and let k be even. We have:*

$$\xi(\upsilon_k) = \prod_{1 \leq i < j \leq k} \frac{k+i+j-1}{i+j-1} = \left[\frac{\Phi(3k-1)\,\Phi(k-1)^3\,(2k-1)!!\,(k-1)!!}{\Phi(2k-1)^3\,(3k-1)!!}\right]^{1/2}.$$

Proof. The product in the RHS is the *Proctor's formula* for the number of (reverse) plane partitions of shape δ_k with entries $\leq k/2$ [Pro, Cases CG]. Let us show that the number also counts excited diagrams of the shape δ_{2k}/δ_k. Indeed, by the proof of Theorem 3.1 in [MPP1], the excited diagrams in this case are in bijection with flagged tableaux of shape δ_k with entries in row i at most $\vartheta_i = k/2 + i$. By subtracting i from the entries in row i of such tableaux we obtain reverse plane partitions of shape δ_k with entries at most $k/2$, as desired. $\qquad\qquad\qquad\qquad\qquad\square$

Remark 8.2. Proctor's formula also counts the number of plane partitions inside a $[k \times k \times k]$ cube whose (matrix) transpose is the same as its complement (see also [K2, Class 6], [S1, Case 6] and [OEIS, A181119]). As shown in [S1], this is because the anti-diagonal consists of the values $k/2$, and the plane partition below of the diagonal is of shape δ_k with entries at most $k/2$. Clearly, this determines the rest of the partition.

From the proposition, we obtain:

$$\log \xi(\upsilon_k) \sim \left(\frac{3 \log 3}{2} - 2 \log 2 \right) n + o(n) = 0.2616n + o(n),$$

and therefore

$$\log e(\upsilon_k) \leq \log F(\upsilon_k) + \log \xi(\upsilon_k) = \frac{1}{2} n \log n + \left(\frac{1}{6} - \frac{7 \log 2}{2} + 2 \log 3 \right) n + o(n)$$

$$\leq \frac{1}{2} n \log n - 0.0621n + o(n).$$

This proves the upper bound in Theorem 1.1.

8.2. **Comparison with general bounds.** Denote by $\mathcal{P}_k$ the poset corresponding to υ_k. We have $M(\mathcal{P}_k) = 2k - 1$, $L(\mathcal{P}_k) = k$, and $r_1 = k, \ldots, r_k = 2k - 1$. Consider partition of the poset $\mathcal{P}_k$ corresponding to υ_k into chains of lengths $1, 2, \ldots, k - 1$ and k chains of length k. Now Theorem 2.1 gives

$$r_1! \cdots r_k! = \frac{\Phi(2k - 1)}{\Phi(k - 1)} \leq e(\upsilon_k) \leq \binom{n}{1, 2, \ldots, k - 1, k, k, \ldots, k} = \frac{n!}{(k!)^{k-1} \Phi(k)}.$$

In other words,

$$\log e(\upsilon_k) \geq \log \frac{\Phi(2k - 1)}{\Phi(k - 1)} = \frac{1}{2} n \log n + \left(\frac{11 \log 2}{6} - \frac{\log 3}{2} - \frac{3}{2} \right) n + o(n)$$

$$\geq \frac{1}{2} n \log n - 0.7785n + o(n), \quad \text{and}$$

$$\log e(\upsilon_k) \leq \log \frac{n!}{(k!)^{k-1} \Phi(k)} = \frac{1}{2} n \log n + \left(\frac{1}{6} - \frac{\log 2}{2} + \frac{\log 3}{2} \right) n + o(n)$$

$$\leq \frac{1}{2} n \log n + 0.3694n + o(n).$$

Now, denote by c the constant in the asymptotics

$$\log e(\upsilon_k) = \frac{1}{2} n \log n + cn + o(n).$$

Our bounds imply $c \in [-0.3237, -0.0621]$. This is much stronger than the bounds $c \in [-0.7785, 0.3694]$ which follows from Theorem 2.1.

Finally, using the HLF for $e(\delta_k)$ (see [OEIS, A005118]), Proposition 2.4 gives:

$$\log e(\upsilon_k) \leq \log \frac{|\delta_{2k}|! \, e(\delta_k)}{|\delta_k|! \, e(\delta_{2k})} = \frac{1}{2} n \log n + \left(\frac{3}{2} - \frac{1}{2} \log 2 \right) n + o(n)$$

$$\leq \frac{1}{2} n \log n + 1.1534n + o(n).$$

Note that this bound has correct first asymptotic term once again, but is weak in the $O(n)$ term.

8.3. **Thick ribbons of subpolynomials depth.** Consider now thick ribbons $v_k = \delta_{k+g(k)}/\delta_k$, where $g(k)$ is a subpolynomial growth function. Note that $n = |v_k| = kg(k) + O(g(k)^2)$.

Theorem 8.3. *Let $g(k)$ be a subpolynomial growth function, and let $\{v_k\}$ be thick strips defined as above. Then*

$$-\log 2 + o(1) \leq \frac{1}{n}\Big(\log e(v_k) - n\log n + n\log g(k)\Big) \leq 0 + o(1) \quad as \ \ k \to \infty,$$

where $n = kg(k)$.

Proof. By a direct computation:

$$\log F(v_k) = n\log n - \log\big((2g(k) - 1)!!\big)^k - O\big(g(k)^2 \log g(k)\big)$$
$$= n\log n - n\log g(k) - n\log 2 - O\big(g(k)^2 \log g(k)\big).$$

The result follows from Theorem 3.3 and Lemma 4.3.   □

Remark 8.4. We conjecture that the lower bound is tight in the theorem. This is supported by the zigzag ribbon shapes calculations (see below). Note also that Proctor's formula applies to all thick ribbons δ_{k+2r}/δ_k and gives $\xi(v_k) = 2^{n(1+o(1))}$ in this case.

The lower bound in Theorem 2.1 gives the lower bound of (-1) as in the theorem. Curiously, the upper bound

$$e(v_k) \leq \frac{n!}{\big(g(k)!\big)^{k-1}\Phi\big(g(k)\big)}$$

given by Theorem 2.1 matches the upper bound in Theorem 8.3.

9. Two dual shapes

9.1. **Inverted hooks.** Consider the *inverted hook shape* $v_k = (k+1)^{k+1}/k^k$, (cf. §8.3 and [MPP1, §3.1]). We have $n = |v_k| = 2k - 1$ in this case. Straight from the definition, we have:

$$e(v_k) = \binom{2k}{k}, \qquad F(v_k) = \frac{(2k+1)!}{(k+1)!^2} = \frac{2k+1}{(k+1)^2}\binom{2k}{k}.$$

Since $\binom{2k}{k} \sim c4^k/\sqrt{k}$, the lower bound in Theorem 3.3 is off only by a $O(k)$ factor. The lower bound in Theorem 2.3 clearly coincides with this bound for a poset $\mathcal{P}$ corresponding to v_k and is the exact for the dual poset $\mathcal{P}^*$. At the same time, the lower bound in Theorem 2.1 is 2^k, which is off by an exponential factor.

For the upper bound, observe that the excited diagrams are exactly the complements of the paths γ from the SW to the NE corner of the $(k+1)$-square $\lambda^{(k)}$. This gives $\xi(v_k) = \binom{2k}{k}$, and the upper bound in Theorem 3.3 gives only $O(16^k/k^2)$ (cf. §9.2). On the other hand, the upper bound in Theorem 2.1 is $\binom{2k-1}{k}$, which is off only by a factor of 2.

9.2. **Inverted thick hooks.** Let $\varkappa_k = (2k)^{2k}/k^k$. We call v_k *inverted thick hooks*; they have strongly stable shape (see Figure 4). The following is the analogue of Proposition 8.1.

Proposition 9.1. *For $\varkappa_k = (2k)^{2k}/k^k$, we have:*

$$\xi\big(\varkappa_k\big) = \prod_{i=1}^{k}\prod_{j=1}^{k}\frac{k+i+j-1}{i+j-1} = \frac{\Phi(k-1)^3\,\Phi(3k-1)}{\Phi(2k-1)^3} \sim C\left(\frac{3\sqrt{3}}{4}\right)^n n^{-1/24},$$

where $n = |\varkappa_k| = 3k^2$ and $C > 0$ is a constant.

As in the proof of Proposition 8.1, the exited diagrams in this case are in bijection with *solid partitions* which fit inside a $k \times k \times k$ box, see [MPP2]. The classical *MacMahon's formula* (see e.g. [S3] and [OEIS, A008793]), gives the proposition. We omit the details.

Now, by the calculation similar to the one in the previous section, Theorem 6.3 gives the following lower and upper bounds:

$$\log e(\varkappa_k) \geq \log F(\varkappa_k) = \frac{1}{2} n \log n - n - c_1 \frac{n}{3} - 2 c_2 \frac{n}{3} + o(n)$$

$$\geq \frac{1}{2} n \log n - 1.4095 n + o(n),$$

$$\log e(\varkappa_k) \leq \log F(\varkappa_k) + \log \xi(\varkappa_k) = \log F(\varkappa_k) + n \log \frac{3\sqrt{3}}{4} + o(n)$$

$$\leq \frac{1}{2} n \log n - 1.1479 n + o(n).$$

Here

$$c_1 = \iint_{\mathcal{C}_0} \log(x + y) \, dx \, dy = 2 \log 2 - \frac{3}{4} \approx -0.1137,$$

$$c_2 = \iint_{\mathcal{C}_0} \log(1 + x + y) \, dx \, dy = \frac{9}{2} \log 3 - 4 \log 2 - \frac{3}{2} \approx 0.6712,$$

and $\mathcal{C}_0 = [0,1]^2$ is the unit square. On the other hand,

$$\log e(\varkappa_k) = \frac{1}{2} n \log n - n - 2 c_1 \frac{n}{3} - c_3 \frac{n}{3} + o(n) = \frac{1}{2} n \log n - 1.2873 n + o(n),$$

where

$$c_3 = \iint_{\mathcal{C}_0} \log(2 + x + y) \, dx \, dy = 18 \log 2 - 9 \log 3 - \frac{3}{2} \approx 1.0891.$$

In other words, the correct asymptotics in this case is near the middle between the upper and lower bounds in Theorem 6.3.

Similarly, Theorem 2.1 implies the following upper and lower bounds:

$$e(\varkappa_k) \geq 2! 4! \cdots (2k)! (2k-1)! \cdots 2! 1! = \frac{\Phi(2k) \Phi(2k-1)}{\Psi(k)},$$

$$e(\varkappa_k) \leq \binom{n}{3k-1, 3k-2, 3k-4, 3k-5, \ldots, 5, 4, 2, 1} = \frac{n! \cdot 3! 6! \cdots (3k)!}{\Phi(3k)},$$

where the upper bound follows from the chain decomposition as in Figure 5. Using the asymptotics in §2.1 and [OEIS, A268504], this gives:

$$\log e(\varkappa_k) \geq \frac{1}{2} n \log n + \left(-\frac{3}{2} - \frac{\log 3}{2} + \log 2 \right) n + o(n) = \frac{1}{2} n \log n - 1.3562 n + o(n),$$

$$\log e(\varkappa_k) \leq \frac{1}{2} n \log n + \left(\frac{1}{2} - \frac{\log 3}{2} \right) n + o(n) = \frac{1}{2} n \log n - 0.0493 n + o(n).$$

In summary, for the constant $c = -1.2873$ in second term of the asymptotics of $e(\varkappa_k)$, Theorem 6.3 gives $c \in [-1.4095, -1.1479]$. Similarly, Theorem 2.1 gives $c \in [-1.3562, -0.0493]$ which is much wider, but has a slightly better lower bound. This is not a surprise: the further skew shape is from the regular shape, the larger is the number $\xi(\varkappa_k)$ of excited diagrams, and thus the worse are the bounds given by Theorem 3.3 (see Example 9.1 for an extreme special case).

Note that if we use a weaker bound $\log \xi(\varkappa_k) \leq (\log 2) n$ given by Lemma 4.3, we get $c \leq -0.7164$, which is still sharper than the bound $c \leq -0.0493$ from Theorem 2.1 (cf. Remark 8.4).

Remark 9.2. There is a strong connection between the cases of thick hooks and thick ribbons, since excited diagrams in both cases enumerate certain families of solid partitions in a box (equivalently, lozenge tilings of a regular hexagon). We refer to [S1] for an early overview and to [K2] for a recent

survey. Now, it may seem puzzling that $\log \xi(\varkappa_k) \sim 2 \log \xi(v_k)$ in these examples. In one direction this is straightforward — simply observe that $\xi(\varkappa_k) \geq \xi(v_k)\xi(v_{k+1})$ (see Figure 5).
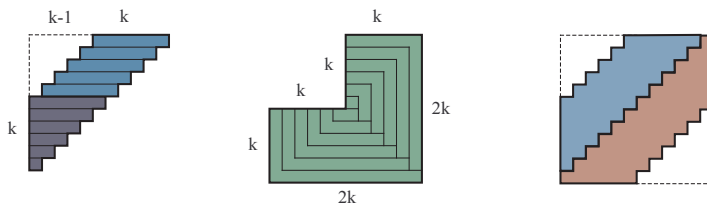

FIGURE 5. Chain partitions of the thick ribbon shape v_k and inverted thick hook $\varkappa_k$, for $k = 6$. Two thick ribbons forming an excited diagram of the inverted thick hook.

10. Ribbon shapes

10.1. **Zigzag shapes.** Let $\mathsf{Alt}(n) = \{\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \ldots\} \subset S_n$ be the set of *alternating permutations*. The number $E_n = |\mathsf{Alt}(n)|$ is the n-th *Euler number* (see [S5] and [OEIS, A000111]), with the asymptotics

$$E_n \sim n! \left(\frac{2}{\pi}\right)^n \frac{4}{\pi} \left(1 + o(1)\right) \quad \text{as} \ \ n \to \infty$$

(see e.g. [FlS, S5]). Consider the *zigzag ribbon hook* $\rho_k = \delta_{k+2}/\delta_k$, $n = |\rho_k| = 2k + 1$ (see Figure 6). Clearly, $e(\rho_k) = E_{2k+1}$. Observe that $\xi(\rho_k) = C_{k+1}$, the $(k+1)$-st Catalan number

$$C_m = \frac{1}{m+1} \binom{2m}{m} \sim 4^m m^{-3/2} \pi^{-1/2}$$

(see [MPP2]). Thus $\xi(\rho_k) = \Theta\left(2^n/n^{3/2}\right)$, so Lemma 4.3 is asymptotically tight for the staircase shape. Also, $d(\rho_k) \sim k/2$, so Lemma 4.4 is weaker than Lemma 4.3 in this case.

Writing $\xi(\rho_k) \sim cn!\eta^n n^\alpha$, let us compare the estimates from different bounds. We have $F(\rho_k) = n!/3^k$, so Theorem 3.3 for ρ_k gives

$$\frac{n!}{3^k} \leq E_n \leq \frac{n! \cdot C_k}{3^k},$$

implying that $1/\sqrt{3} \leq \eta \leq 2/\sqrt{3}$. Note that $1/\sqrt{3} \approx 0.577$, $2/\pi \approx 0.636$ and $2/\sqrt{3} \approx 1.155$. This example shows that the lower bound in Theorem 3.3 is nearly tight, while the upper bound is vacuous for large n (we always have $e(\mathcal{P}) \leq n!$, of course).

Similarly, Theorem 2.1 for ρ_k gives

$$k!(k+1)! \leq E_n \leq \frac{n!}{2^k},$$

implying $0.5 \leq \eta \leq 1/\sqrt{2} \approx 0.707$. Thus, in this case the general poset lower bound is weaker, while the upper bound is much sharper than $(*)$ in Theorem 3.3.

Remark 10.1. The asymptotics of $e(\lambda/\mu)$ is known for other ribbons λ/μ with a periodic pattern (see Figure 6). For example, for *two up, two right ribbons* ρ_n we have $e(\rho_n) = \Theta(n!\alpha^n)$, where $\alpha \approx 0.533$ is the smallest positive root of $\left(\cos \frac{1}{\alpha}\right)\left(\cosh \frac{1}{\alpha}\right) = -1$, see [CS] and [OEIS, A131454]. It would be interesting to find the asymptotics for more general ribbons, e.g. ribbons along a curve as in the figure.
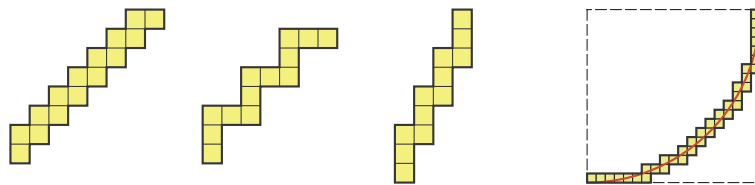
FIGURE 6. Zigzag ribbon $\rho_7 = \delta_9/\delta_7$, "two up two right" ribbon, ribbon $\rho(4,3)$, and a ribbon along the circle segment.

10.2. **Ribbons with subpolynomial depth.** Let $n = km$, and let $\rho(k,m)$ be the unique ribbon hook where all columns have length m (see Figure 6). We have $e\big(\rho(k,m)\big)$ is the number of "$(m-1)$ up, 1 down" permutations in S_n.

For fixed m, both the g.f. and the asymptotics for $a(k,m)$ are well understood (see e.g. [GJ, S5]). There is also a formula for $\xi(\rho_n)$ in this case in terms of the *Fuss–Catalan numbers*:

$$\xi(\rho_n) = \frac{1}{n-k+1}\binom{n}{k} = \frac{1}{(m-1)k+1}\binom{mk}{k}$$

(see e.g. [GJ, S6]).

Consider the case $m = g(k)$, where $g(k)$ is a subpolynomial function. The ribbon $\rho(k,m)$ has a subpolynomial depth, so Theorem 7.1 applies:

$$\log e\big(\rho(k,g(k))\big) = n\log n - \Theta\big(n\log g(k)\big).$$

The following result gives a sharper estimate (cf. §8.3).

Theorem 10.2. *Let $n = kg(k)$, where $g(k) \to \infty$, $g(k) = k^{o(1)}$ as $k \to \infty$. Then:*

$$\log e\big(\rho(k,g(k))\big) = n\log n - n\log g(k) - \frac{n\log g(k)}{2g(k)} - \frac{n}{g(k)} + O\left(\frac{n}{g(k)^2}\right) \quad as \quad n \to \infty.$$

Proof. Denote $a(k,m) = e\big(\rho(k,m)\big)$. By the lower bound in Theorem 3.3, we have:

$$a\big(k,g(k)\big) \geq F(\tau_n) \sim \frac{n!}{(m-1)!^k(m+1)^k} \quad as \quad k \to \infty,$$

where $m = g(k)$. On the other hand, by the upper bound in Theorem 2.1, we have:

$$a\big(k,g(k)\big) \leq \binom{n}{m,m,\ldots,m} = \frac{n!}{m!^k}.$$

We conclude:

$$\log a\big(k,g(k)\big) = \log F(\tau_n) + kO\big(\log(1+1/m)\big) = n\log n - k\log m! + O(k/m)$$

$$= n\log n - n - O(\log n) - k\left[m\log m - m + \frac{1}{2}\log m + 1 + O\left(\frac{1}{m}\right)\right] + O(k/m)$$

$$= n\log n - n\log g(k) - \frac{n\log g(k)}{2g(k)} - \frac{n}{g(k)} + O\left(\frac{n}{g(k)^2}\right),$$

as desired. $\square$

11. Slim shapes

11.1. Number of excited diagrams. Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be such that $\lambda_\ell \geq \mu_1 + \ell$. Such partitions are called *slim*. The following result improve the bound in Lemma 4.4 in this case.

Proposition 11.1. *In the notation above, let $\mu \vdash m$ be fixed and let $\{\lambda^{(k)}\}$ be a family of slim partitions, such that $\ell(k) := \ell\big(\lambda_1^{(k)}\big) \to \infty$ as $k \to \infty$. We have:*

$$\xi\big(\lambda^{(k)}/\mu\big) \sim \frac{\ell(k)^m}{\prod_{x \in \mu} h(x)} \quad as \quad k \to \infty .$$

Note that there no direct dependence on $n = \big|\lambda^{(k)}/\mu\big|$ in this case.

Proof. Consider random position of squares of μ on the corresponding diagonals inside λ, which all have lengths between $\ell - O(1)$ and ℓ. For points sampled from the unit interval, the probability the order relations are satisfied is exactly

$$\frac{|\operatorname{SYT}(\mu)|}{m!} = \prod_{x \in \mu} \frac{1}{h(x)} .$$

This is equivalent to calculation of the volume of the *poset polytope* (see e.g. [BR, S3]). From above, the number of such diagrams is $\big(\ell(k) - O(1)\big)^m$, which implies the result. $\square$

11.2. Complementary shapes in large slim rectangles. The case of a fixed μ and λ a rectangle is especially interesting (cf. §12.1 below).

Theorem 11.2 ([RV]). *Let μ be fixed and $\lambda = (k^\ell)$ with $\ell, k/\ell \to \infty$. We have:*

$$\frac{e(\lambda/\mu)}{e(\lambda)} \sim \prod_{x \in \mu} \frac{1}{h(x)} , \quad as \ \ell, k/\ell \to \infty.$$

This theorem is obtained by Regev and Vershik in [RV] by a calculation based on [OO]. Let us show that the upper bound in the theorem follows from the upper bound in Theorem 3.3.

Let $m = |\mu|$. We have in this case:

$$\frac{F(\lambda/\mu)}{e(\lambda)} = \frac{(k\ell - m)!}{(k\ell)!} \left[\prod_{x \in \mu} h(x) \right],$$

where $h(x)$ is a hook-length in λ. Letting $k/\ell \to \infty$, we get

$$\frac{F(\lambda/\mu)}{e(\lambda)} \sim \frac{(k\ell - m)! \, k^m}{(k\ell)!} \sim \frac{1}{\ell^m} .$$

Now the upper bound in Theorem 3.3 and Proposition 11.1 give:

$$\frac{e(\lambda/\mu)}{e(\lambda)} \lesssim \frac{\xi(\lambda/\mu) \, F(\lambda/\mu)}{e(\lambda)} \sim \prod_{x \in \mu} \frac{1}{h(x)} , \quad as \ \ell, k/\ell \to \infty.$$

This implies that the upper bound in Theorem 3.3 is asymptotically tight in this case, while the lower bound is off by a $\Theta\big(\ell(\lambda)^{|\mu|}\big)$ factor.

12. FURTHER EXAMPLES AND APPLICATIONS

12.1. **Dual shapes.** Let $\lambda = (k^\ell)$ be a rectangle and $\mu \subset \lambda$. Denote by $\nu = (\lambda/\mu)^*$ the partition obtained by the 180 degree rotation of λ/μ. Thus we can apply the lower bound in Theorem 3.3 to obtain the following unusual result.

Proposition 12.1. *Let ν be a partition of n. Denote by $h^*(x)$, $x = (i,j) \in \nu$, the* dual hooks *defined by $h^*(i,j) = i + j - 1$, and let*

$$H(\nu) = \prod_{x \in \nu} h(x), \qquad H^*(\nu) = \prod_{x \in \nu} h^*(x).$$

Then $H(\nu) \leq H^*(\nu)$, and the equality holds only when ν is a rectangle.

Note that this result cannot be proved by a simple monotonicity argument, since

$$\sum_{x \in \nu} h(x) = \sum_{x \in \nu} h^*(x) = n + \sum_i \binom{\nu_i}{2} + \sum_j \binom{\nu'_j}{2}.$$

Proof. Observe that $e(\lambda/\mu) = e(\nu) = n!/H(\nu)$ and $F(\lambda/\mu) = n!/H^*(\nu)$, where $n = |\lambda/\mu| = |\nu|$. Now the inequality $e(\lambda/\mu) \geq F(\lambda/\mu)$ implies the first part. For the second part, recall from the proof of Theorem 3.3 that $e(\lambda/\mu) = F(\lambda/\mu)$ only if $\xi(\lambda/\mu) = 1$, i.e. when no excited moves are allowed. It is easy to see that this can happen only when ν is a rectangle. $\square$

Remark 12.2. After the proposition was obtained, in response to second author's `MathOverflow` question, F. Petrov found a generalization of this result to all concave functions.[4] In particular, his proof implies that the variance of hooks is larger than that of complementary hooks in all Young diagrams.

12.2. **Regev–Vershik shapes.** Let $\sigma \subset \tau$, where $\tau = (\ell^k)$ is a rectangle. As above, denote by σ^* the 180 degree rotation of σ. Consider a skew shape λ/μ obtained by attaching two copies of σ^* above and to the left of τ, and removing σ^* from τ (see Figure 7). The theorem by Regev and Vershik in [RV] states that

$$(\diamond) \qquad \prod_{x \in \lambda/\mu} h(x) = \left[\prod_{x \in \mu} h(x) \right] \cdot \left[\prod_{x \in \tau} h(x) \right].$$

Proposition 12.3. *In the notation above, let $s := |\sigma|$, $t := |\tau| = k\ell$. Then:*

$$e(\lambda/\mu) \geq \binom{s+t}{s} e(\sigma) e(\tau).$$

The inequality is trivially tight for $\sigma = \varnothing$ or $\sigma = \tau$, when the skew shapes coincide: $\lambda/\mu = \sigma \circ \tau$.

Proof. Note that $|\lambda/\mu| = |\sigma| + |\tau| = s + t$. By the Regev–Vershik theorem $(\diamond)$, we have

$$\frac{F(\lambda/\mu)}{(s+t)!} = \frac{e(\sigma)}{s!} \cdot \frac{e(\tau)}{t!}.$$

By the lower bound in Theorem 3.3, we have:

$$e(\lambda/\mu) \geq F(\lambda/\mu) = \binom{s+t}{t} e(\sigma) e(\tau),$$

as desired. $\square$

Remark 12.4. The Regev–Vershik theorem has been reproved bijectively and generalized in a number of directions by Janson, Regev and Zeilberger, Bessenrodt, Krattenthaler, Goulden and Yong, and others. We refer to [K1] for the "master bijection" and to [AR, §12] for further references.
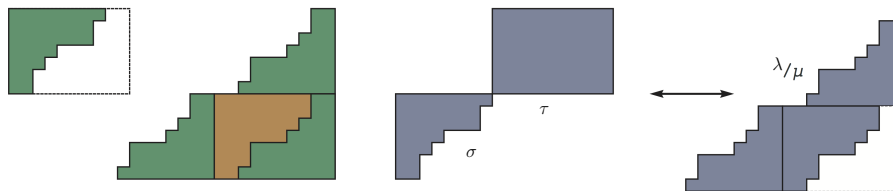
---

[4]See http://mathoverflow.net/q/243846.

FIGURE 7. Construction of the Regev–Vershik shape with $\sigma = (87^2432^2)$, $\tau = (10^7)$. Now ($\diamond$) says that two skew shapes colored blue have the same multiset of hooks.

13. FINAL REMARKS AND OPEN PROBLEMS

13.1.   Computing $e(\mathcal{P})$ is known to be #P-complete [BW], which partly explains relatively few good general bounds (see [ERZ, Tro]). Note, however, a nice counterpart of the upper bound in Theorem 2.1, relating $\log e(\mathcal{P})$ and the entropy, defined in terms of the $M(\mathcal{P})$, see [C+, KK]. Unfortunately this bound is not sharp enough for bounds on $\log e(\lambda/\mu)$ as it is off by a multiplicative constant.

Note that this approach was used to improve the bounds on the second term in the asymptotic expansion of $e(B_k)$, the number linear extensions of the Boolean lattice [BT] (see also [SK, Bri]. Namely, for even k, Theorem 2.1 gives:

$$\log_2 \binom{k}{k/2} + \frac{3}{2}\log_2 e + o(1) \leq \frac{\log_2 e(B_k)}{2^k} \leq \log_2 \binom{k}{k/2}.$$

Brightwell and Tetali in [BT] show that the lower bound is tight.

13.2.   We show that in many special cases of interest our bounds give the first or the first two terms in the asymptotic expansion. Although the number $\xi(\lambda/\mu)$ of excited diagrams can be large, even exponential in many cases, this is dwarfed by the number of standard Young tableaux, implying that the "naive HLF" $F(\lambda/\mu)$ is indeed a good estimate in these cases.

13.3.   The arXiv version of [MPP1] contains an extended survey and comparison between NHLF and other known formulas for $f^{\lambda/\mu}$. Of course, the Jacobi–Trudy and other determinant formulas (see e.g. [HG, LP, S3]) are computationally efficient, but difficult to use for asymptotic estimates. While the Littlewood–Richardson rule is positive, the LR–coefficients have a complicated combinatorial structure and are hard to compute [N1] and estimate (cf. Proposition 2.4 and §13.5). Finally, the *Okounkov–Olshanski formula* [OO] mentioned earlier seems to be weak for large μ (cf. [MPP3].

13.4.   The notion of stable limit shape goes back to Erdős and Szekeres in the context of random partitions, and to Vershik and Kerov in the context of Young tableaux and asymptotic representation theory (see e.g. [Bia, Rom]). Our notion of strong stable shapes is more restrictive as need faster convergence for the proof of Theorem 6.3.

13.5.   Most recently, Stanley announced in [S3, Supp. Exc. 7.79] a curious and asymptotically sharp upper bound $\left(c_{\mu\nu}^\lambda\right)^2 \leq p_2(N)$, for all $|\lambda| = |\mu| + |\nu| = N$, where $p_2(N)$ is the number of *bicolored partitions* of N (see e.g. [OEIS, A070933]).

Proposition 13.1. *For all $|\lambda/\mu| = n$ and $\lambda \vdash N$, we have:*

$$\log f^{\lambda/\mu} \leq \frac{1}{2}\Big(n\log n - n + (\log 2)N\Big) + O(\sqrt{N}), \quad as \quad N \to \infty.$$

Proof. Recall that

$$p_2(N) = \sum_{\lambda \vdash N} 2^{\ell(\lambda)} = O(2^N).$$

Using Stanley's bound and $f^\nu \le \sqrt{n!}$, we have:

$$f^{\lambda/\mu} = \sum_{\nu \vdash n} c^\lambda_{\mu\nu} f^\nu \le p(n) \sqrt{p_2(N)} \sqrt{n!},$$

which implies the result. □

This bound is quite weak for $n = |\lambda/\mu| = o(N/\log N)$, but gives good bounds in the stable shape case. We plan to explore this approach in our future work.

13.6. There is a shifted version of the NHLF also obtained by Naruse [N2], which should give similar asymptotic results for the number of SYT of shifted shapes (cf. [AR, §5.3]). In fact, the results should follow verbatim for the stable shape limit case. Let us also mention that we can do similar calculations for thick ribbons (see Section 8) can be translated to this case; the excited diagrams in the shifted case correspond to *type B* and have been extensively studied. We refer to [MPP2, MPP3] for details and further references.

13.7. Note that the shifted skew diagrams greatly increases the number of limit shapes of area 1, for which we have

$$\log e(\upsilon_n) \sim \frac{1}{2} n \log n \quad \text{as} \ \ n \to \infty.$$

As we mentioned in Remark 2.6, this holds also for some "truncated shapes" (see [AR, Pan]). We generalize this result in a forthcoming [Pak] to all piecewise linear shapes in the plane.

References

[AR]   R. Adin and Y. Roichman, Standard Young tableaux, in *Handbook of Enumerative Combinatorics* (M. Bóna, editor), CRC Press, Boca Raton, 2015, 895–974.

[BR]   Y. Baryshnikov and D. Romik, Enumeration formulas for Young tableaux in a diagonal strip, *Israel J. Math.* **178** (2010), 157–186.

[Bia]   P. Biane, Representations of symmetric groups and free probability, *Adv. Math.* **138** (1998), 126–181.

[Bri]   G. R. Brightwell, The number of linear extensions of ranked posets, LSE CDAM Res. Report **18** (2003), 6 pp.

[BT]   G. R. Brightwell and P. Tetali, The number of linear extensions of the Boolean lattice, *Order* **20** (2003), 333–345.

[BW]   G. R. Brightwell and P. Winkler, Counting linear extensions, *Order* **8** (1991), 225–242.

[C+]   J. Cardinal, S. Fiorini, G. Joret, R. M. Jungers and J. I. Munro, Sorting under partial information (without the ellipsoid algorithm), *Combinatorica* **33** (2013), 655–697.

[CS]   L. Carlitz and R. Scoville, Enumeration of rises and falls by position, *Discrete Math.* **5** (1973), 45–59.

[CGS]   S. Corteel, A. Goupil and G. Schaeffer, Content evaluation and class symmetric functions, *Adv. Math.* **188** (2004), 315–336.

[DVZ]   A. Dembo, A. Vershik and O. Zeitouni, Large deviations for integer partitions, *Markov Process. Related Fields* **6** (2000), 147–179.

[ERZ]   K. Ewacha, I. Rival and N. Zaguia, Approximating the number of linear extensions, *Theoret. Comput. Sci.* **175** (1997), 271–282.

[Feit] W. Feit, The degree formula for the skew-representations of the symmetric group, *Proc. AMS* **4** (1953), 740–744.

[FeS]   V. Féray and P. Śniady, Asymptotics of characters of symmetric groups related to Stanley character formula, *Ann. of Math.* **173** (2011), 887–906.

[FlS]   P. Flajolet and R. Sedgewick, *Analytic combinatorics*, Cambridge Univ. Press, Cambridge, 2009.

[FRT]   J. S. Frame, G. de B. Robinson and R. M. Thrall, The hook graphs of the symmetric group, *Canad. J. Math.* **6** (1954), 316–324.

[GV]   I. M. Gessel and X. G. Viennot, Determinants, paths and plane partitions, preprint (1989); available at tinyurl.com/zv3wvyh.

[GM]   A. Giambruno and S. Mishchenko, Degrees of irreducible characters of the symmetric group and exponential growth, *Proc. AMS* **144** (2016), 943–953.

[GJ]   I. P. Goulden and D. M. Jackson, *Combinatorial enumeration*, John Wiley, New York, 1983.

[GP]   R. Grigorchuk and I. Pak, Groups of Intermediate Growth: an Introduction for Beginners, *Ens. Math.* **54** (2008), 251–272.

[HG]   A. M. Hamel and I. P. Goulden, Planar decompositions of tableaux and Schur function determinants, *Europ. J. Combin.* **16** (1995), 461–477.

[HP]   A. Hammett and B. Pittel, How often are two permutations comparable? *Trans. AMS* **360** (2008), 4541–4568.

[KK]   J. Kahn and J. H. Kim, Entropy and sorting, *J. Comput. Syst. Sci.* **51** (1995), 390–399.

[K1]   C. Krattenthaler, Bijections for hook pair identities, *Electron. J. Combin.* **7** (2000),3 RP 27, 13 pp.

[K2]   C. Krattenthaler, Plane partitions in the work of Richard Stanley and his school, `arXiv:1503.05934`.

[Kre]   V. Kreiman, Schubert classes in the equivariant K-theory and equivariant cohomology of the Grassmannian, `arXiv:math.AG/0512204`.

[LP]   A. Lascoux and P. Pragacz, Ribbon Schur functions, *Europ. J. Combin.* **9** (1988), 561–574.

[LasS] A. Lascoux and M.-P. Schützenberger, Polynômes de Schubert (in French), *C.R. Acad. Sci. Paris Sér. I Math.* **294** (1982), 447–450.

[MPP1] A. H. Morales, I. Pak and G. Panova, Hook formulas for skew shapes I. *q*-analogues and bijections; `arXiv:1512.08348`.

[MPP2] A. H. Morales, I. Pak and G. Panova, Hook formulas for skew shapes II. Combinatorial proofs and enumerative applications, `arXiv:1610.04744`.

[MPP3] A. H. Morales, I. Pak and G. Panova, Hook formulas for skew shapes III. Multivariate formulas from Schubert calculus, in preparation.

[N1]   H. Narayanan, Estimating deep Littlewood–Richardson coefficients, in *Proc. FPSAC 2014*, DMTCS, Nancy, 2014, 321–332

[N2]   H. Naruse, Schubert calculus and hook formula, talk slides at *73rd Sém. Lothar. Combin.*, Strobl, Austria, 2014; available at tinyurl.com/z6paqzu.

[OO]   A. Okounkov and G. Olshanski, Shifted Schur functions, *St. Petersburg Math. J.* **9** (1998), 239–300.

[Pak]   I. Pak, On the number of tableaux of large diagrams, in preparation.

[Pan]   G. Panova, Tableaux and plane partitions of truncated shapes, *Adv. Appl. Math.* **49** (2012), 196–217.

[Pro]   R. A. Proctor, New symmetric plane partition identities from invariant theory work of De Concini and Procesi, *Europ. J. Combin.* **11** (1990), 289–300.

[RV]   A. Regev and A. Vershik, Asymptotics of Young diagrams and hook numbers, *Electron. J. Combin.* **4** (1997), no. 1, RP 22, 12 pp.

[Rom]   D. Romik, *The surprising mathematics of longest increasing subsequences*, Cambridge Univ. Press, New York, 2015.

[Sag]   B. E. Sagan, *The Symmetric Group*, Springer, New York, 2001.

[SK]   J. Sha and D. J. Kleitman, The number of linear extensions of subset ordering, *Discrete Math.* **63** (1987), 271–278.

[OEIS] N. J. A. Sloane, The Online Encyclopedia of Integer Sequences, `oeis.org`.

[S1]   R. P. Stanley, Symmetries of plane partitions, *J. Combin. Theory, Ser. A* **43** (1986), 103–113.

[S2]   R. P. Stanley, On the enumeration of skew Young tableaux, *Adv. Appl. Math.* **30** (2003), 283–294.

[S3]   R. P. Stanley, *Enumerative Combinatorics*, vol. 1 (second ed.) and vol. 2, Cambridge Univ. Press, 2012 and 1999; supplementary excercies available at http://www-math.mit.edu/~rstan/ec.

[S4]   R. P. Stanley, Increasing and decreasing subsequences and their variants, in *Proc. ICM*, Vol. I, EMS, Zürich, 2007, 545–579.

[S5]   R. P. Stanley, A survey of alternating permutations, in *Combinatorics and graphs*, AMS, Providence, RI, 2010, 165–196.

[S6]   R. P. Stanley, *Catalan Numbers*, Cambridge Univ. Press, 2015.

[Tro]   W. T. Trotter, Partially ordered sets, in *Handbook of combinatorics*, Vol. 1, Elsevier, Amsterdam, 1995, 433–480.

[VK]   A. M. Vershik and S. V. Kerov, The asymptotic character theory of the symmetric group, *Funct. Anal. Appl.* **15** (1981), 246–255.

[Wac]   M. Wachs, Flagged Schur functions, Schubert polynomials, and symmetrizing operators, *J. Combin. Theory, Ser. A* **40** (1985), 276–289.